Approximation of the Euclidean ball by polytopes

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Abstract

There is a constant c such that for every $n \in \mathbb{N}$, there is a N_n so that for every $N \geq N_n$ there is a polytope P in $\mathbb{R}^n$ with N vertices and

$$\mathrm{vol}_n(B_2^n \triangle P) \leq c \ \mathrm{vol}_n(B_2^n) N^{-\frac{2}{n-1}}$$

where B_2^n denotes the Euclidean unit ball of dimension n.

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1 Main results

Let C and K be two convex bodies in $\mathbb{R}^n$. The Euclidean ball with center 0 and radius r is denoted by $B_2^n(r)$. The ball $B_2^n(1)$ is denoted by B_2^n. Let K be a convex body in $\mathbb{R}^n$ with C^2-boundary ∂K and everywhere strictly positive curvature κ. Then

$$\lim_{N\to\infty} \frac{\inf\{\mathrm{vol}_n(K\setminus P) | P \subseteq K \text{ and } P \text{ has at most N vertices}\}}{N^{-\frac{2}{n-1}}} = \frac{1}{2}\mathrm{del}_{n-1}\left(\int_{\partial K}\kappa(x)^{\frac{1}{n+1}}d\mu_{\partial K}(x)\right)^{\frac{n+1}{n-1}} \tag{1}$$

where $\mu_{\partial K}$ denotes the surface measure of ∂K. This theorem gives asymptotically the order of best approximation of a convex body K by polytopes contained in K with a fixed number of vertices. It was proved by McClure and Vitale [McV] in dimension 2 and by Gruber [Gr2] for general n. The constant del_{n-1} is positive and depends on the dimension n only. Its order of magnitude can be computed by considering the case $K = B_2^n$. This has been done in [GRS1] and [GRS2] by Gordon, Reisner and Schütt, namely there are numerical constants a and b such that

$$a\cdot n \le \mathrm{del}_{n-1} \le b\cdot n.$$

The constant del_{n-1} was determined more precisely by Mankiewicz and Schütt [MaS1], [MaS2]. It was shown there

$$\frac{n-1}{n+1}\mathrm{vol}_{n-1}(B_2^{n-1})^{-\frac{2}{n-1}} \le \mathrm{del}_{n-1} \le \left(1+\frac{c\ln n}{n}\right)\frac{n-1}{n+1}\mathrm{vol}_{n-1}(B_2^{n-1})^{-\frac{2}{n-1}}, \tag{2}$$

where c is a numerical constant. In particular,

$$\lim_{n\to\infty}\frac{\mathrm{del}_{n-1}}{n} = \frac{1}{2\pi e}.$$

What happens if we drop the condition that the polytopes have to be contained in the convex body and allow all polytopes having at most N vertices? How much better can we approximate the Euclidean ball?

In [Lud] it was shown that for all convex bodies K whose boundary is twice continuously differentiable and whose curvature is everywhere strictly positive

$$\lim_{N\to\infty} \frac{\inf\{\mathrm{vol}_n(K\triangle P) | P \text{ is a polytope with at most } N \text{ vertices}\}}{N^{-\frac{2}{n-1}}} = \frac{1}{2}\mathrm{ldel}_{n-1}\left(\int_{\partial K}\kappa(x)^{\frac{1}{n+1}}d\mu_{\partial K}(x)\right)^{\frac{n+1}{n-1}}.$$

The constant ldel_{n-1} is positive and depends only on n. Clearly, from the above mentioned results it follows that $\mathrm{ldel}_{n-1} \leq c \cdot n$. On the other hand, it has been shown in [Bö] that for a polytope P with at most N vertices

$$\mathrm{vol}_n(B_2^n \triangle P) \geq \frac{1}{67e^2\pi}\frac{1}{n}\mathrm{vol}_n(B_2^n)N^{-\frac{2}{n-1}}.$$

Thus between the upper and lower estimate for ldel_{n-1} there is a gap of order n^2. In this paper we narrow this gap by showing that $\mathrm{ldel}_{n-1} \leq c$ where c is a numerical constant.

Theorem 1 *There is a constant c such that for every $n \in \mathbb{N}$ there is a N_n so that for every $N \geq N_n$ there is a polytope P in $\mathbb{R}^n$ with N vertices such that*

$$\mathrm{vol}_n(B_2^n \triangle P) \leq c\ \mathrm{vol}_n(B_2^n)N^{-\frac{2}{n-1}}. \tag{3}$$

Gruber also showed [Gr2]

$$\lim_{N\to\infty} \frac{\inf\{\mathrm{vol}_n(K \triangle P) | K \subseteq P \text{ and } P \text{ is a polytope with at most } N \text{ facets}\}}{N^{-\frac{2}{n-1}}}$$
$$= \frac{1}{2}\mathrm{div}_{n-1}\left(\int_{\partial K} \kappa(x)^{\frac{1}{n+1}} d\mu_{\partial K}(x)\right)^{\frac{n+1}{n-1}}$$

where div_{n-1} is a positive constant that depends on n only. It is easy to show [Lud, MaS1] that there are numerical constants a and b such that $a \cdot n \leq \mathrm{div}_{n-1} \leq b \cdot n$.

Ludwig [Lud] showed that for general polytopes

$$\lim_{N\to\infty} \frac{\inf\{\mathrm{vol}_n(K \triangle P) |\ P \text{ is a polytope with at most } N \text{ facets}\}}{N^{-\frac{2}{n-1}}}$$
$$= \frac{1}{2}\mathrm{ldiv}_{n-1}\left(\int_{\partial K} \kappa(x)^{\frac{1}{n+1}} d\mu_{\partial K}(x)\right)^{\frac{n+1}{n-1}}$$

where ldiv_{n-1} is a positive constant that depends on n only. Clearly, $\mathrm{ldiv}_{n-1} \leq c \cdot n$ and Böröczky [Bö] showed that for polytopes P with N facets

$$\mathrm{vol}_n(B_2^n \triangle P) \geq \frac{1}{67e^2\pi}\frac{1}{n}\mathrm{vol}_n(B_2^n)N^{-\frac{2}{n-1}}.$$

Thus again, there is a gap between the upper and lower estimates for ldiv_{n-1} of order n^2. We narrow this gap by a factor of n.

Theorem 2 *There is a constant c such that for every $n \in \mathbb{N}$ and for every $M \geq 10^{\frac{n-1}{2}}$ and all polytopes P in $\mathbb{R}^n$ with M facets we have*

$$\mathrm{vol}_n(B_2^n \triangle P) \geq c\ \mathrm{vol}_n(B_2^n) M^{-\frac{2}{n-1}}. \tag{4}$$

2 Proof of Theorem 1

We need the following lemmas.

Lemma 3 *(Stirling's formula) For all $x > 0$*

$$\sqrt{2\pi x} x^x e^{-x} < \Gamma(x+1) < \sqrt{2\pi x} x^x e^{-x} e^{\frac{1}{12x}}.$$

The following lemma is due to J. Müller [Mü].

Lemma 4 *[Mü] Let $\mathbb{E}(\partial B_2^n, N)$ be the expected volume of a random polytope of N points that are independently chosen on the boundary of the Euclidean ball B_2^n with respect to the normalized surface measure. Then*

$$\lim_{N \to \infty} \frac{\mathrm{vol}_n(B_2^n) - \mathbb{E}(\partial B_2^n, N)}{N^{-\frac{2}{n-1}}} = \frac{(n-1)^{\frac{n+1}{n-1}} (\mathrm{vol}_{n-1}(\partial B_2^n))^{\frac{n+1}{n-1}}}{(\mathrm{vol}_{n-2}(\partial B_2^{n-1}))^{\frac{2}{n-1}}} \frac{\Gamma\left(n+1+\frac{2}{n-1}\right)}{2(n+1)!}.$$

The following lemma can be found in [Mil], ([SW], p. 317), and [Zä].

Lemma 5 *[Mil]*

$$\begin{aligned} &d\mu_{\partial B_2^n}(x_1) \cdots d\mu_{\partial B_2^n}(x_n) \\ &\quad = (n-1)! \frac{vol_{n-1}([x_1, \dots, x_n])}{(1-p^2)^{\frac{n}{2}}} d\mu_{\partial B_2^n \cap H}(x_1) \cdots d\mu_{\partial B_2^n \cap H}(x_n) dp d\mu_{\partial B_2^n}(\xi) \end{aligned} \tag{5}$$

where ξ is the normal to the plane H through $x_1, \dots, x_n$ and p is the distance of the plane H to the origin.

Lemma 6 *[Mil]*

$$\int_{\partial B_2^n(r)} \cdots \int_{\partial B_2^n(r)} (vol_n([x_1, \dots, x_{n+1}]))^2 d\mu_{\partial B_2^n(r)}(x_1) \cdots d\mu_{\partial B_2^n(r)}(x_{n+1})$$
$$= \frac{(n+1) r^{n^2+2n-1}}{n! n^n} (vol_{n-1}(\partial B_2^n))^{n+1} \tag{6}$$

For a given hyperplane H that does not contain the origin we denote by H^+ the halfspace containing the origin and H^- the halfspace not containing the origin. A cap C of the Euclidean ball B_2^n is the intersection of a half space H^- with B_2^n. The radius of such a cap is the radius of the $(n-1)$-dimensional ball $B_2^n \cap H$.

Lemma 7 *[SW] Let H be a hyperplane, p its distance from the origin and s the normalized surface area of $\partial B_2^n \cap H^-$, i.e.*

$$s = \frac{\text{vol}_{n-1}(\partial B_2^n \cap H^-)}{\text{vol}_{n-1}(\partial B_2^n)}.$$

Then

$$\frac{dp}{ds} = -\frac{1}{(1-p^2)^{\frac{n-3}{2}}} \frac{\text{vol}_{n-1}(\partial B_2^n)}{\text{vol}_{n-2}(\partial B_2^{n-1})}. \tag{7}$$

The following lemma is Lemma 3.13 from [SW].

Lemma 8 *[SW] Let C be a cap of a Euclidean ball with radius 1. Let u be the surface area of this cap and r its radius. Then we have*

$$\left(\frac{u}{\text{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{1}{n-1}} - \frac{1}{2(n+1)} \left(\frac{u}{\text{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{3}{n-1}} - c \left(\frac{u}{\text{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{5}{n-1}}$$
$$\leq r(u) \leq \left(\frac{u}{\text{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{1}{n-1}} \tag{8}$$

where c is a numerical constant.

The right hand inequality is immediately verified, since $u \geq r^{n-1} \text{vol}_{n-1}(B_2^{n-1})$.

Proof of Theorem 1. The approximating polytope is obtained in a probabilistic way. We are considering a Euclidean ball that is slightly bigger than the Euclidean ball with radius 1. The factor by which the bigger ball is bigger than the smaller is important and carefully chosen. We are choosing N points randomly on the bigger ball and we are taking the convex hull of these points. With large probability there is a random polytope that fits our requirements.

For technical reasons we are choosing random points on a Euclidean ball of radius 1 and we are approximating a slightly smaller Euclidean ball, say with radius $1-c$ where $c = c_{n,N}$ depends on n and N only.

We compute now the expected volume difference between $B_2^n(1-c)$ and a random polytope $[x_1, \dots, x_N]$ whose vertices are chosen randomly from the boundary of B_2^n. Please note that random polytopes are simplicial with probability 1. We want to estimate the expected volume difference

$$\begin{aligned} &\mathbb{E}\mathrm{vol}_n(B_2^n(1-c)\triangle P_N) \\ &= \int_{\partial B_2^n} \cdots \int_{\partial B_2^n} \mathrm{vol}_n\left(B_2^n(1-c)\triangle[x_1, \dots, x_N]\right) d\mathbb{P}(x_1)\cdots d\mathbb{P}(x_N), \end{aligned} \tag{9}$$

where $\mathbb{P}$ denotes the uniform probability measure on ∂B_2^n. Since the volume difference between $B_2^n(1-c)$ and a polytope $P_N = [x_1, \dots, x_N]$ is

$$\begin{aligned} &\mathrm{vol}_n(B_2^n(1-c)\triangle P_N) \\ &= \mathrm{vol}_n(B_2^n \setminus B_2^n(1-c)) - \mathrm{vol}_n(B_2^n \setminus P_N) + 2\mathrm{vol}_n(B_2^n(1-c) \cap P_N^c), \end{aligned}$$

the above expression equals

$$\begin{aligned} &\mathrm{vol}_n(B_2^n \setminus B_2^n(1-c)) \\ &- \int_{\partial B_2^n} \cdots \int_{\partial B_2^n} \mathrm{vol}_n(B_2^n \setminus [x_1, \dots, x_N]) d\mathbb{P}(x_1)\cdots d\mathbb{P}(x_N) \\ &+2 \int_{\partial B_2^n} \cdots \int_{\partial B_2^n} \mathrm{vol}_n(B_2^n(1-c) \cap [x_1, \dots, x_N]^c) d\mathbb{P}(x_1)\cdots d\mathbb{P}(x_N). \end{aligned}$$

For given N we are choosing c such that

$$\mathrm{vol}_n(B_2^n \setminus B_2^n(1-c)) = \int_{\partial B_2^n} \cdots \int_{\partial B_2^n} \mathrm{vol}_n(B_2^n \setminus [x_1, \dots, x_N]) d\mathbb{P}(x_1)\cdots d\mathbb{P}(x_N). \tag{10}$$

For this particular c we have

$$\begin{aligned} &\int_{\partial B_2^n} \cdots \int_{\partial B_2^n} \mathrm{vol}_n\left(B_2^n(1-c)\triangle[x_1, \dots, x_N]\right) d\mathbb{P}(x_1)\cdots d\mathbb{P}(x_N) \\ &= 2 \int_{\partial B_2^n} \cdots \int_{\partial B_2^n} \mathrm{vol}_n(B_2^n(1-c) \cap [x_1, \dots, x_N]^c) d\mathbb{P}(x_1)\cdots d\mathbb{P}(x_N). \end{aligned}$$

By Lemma 4 the quantity c is for large N asymptotically equal to

$$N^{-\frac{2}{n-1}}(n-1)^{\frac{n+1}{n-1}}\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-2}(\partial B_2^{n-1})}\right)^{\frac{2}{n-1}}\frac{\Gamma(n+1+\frac{2}{n-1})}{2(n+1)!}. \tag{11}$$

In particular, for large enough N

$$c\leq\left(1+\frac{1}{n^2}\right)N^{-\frac{2}{n-1}}(n-1)^{\frac{n+1}{n-1}}\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-2}(\partial B_2^{n-1})}\right)^{\frac{2}{n-1}}\frac{\Gamma(n+1+\frac{2}{n-1})}{2(n+1)!} \tag{12}$$

and

$$\left(1-\frac{1}{n^2}\right)N^{-\frac{2}{n-1}}(n-1)^{\frac{n+1}{n-1}}\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-2}(\partial B_2^{n-1})}\right)^{\frac{2}{n-1}}\frac{\Gamma(n+1+\frac{2}{n-1})}{2(n+1)!}\leq c. \tag{13}$$

Thus there are constants a and b such that

$$aN^{-\frac{2}{n-1}}\leq c\leq bN^{-\frac{2}{n-1}}. \tag{14}$$

We continue the computation of the expected volume difference

$$\begin{aligned}
&\int_{\partial B_2^n}\cdots\int_{\partial B_2^n}\mathrm{vol}_n\left(B_2^n(1-c)\triangle[x_1,\dots,x_N]\right)d\mathbb{P}(x_1)\cdots d\mathbb{P}(x_N)\\
&=2\int_{\partial B_2^n}\cdots\int_{\partial B_2^n}\mathrm{vol}_n(B_2^n(1-c)\cap[x_1,\dots,x_N]^c)\chi_{\{0\in[x_1,\dots,x_N]^\circ\}}d\mathbb{P}(x_1)\cdots d\mathbb{P}(x_N)\\
&+2\int_{\partial B_2^n}\cdots\int_{\partial B_2^n}\mathrm{vol}_n(B_2^n(1-c)\cap[x_1,\dots,x_N]^c)\chi_{\{0\notin[x_1,\dots,x_N]^\circ\}}d\mathbb{P}(x_1)\cdots d\mathbb{P}(x_N)\\
&\leq 2\int_{\partial B_2^n}\cdots\int_{\partial B_2^n}\mathrm{vol}_n(B_2^n(1-c)\cap[x_1,\dots,x_N]^c)\chi_{\{0\in[x_1,\dots,x_N]^\circ\}}d\mathbb{P}(x_1)\cdots d\mathbb{P}(x_N)\\
&+\mathrm{vol}_n\left(B_2^n\right)\int_{\partial B_2^n}\cdots\int_{\partial B_2^n}\chi_{\{0\notin[x_1,\dots,x_N]^\circ\}}d\mathbb{P}(x_1)\cdots d\mathbb{P}(x_N).
\end{aligned}$$

By a result of [Wen] the second summand equals

$$\mathrm{vol}_n\left(B_2^n\right)2^{-N+1}\sum_{k=0}^{n-1}\binom{N-1}{k}\leq\mathrm{vol}_n\left(B_2^n\right)2^{-N+1}nN^n.$$

The second summand is of much smaller order than the first summand: The second summand is essentially of the order 2^{-N}, while, as we shall see, the first is of the order of $N^{-\frac{2}{n-1}}$. Therefore, we consider in what follows the first summand.

We introduce $\Phi_{j_1,\dots,j_n}:\partial B_2^n\times\cdots\times\partial B_2^n\to\mathbb{R}$ where

$$\Phi_{j_1,\dots,j_n}(x_1,\dots,x_N)=0$$

if $[x_{j_1},\dots,x_{j_n}]$ is not a $(n-1)$-dimensional face of $[x_1,\dots,x_N]$ or if 0 is not an element of $[x_1,\dots,x_N]$, and

$$\begin{aligned}&\Phi_{j_1,\dots,j_n}(x_1,\dots,x_N)\\&=\mathrm{vol}_n(B_2^n(1-c)\cap[x_1,\dots,x_N]^c\cap\mathrm{cone}(x_{j_1},\dots,x_{j_n}))\chi_{\{0\in[x_1,\dots,x_N]^\circ\}}\end{aligned}$$

if $[x_{j_1},\dots,x_{j_n}]$ is a facet of $[x_1,\dots,x_N]$ and if 0 is an element of $[x_1,\dots,x_N]$. Here

$$\mathrm{cone}(x_1,\dots,x_n)=\left\{\sum_{i=1}^n a_ix_i\,\middle|\,\forall i:0\le a_i\right\}.$$

For all random polytopes $[x_1,\dots,x_N]$ that contain 0 as an interior point

$$\mathbb{R}^n=\bigcup_{[x_{j_1},\dots,x_{j_n}]\text{ is facet of }[x_1,\dots,x_n]}\mathrm{cone}(x_{j_1},\dots,x_{j_n}).$$

With this we get

$$\begin{aligned}&\int_{\partial B_2^n}\cdots\int_{\partial B_2^n}\mathrm{vol}_n(B_2^n(1-c)\cap[x_1,\dots,x_N]^c)\chi_{\{0\in[x_1,\dots,x_N]^\circ\}}d\mathbb{P}(x_1)\cdots d\mathbb{P}(x_N)\\&=\int_{\partial B_2^n}\cdots\int_{\partial B_2^n}\sum_{\{j_1,\dots,j_n\}\subseteq\{1,\dots,N\}}\Phi_{j_1,\dots,j_n}(x_1,\dots,x_N)d\mathbb{P}(x_1)\cdots d\mathbb{P}(x_N)\end{aligned}$$

where we sum over all different subsets $\{j_1,\dots,j_n\}$. The latter expression equals

$$\binom{N}{n}\int_{\partial B_2^n}\cdots\int_{\partial B_2^n}\Phi_{1,\dots,n}(x_1,\dots,x_N)d\mathbb{P}(x_1)\cdots d\mathbb{P}(x_N).$$

Let H be the hyperplane containing the points $x_1,\dots,x_n$. The set of points where H is not well defined has measure 0 and

$$\begin{aligned}&\mathbb{P}^{N-n}(\{(x_{n+1},\dots,x_N)|[x_1,\dots,x_n]\text{ is facet of }[x_1,\dots,x_N]\text{ and }0\in[x_1,\dots,x_N]\})\\&\quad=\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n\cap H^+)}{\mathrm{vol}_{n-1}(\partial B_2^n)}\right)^{N-n}.\end{aligned}$$

Therefore the above expression equals

$$\begin{aligned}&\binom{N}{n}\int_{\partial B_2^n}\cdots\int_{\partial B_2^n}\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n\cap H^+)}{\mathrm{vol}_{n-1}(\partial B_2^n)}\right)^{N-n}\\&\qquad\mathrm{vol}_n(B_2^n(1-c)\cap H^-\cap\mathrm{cone}(x_1,\dots,x_n))\chi_{\{0\in[x_1,\dots,x_N]^\circ\}}d\mathbb{P}(x_1)\cdots d\mathbb{P}(x_n).\end{aligned}$$

Since H^- does not contain 0 this can be estimated by

$$\binom{N}{n}\int_{\partial B_2^n}\cdots\int_{\partial B_2^n}\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n\cap H^+)}{\mathrm{vol}_{n-1}(\partial B_2^n)}\right)^{N-n}$$
$$\mathrm{vol}_n(B_2^n(1-c)\cap H^-\cap\mathrm{cone}(x_1,\ldots,x_n))d\mathbb{P}(x_1)\cdots d\mathbb{P}(x_n).$$

By Lemma 5 the latter expression equals

$$\binom{N}{n}\frac{(n-1)!}{(\mathrm{vol}_{n-1}(\partial B_2^n))^n}\int_{\partial B_2^n}\int_0^1\int_{\partial B_2^n\cap H}\cdots\int_{\partial B_2^n\cap H}\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n\cap H^+)}{\mathrm{vol}_{n-1}(\partial B_2^n)}\right)^{N-n}$$
$$\mathrm{vol}_n(B_2^n(1-c)\cap H^-\cap\mathrm{cone}(x_1,\ldots,x_n))$$
$$\frac{\mathrm{vol}_{n-1}([x_1,\ldots,x_n])}{(1-p^2)^{n/2}}d\mu_{\partial B_2^n\cap H}(x_1)\cdots d\mu_{\partial B_2^n\cap H}(x_n)dpd\mu_{\partial B_2^n}(\xi).$$

This in turn can be estimated by

$$\binom{N}{n}\frac{(n-1)!}{(\mathrm{vol}_{n-1}(\partial B_2^n))^n}\int_{\partial B_2^n}\int_{1-\frac{1}{n}}^1\int_{\partial B_2^n\cap H}\cdots\int_{\partial B_2^n\cap H}\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n\cap H^+)}{\mathrm{vol}_{n-1}(\partial B_2^n)}\right)^{N-n}$$
$$\mathrm{vol}_n(B_2^n(1-c)\cap H^-\cap\mathrm{cone}(x_1,\ldots,x_n)) \tag{15}$$
$$\frac{\mathrm{vol}_{n-1}([x_1,\ldots,x_n])}{(1-p^2)^{n/2}}d\mu_{\partial B_2^n\cap H}(x_1)\cdots d\mu_{\partial B_2^n\cap H}(x_n)dpd\mu_{\partial B_2^n}(\xi)$$

times a factor that is less than 2 provided that N is sufficiently big. Indeed, for $p\le 1-\frac{1}{n}$

$$\begin{aligned}\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n\cap H^+)}{\mathrm{vol}_{n-1}(\partial B_2^n)}\right)^{N-n} &\le\exp\left(-(N-n)\frac{\mathrm{vol}_{n-1}(\partial B_2^n\cap H^-)}{\mathrm{vol}_{n-1}(\partial B_2^n)}\right)\\ &\le\exp\left(-(N-n)\left(\frac{2}{n}-\frac{1}{n^2}\right)^{\frac{n-1}{2}}\frac{\mathrm{vol}_{n-1}(B_2^{n-1})}{n\ \mathrm{vol}_n(B_2^n)}\right)\\ &\le\exp\left(-\frac{N-n}{n^{(n+1)/2}}\right)\end{aligned}$$

and the rest of the expression is bounded. We have

$$\begin{aligned}&\mathrm{vol}_n(B_2^n(1-c)\cap H^-\cap\mathrm{cone}(x_1,\ldots,x_n))\\ &\le\frac{p}{n}\max\left\{0,\left(\frac{1-c}{p}\right)^n-1\right\}\mathrm{vol}_{n-1}([x_1,\ldots,x_n]).\end{aligned}$$

This holds since the set $B_2^n(1-c)\cap H^-\cap\mathrm{cone}(x_1,\ldots,x_n)$ is contained in the cone $\mathrm{cone}(x_1,\ldots,x_n)$, truncated between H and the hyperplane parallel to H at distance $1-c$ from 0. Therefore, as $p \leq 1$ the above is smaller or equal than

$$\binom{N}{n} \frac{(n-1)!}{(\mathrm{vol}_{n-1}(\partial B_2^n))^n} \int_{\partial B_2^n} \int_{1-\frac{1}{n}}^{1} \int_{\partial B_2^n \cap H} \cdots \int_{\partial B_2^n \cap H} \left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n \cap H^+)}{\mathrm{vol}_{n-1}(\partial B_2^n)} \right)^{N-n}$$
$$\frac{1}{n} \max \left\{ 0, \left(\frac{1-c}{p} \right)^n - 1 \right\} \frac{(\mathrm{vol}_{n-1}([x_1, \dots, x_n]))^2}{(1-p^2)^{n/2}}$$
$$d\mu_{\partial B_2^n \cap H}(x_1) \cdots d\mu_{\partial B_2^n \cap H}(x_n) dp d\mu_{\partial B_2^n}(\xi).$$

With Lemma 6 this equals

$$\binom{N}{n} \frac{(\mathrm{vol}_{n-2}(\partial B_2^{n-1}))^n}{(\mathrm{vol}_{n-1}(\partial B_2^n))^n} \frac{n}{(n-1)^{n-1}} \int_{\partial B_2^n} \int_{1-\frac{1}{n}}^{1} \left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n \cap H^+)}{\mathrm{vol}_{n-1}(\partial B_2^n)} \right)^{N-n}$$
$$\frac{1}{n} \max \left\{ 0, \left(\frac{1-c}{p} \right)^n - 1 \right\} \frac{r^{n^2-2}}{(1-p^2)^{n/2}} dp d\mu_{\partial B_2^n}(\xi)$$

where r denotes the radius of $B_2^n \cap H$. Since the integral does not depend on the direction ξ and $r^2 + p^2 = 1$ this is

$$\binom{N}{n} \frac{(\mathrm{vol}_{n-2}(\partial B_2^{n-1}))^n}{(\mathrm{vol}_{n-1}(\partial B_2^n))^{n-1}} \frac{n}{(n-1)^{n-1}}$$
$$\int_{1-\frac{1}{n}}^{1} \left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n \cap H^+)}{\mathrm{vol}_{n-1}(\partial B_2^n)} \right)^{N-n} \frac{1}{n} \max \left\{ 0, \left(\frac{1-c}{p} \right)^n - 1 \right\} r^{n^2-n-2} dp.$$

This equals

$$\binom{N}{n} \frac{(\mathrm{vol}_{n-2}(\partial B_2^{n-1}))^n}{(\mathrm{vol}_{n-1}(\partial B_2^n))^{n-1}} \frac{n}{(n-1)^{n-1}} \tag{16}$$
$$\int_{1-\frac{1}{n}}^{1-c} \left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n \cap H^+)}{\mathrm{vol}_{n-1}(\partial B_2^n)} \right)^{N-n} \frac{1}{n} \left\{ \left(\frac{1-c}{p} \right)^n - 1 \right\} r^{n^2-n-2} dp.$$

Since $p \geq 1 - \frac{1}{n}$ and c is of the order $N^{-\frac{2}{n-1}}$ we have for sufficiently big N

$$\frac{1}{n} \left\{ \left(\frac{1-c}{p} \right)^n - 1 \right\} \leq 3(1-c-p).$$

Therefore, the previous expression can be estimated by an absolute constant times

$$\binom{N}{n} \frac{(\mathrm{vol}_{n-2}(\partial B_2^{n-1}))^n}{(\mathrm{vol}_{n-1}(\partial B_2^n))^{n-1}} \frac{n}{(n-1)^{n-1}} \tag{17}$$
$$\int_{1-\frac{1}{n}}^{1-c} \left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n \cap H^+)}{\mathrm{vol}_{n-1}(\partial B_2^n)} \right)^{N-n} (1-c-p) r^{n^2-n-2} dp.$$

We choose

$$s = \frac{\mathrm{vol}_{n-1}(\partial B_2^n \cap H^-)}{\mathrm{vol}_{n-1}(\partial B_2^n)}$$

as our new variable under the integral. We apply Lemma 7 in order to change the variable under the integral

$$\binom{N}{n} \frac{(\mathrm{vol}_{n-2}(\partial B_2^{n-1}))^{n-1}}{(\mathrm{vol}_{n-1}(\partial B_2^n))^{n-2}} \frac{n}{(n-1)^{n-1}} \int_{s(1-c)}^{\frac{1}{2}} (1-s)^{N-n}(1-c-p) r^{(n-1)^2} ds \quad (18)$$

where the normalized surface area s of the cap is a function of the distance p of the hyperplane to 0. Before we proceed we want to estimate $s(1-c)$. The radius r and the distance p satisfy $1 = p^2 + r^2$. We have

$$r^{n-1} \frac{\mathrm{vol}_{n-1}(B_2^{n-1})}{\mathrm{vol}_{n-1}(\partial B_2^n)} \leq s\left(\sqrt{1-r^2}\right) \leq \frac{1}{\sqrt{1-r^2}} r^{n-1} \frac{\mathrm{vol}_{n-1}(B_2^{n-1})}{\mathrm{vol}_{n-1}(\partial B_2^n)}.$$

We show this. We compare s with the surface area of the intersection $B_2^n \cap H$ of the Euclidean ball and the hyperplane H. We have

$$\frac{\mathrm{vol}_{n-1}(B_2^n \cap H)}{\mathrm{vol}_{n-1}(\partial B_2^n)} = r^{n-1} \frac{\mathrm{vol}_{n-1}(B_2^{n-1})}{\mathrm{vol}_{n-1}(\partial B_2^n)}.$$

Since the orthogonal projection onto H maps $\partial B_2^n \cap H^-$ onto $B_2^n \cap H$ the left hand inequality follows.

The right hand inequality follows again by considering the orthogonal projection onto H. The surface area of a surface element of $\partial B_2^n \cap H^-$ equals the surface area of the one it is mapped to in $B_2^n \cap H$ divided by the cosine of the angle between the normal to H and the normal to ∂B_2^n at the given point. The cosine is always greater than $\sqrt{1-r^2}$.

For $p = 1-c$ we have $r = \sqrt{2c - c^2} \leq \sqrt{2c}$. Therefore we get by (12)

$$\begin{aligned} s(1-c) &\leq \frac{e^{\frac{1}{n}}}{1-c} \frac{\mathrm{vol}_{n-1}(B_2^{n-1})}{\mathrm{vol}_{n-1}(\partial B_2^n)} \\ &\quad \left\{ 2N^{-\frac{2}{n-1}} (n-1)^{\frac{n+1}{n-1}} \left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-2}(\partial B_2^{n-1})} \right)^{\frac{2}{n-1}} \frac{\Gamma(n+1+\frac{2}{n-1})}{2(n+1)!} \right\}^{\frac{n-1}{2}} \\ &= \frac{e^{\frac{1}{n}}}{1-c} \frac{1}{N} \left\{ \frac{\Gamma(n+1+\frac{2}{n-1})(n-1)}{(n+1)!} \right\}^{\frac{n-1}{2}}. \end{aligned} \quad (19)$$

The quantity c is of the order $N^{-\frac{2}{n-1}}$, therefore $1/(1-c)$ is as close to 1 as we desire for N large enough. Moreover, for large n

$$\left(\frac{n-1}{n+1}\right)^{\frac{n-1}{2}}$$

is asymptotically equal to $1/e$. Therefore, for both n and N large enough

$$s(1-c) \leq e^{\frac{1}{12}} \frac{1}{eN} \left\{ \frac{\Gamma(n+1+\frac{2}{n-1})}{n!} \right\}^{\frac{n-1}{2}}.$$

For n sufficiently big

$$\left\{ \frac{\Gamma(n+1+\frac{2}{n-1})}{n!} \right\}^{\frac{n-1}{2}} \leq e^{\frac{1}{12}} n.$$

We verify the estimate. By Lemma 3

$$\frac{\Gamma(n+1+\frac{2}{n-1})}{n!} \leq \left(1+\frac{2}{n(n-1)}\right)^{n+\frac{1}{2}} \left(n+\frac{2}{n-1}\right)^{\frac{2}{n-1}} e^{-\frac{2}{n-1}} e^{\frac{1}{12(n+\frac{2}{n-1})}}$$

and

$$\left(\frac{\Gamma(n+1+\frac{2}{n-1})}{n!}\right)^{\frac{n-1}{2}} \leq \frac{1}{e} \left(1+\frac{2}{n(n-1)}\right)^{\frac{n-1}{2}(n+\frac{1}{2})} \left(n+\frac{2}{n-1}\right) e^{\frac{n-1}{24(n+\frac{2}{n-1})}}.$$

The right hand expression is asymptotically equal to $ne^{1/24}$.

Altogether,

$$s(1-c) \leq e^{\frac{1}{6}} \frac{n}{eN}. \tag{20}$$

Since $p = \sqrt{1-r^2}$ we get for all r with $0 \leq r \leq 1$

$$1-c-p = 1-c-\sqrt{1-r^2} \leq \frac{1}{2}r^2 + r^4 - c.$$

(The estimate is equivalent to $1-\frac{1}{2}r^2 - r^4 \leq \sqrt{1-r^2}$. The left hand side is negative for $r \geq .9$ and thus the inequality holds for r with $.9 \leq r \leq 1$. For r with $0 \leq r \leq .9$ we square both sides.) Thus (18) is smaller than or equal to

$$\binom{N}{n} \frac{(\mathrm{vol}_{n-2}(\partial B_2^{n-1}))^{n-1}}{(\mathrm{vol}_{n-1}(\partial B_2^n))^{n-2}} \frac{n}{(n-1)^{n-1}} \int_{s(1-c)}^1 (1-s)^{N-n} \left(\frac{1}{2}r^2 + r^4 - c\right) r^{(n-1)^2} ds. \tag{21}$$

Now we evaluate the integral. We use Lemma 8. By differentiation we verify that $(\frac{1}{2}r^2+r^4-c)r^{(n-1)^2}$ is a monotone function of r. Here we use that $\frac{1}{2}r^2+r^4-c$ is nonnegative.

$$\begin{aligned}
&\int_{s(1-c)}^{1}(1-s)^{N-n}\left(\frac{1}{2}r^2+r^4-c\right)r^{(n-1)^2}ds\\
&\quad\le\frac{1}{2}\int_0^1(1-s)^{N-n}\left(s\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{n-1+\frac{2}{n-1}}ds\\
&\quad+\int_0^1(1-s)^{N-n}\left(s\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{n-1+\frac{4}{n-1}}ds\\
&\quad-\int_0^1(1-s)^{N-n}c\left(s\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{n-1}ds\\
&\quad+\int_0^{s(1-c)}(1-s)^{N-n}c\left(s\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{n-1}ds.
\end{aligned}$$

By (13)

$$\begin{aligned}
&\int_{s(1-c)}^{1}(1-s)^{N-n}\left(\frac{1}{2}r^2+r^4-c\right)r^{(n-1)^2}ds\\
&\quad\le\frac{1}{2}\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{n-1+\frac{2}{n-1}}\frac{\Gamma(N-n+1)\Gamma(n+\frac{2}{n-1})}{\Gamma(N+1+\frac{2}{n-1})}\\
&\quad+\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{n-1+\frac{4}{n-1}}\frac{\Gamma(N-n+1)\Gamma(n+\frac{4}{n-1})}{\Gamma(N+1+\frac{4}{n-1})}\\
&\quad-\left(1-\frac{1}{n^2}\right)\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{n-1}\frac{\Gamma(N-n+1)\Gamma(n)}{\Gamma(N+1)}\\
&\qquad\frac{(n-1)^{\frac{n+1}{n-1}}(\mathrm{vol}_{n-1}(\partial B_2^n))^{\frac{2}{n-1}}}{(\mathrm{vol}_{n-2}(\partial B_2^{n-1}))^{\frac{2}{n-1}}}\frac{\Gamma\left(n+1+\frac{2}{n-1}\right)}{2(n+1)!}N^{-\frac{2}{n-1}}\\
&\quad+c\cdot s(1-c)\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{n-1}\max_{s\in[0,s(1-c)]}(1-s)^{N-n}s^{n-1}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\int_{s(1-c)}^{1}(1-s)^{N-n}\left(\frac{1}{2}r^2+r^4-c\right)r^{(n-1)^2}ds \qquad (22)\\
&\quad\le\frac{1}{2}\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{n-1+\frac{2}{n-1}}\frac{\Gamma(N-n+1)\Gamma(n+\frac{2}{n-1})}{\Gamma(N+1+\frac{2}{n-1})}
\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{n-1+\frac{4}{n-1}} \frac{\Gamma(N-n+1)\Gamma(n+\frac{4}{n-1})}{\Gamma(N+1+\frac{4}{n-1})} \\
&-\frac{1}{2}\left(1-\frac{1}{n^2}\right)\frac{n-1}{(n+1)n}\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{n-1+\frac{2}{n-1}} \\
&\qquad \frac{\Gamma(N-n+1)\Gamma\left(n+1+\frac{2}{n-1}\right)}{\Gamma(N+1)} N^{-\frac{2}{n-1}} \\
&+c\cdot s(1-c)\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{n-1} \max_{s\in[0,s(1-c)]}(1-s)^{N-n}s^{n-1}.
\end{aligned}
$$

The second summand is asymptotically equal to

$$
\begin{aligned}
&\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{n-1+\frac{4}{n-1}} \frac{(N-n)!(n-1)!n^{\frac{4}{n-1}}}{N!(N+1)^{\frac{4}{n-1}}} \\
&\qquad = \left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{n-1+\frac{4}{n-1}} \frac{n^{-1+\frac{4}{n-1}}}{\binom{N}{n}(N+1)^{\frac{4}{n-1}}}.
\end{aligned} \tag{23}
$$

This summand is of the order $N^{-\frac{4}{n-1}}$ while the others are of the order $N^{-\frac{2}{n-1}}$.

We consider the sum of the first and third summands:

$$
\begin{aligned}
&\frac{1}{2}\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{n-1+\frac{2}{n-1}} \frac{\Gamma(N-n+1)\Gamma(n+\frac{2}{n-1})}{\Gamma(N+1+\frac{2}{n-1})} \\
&\qquad \left(1-\left(1-\frac{1}{n^2}\right)\frac{(n-1)\Gamma(n+1+\frac{2}{n-1})\Gamma(N+1+\frac{2}{n-1})}{n(n+1)\Gamma(n+\frac{2}{n-1})\Gamma(N+1)N^{\frac{2}{n-1}}}\right)
\end{aligned}
$$

Since $\Gamma(n+1+\frac{2}{n-1})=(n+\frac{2}{n-1})\Gamma(n+\frac{2}{n-1})$ the latter expression equals

$$
\begin{aligned}
&\frac{1}{2}\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{n-1+\frac{2}{n-1}} \frac{\Gamma(N-n+1)\Gamma(n+\frac{2}{n-1})}{\Gamma(N+1+\frac{2}{n-1})} \\
&\qquad \left(1-\left(1-\frac{1}{n^2}\right)\frac{(n-1)(n+\frac{2}{n-1})\Gamma(N+1+\frac{2}{n-1})}{n(n+1)\Gamma(N+1)N^{\frac{2}{n-1}}}\right).
\end{aligned}
$$

Since $\Gamma(N+1+\frac{2}{n-1})$ is asymptotically equal to $(N+1)^{\frac{2}{n-1}}\Gamma(N+1)$ the sum of the first and third summand is for large N of the order

$$
\frac{1}{n}\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{n-1+\frac{2}{n-1}} \frac{\Gamma(N-n+1)\Gamma(n+\frac{2}{n-1})}{\Gamma(N+1+\frac{2}{n-1})} \tag{24}
$$

which in turn is of the order

$$\frac{1}{n^2}\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{n-1+\frac{2}{n-1}}\binom{N}{n}^{-1}N^{-\frac{2}{n-1}}. \tag{25}$$

We consider now the fourth summand. By (14) and (20) the fourth summand is less than

$$bN^{-\frac{2}{n-1}}\frac{n}{e^{5/6}N}\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{n-1}\max_{s\in[0,s(1-c)]}(1-s)^{N-n}s^{n-1}. \tag{26}$$

The maximum of the function $(1-s)^{N-n}s^{n-1}$ is attained at $(n-1)/(N-1)$ and the function is increasing on the interval $[0,(n-1)/(N-1)]$. Therefore, by (20) we have $s(1-c)<(n-1)/(N-1)$ and the maximum of this function over the interval $[0,s(1-c)]$ is attained at $s(1-c)$. By (20) we have $s(1-c)\le e^{\frac{1}{6}}\frac{n}{eN}$ and thus for N sufficiently big

$$\begin{aligned}\max_{s\in[0,s(1-c)]}(1-s)^{N-n}s^{n-1} &\le \left(1-\frac{n}{e^{5/6}N}\right)^{N-n}\left(e^{\frac{1}{6}}\frac{n}{eN}\right)^{n-1}\\ &\le \exp\left(\frac{n-1}{6}-\frac{n(N-n)}{e^{5/6}N}\right)\left(\frac{n}{eN}\right)^{n-1}\\ &\le \exp\left(-\frac{n}{6}\right)\left(\frac{n}{eN}\right)^{n-1}.\end{aligned}$$

Thus we get for (26) with a new constant b

$$bN^{-\frac{2}{n-1}}\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{n-1}e^{-\frac{n}{6}}\frac{n^ne^{-n}}{N^n}.$$

This is asymptotically equal to

$$bN^{-\frac{2}{n-1}}\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{n-1}e^{-\frac{n}{6}}\frac{1}{\binom{N}{n}\sqrt{2\pi n}}. \tag{27}$$

Altogether, (15) for N sufficiently big can be estimated by

$$\begin{aligned}&\binom{N}{n}\frac{(\mathrm{vol}_{n-2}(\partial B_2^{n-1}))^{n-1}}{(\mathrm{vol}_{n-1}(\partial B_2^n))^{n-2}}\frac{n}{(n-1)^{n-1}}\\ &\quad\left\{\frac{1}{n^2}\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{n-1+\frac{2}{n-1}}\binom{N}{n}^{-1}N^{-\frac{2}{n-1}}\right.\\ &\quad\left.+bN^{-\frac{2}{n-1}}\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{n-1}e^{-\frac{n}{6}}\frac{1}{\binom{N}{n}\sqrt{2\pi n}}\right\}.\end{aligned}$$

This can be estimated by a constant times

$$(\mathrm{vol}_{n-1}(\partial B_2^n))n \left\{ \frac{1}{n^2} N^{-\frac{2}{n-1}} + bN^{-\frac{2}{n-1}} e^{-\frac{n}{6}} \frac{1}{\sqrt{2\pi n}} \right\}. \tag{28}$$

Finally, it should be noted that we have been estimating the approximation of $B_2^n(1-c)$ and not that of B_2^n. Therefore, we need to multiply (28) by $(1-c)^{-n}$. By (14)

$$(1-c)^n \geq 1 - b\frac{n}{N^{\frac{2}{n-1}}}$$

so that we have for all N with $N \geq (2bn)^{\frac{n-1}{2}}$

$$(1-c)^{-n} \leq 2$$

□

3 Proof of Theorem 2

We need another lemma.

Lemma 9 *Let P_M be a polytope with M facets $F_1, \dots, F_M$ that is best approximating for a convex body K in $\mathbb{R}^n$ with respect to the symmetric difference metric. For $k = 1, \dots, M$, let*

$$F_k^i = F_k \cap K, \qquad F_k^a = F_k \cap K^c.$$

Then, for all $j = 1, \dots, M$

$$vol_{n-1}\left(F_j^i\right) = vol_{n-1}\left(F_j^a\right).$$

Proof. Let H_j, $j = 1, \dots, M$ be the hyperplane containing the face F_j. Then

$$P_M = \bigcap_{j=1}^{M} H_j^+.$$

Suppose $H_k = H(x_k, \xi_k)$, i.e. H_k is the hyperplane containing x_k and being orthogonal to ξ_k. We consider

$$P_t = \bigcap_{j \neq k} H_j^+ \cap H^+\left(x_k + \frac{t}{\|x_k\|}x_k, \xi_k\right).$$

We have

$$\mathrm{vol}_{n-1}(P_t \triangle K) = \mathrm{vol}_{n-1}(P_M \triangle K) + t\left(\mathrm{vol}_{n-1}\left(F_k^a\right) - \mathrm{vol}_{n-1}\left(F_k^i\right)\right) + \psi(t)$$

where $\psi(t)/t^2$ is a bounded function. $\square$

Proof of Theorem 2. Let P_M be a best approximating polytope with M facets $F_1, \dots, F_M$ for B_2^n with respect to the symmetric difference metric. For $k = 1, \dots, M$, let

$$F_k^i = F_k \cap B_2^n, \qquad F_k^a = F_k \cap (B_2^n)^c,$$

let H_k be the hyperplane containing the facet F_k and let C_k be the cap of B_2^n with base $H_k \cap B_2^n$. (There are actually two caps, we take the one whose interior has empty intersection with P_M.) We put, for $k = 1, \dots, M$

$$h_k = \begin{cases} \text{height of the cap } C_k, & \text{if } F_k \cap (B_2^n)^\circ \neq \emptyset \\ 0, & \text{if } F_k \cap (B_2^n)^\circ = \emptyset. \end{cases}$$

Then

$$\mathrm{vol}_{n-1}(P_M \triangle B_2^n) \geq \frac{1}{n} \sum_{k=1}^{M} h_k \ \mathrm{vol}_{n-1}\left(F_k^i\right). \tag{29}$$

Let r_k be such that $\mathrm{vol}_{n-1}\left(r_k B_2^{n-1}\right) = \mathrm{vol}_{n-1}\left(F_k^i\right)$. Thus

$$r_k = \left(\frac{\mathrm{vol}_{n-1}\left(F_k^i\right)}{\mathrm{vol}_{n-1}\left(B_2^{n-1}\right)}\right)^{\frac{1}{n-1}}.$$

Let $\tilde{h}_k$ be the height of the cap of B_2^n with base $r_k B_2^{n-1}$. Then

$$\tilde{h}_k \leq h_k, \quad \text{for all} \quad k \tag{30}$$

and

$$\tilde{h}_k \geq \frac{1}{2}\, r_k^2 \geq \frac{1}{2} \left(\frac{\mathrm{vol}_{n-1}\left(F_k^i\right)}{\mathrm{vol}_{n-1}\left(B_2^{n-1}\right)}\right)^{\frac{2}{n-1}}.$$

Thus we get from (29) with (30)

$$\begin{aligned} \mathrm{vol}_{n-1}(P_M \triangle B_2^n) &\geq \frac{1}{2\,n} \sum_{k=1}^{M} \frac{\left(\mathrm{vol}_{n-1}\left(F_k^i\right)\right)^{\frac{n+1}{n-1}}}{\left(\mathrm{vol}_{n-1}\left(B_2^{n-1}\right)\right)^{\frac{2}{n-1}}} \\ &\geq \frac{1}{8\pi e} \sum_{k=1}^{M} \left(\mathrm{vol}_{n-1}\left(F_k^i\right)\right)^{\frac{n+1}{n-1}}. \end{aligned} \tag{31}$$

We consider two cases. The first case is

$$\sum_{k=1}^{M} \mathrm{vol}_{n-1}\left(F_k^i\right) + \sum_{k=1}^{M} \mathrm{vol}_{n-1}\left(F_k^a\right) \geq c \,\mathrm{vol}_{n-1}\left(\partial B_2^n\right), \tag{32}$$

where $M \geq 10^{\frac{n-1}{2}}$ and $c = \frac{9}{10}$. It then follows from Lemma 9 that

$$\sum_{k=1}^{M} \mathrm{vol}_{n-1}\left(F_k^i\right) \geq \frac{c}{2} \,\mathrm{vol}_{n-1}\left(\partial B_2^n\right). \tag{33}$$

By Hölder's inequality

$$\sum_{k=1}^{M} \mathrm{vol}_{n-1}\left(F_k^i\right) \leq \left(\sum_{k=1}^{M} \left(\mathrm{vol}_{n-1}\left(F_k^i\right)\right)^p\right)^{\frac{1}{p}} (M)^{\frac{1}{p'}}.$$

Therefore we get from (31) and (33) with $p = \frac{n+1}{n-1}$ that

$$\mathrm{vol}_{n-1}(P_M \triangle B_2^n) \geq \frac{(c/2)^{\frac{n+1}{n-1}}}{8\pi e} \frac{1}{M^{\frac{2}{n-1}}} \left(n \,\mathrm{vol}_n\left(B_2^n\right)\right)^{\frac{n+1}{n-1}} \geq \frac{c^{\frac{n+1}{n-1}}}{8 \, M^{\frac{2}{n-1}}} \mathrm{vol}_n\left(B_2^n\right).$$

The second case is that (32) does not hold. Thus

$$\sum_{k=1}^{M} \mathrm{vol}_{n-1}\left(F_k\right) = \sum_{k=1}^{M} \mathrm{vol}_{n-1}\left(F_k^i\right) + \sum_{k=1}^{M} \mathrm{vol}_{n-1}\left(F_k^a\right) < c \,\mathrm{vol}_{n-1}\left(\partial B_2^n\right).$$

Then, by the isoperimetric inequality

$$\mathrm{vol}_n\left(P_M\right) \leq \left(\frac{\sum_{k=1}^{M} \mathrm{vol}_{n-1}\left(F_k\right)}{\mathrm{vol}_{n-1}\left(\partial B_2^n\right)}\right)^{\frac{n}{n-1}} \mathrm{vol}_n\left(B_2^n\right) \; < \; c^{\frac{n}{n-1}} \,\mathrm{vol}_n\left(B_2^n\right)$$

and thus

$$\mathrm{vol}_n(P_M \triangle B_2^n) \geq \left(1 - c^{\frac{n}{n-1}}\right) \mathrm{vol}_n\left(B_2^n\right).$$

Since $c = \frac{9}{10}$, then this last expression is greater than $M^{-\frac{2}{n-1}} \mathrm{vol}_n\left(B_2^n\right)$, provided $M \geq 10^{\frac{n-1}{2}}$, which holds by assumption. □

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