

# A Characterization of $L_p$ Intersection Bodies

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*Dedicated to Prof. Peter M. Gruber on the occasion of his sixty-fifth birthday*

## Abstract

All  $GL(n)$  covariant  $L_p$  radial valuations on convex polytopes are classified for every  $p > 0$ . It is shown that for  $0 < p < 1$  there is a unique non-trivial such valuation with centrally symmetric images. This establishes a characterization of  $L_p$  intersection bodies.

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## 1 Introduction

Let  $L \subset \mathbb{R}^n$  be a star body, that is, a compact set which is star-shaped with respect to the origin and has a continuous radial function,  $\rho(L, u) = \max\{r \geq 0 : ru \in L\}$ ,  $u \in S^{n-1}$ . The *intersection body*,  $IL$ , of  $L$  is the star body whose radial function in the direction  $u \in S^{n-1}$  is equal to the  $(n-1)$ -dimensional volume of the section of  $L$  by  $u^\perp$ , the hyperplane orthogonal to  $u$ . So, for  $u \in S^{n-1}$ ,

$$\rho(IL, u) = \text{vol}(L \cap u^\perp),$$

where  $\text{vol}$  denotes  $(n-1)$ -dimensional volume.

Intersection bodies which arise from centrally symmetric convex bodies first appeared in Busemann [5]. They are important in the theory of area in Finsler spaces. Intersection bodies of star bodies were defined and named by Lutwak [27]. The class of intersection bodies (as defined in [27]) turned out to be critical for the solution of the Busemann-Petty problem (see [7], [9], [39]) and are fundamental in geometric tomography (see e.g. [8]), in affine isoperimetric inequalities (see e.g. [19], [36]) and the geometry of Banach spaces (see e.g. [18], [37]).

Valuations allow us to obtain characterizations of many important functionals and operators on convex sets by their invariance or covariance properties with respect to suitable groups of transformations (see [12], [16], [33], [34] for information on the classical theory and [1]–[4], [14], [15], [20]–[22], [25] for some of the recent results). Here a function  $Z : \mathcal{L} \rightarrow \langle \mathcal{G}, + \rangle$ , where  $\mathcal{L}$  is a class of subsets of  $\mathbb{R}^n$  and  $\langle \mathcal{G}, + \rangle$  is an abelian semigroup, is called a *valuation* if

$$ZK + ZL = Z(K \cup L) + Z(K \cap L),$$

whenever  $K, L, K \cup L, K \cap L \in \mathcal{L}$ .

In [24], intersection bodies were characterized as  $\text{GL}(n)$  covariant valuations. To state this result, we need some additional definitions. Let  $\mathcal{P}_0^n$  denote the set of convex polytopes in  $\mathbb{R}^n$  that contain the origin in their interiors and let  $P^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for every } y \in P\}$  denote the polar body of  $P \in \mathcal{P}_0^n$ . We write  $\mathcal{S}^n$  for the set of star bodies in  $\mathbb{R}^n$ . For  $p > 0$ , the  $L_p$ -radial sum  $K \tilde{+}_p L$  of  $K, L \in \mathcal{S}^n$  is defined by

$$\rho(K \tilde{+}_p L, \cdot)^p = \rho(K, \cdot)^p + \rho(L, \cdot)^p.$$

An operator  $Z : \mathcal{P}_0^n \rightarrow \langle \mathcal{S}^n, \tilde{+}_p \rangle$  is called *trivial*, if it is a linear combination with respect to  $\tilde{+}_p$  of the identity and central reflection. An operator  $Z$  is called  $\text{GL}(n)$  *covariant of weight  $q$* ,  $q \in \mathbb{R}$ , if for all  $\phi \in \text{GL}(n)$  and all bodies  $Q$ ,

$$Z(\phi Q) = |\det \phi|^q \phi Z Q,$$

where  $\det \phi$  is the determinant of  $\phi$ . An operator  $Z$  is called  $\text{GL}(n)$  *covariant*, if  $Z$  is  $\text{GL}(n)$  covariant of weight  $q$  for some  $q \in \mathbb{R}$ .

**Theorem** ([24]). *An operator  $Z : \mathcal{P}_0^n \rightarrow \langle \mathcal{S}^n, \tilde{+}_1 \rangle$  is a non-trivial  $\text{GL}(n)$  covariant valuation if and only if there is a constant  $c \geq 0$  such that*

$$Z P = c I P^*$$

for every  $P \in \mathcal{P}_0^n$ .

This theorem establishes a classification of  $\text{GL}(n)$  covariant valuations within the dual Brunn-Minkowski theory. Intersection bodies were introduced in [27] as an analogue in this dual theory of the classical projection bodies in the Brunn-Minkowski theory. In recent years, the Brunn-Minkowski theory was extended using Firey's  $L_p$  Minkowski addition (see [28], [29]). In particular, Lutwak, Yang, and Zhang introduced  $L_p$  projection bodies and obtained important affine isoperimetric inequalities (see [30], [31]). In [23], a valuation theoretic characterization of  $L_p$  projection bodies was obtained.

Here we ask the corresponding question within the dual Brunn-Minkowski theory. The notion corresponding to  $L_p$  Minkowski addition is  $L_p$  radial addition. So we ask for a classification of  $L_p$  radial valuations. A complete answer for the planar case is given in Theorem 3 in Section 3.3. For  $n \geq 3$ , we obtain the following result.

**Theorem 1.** *For  $0 < p < 1$ , an operator  $Z : \mathcal{P}_0^n \rightarrow \langle \mathcal{S}^n, \tilde{+}_p \rangle$  is a non-trivial  $\text{GL}(n)$  covariant valuation if and only if there are constants  $c_1, c_2 \geq 0$  such that*

$$Z P = c_1 I_p^+ P^* \tilde{+}_p c_2 I_p^- P^*$$

for every  $P \in \mathcal{P}_0^n$ . For  $p > 1$ , all  $\text{GL}(n)$  covariant valuations  $Z : \mathcal{P}_0^n \rightarrow \langle \mathcal{S}^n, \tilde{+}_p \rangle$  are trivial.

Here, for  $Q \in \mathcal{P}_0^n$ , the star body  $I_p^+ Q$  is defined for  $u \in S^{n-1}$  by

$$\rho(I_p^+ Q, u)^p = \int_{Q \cap u^+} |u \cdot x|^{-p} dx,$$

where  $u^+ = \{x \in \mathbb{R}^n : u \cdot x \geq 0\}$ . We define  $I_p^- Q = I_p^+(-Q)$ .

As a consequence, we obtain the following characterization of  $L_p$  intersection bodies. For  $p < 1$ , we call the centrally symmetric star body  $I_p Q = I_p^+ Q \tilde{\dagger}_p I_p^- Q$  the  $L_p$  intersection body of  $Q \in \mathcal{P}_0^n$ . So, for  $u \in S^{n-1}$ ,

$$\rho(I_p Q, u)^p = \int_Q |u \cdot x|^{-p} dx. \quad (1)$$

Since

$$\text{vol}(Q \cap u^\perp) = \lim_{\varepsilon \rightarrow 0^+} \frac{\varepsilon}{2} \int_Q |u \cdot x|^{-1+\varepsilon} dx$$

(cf. [18], p. 9),

$$\rho(IQ, u) = \lim_{p \rightarrow 1^-} \frac{1-p}{2} \rho(I_p Q, u)^p,$$

that is, the intersection body of  $Q$  is obtained as a limit of  $L_p$  intersection bodies of  $Q$ . Also note that a change to polar coordinates in (1) shows that up to a normalization factor  $\rho(I_p Q, u)^p$  equals the  $L_p$  cosine transform of  $\rho(Q, \cdot)^{n-p}$ .

We denote by  $\mathcal{S}_c^n$  the set of centrally symmetric star bodies in  $\mathbb{R}^n$  and classify  $\text{GL}(n)$  covariant valuations  $Z : \mathcal{P}_0^n \rightarrow \langle \mathcal{S}_c^n, \tilde{\dagger}_p \rangle$ . The planar case is contained in Theorem 4 in Section 3.3. For  $n \geq 3$ , we obtain the following result.

**Theorem 2.** *For  $0 < p < 1$ , an operator  $Z : \mathcal{P}_0^n \rightarrow \langle \mathcal{S}_c^n, \tilde{\dagger}_p \rangle$  is a non-trivial  $\text{GL}(n)$  covariant valuation if and only if there is a constant  $c \geq 0$  such that*

$$ZP = c I_p P^*$$

for every  $P \in \mathcal{P}_0^n$ . For  $p > 1$ , all  $\text{GL}(n)$  covariant valuations  $Z : \mathcal{P}_0^n \rightarrow \langle \mathcal{S}_c^n, \tilde{\dagger}_p \rangle$  are trivial.

Up to multiplication with a suitable power of the volume of  $Q$ , the  $L_p$  intersection body of  $Q$  is just the polar  $L_q$  centroid body of  $Q$ . For  $q > 1$ ,  $L_q$  centroid bodies were introduced by Lutwak and Zhang [32]. They led to important affine isoperimetric inequalities (see [6], [10], [30], [32]). Yaskin and Yaskina [38] introduced polar  $L_q$  centroid bodies for  $-1 < q < 1$  and solved the corresponding Busemann-Petty problem. For applications connected with embeddings in  $L_q$  spaces, see [13], [17], and [35]. For a detailed discussion of the operators  $I_p^+$  and  $I_p$ , we refer to [11].

## 2 Notation and Preliminaries

We work in  $n$ -dimensional Euclidean space  $\mathbb{R}^n$  and write  $x = (x_1, x_2, \dots, x_n)$  for vectors  $x \in \mathbb{R}^n$ . The standard basis in  $\mathbb{R}^n$  will be denoted by  $e_1, e_2, \dots, e_n$ . We use  $x \cdot y$  to denote the usual scalar product  $x_1 y_1 + x_2 y_2 + \dots + x_n y_n$  of two vectors  $x, y \in \mathbb{R}^n$ , and to define the norm  $\|x\| = \sqrt{x \cdot x}$ . The unit sphere  $\{x \in \mathbb{R}^n : \|x\| = 1\}$  is denoted by  $S^{n-1}$ . Given  $A, A_1, A_2, \dots, A_k \subset \mathbb{R}^n$ , we write  $[A_1, A_2, \dots, A_k]$  for the convex hull of  $A_1, A_2, \dots, A_k$ , we write  $\text{lin } A$  for the linear hull of  $A$ , and set  $A^\perp = \{x \in \mathbb{R}^n : x \cdot y = 0 \text{ for all } y \in A\}$ .

For  $L \in \mathcal{S}^n$ , we extend the radial function to a homogeneous function defined on  $\mathbb{R}^n \setminus \{0\}$  by  $\rho(L, x) = \|x\|^{-1} \rho(L, x/\|x\|)$ . Then it follows immediately from the definition that

$$\rho(\phi L, x) = \rho(L, \phi^{-1}x), \quad x \in \mathbb{R}^n \setminus \{0\} \quad (2)$$

for  $\phi \in \text{GL}(n)$ .

We call a valuation  $Z : \mathcal{P}_0^n \rightarrow \langle \mathcal{S}^n, \tilde{+}_p \rangle$  an  $L_p$  radial valuation. A valuation  $Z$  with  $ZP = \{0\}$  for every  $P$  having dimension less than  $n$  is called *simple*. A valuation is called  $\text{GL}(n)$  contravariant of weight  $q$ ,  $q \in \mathbb{R}$ , if

$$Z\phi P = |\det \phi|^q \phi^{-t} ZP$$

for every  $\phi \in \text{GL}(n)$  and every  $P \in \mathcal{P}_0^n$ . Here  $\phi^{-t}$  denotes the transpose of the inverse of  $\phi$ . For  $0 < p < 1$ , the operators  $I_p^\pm : \mathcal{P}_0^n \rightarrow \mathcal{S}^n$  and  $I_p : \mathcal{P}_0^n \rightarrow \mathcal{S}^n$  are  $L_p$  radial valuations and  $\text{GL}(n)$  contravariant operators of weight  $1/p$ .

The following lemma guarantees that a classification of all  $L_p$  radial valuations which are  $\text{GL}(n)$  covariant with negative weight follows from a classification of all  $L_p$  radial valuations which are  $\text{GL}(n)$  contravariant with positive weight. Moreover, if we know all  $L_p$  radial valuations,  $\text{GL}(n)$  covariant of arbitrary weight, we will know all  $\text{GL}(n)$  contravariant  $L_p$  radial valuations and vice versa.

**Lemma 1.** *Let  $Z$  be an  $L_p$  radial valuation and define another  $L_p$  radial valuation  $Z^*$  by  $Z^*P = ZP^*$  for every  $P \in \mathcal{P}_0^n$ . Then  $Z$  is  $\text{GL}(n)$  covariant of weight  $q$  if and only if  $Z^*$  is  $\text{GL}(n)$  contravariant of weight  $-q$ .*

*Proof.* That  $Z^*$  satisfies the valuation property is a consequence of

$$(P \cup Q)^* = P^* \cap Q^*, \quad (P \cap Q)^* = P^* \cup Q^*$$

for polytopes  $P, Q \in \mathcal{P}_0^n$  having convex union (see, for example, [36]). The statement of the lemma follows from the fact that  $(\phi P)^* = \phi^{-t} P^*$  holds for every  $P \in \mathcal{P}_0^n$  and every  $\phi \in \text{GL}(n)$ .  $\square$

## 2.1 Extension

Given an  $L_p$  radial valuation  $Z$ , we define another valuation  $Y : \mathcal{P}_0^n \rightarrow \mathcal{C}_+(S^{n-1})$  by  $YP(\cdot) = \rho(ZP, \cdot)^p$ . Here  $\mathcal{C}_+(S^{n-1})$  is the set of non-negative continuous functions on the sphere. We want to extend this valuation to the set  $\overline{\mathcal{P}}_0^n$  of convex polytopes which are either in  $\mathcal{P}_0^n$  or are intersections of a polytope in  $\mathcal{P}_0^n$  and a polyhedral cone with at most  $n$  facets having its apex at the origin. The following preparations will show when such extensions exist. For  $1 \leq j \leq n$ , let  $\overline{\mathcal{P}}_j^n$  denote the set of polytopes which are intersections of polytopes in  $\mathcal{P}_0^n$  and  $j$  halfspaces bounded by hyperplanes  $H_1, \dots, H_j$  containing the origin and having linearly independent normals. We need some more notation. For a hyperplane  $H \subset \mathbb{R}^n$ ,  $\mathcal{P}_0^n(H)$  is the set of convex polytopes in  $H$  containing the origin in their interiors relative to  $H$ . Let  $\overline{\mathcal{C}}_+(S^{n-1})$  denote the superset of  $\mathcal{C}_+(S^{n-1})$  consisting of all non-negative functions defined almost everywhere (with respect to spherical Lebesgue measure) on  $S^{n-1}$  which are continuous almost everywhere. We write  $H^+, H^-$  for the closed halfspaces bounded by  $H$ .

For  $P \in \mathcal{P}_0^n(H)$  and  $A \subset S^{n-1}$ , we say that  $Y : \mathcal{P}_0^n \rightarrow \mathcal{C}_+(S^{n-1})$  *vanishes on  $A$  at  $P$*  if for  $u \in H^- \setminus H$ ,  $v \in H^+ \setminus H$  and every  $w \in A$ , there exists a neighbourhood  $A(w)$  of  $w$  such that

$$\lim_{u, v \rightarrow 0} Y[P, u, v] = 0 \quad \text{uniformly on } A(w)$$

holds. If there exists a constant  $c \in \mathbb{R}$  such that  $Y[P, u, v] \leq c$  for  $\|u\|, \|v\| \leq 1$ ,  $u \in H^- \setminus H$ ,  $v \in H^+ \setminus H$  and  $[P, u, v] = [P, u] \cup [P, v]$ , then we say that  $Y : \mathcal{P}_0^n \rightarrow \mathcal{C}_+(S^{n-1})$  is *bounded at  $P$* .

Now we are able to formulate the following lemma proved in [24].

**Lemma 2.** *Let  $Y : \mathcal{P}_0^n \rightarrow \mathcal{C}_+(S^{n-1})$  be a valuation.*

1. *If  $Y$  vanishes on  $S^{n-1}$  at  $P$  for every hyperplane  $H$  containing the origin and every  $P \in \mathcal{P}_0^n(H)$ , then  $Y$  can be extended to a simple valuation  $\bar{Y} : \bar{\mathcal{P}}_0^n \rightarrow \mathcal{C}_+(S^{n-1})$ .*
2. *If  $Y$  is bounded and vanishes on  $S^{n-1} \setminus H$  at  $P$  for every hyperplane  $H$  containing the origin and every  $P \in \mathcal{P}_0^n(H)$ , then  $Y$  can be extended to a simple valuation  $\bar{Y} : \bar{\mathcal{P}}_0^n \rightarrow \bar{\mathcal{C}}_+(S^{n-1})$  and for  $P \in \bar{\mathcal{P}}_0^n$  bounded by hyperplanes  $H_1, H_2, \dots, H_n$  containing the origin,  $\bar{Y} P$  is continuous and bounded on  $S^{n-1} \setminus (H_1 \cup \dots \cup H_n)$ .*
3. *If  $Y$  is bounded and vanishes on  $S^{n-1} \setminus H^\perp$  at  $P$  for every hyperplane  $H$  containing the origin and every  $P \in \mathcal{P}_0^n(H)$ , then  $Y$  can be extended to a simple valuation  $\bar{Y} : \bar{\mathcal{P}}_0^n \rightarrow \bar{\mathcal{C}}_+(S^{n-1})$  and for  $P \in \bar{\mathcal{P}}_0^n$  bounded by hyperplanes  $H_1, H_2, \dots, H_n$  containing the origin,  $\bar{Y} P$  is continuous and bounded on  $S^{n-1} \setminus (H_1^\perp \cup \dots \cup H_n^\perp)$ .*
4. *If  $Y$  vanishes on  $S^{n-1} \setminus H^\perp$  at  $P$  for every hyperplane  $H$  containing the origin and every  $P \in \mathcal{P}_0^n(H)$ , then  $Y$  can be extended to a simple valuation  $\bar{Y} : \bar{\mathcal{P}}_0^n \rightarrow \bar{\mathcal{C}}_+(S^{n-1})$  and for  $P \in \bar{\mathcal{P}}_0^n$  bounded by hyperplanes  $H_1, H_2, \dots, H_n$  containing the origin,  $\bar{Y} P$  is continuous on  $S^{n-1} \setminus (H_1^\perp \cup \dots \cup H_n^\perp)$ .*

The extension is defined inductively for  $j = 1, \dots, n$ , and convex polytopes  $P = P_0 \cap H_1^+ \cap \dots \cap H_j^+$  with  $P_0 \in \mathcal{P}_0^n$  and hyperplanes having linearly independent normals: For  $u \in H_1 \cap \dots \cap H_{j-1}$ ,  $u \in H_j^- \setminus H$ , set

$$\bar{Y} P = \lim_{u \rightarrow 0} \bar{Y}[P, u]$$

on  $S^{n-1}$ ,  $S^{n-1} \setminus (H_1 \cup \dots \cup H_j)$  or  $S^{n-1} \setminus (H_1^\perp \cup \dots \cup H_j^\perp)$  if  $Y$  vanishes on  $S^{n-1}$ ,  $S^{n-1} \setminus H$  or  $S^{n-1} \setminus H^\perp$ , respectively.

The proof of the following lemma is omitted since it is nearly the same as the proof of Lemma 5 and Lemma 8 in [24].

**Lemma 3.** *Let  $Z : \mathcal{P}_0^n \rightarrow \mathcal{S}^n$  be an  $L_p$  radial valuation and define  $Y : \mathcal{P}_0^n \rightarrow \mathcal{C}_+(S^{n-1})$  by  $Y P(\cdot) = \rho(Z P, \cdot)^p$ .*

1. *If  $Z$  is  $\text{GL}(n)$  covariant of weight  $q$ , then for every hyperplane  $H$  containing the origin and every  $P \in \mathcal{P}_0(H)$ , the following holds: If  $q = 0$ , then  $Y$  vanishes on  $S^{n-1} \setminus H$  at  $P$  and if  $q > 0$ , then  $Y$  vanishes on  $S^{n-1}$  at  $P$ . In both cases,  $Y$  is bounded at  $P$ .*
2. *If  $Z$  is  $\text{GL}(n)$  contravariant of weight  $q$ , then for every hyperplane  $H$  containing the origin and every  $P \in \mathcal{P}_0(H)$ , the following holds: If  $q > 0$ , then  $Y$  vanishes on  $S^{n-1} \setminus H^\perp$  at  $P$  and if  $q > 1$ , then  $Y$  vanishes on  $S^{n-1}$  at  $P$ . For  $q \geq 1$ ,  $Y$  is bounded at  $P$ .*

Let  $Z$  be an  $L_p$  radial valuation which is  $\text{GL}(n)$  contravariant of weight  $q$ . For  $q > 0$ , Lemma 2 and Lemma 3 guarantee the existence of an extension of  $YP(\cdot) = \rho(ZP, \cdot)^p$  to  $\overline{\mathcal{P}}_0^n$  for which we write  $\overline{Y}$ . We extend these functions from  $S^{n-1}$  to  $\mathbb{R}^n \setminus \{0\}$  by making them homogeneous of degree  $-p$ . From the definition of this extension it follows for  $\phi \in \text{GL}(n)$  and  $P \in \overline{\mathcal{P}}_0^n$  bounded by hyperplanes  $H_1, H_2, \dots, H_j$  that

$$\overline{Y} \phi P(x) = |\det \phi|^{pq} \overline{Y} P(\phi^t x) \quad (3)$$

on  $S^{n-1} \setminus \phi^{-t}(H_1^\perp \cup \dots \cup H_j^\perp)$  for  $0 < q \leq 1$  and on  $S^{n-1}$  for  $q > 1$ .

If  $Z$  is an  $L_p$  radial valuation which is  $\text{GL}(n)$  covariant of weight  $q$ , we proceed as above. For  $q \geq 0$ , Lemma 2 and Lemma 3 guarantee the existence of an extension of  $YP(\cdot) = \rho(ZP, \cdot)^p$  to  $\overline{\mathcal{P}}_0^n$  for which we write  $\overline{Y}$  and which we extend from  $S^{n-1}$  to  $\mathbb{R}^n \setminus \{0\}$  by making it homogeneous of degree  $-p$ . From the definition of this extension it follows for  $\phi \in \text{GL}(n)$  and  $P \in \overline{\mathcal{P}}_0^n$  bounded by hyperplanes  $H_1, H_2, \dots, H_j$  that

$$\overline{Y} \phi P(x) = |\det \phi|^{pq} \overline{Y} P(\phi^{-1} x) \quad (4)$$

on  $S^{n-1} \setminus \phi(H_1 \cup \dots \cup H_j)$  for  $q = 0$  and on  $S^{n-1}$  for  $q > 0$ .

### 3 Proof of the Classification Results

We first establish a classification of valuations which are  $\text{GL}(n)$  contravariant of weight  $q > 0$  and then a classification of valuations which are  $\text{GL}(n)$  covariant of weight  $q \geq 0$ . By Lemma 1, combining these results gives a classification of  $\text{GL}(n)$  covariant valuations. The classification result for  $n \geq 3$  is contained in Theorem 1. The result for  $n = 2$  is stated in Section 3.3.

Let  $Z : \mathcal{P}_0^n \rightarrow \langle \mathcal{S}^n, \tilde{+}_p \rangle$  be a valuation which is  $\text{GL}(n)$  contravariant of weight  $q > 0$  and let  $\overline{Z}$  denote its extension to  $\overline{\mathcal{P}}_0^n$ . Let  $T^n = [0, e_1, \dots, e_n]$  be the standard simplex in  $\mathbb{R}^n$ . First, we show that  $\overline{Z}$  is determined on  $\overline{\mathcal{P}}_0^n$  by its value on  $T^n$ .

Since  $\overline{Z}$  is a simple valuation on  $\overline{\mathcal{P}}_0^n$ , it suffices to show the statement for a polytope  $P \in \overline{\mathcal{P}}_0^n$  contained in a simplicial cone  $C$  bounded by  $n$  hyperplanes containing the origin and with linearly independent normal vectors. We dissect  $P = T_1 \cup \dots \cup T_k$ , where  $T_i \in \overline{\mathcal{P}}_0^n$  are  $n$ -dimensional simplices with pairwise disjoint interiors. Let  $H$  be a suitable affine hyperplane such that  $D = C \cap H$  and  $S_i = T_i \cap H$  are  $(n-1)$ -dimensional simplices. We need the following notions (see [26]). A finite set of  $(n-1)$ -dimensional simplices  $\alpha D$  is called a *triangulation* of  $D$  if the simplices have pairwise disjoint interiors and their union equals  $D$ . An *elementary move* applied to  $\alpha D$  is one of the two following operations: a simplex  $S \in \alpha D$  is dissected into two  $(n-1)$ -dimensional simplices  $S_1, S_2$  by an  $(n-2)$ -dimensional plane containing an  $(n-3)$ -dimensional face of  $S$ ; or the reverse, that is, two simplices  $S_1, S_2 \in \alpha D$  are replaced by  $S = S_1 \cup S_2$  if  $S$  is again a simplex. It is shown in [26] that for every triangulation  $\alpha D$  there are finitely many elementary moves that transform  $\alpha D$  into the trivial triangulation  $\{D\}$ . Note that to each  $(n-1)$ -dimensional simplex  $S \in \alpha D$ , there corresponds a polytope  $Q \in \overline{\mathcal{P}}_0^n$  such that  $Q \cap H = S$ . If  $S$  is dissected by an  $(n-2)$ -dimensional plane  $E \subset H$  corresponding to an elementary move into  $S_1, S_2$ , then  $Q$  is dissected by the cone generated by  $E$  into  $Q_1, Q_2 \in \overline{\mathcal{P}}_0^n$ . Since  $\overline{Z}$  is a simple valuation on  $\overline{\mathcal{P}}_0^n$ , we obtain  $\overline{Z} Q = \overline{Z} Q_1 \tilde{+}_p \overline{Z} Q_2$ . The same

argument applies for the reverse move. Thus, after finitely many steps, we obtain that  $\bar{Z}P = \bar{Z}T_1 \tilde{\mp}_p \cdots \tilde{\mp}_p \bar{Z}T_k$ . Since  $\bar{Z}$  is  $\text{GL}(n)$  contravariant, this proves that  $\bar{Z}$  is determined on  $\bar{\mathcal{P}}_0^n$  by  $\bar{Z}T^n$ .

We set

$$f(x) = \rho(\bar{Z}T^n, x)^p$$

almost everywhere on  $\mathbb{R}^n$ . Since  $Z$  is  $\text{GL}(n)$  contravariant and  $T^n$  does not change when the coordinates are permuted, we obtain

$$f(x_1, \dots, x_n) = f(x_{k_1}, \dots, x_{k_n}) \quad (5)$$

for every permutation  $(k_1, \dots, k_n)$  of  $(1, \dots, n)$ . We derive a family of functional equations for  $f$ .

For  $0 < \lambda_j < 1$  and  $j = 2, 3, \dots, n$ , we define two families of linear maps by

$$\begin{aligned} \phi_j e_j &= \lambda_j e_j + (1 - \lambda_j) e_1, & \phi_j e_k &= e_k \text{ for } k \neq j, \\ \psi_j e_1 &= \lambda_j e_j + (1 - \lambda_j) e_1, & \psi_j e_k &= e_k \text{ for } k \neq 1. \end{aligned}$$

Note that

$$\begin{aligned} \phi_j^{-1} e_j &= \frac{1}{\lambda_j} e_j - \frac{1 - \lambda_j}{\lambda_j} e_1, & \phi_j^{-1} e_k &= e_k \text{ for } k \neq j, \\ \psi_j^{-1} e_1 &= -\frac{\lambda_j}{1 - \lambda_j} e_j + \frac{1}{1 - \lambda_j} e_1, & \psi_j^{-1} e_k &= e_k \text{ for } k \neq 1. \end{aligned}$$

Let  $H_j$  be the hyperplane through 0 with normal vector  $\lambda_j e_1 - (1 - \lambda_j) e_j$ . Then we have  $T^n \cap H_j^+ = \phi_j T^n$  and  $T^n \cap H_j^- = \psi_j T^n$ . Since  $\bar{Z}$  is a simple valuation, it follows that

$$\bar{Z}T^n = \bar{Z}(\phi_j T^n) \tilde{\mp}_p \bar{Z}(\psi_j T^n).$$

Since  $\bar{Z}$  is  $\text{GL}(n)$  contravariant, this and (3) imply

$$f(x) = \lambda_j^{pq} f(\phi_j^t x) + (1 - \lambda_j)^{pq} f(\psi_j^t x) \quad (6)$$

almost everywhere on  $\mathbb{R}^n$  where the set of exception depends on the value of  $q$ .

Similar observations can be made if the valuation  $Z$  is  $\text{GL}(n)$  covariant of weight  $q \geq 0$ . Then we have by (4)

$$f(x) = \lambda_j^{pq} f(\phi_j^{-1} x) + (1 - \lambda_j)^{pq} f(\psi_j^{-1} x) \quad (7)$$

almost everywhere. Note that (5) holds in the covariant case, too.

### 3.1 The 2-dimensional Contravariant Case

**Lemma 4.** *Let  $Z : \mathcal{P}_0^2 \rightarrow \langle \mathcal{S}^2, \tilde{\mp}_p \rangle$  be a valuation which is  $\text{GL}(2)$  contravariant of weight  $q = 1$ . Then there exists a constant  $c \geq 0$  such that*

$$ZP = c \psi_{\frac{\pi}{2}}(P \tilde{\mp}_p (-P))$$

for every  $P \in \mathcal{P}_0^2$ , where  $\psi_{\frac{\pi}{2}}$  denotes the rotation by an angle  $\frac{\pi}{2}$ .

*Proof.* Since  $\rho(P \cup Q, \cdot) = \max\{\rho(P, \cdot), \rho(Q, \cdot)\}$ ,  $\rho(P \cap Q, \cdot) = \min\{\rho(P, \cdot), \rho(Q, \cdot)\}$ , (2) implies that the function  $P \mapsto c\psi_{\frac{\pi}{2}}(P \tilde{+}_p (-P))$  is in fact an  $L_p$  radial valuation. Since

$$\psi_{\frac{\pi}{2}}\phi\psi_{\frac{\pi}{2}}^{-1} = (\det \phi)\phi^{-t} \quad (8)$$

holds for every  $\phi \in \text{GL}(2)$ , we obtain by using (2)

$$\begin{aligned} \rho(c\psi_{\frac{\pi}{2}}(\phi P \tilde{+}_p (-\phi P)), x)^p &= c^p \rho((\det \phi)P, \psi_{\frac{\pi}{2}}^{-1}\phi^t x)^p + c^p \rho(-(\det \phi)P, \psi_{\frac{\pi}{2}}^{-1}\phi^t x)^p \\ &= c^p \rho(|\det \phi|P, \psi_{\frac{\pi}{2}}^{-1}\phi^t x)^p + c^p \rho(-|\det \phi|P, \psi_{\frac{\pi}{2}}^{-1}\phi^t x)^p \\ &= \rho(|\det \phi|\phi^{-t}c\psi_{\frac{\pi}{2}}(P \tilde{+}_p (-P)), x)^p. \end{aligned}$$

This proves the contravariance of weight 1.

From Lemma 2, Lemma 3, and (6) we know that

$$f(x) = \lambda_2^p f(\phi_2^t x) + (1 - \lambda_2)^p f(\psi_2^t x) \quad (9)$$

holds for every  $x \in \mathbb{R}^2$  which does not lie in the linear hull of  $e_1, e_2$  or  $\lambda_2 e_1 - (1 - \lambda_2)e_2$ . Thus it follows by induction that for  $k = 1, 2, \dots$ ,

$$f((\psi_2^{-t})^k x) = \lambda_2^p \sum_{i=1}^k (1 - \lambda_2)^{p(k-i)} f(\phi_2^t (\psi_2^{-t})^i x) + (1 - \lambda_2)^{kp} f(x) \quad (10)$$

holds on  $\mathbb{R}^2$  except on a set consisting of countably many lines. For suitable  $\varepsilon > 0$ , we can evaluate (10) at  $x = e_1 - \varepsilon e_2$ . From this we obtain, using the homogeneity and the non-negativity of  $f$ , that

$$f(e_1 - (1 - \lambda_2)^k \varepsilon (\psi_2^{-t})^k e_2) \geq \lambda_2^p \sum_{i=1}^k f(\phi_2^t (e_1 - (1 - \lambda_2)^i \varepsilon (\psi_2^{-t})^i e_2)). \quad (11)$$

Note that  $(\psi_2^{-t})^k e_2 = -\lambda_2 \sum_{i=0}^{k-1} (1 - \lambda_2)^{i-k} e_1 + e_2$ . Thus  $\|e_1 - (1 - \lambda_2)^k \varepsilon (\psi_2^{-t})^k e_2\| \geq 1$ . Let  $k \rightarrow \infty$  in (11). By Lemma 2,  $f$  is uniformly bounded on  $S^1 \setminus \{\pm e_1, \pm e_2\}$ . So  $f(\phi_2^t (e_1 - (1 - \lambda_2)^i \varepsilon (\psi_2^{-t})^i e_2)) \rightarrow 0$  as  $i \rightarrow \infty$ . It follows from the continuity properties of  $f$ , that  $f((1 + \varepsilon)(e_1 + (1 - \lambda_2)e_2)) = 0$ . Taking the limit  $\varepsilon \rightarrow 0$ , we obtain

$$f(1, x_2) = 0, \quad \text{for } 0 < x_2 < 1. \quad (12)$$

By (5), this implies

$$f(x_1, 1) = 0, \quad \text{for } 0 < x_1 < 1. \quad (13)$$

Relations (12), (13), and the homogeneity of  $f$  imply

$$f(x_1, x_2) = 0, \quad \text{for } x_1, x_2 > 0. \quad (14)$$

By evaluating (10) at  $-e_1 - \varepsilon e_2$  we get in a similar way

$$f(-x_1, -x_2) = 0, \quad \text{for } x_1, x_2 > 0. \quad (15)$$

Formula (9) gives

$$f(-1, 1) = \lambda_2^p f(-1, -1 + 2\lambda_2) + (1 - \lambda_2)^p f(-1 + 2\lambda_2, 1).$$



In combination with (14) and (15) we obtain

$$\begin{aligned} f(-1, 1) &= \lambda_2^p f(-1, -1 + 2\lambda_2) \quad \text{for } \frac{1}{2} < \lambda_2 < 1, \\ f(-1, 1) &= (1 - \lambda_2)^p f(-1 + 2\lambda_2, 1) \quad \text{for } 0 < \lambda_2 < \frac{1}{2}. \end{aligned}$$

Hence

$$\begin{aligned} f(-1, x_2) &= \frac{c^p}{(1 + x_2)^p} \quad \text{for } 0 < x_2 < 1, \\ f(-x_1, 1) &= \frac{c^p}{(1 + x_1)^p} \quad \text{for } 0 < x_1 < 1, \end{aligned}$$

with  $c^p = 2^p f(-1, 1)$ . Since  $f$  is homogeneous of degree  $-p$ , we get

$$f(-x_1, x_2) = \frac{c^p}{(x_1 + x_2)^p} \quad \text{for } x_1, x_2 > 0,$$

and by (5)

$$f(x_1, -x_2) = \frac{c^p}{(x_1 + x_2)^p} \quad \text{for } x_1, x_2 > 0.$$

Combining these results finally yields

$$f(x) = c^p \rho(\psi_{\frac{\pi}{2}} T^2, x)^p + c^p \rho(\psi_{\frac{\pi}{2}}(-T^2), x)^p$$

almost everywhere on  $\mathbb{R}^2$ . □

For given  $p, q \in \mathbb{R}$ , we define the function  $g_{p,q}$  on  $\mathbb{R}^2$  by

$$g_{p,q}(x_1, x_2) = \begin{cases} (x_1^{pq-p} - x_2^{pq-p}) / (x_1 - x_2)^{pq} & \text{for } 0 \leq x_2 < x_1, \\ x_1^{pq-p} / (x_1 - x_2)^{pq} & \text{for } x_1 > 0, x_2 < 0, \\ 0 & \text{otherwise.} \end{cases}$$

Define the linear transformations  $\gamma_i$ ,  $i = 0, 1, 2$ , by

$$\gamma_0(x_1, x_2) = (-x_1, -x_2), \quad \gamma_1(x_1, x_2) = (x_2, x_1), \quad \gamma_2(x_1, x_2) = (-x_2, -x_1),$$

that is,  $\gamma_0$ ,  $\gamma_1$ , and  $\gamma_2$  are the reflections with respect to the origin, the first median, and the second median, respectively.

**Lemma 5.** *Let  $f : \mathbb{R}^2 \setminus \{0\} \rightarrow \mathbb{R}$  be a function positively homogeneous of degree  $-p$  such that*

$$f(x_1, x_2) = \lambda^{pq} f(x_1, (1 - \lambda)x_1 + \lambda x_2) + (1 - \lambda)^{pq} f((1 - \lambda)x_1 + \lambda x_2, x_2) \quad (16)$$

*holds on  $\mathbb{R}^2 \setminus \{0\}$  for every  $0 < \lambda < 1$ . Then*

$$f = f(1, 0) g_{p,q} + f(-1, 0) g_{p,q} \circ \gamma_0 + f(0, 1) g_{p,q} \circ \gamma_1 + f(0, -1) g_{p,q} \circ \gamma_2 \quad (17)$$

*on  $\mathbb{R}^2 \setminus \{(x_1, x_2) : x_1 = x_2\}$ .*

*Proof.* Equation (16) evaluated at the points  $\pm(1, 0)$ ,  $\pm(0, 1)$ ,  $\pm(-\lambda, 1 - \lambda)$  and the homogeneity of  $f$  yield

$$f(1, 1 - \lambda) = \frac{1 - (1 - \lambda)^{pq-p}}{\lambda^{pq}} f(1, 0), \quad (18)$$

$$f(-1, \lambda - 1) = \frac{1 - (1 - \lambda)^{pq-p}}{\lambda^{pq}} f(-1, 0), \quad (19)$$

$$f(\lambda, 1) = \frac{1 - \lambda^{pq-p}}{(1 - \lambda)^{pq}} f(0, 1), \quad (20)$$

$$f(-\lambda, -1) = \frac{1 - \lambda^{pq-p}}{(1 - \lambda)^{pq}} f(0, -1), \quad (21)$$

$$f(-\lambda, 1 - \lambda) = \lambda^{pq-p} f(-1, 0) + (1 - \lambda)^{pq-p} f(0, 1), \quad (22)$$

$$f(\lambda, \lambda - 1) = \lambda^{pq-p} f(1, 0) + (1 - \lambda)^{pq-p} f(0, -1). \quad (23)$$

First, suppose that  $x_1 > x_2 \geq 0$ . If  $x_2 = 0$ , it follows from the homogeneity of  $f$  that  $f(x_1, 0) = x_1^{-p} f(1, 0) = f(1, 0) g_{p,q}(x_1, 0)$ . For  $x_1 > x_2 > 0$  we obtain by (18)

$$\begin{aligned} f(x_1, x_2) &= x_1^{-p} f(1, 1 - (1 - x_2/x_1)) = x_1^{-p} \frac{1 - (x_2/x_1)^{pq-p}}{(1 - (x_2/x_1))^{pq}} f(1, 0) \\ &= \frac{x_1^{pq-p} - x_2^{pq-p}}{(x_1 - x_2)^{pq}} f(1, 0) = f(1, 0) g_{p,q}(x_1, x_2). \end{aligned}$$

Since  $g_{p,q} \circ \gamma_0$ ,  $g_{p,q} \circ \gamma_1$ , and  $g_{p,q} \circ \gamma_2$  are zero for  $x_1 > x_2 \geq 0$ , (17) holds in this part of the plane. (19) gives

$$\begin{aligned} f(-x_1, -x_2) &= x_1^{-p} f(-1, (1 - x_2/x_1) - 1) = x_1^{-p} \frac{1 - (x_2/x_1)^{pq-p}}{(1 - (x_2/x_1))^{pq}} f(-1, 0) \\ &= \frac{x_1^{pq-p} - x_2^{pq-p}}{(x_1 - x_2)^{pq}} f(-1, 0) = f(-1, 0) (g_{p,q} \circ \gamma_0)(-x_1, -x_2). \end{aligned}$$

But  $g_{p,q}$ ,  $g_{p,q} \circ \gamma_1$  as well as  $g_{p,q} \circ \gamma_2$  vanish for  $x_1 < x_2 < 0$  and therefore (17) is true if  $x_1 < x_2 < 0$ . Using the homogeneity we obtain that (17) is correct for  $x_1 < 0$ ,  $x_2 = 0$ .

Now, assume  $x_2 > x_1 \geq 0$ . If  $x_1 = 0$ , then we have

$$\begin{aligned} f(0, x_2) &= x_2^{-p} f(0, 1) = f(0, 1) (g_{p,q} \circ \gamma_1)(0, x_2), \\ f(0, -x_2) &= x_2^{-p} f(0, -1) = f(0, -1) (g_{p,q} \circ \gamma_2)(0, -x_2). \end{aligned}$$

Formulae (20) and (21) for  $x_2 > x_1 > 0$  yield

$$\begin{aligned} f(x_1, x_2) &= x_2^{-p} f(x_1/x_2, 1) = x_2^{-p} \frac{1 - (x_1/x_2)^{pq-p}}{(1 - (x_1/x_2))^{pq}} f(0, 1) \\ &= \frac{x_2^{pq-p} - x_1^{pq-p}}{(x_2 - x_1)^{pq}} f(0, 1) = f(0, 1) (g_{p,q} \circ \gamma_1)(x_1, x_2), \\ f(-x_1, -x_2) &= x_2^{-p} f(-x_1/x_2, -1) = x_2^{-p} \frac{1 - (x_1/x_2)^{pq-p}}{(1 - (x_1/x_2))^{pq}} f(0, -1) \\ &= \frac{x_2^{pq-p} - x_1^{pq-p}}{(x_2 - x_1)^{pq}} f(0, -1) = f(0, -1) (g_{p,q} \circ \gamma_2)(-x_1, -x_2). \end{aligned}$$

Since  $g_{p,q}$ ,  $g_{p,q} \circ \gamma_0$ ,  $g_{p,q} \circ \gamma_2$  are zero for  $x_2 > x_1 \geq 0$  and  $g_{p,q}$ ,  $g_{p,q} \circ \gamma_0$ ,  $g_{p,q} \circ \gamma_1$  vanish for  $x_2 < x_1 \leq 0$ , it remains to prove identity (17) if the coordinates have different signs.

Finally, let  $x_1$  and  $x_2$  be greater than zero. By (22) and (23) we have

$$\begin{aligned}
f(-x_1, x_2) &= (x_1 + x_2)^{-p} f(-x_1/(x_1 + x_2), 1 - x_1/(x_1 + x_2)) \\
&= \frac{x_2^{pq-p}}{(x_1 + x_2)^{pq}} f(0, 1) + \frac{x_1^{pq-p}}{(x_1 + x_2)^{pq}} f(-1, 0) \\
&= f(0, 1)(g_{p,q} \circ \gamma_1)(-x_1, x_2) + f(-1, 0)(g_{p,q} \circ \gamma_0)(-x_1, x_2), \\
f(x_1, -x_2) &= (x_1 + x_2)^{-p} f(x_1/(x_1 + x_2), x_1/(x_1 + x_2) - 1) \\
&= \frac{x_2^{pq-p}}{(x_1 + x_2)^{pq}} f(0, -1) + \frac{x_1^{pq-p}}{(x_1 + x_2)^{pq}} f(1, 0) \\
&= f(0, -1)(g_{p,q} \circ \gamma_2)(x_1, -x_2) + f(1, 0)g_{p,q}(x_1, -x_2).
\end{aligned}$$

The fact that  $g_{p,q}$  and  $g_{p,q} \circ \gamma_2$  are zero in the second quadrant and  $g_{p,q} \circ \gamma_0$ ,  $g_{p,q} \circ \gamma_1$  are zero in the fourth quadrant completes the proof.  $\square$

In the following, we have  $q > 1$ . Therefore Lemma 2 and Lemma 3 imply that  $f$  is continuous on  $S^{n-1}$ . Thus (6) holds on  $\mathbb{R} \setminus \{0\}$  and  $f$  satisfies the conditions of Lemma 5. Combined with (5) this implies that

$$f = f(1, 0)(g_{p,q} + g_{p,q} \circ \gamma_1) + f(-1, 0)(g_{p,q} \circ \gamma_0 + g_{p,q} \circ \gamma_2) \quad (24)$$

on  $\mathbb{R}^2 \setminus \{0\}$ .

**Lemma 6.** *Let  $Z : \mathcal{P}_0^2 \rightarrow \langle \mathcal{S}^2, \tilde{\dagger}_p \rangle$  be a valuation which is  $\mathrm{GL}(2)$  contravariant of weight  $q$ . Let  $p > 1$ ,  $q > 1$  or  $0 < p < 1$ ,  $q > 1/p$ . Then*

$$ZP = \{0\}$$

for every  $P \in \mathcal{P}_0^2$ .

*Proof.* For  $x_2 > 0$  fixed, we obtain by (24) that

$$\lim_{x_1 \rightarrow x_2^+} f(x_1, x_2) = \lim_{x_1 \rightarrow x_2^+} \frac{x_1^{pq-p} - x_2^{pq-p}}{(x_1 - x_2)^{pq}} f(1, 0)$$

has to be finite. This implies that  $f(1, 0)$  has to be zero.

Considering  $\lim_{x_1 \rightarrow x_2^+} f(-x_1, -x_2)$  proves  $f(-1, 0) = 0$ . So by (24),  $f$  vanishes on  $\mathbb{R}^2 \setminus \{0\}$ .  $\square$

**Lemma 7.** *For  $p < 1$ , let  $Z : \mathcal{P}_0^2 \rightarrow \langle \mathcal{S}^2, \tilde{\dagger}_p \rangle$  be a valuation which is  $\mathrm{GL}(2)$  contravariant of weight  $q = 1/p$ . Then there exist constants  $c_1, c_2 \geq 0$  such that*

$$ZP = c_1 I_p^+ P \tilde{\dagger}_p c_2 I_p^- P$$

for every  $P \in \mathcal{P}_0^2$ .

*Proof.* A simple calculation shows

$$\rho(\mathbb{I}_p^+ T^2, \cdot)^p = (p^2 - 3p + 2)^{-1}(g_{p,1/p} + g_{p,1/p} \circ \gamma_1) \quad (25)$$

almost everywhere. Therefore

$$\rho(\mathbb{I}_p^- T^2, \cdot)^p = \rho(\mathbb{I}_p^+ T^2, \gamma_0(\cdot))^p = (p^2 - 3p + 2)^{-1}(g_{p,1/p} \circ \gamma_0 + g_{p,1/p} \circ \gamma_2). \quad (26)$$

Combined with (24), these equations complete the proof.  $\square$

Finally, we consider the case  $p < 1$  and  $q \in (1, 1/p)$ . We define

$$\rho(\mathbb{I}_{p,q}^+ T^2, x)^p = (g_{p,q} + g_{p,q} \circ \gamma_1)(x).$$

The restrictions on  $q$  show that  $\rho(\mathbb{I}_{p,q}^+ T^2, \cdot)$  is continuous and non-negative on  $\mathbb{R}^2 \setminus \{0\}$ . By definition,  $\rho(\mathbb{I}_{p,q}^+ T^2, \cdot)$  is positively homogeneous of degree  $-1$  and thus is the radial function of a star body.

We extend this definition to all simplices in  $\mathbb{R}^2$  having one vertex at the origin (we denote this set by  $\mathcal{T}_0^2$ ):

$$\mathbb{I}_{p,q}^+ S = \begin{cases} |\det \phi|^q \phi^{-t} \mathbb{I}_{p,q}^+ T^2 & \text{if } S \text{ is 2-dimensional and } S = \phi T^2, \\ \{0\} & \text{otherwise.} \end{cases}$$

Note that  $\mathbb{I}_{p,q}^+$  is well defined on  $\mathcal{T}_0^2$  since  $\rho(\mathbb{I}_{p,q}^+ T^2, \cdot)$  does not change if the coordinates are interchanged. We claim that  $\mathbb{I}_{p,q}^+$  is a valuation on  $\mathcal{T}_0^2$ . To prove this, it suffices to check the valuation property if the two involved simplices coincide in an edge. Since by definition  $\mathbb{I}_{p,q}^+$  is  $\text{GL}(2)$  contravariant, it suffices to check the valuation property for the standard simplex. Thus it suffices to show that

$$\mathbb{I}_{p,q}^+ T^2 = \mathbb{I}_{p,q}^+(T^2 \cap H^+) \dot{+}_p \mathbb{I}_{p,q}^+(T^2 \cap H^-)$$

where  $H$  is the line with normal vector  $\lambda e_1 - (1 - \lambda)e_2$ ,  $0 < \lambda < 1$ . Therefore we have to prove

$$\begin{aligned} \rho(\mathbb{I}_{p,q}^+ T^2, (x_1, x_2))^p &= \lambda^{pq} \rho(\mathbb{I}_{p,q}^+ T^2, (x_1, (1 - \lambda)x_1 + \lambda x_2))^p \\ &\quad + (1 - \lambda)^{pq} \rho(\mathbb{I}_{p,q}^+ T^2, ((1 - \lambda)x_1 + \lambda x_2, x_2))^p. \end{aligned} \quad (27)$$

The case  $x_1, x_2 < 0$  is trivial. So assume  $x_1 > x_2 \geq 0$ . Then  $x_1 > (1 - \lambda)x_1 + \lambda x_2 \geq 0$ ,  $(1 - \lambda)x_1 + \lambda x_2 > x_2 \geq 0$ , and the right hand side of (27) equals

$$\lambda^{pq} \frac{x_1^{pq-p} - ((1 - \lambda)x_1 + \lambda x_2)^{pq-p}}{(x_1 - (1 - \lambda)x_1 - \lambda x_2)^{pq}} + (1 - \lambda)^{pq} \frac{((1 - \lambda)x_1 + \lambda x_2)^{pq-p} - x_2^{pq-p}}{((1 - \lambda)x_1 + \lambda x_2 - x_2)^{pq}}$$

which is nothing else than  $\rho(\mathbb{I}_{p,q}^+ T^2, (x_1, x_2))^p$ . Similar, we obtain (27) for points  $x_2 > x_1 \geq 0$ . To check (27) for  $(x_1, -x_2)$ ,  $x_1, x_2 > 0$ , we first assume that  $(1 - \lambda)x_1 - \lambda x_2 > 0$ . Then  $0 < (1 - \lambda)x_1 - \lambda x_2 < x_1$  and the sum appearing in (27) equals

$$\lambda^{pq} \frac{x_1^{pq-p} - ((1 - \lambda)x_1 - \lambda x_2)^{pq-p}}{(x_1 - (1 - \lambda)x_1 + \lambda x_2)^{pq}} + (1 - \lambda)^{pq} \frac{((1 - \lambda)x_1 - \lambda x_2)^{pq-p}}{((1 - \lambda)x_1 - \lambda x_2 + x_2)^{pq}}.$$

If  $(1 - \lambda)x_1 - \lambda x_2 < 0$ , the right hand side of (27) is

$$\lambda^{pq} \frac{x_1^{pq-p}}{(x_1 - (1 - \lambda)x_1 + \lambda x_2)^{pq}}.$$

These two expressions are equal to  $\rho(\mathbb{I}_{p,q}^+ T^2, (x_1, -x_2))^p$ . The case  $(1 - \lambda)x_1 - \lambda x_2 = 0$  is simple and the remaining part can be treated in an analogous way.

Now, we extend the valuation  $\mathbb{I}_{p,q}^+$  to  $\overline{\mathcal{P}}_0^2$  by setting

$$\rho(\mathbb{I}_{p,q}^+ P, x)^p = \sum_{i \in I} \rho(\mathbb{I}_{p,q}^+ S_i, x)^p,$$

where  $\{S_i : i \in I, \dim S_i = 2\} \subset \mathcal{T}_0^2$  is a dissection of  $P$ , that is,  $I$  is finite,  $P = \bigcup_{i \in I} S_i$  and no pair of simplices intersects in a set of dimension 2.

Given two different dissections, it is always possible to obtain one from the other by a finite number of the following operations: a simplex is dissected into two 2-dimensional simplices by a line through the origin, or the converse, that is, two simplices whose union is again a simplex are replaced by their union (We remark that the corresponding result holds true for  $n \geq 3$ , see [26]). Since  $\mathbb{I}_{p,q}^+$  is a valuation on  $\mathcal{T}_0^2$ , this shows that  $\mathbb{I}_{p,q}^+$  is well defined on  $\overline{\mathcal{P}}_0^2$ .

We have to prove that  $\mathbb{I}_{p,q}^+$  is a valuation. To do so, let  $P, Q \in \overline{\mathcal{P}}_0^2$  be two 2-dimensional convex polytopes such that their union is again convex. We dissect  $\mathbb{R}^2$  into 2-dimensional convex cones with apex 0 in such a way that each vertex of  $P, Q, P \cap Q, P \cup Q$  lies on the boundary of some cone in this dissection. The intersections of such a cone with the boundary of  $P$  and  $Q$  are line segments which are either identical, do not intersect, or intersect in their endpoints only. Therefore  $\mathbb{I}_{p,q}^+$  is a valuation and obviously it is  $\text{GL}(2)$  contravariant of weight  $q$ .

We define the  $L_p$  radial valuation  $\mathbb{I}_{p,q}^-$  by setting  $\mathbb{I}_{p,q}^- P = \mathbb{I}_{p,q}^+(-P)$ . Now, (24) implies the following result.

**Lemma 8.** *For  $p < 1$ , let  $Z : \mathcal{P}_0^2 \rightarrow \langle \mathcal{S}^2, \tilde{\dagger}_p \rangle$  be a valuation which is  $\text{GL}(2)$  contravariant of weight  $1 < q < 1/p$ . Then there exist constants  $c_1, c_2 \geq 0$  such that*

$$ZP = c_1 \mathbb{I}_{p,q}^+ P \tilde{\dagger}_p c_2 \mathbb{I}_{p,q}^- P$$

for every  $P \in \mathcal{P}_0^2$ .

### 3.2 The 2-dimensional Covariant Case

Let  $Z : \mathcal{P}_0^2 \rightarrow \langle \mathcal{S}^2, \tilde{\dagger}_p \rangle$  be a valuation which is  $\text{GL}(2)$  covariant of weight  $q$ . Let  $q > 0$ . As before, let  $Y P(\cdot) = \rho(ZP, \cdot)^p$  and denote the extension of  $Y$  to  $\overline{\mathcal{P}}_0^2$  by  $\overline{Y}$ . Note that Lemma 2 and Lemma 3 imply that  $\overline{Y}$  is continuous on  $S^1$ . We define the valuation  $\hat{Y}$  by  $\hat{Y} P(\cdot) = \overline{Y} P(\psi_{\frac{\pi}{2}}^{-1}(\cdot))$  for every  $P \in \overline{\mathcal{P}}_0^2$ . From (4) and (8) it follows that for  $\phi \in \text{GL}(2)$  with  $\det \phi > 0$

$$\hat{Y} \phi P(x) = |\det \phi|^{pq} \overline{Y} P(\phi^{-1} \psi_{\frac{\pi}{2}}^{-1} x) = |\det \phi|^{pq+p} \hat{Y} P(\phi^t x)$$

for every  $P \in \overline{\mathcal{P}}_0^2$ . So  $\hat{Y}T^2$  satisfies (16) with  $q + 1$  instead of  $q$ . From the  $\text{GL}(2)$  covariance it follows that  $\hat{Y}T^2(x_1, x_2) = \hat{Y}T^2(-x_2, -x_1)$ . Thus Lemma 5 shows that

$$\hat{Y}T^2 = \hat{Y}T^2(1, 0)(g_{p,q+1} + g_{p,q+1} \circ \gamma_2) + \hat{Y}T^2(0, 1)(g_{p,q+1} \circ \gamma_0 + g_{p,q+1} \circ \gamma_1). \quad (28)$$

Considering the limit

$$\lim_{x_1 \rightarrow x_2+} \frac{x_1^{pq} - x_2^{pq}}{(x_1 - x_2)^{pq+p}}$$

for fixed  $x_2 > 0$ , we derive for  $p > 1$  that  $\hat{Y}T^2(1, 0) = \hat{Y}T^2(0, 1) = 0$  since  $\hat{Y}T^2$  is continuous on  $\mathbb{R}^2 \setminus \{0\}$  and has to be finite on the first median. This limit also proves that  $\hat{Y}T^2(1, 0) = \hat{Y}T^2(0, 1) = 0$  for  $p < 1$  and  $q > 1/p - 1$ . Now, (28) implies the following result.

**Lemma 9.** *Let  $Z : \mathcal{P}_0^2 \rightarrow \langle \mathcal{S}^2, \tilde{\dagger}_p \rangle$  be a valuation which is  $\text{GL}(2)$  covariant of weight  $q > 0$ . Let  $p > 1$ ,  $q > 0$  or  $0 < p < 1$ ,  $q > 1/p - 1$ . Then*

$$ZP = \{0\}$$

for every  $P \in \mathcal{P}_0^2$ .

For  $p < 1$  and  $q \in (0, 1/p - 1)$ , we define  $J_{p,q}^+$  by

$$\rho(J_{p,q}^+ T^2, x)^p = (g_{p,q+1} + g_{p,q+1} \circ \gamma_2)(\psi_{\frac{\pi}{2}} x).$$

Similar to the contravariant case,  $J_{p,q}^+$  can be extended to a covariant valuation on  $\mathcal{P}_0^2$ . We define  $J_{p,q}^-$  by  $J_{p,q}^- P = J_{p,q}^+(-P)$ . Now, (28) implies the following result.

**Lemma 10.** *For  $p < 1$ , let  $Z : \mathcal{P}_0^2 \rightarrow \langle \mathcal{S}^2, \tilde{\dagger}_p \rangle$  be a valuation which is  $\text{GL}(2)$  covariant of weight  $0 < q < 1/p - 1$ . Then there exist constants  $c_1, c_2 \geq 0$  such that*

$$ZP = c_1 J_{p,q}^+ P \tilde{\dagger}_p c_2 J_{p,q}^- P$$

for every  $P \in \mathcal{P}_0^2$ .

For  $q = 1/p - 1$ , the continuity of  $\hat{Y}T^2$  at the first median and (28) yield that  $\hat{Y}T^2(1, 0) = \hat{Y}T^2(0, 1)$ . Therefore we obtain the following lemma by using (25), (26) and the identity  $\psi_{\frac{\pi}{2}} I_p P = \psi_{\frac{\pi}{2}}^{-1} I_p P$ .

**Lemma 11.** *Let  $Z : \mathcal{P}_0^2 \rightarrow \langle \mathcal{S}^2, \tilde{\dagger}_p \rangle$  be a valuation which is  $\text{GL}(2)$  covariant of weight  $q = 1/p - 1$ . Then there exists a constant  $c \geq 0$  such that*

$$ZP = c \psi_{\frac{\pi}{2}} I_p P$$

for every  $P \in \mathcal{P}_0^2$ .

### 3.3 The 2-dimensional Classification Theorems

Using the lemmas of the preceding sections and the planar case of Lemma 12 and Lemma 17, we obtain the following result.

**Theorem 3.** *For  $0 < p < 1$ , an operator  $Z : \mathcal{P}_0^2 \rightarrow \langle \mathcal{S}^2, \tilde{\dagger}_p \rangle$  is a non-trivial valuation which is  $\text{GL}(2)$  covariant of weight  $q$  if and only if there are constants  $c_1, c_2 \geq 0$  such that*

$$ZP = \begin{cases} c_1 I_p^+ P^* \tilde{\dagger}_p c_2 I_p^- P^* & \text{for } q = -1/p \\ c_1 I_{p,q}^+ P^* \tilde{\dagger}_p c_2 I_{p,q}^- P^* & \text{for } -1/p < q < -1 \\ c_1 \psi_{\frac{\pi}{2}}(P^* \tilde{\dagger}_p (-P^*)) & \text{for } q = -1 \\ c_1 J_{p,q}^+ P \tilde{\dagger}_p c_2 J_{p,q}^- P & \text{for } 0 < q < 1/p - 1 \\ c_1 \psi_{\frac{\pi}{2}} I_p P & \text{for } q = 1/p - 1 \end{cases}$$

for every  $P \in \mathcal{P}_0^2$ . For  $p > 1$ , an operator  $Z : \mathcal{P}_0^2 \rightarrow \langle \mathcal{S}^2, \tilde{\dagger}_p \rangle$  is a non-trivial  $\text{GL}(2)$  covariant valuation if and only if there is a constant  $c \geq 0$  such that

$$ZP = c \psi_{\frac{\pi}{2}}(P^* \tilde{\dagger}_p (-P^*))$$

for every  $P \in \mathcal{P}_0^2$ .

Next, we consider an operator  $Z$  with centrally symmetric images. Note that in this case also the extended operator  $\bar{Z}$  has centrally symmetric images. Using again the lemmas of the preceding sections and the planar case of Lemma 12 and Lemma 17, we obtain the following result. Here  $I_{p,q} P = I_{p,q}^+ P \tilde{\dagger}_p I_{p,q}^- P$ .

**Theorem 4.** *For  $0 < p < 1$ , an operator  $Z : \mathcal{P}_0^2 \rightarrow \langle \mathcal{S}_c^2, \tilde{\dagger}_p \rangle$  is a non-trivial valuation which is  $\text{GL}(2)$  covariant of weight  $q$  if and only if there is a constant  $c \geq 0$  such that*

$$ZP = \begin{cases} c I_p P^* & \text{for } q = -1/p \\ c I_{p,q} P^* & \text{for } -1/p < q < -1 \\ c \psi_{\frac{\pi}{2}}(P^* \tilde{\dagger}_p (-P^*)) & \text{for } q = -1 \\ c \psi_{\frac{\pi}{2}} I_{p,q} P & \text{for } 0 < q < 1/p - 1 \\ c \psi_{\frac{\pi}{2}} I_p P & \text{for } q = 1/p - 1 \end{cases}$$

for every  $P \in \mathcal{P}_0^2$ . For  $p > 1$ , an operator  $Z : \mathcal{P}_0^2 \rightarrow \langle \mathcal{S}_c^2, \tilde{\dagger}_p \rangle$  is a non-trivial  $\text{GL}(2)$  covariant valuation if and only if there is a constant  $c \geq 0$  such that

$$ZP = c \psi_{\frac{\pi}{2}}(P^* \tilde{\dagger}_p (-P^*))$$

for every  $P \in \mathcal{P}_0^2$ .

### 3.4 The Contravariant Case for $n \geq 3$

**Lemma 12.** *Let  $Z : \mathcal{P}_0^n \rightarrow \langle \mathcal{S}^n, \tilde{\dagger}_p \rangle$ ,  $n \geq 2$ , be a valuation which is  $\text{GL}(n)$  contravariant of weight  $0 < q < 1$ . Then*

$$ZP = \{0\}$$

for every  $P \in \mathcal{P}_0^n$ .

*Proof.* From (6) we deduce that for  $x \notin \text{lin } e_1 \cup \dots \cup \text{lin } e_n \cup \text{lin}(\lambda_j e_1 - (1 - \lambda_j)e_j)$

$$f(x) = \lambda_j^{pq} f(\phi_j^t x) + (1 - \lambda_j)^{pq} f(\psi_j^t x) \quad (29)$$

holds. First, we want to show that  $f$  is uniformly bounded on  $S^{n-1} \setminus \{\pm e_1, \dots, \pm e_n\}$ . To do so, note that since  $f$  is positive, equation (29) for  $j = 2$  at  $(x_1, 1 - \lambda_2, x_3, \dots, x_n)$  and  $(x_1, -(1 - \lambda_2), x_3, \dots, x_n)$  gives

$$f(x_1, (1 - \lambda_2)(x_1 + \lambda_2), x_3, \dots, x_n) \leq \lambda_2^{-pq} f(x_1, 1 - \lambda_2, x_3, \dots, x_n), \quad (30)$$

$$f(x_1, (1 - \lambda_2)(x_1 - \lambda_2), x_3, \dots, x_n) \leq \lambda_2^{-pq} f(x_1, -(1 - \lambda_2), x_3, \dots, x_n). \quad (31)$$

Let  $x_1 \rightarrow -\lambda_2$ ,  $x_2, \dots, x_n \rightarrow 0$  in (30). Since  $f$  is continuous on  $S^{n-1} \setminus \{\pm e_1, \dots, \pm e_n\}$  and homogeneous, it is bounded in a suitable neighbourhood of  $-\lambda_2 e_1 + (1 - \lambda_2)e_2$ . Thus  $f$  is also bounded in a suitable neighbourhood of  $-\lambda_2 e_1$ . From (5) we conclude that  $f$  is bounded in a neighbourhood of every  $-e_i$ ,  $i = 1, \dots, n$ . Proceeding in an analogous way but taking the limit  $x_1 \rightarrow \lambda_2$  and taking (31) into account we obtain the boundedness in suitable neighbourhoods of  $e_i$ ,  $i = 1, \dots, n$ .

From (29) we know that

$$f(\phi_2^{-t} x) = \lambda_2^{pq} f(x) + (1 - \lambda_2)^{pq} f(\psi_2^t \phi_2^{-t} x)$$

for  $x \notin \text{lin } e_1 \cup \dots \cup \text{lin } e_n \cup \text{lin}(e_1 + (1 - \lambda_2)e_2)$ . Thus we obtain for  $(-1, 1, x_3, \dots, x_n)$  by using the homogeneity and the non-negativity of  $f$  that

$$\lambda_2^{pq-p} f(-1, 1, x_3, \dots, x_n) \leq f(-\lambda_2, 2 - \lambda_2, \lambda_2 x_3, \dots, \lambda_2 x_n).$$

Since  $pq - p < 0$  and  $f$  is bounded, this yields

$$f(-1, 1, x_3, \dots, x_n) = 0, \quad x_3, \dots, x_n \in \mathbb{R}.$$

Evaluating (29) at  $(-1, 1, x_3, \dots, x_n)$  proves

$$0 = \lambda_2^{pq} f(-1, 2\lambda_2 - 1, x_3, \dots, x_n) + (1 - \lambda_2)^{pq} f(2\lambda_2 - 1, 1, x_3, \dots, x_n).$$

for  $\lambda_2 \neq 1/2$ . Since  $f$  is non-negative,

$$f(-1, x_2, x_3, \dots, x_n) = 0, \quad -1 < x_2 < 1, \quad x_2 \neq 0, \quad x_3, \dots, x_n \in \mathbb{R},$$

$$f(x_1, 1, x_3, \dots, x_n) = 0, \quad -1 < x_1 < 1, \quad x_1 \neq 0, \quad x_3, \dots, x_n \in \mathbb{R}.$$

Because of (3) we also have for  $-1 < x_1 < 1$ ,  $-1 < x_2 < 1$ ,  $x_1, x_2 \neq 0$  and arbitrary  $x_3, \dots, x_n$

$$f(x_1, -1, x_3, \dots, x_n) = 0,$$

$$f(1, x_2, x_3, \dots, x_n) = 0.$$

These last four equations prove that  $f$  is equal to zero almost everywhere on  $\mathbb{R}^n$ .  $\square$



**Lemma 13.** Let  $Z : \mathcal{P}_0^n \rightarrow \langle \mathcal{S}^n, \tilde{\dagger}_p \rangle$ ,  $n \geq 3$ , be a valuation which is  $\text{GL}(n)$  contravariant of weight  $q = 1$ . Then

$$ZP = \{0\}$$

for every  $P \in \mathcal{P}_0^n$ .

*Proof.* By Lemma 2 and Lemma 3,  $f$  is continuous and uniformly bounded on  $S^{n-1}$  except on  $\text{lin } e_1 \cup \dots \cup \text{lin } e_n$ . By (6), we have for  $2 \leq j \leq n$

$$f(x) = \lambda_j^p f(\phi_j^t x) + (1 - \lambda_j)^p f(\psi_j^t x) \quad (32)$$

on  $\mathbb{R}^n$  except on a finite union of lines. Using this repeatedly, we get

$$\begin{aligned} f(x) &= \lambda_2^p \cdots \lambda_n^p f(\phi_n^t \cdots \phi_2^t x) + \sum_{j=3}^n \lambda_2^p \cdots \lambda_{j-1}^p (1 - \lambda_j)^p f(\psi_j^t \phi_{j-1}^t \cdots \phi_2^t x) \\ &\quad + (1 - \lambda_2)^p f(\psi_2^t x) \\ &\geq \lambda_2^p \cdots \lambda_n^p f(\phi_n^t \cdots \phi_2^t x) + (1 - \lambda_2)^p f(\psi_2^t x). \end{aligned}$$

This implies for  $k = 1, 2, \dots$ ,

$$f((\psi_2^{-t})^k x) \geq \lambda_2^p \cdots \lambda_n^p \sum_{i=1}^k (1 - \lambda_2)^{p(k-i)} f(\phi_n^t \cdots \phi_2^t (\psi_2^{-t})^i x) \quad (33)$$

except on countably many lines. Define  $x' = x_3 e_3 + \dots + x_n e_n$ . Evaluating (33) at suitable  $e_1 + x'$  and multiplying by  $(1 - \lambda_2)^{-pk}$  shows that

$$f(e_1 + (1 - \lambda_2)^k x') \geq \lambda_2^p \cdots \lambda_n^p \sum_{i=1}^k f(\phi_n^t \cdots \phi_2^t (e_1 + (1 - \lambda_2)^i x')).$$

Let  $k \rightarrow \infty$ . Since  $f$  is uniformly bounded and continuous at  $\phi_n^t \cdots \phi_2^t e_1 = e_1 + (1 - \lambda_2)e_2 + \dots + (1 - \lambda_n)e_n$ , it follows that  $f(\phi_n^t \cdots \phi_2^t e_1) = 0$ . So we get

$$f(1, x_2, \dots, x_n) = 0, \quad 0 < x_2, \dots, x_n < 1.$$

From (5) we obtain (using the homogeneity of  $f$ ) that

$$f(1, x_2, \dots, x_n) = 0, \quad x_2, \dots, x_k > 0, \quad 0 < x_{k+1}, \dots, x_n < 1.$$

So  $f(x_1, \dots, x_n) = 0$  for  $x_1, \dots, x_n > 0$ . Considering  $-e_1 + x'$  and (33) like before shows  $f(-x_1, \dots, -x_n) = 0$  for  $x_1, \dots, x_n > 0$ .

Note that (32) for  $j = 2$  and arbitrary  $c \geq 1$  at  $(c, c, -1, c, \dots, c)$  proves (since  $p \neq 1$ ) that  $f(c, c, -1, c, \dots, c) = 0$ . Let  $x_1 < 0$ ,  $x_2, \dots, x_n > 0$ , and  $(1 - \lambda_j)x_1 + \lambda_j x_j > 0$ . By (32) and the fact that  $f$  vanishes at points having all coordinates greater than zero we get

$$f(\phi_n^t \cdots \phi_2^t x) = \frac{1}{\lambda_2^p \cdots \lambda_n^p} f(x)$$

except on finitely many lines. Thus we obtain

$$\begin{aligned} &\lambda_2^{-p} \cdots \lambda_n^{-p} f(-1, c - \varepsilon, c, \dots, c) \\ &= f(\phi_n^t \cdots \phi_2^t (-1, c - \varepsilon, c, \dots, c)) \\ &= f(-1, -1 + \lambda_2(1 + c - \varepsilon), -1 + \lambda_3(1 + c), \dots, -1 + \lambda_n(1 + c)) \end{aligned}$$

for suitable  $\varepsilon > 0$  and  $\lambda_2, \dots, \lambda_n > 1/(1 + c - \varepsilon)$ . The continuity of  $f$  shows

$$f(-1, x_2, \dots, x_n) = 0, \quad 0 < x_2, \dots, x_n < c.$$

But  $c \geq 1$  was arbitrary, so  $f(-1, x_2, \dots, x_n) = 0$  for  $x_2, \dots, x_n > 0$ . The homogeneity yields  $f(-x_1, x_2, \dots, x_n) = 0$  for  $x_1, x_2, \dots, x_n > 0$ . In conclusion,  $f(x_1, \dots, x_n) = 0$  if at most one coordinate is negative. Suppose  $f(x_1, \dots, x_n) = 0$  where at most  $1 \leq k < n - 1$  coordinates are negative. Let  $x$  be chosen such that  $x_1, \dots, x_{k+1} < 0$  and  $x_{k+2}, \dots, x_n > 0$ . Suppose  $x_2 < x_1 < 0$ . Choose  $\lambda_2$  with  $0 < x_1/x_2 < \lambda_2 < 1$ . Then

$$\begin{aligned} (\psi_2^{-t}x)_1 &= (\phi_2^t \psi_2^{-t}x)_1 = \frac{x_1}{1 - \lambda_2} - \frac{\lambda_2}{1 - \lambda_2} x_2 > 0, \\ (\psi_2^{-t}x)_i &= (\phi_2^t \psi_2^{-t}x)_i > 0, \quad i = k + 2, \dots, n. \end{aligned}$$

Since  $f(\psi_2^{-t}x) = \lambda_2^{pq} f(\phi_2^t \psi_2^{-t}x) + (1 - \lambda_2)^{pq} f(x)$  we obtain  $f(x) = 0$ . By (5) we conclude  $f(x) = 0$  for the case  $x_1 < x_2 < 0$ .  $\square$

In the following, we have  $q > 1$ . Therefore Lemma 2 and Lemma 3 imply that  $f$  is continuous on  $\mathbb{R}^n \setminus \{0\}$ . In the proof of Lemmas 14 to 16, we use the following remark. Suppose we have two functions  $f_1, f_2$  which are continuous on  $\mathbb{R}^n \setminus \{0\}$  satisfying (5) and such that for  $0 < \lambda_j < 1$ ,  $j = 2, \dots, n$ ,

$$f_i(x) = \lambda_j^{pq} f_i(\phi_j^t x) + (1 - \lambda_j)^{pq} f_i(\psi_j^t x)$$

holds on  $\mathbb{R}^n$ . Further assume that these functions are equal for all points where at most two coordinates do not vanish. Then an argument similar to that at the end of the last proof shows that these functions have to be equal.

**Lemma 14.** *For  $p > 1$ , let  $Z : \mathcal{P}_0^n \rightarrow \langle \mathcal{S}^n, \tilde{\dagger}_p \rangle$  be a valuation which is  $\text{GL}(n)$  contravariant of weight  $q > 1$ . Then*

$$ZP = \{0\}$$

for every  $P \in \mathcal{P}_0^n$ .

*Proof.* Define  $\tilde{f}(x_1, x_2) = f(x_1 e_1 + x_2 e_2)$ . Then  $\tilde{f}$  is continuous and satisfies the conditions of Lemma 5. The proof of Lemma 6 shows  $\tilde{f} = 0$ . By (5) this implies that  $f(x_i e_i + x_j e_j) = 0$  for arbitrary  $1 \leq i, j \leq n$ . Thus  $f$  vanishes on  $\mathbb{R}^n \setminus \{0\}$ .  $\square$

**Lemma 15.** *For  $p < 1$ , let  $Z : \mathcal{P}_0^n \rightarrow \langle \mathcal{S}^n, \tilde{\dagger}_p \rangle$  be a valuation which is  $\text{GL}(n)$  contravariant of weight  $q = 1/p$ . Then there exist constants  $c_1, c_2 \geq 0$  such that*

$$ZP = c_1 I_p^+ P \tilde{\dagger}_p c_2 I_p^- P$$

for every  $P \in \mathcal{P}_0^n$ .

*Proof.* For  $x = x_1 e_1 + x_2 e_2$ , note that  $\rho(I_p^\pm T^n, x)$  is a multiple of  $\rho(I_p^\pm T^2, (x_1, x_2))$ . This and an analogous argument as before proves the lemma.  $\square$

**Lemma 16.** For  $p < 1$ , let  $Z : \mathcal{P}_0^n \rightarrow \langle \mathcal{S}^n, \tilde{\dagger}_p \rangle$  be a valuation which is  $\text{GL}(n)$  contravariant of weight  $q > 1$ ,  $q \neq 1/p$ . Then

$$ZP = \{0\}$$

for every  $P \in \mathcal{P}_0^n$ .

*Proof.* By (6) we have

$$f(x) = \lambda^{pq} f(\phi_2^{\dagger} x) + (1 - \lambda)^{pq} f(\psi_2^{\dagger} x)$$

on  $\mathbb{R}^n \setminus \{0\}$ . Since  $e_3$  is an eigenvector of  $\phi_2^{\dagger}$  and  $\psi_2^{\dagger}$  with eigenvalue 1, we get  $f(\pm e_i) = 0$  for  $i = 1, \dots, n$ . For  $\tilde{f}(x_1, x_2) = f(x_1 e_1 + x_2 e_2)$  this implies  $\tilde{f}(1, 0) = \tilde{f}(-1, 0) = 0$ . Lemma 5 proves  $\tilde{f} = 0$ .  $\square$

### 3.5 The Covariant Case for $n \geq 3$

**Lemma 17.** Let  $Z : \mathcal{P}_0^n \rightarrow \langle \mathcal{S}^n, \tilde{\dagger}_p \rangle$ ,  $n \geq 2$ , be a valuation which is  $\text{GL}(n)$  covariant of weight  $q = 0$ . Then there exist constants  $c_1, c_2 \geq 0$  such that

$$ZP = c_1 P \tilde{\dagger}_p c_2(-P)$$

for every  $P \in \mathcal{P}_0^n$ .

*Proof.* By Lemma 2 and Lemma 3,  $f$  is continuous and uniformly bounded on  $S^{n-1} \setminus (e_1^{\perp} \cup \dots \cup e_n^{\perp})$ . By (7), the equation

$$f(x) = f(\phi_j^{-1} x) + f(\psi_j^{-1} x) \quad (34)$$

holds for  $x \notin e_1^{\perp} \cup \dots \cup e_n^{\perp} \cup (\lambda_j e_1 - (1 - \lambda_j) e_j)^{\perp}$ . Using this, we get by induction for  $k = 1, 2, \dots$ ,

$$f(\phi_2^k x) = \sum_{i=1}^k f(\psi_2^{-1} \phi_2^i x) + f(x) \quad (35)$$

for  $x \notin e_2^{\perp} \cup \dots \cup e_n^{\perp} \cup \bigcup_{i=1}^{\infty} (e_1 + a_i e_2)^{\perp}$  and a suitable sequence  $(a_i)$ . Define  $x' = x_1 e_1 + x_3 e_3 + x_4 e_4 + \dots + x_n e_n$  where  $x_1, x_3, x_4, \dots, x_n \neq 0$  and  $x_1 \neq 1 - a_i$  for every  $i$ . Then (35) at  $e_2 - e_1 + x'$  and the non-negativity of  $f$  show

$$f(\lambda_2^k (e_2 - e_1) + x') \geq \sum_{i=1}^k f(\psi_2^{-1} (\lambda_2^i (e_2 - e_1) + x')).$$

Let  $k \rightarrow \infty$ . Since  $f$  is uniformly bounded,  $\lim_{i \rightarrow \infty} f(\psi_2^{-1} (\lambda_2^i (e_2 - e_1) + x')) = 0$ . The continuity properties of  $f$  yield

$$f\left(\frac{x_1}{1 - \lambda_2}, \frac{-\lambda_2 x_1}{1 - \lambda_2}, x_3, \dots, x_n\right) = 0, \quad \text{for } x_1, x_3, \dots, x_n \neq 0. \quad (36)$$

From (36) we obtain that

$$f(x_1, x_2, \dots, x_n) = 0, \quad \text{for } x_1, x_2, \dots, x_n \neq 0 \text{ and not all } x_i \text{ have the same sign.}$$

For  $j = 2, 3, \dots, n$  and  $x = x_1e_1 + x_2e_2 + \dots + x_n e_n$  where all the  $x_i$  have the same sign, it follows that at least two coordinates of  $\psi_j^{-1}\phi_j x$  have different signs. Thus (34) gives

$$f(\phi_n \cdots \phi_2 x) = f(x), \quad \text{for } x \notin e_1^\perp \cup \cdots \cup e_n^\perp \cup \bigcup_{k=0}^{n-2} (e_1 + \sum_{i=0}^k (1 - \lambda_{2+i})e_{2+i})^\perp. \quad (37)$$

Evaluating (37) at  $(1, \dots, 1)$  gives

$$f(1 + (1 - \lambda_2) + \cdots + (1 - \lambda_n), \lambda_2, \dots, \lambda_n) = f(1, \dots, 1),$$

from which we conclude

$$f(x_1, \dots, x_n) = f(1, \dots, 1), \quad \text{for } 0 < x_2, \dots, x_n < 1, \quad x_1 = n - x_2 - \cdots - x_n. \quad (38)$$

But (37) for positive  $x_2, \dots, x_n$  is nothing else than

$$f(x_1, \dots, x_n) = f(x_1 + (1 - \lambda_2)x_2 + \cdots + (1 - \lambda_n)x_n, \lambda_2 x_2, \dots, \lambda_n x_n).$$

Choosing sufficiently small  $\lambda_2, \dots, \lambda_n$ , we obtain by (38)

$$f(x_1, \dots, x_n) = f(1, \dots, 1), \quad \text{for } x_1, \dots, x_n > 0, \quad x_1 = n - x_2 - \cdots - x_n.$$

Similarly, we derive

$$f(-x_1, \dots, -x_n) = f(-1, \dots, -1), \quad \text{for } x_1, \dots, x_n > 0, \quad x_1 = n - x_2 - \cdots - x_n.$$

This shows that  $f(x) = c_1 \rho(T^n, x)^p + c_2 \rho(-T^n, x)^p$ .  $\square$

**Lemma 18.** *Let  $Z : \mathcal{P}_0^n \rightarrow \langle \mathcal{S}^n, \tilde{\tau}_p \rangle$ ,  $n \geq 3$ , be a valuation which is  $\text{GL}(n)$  covariant of weight  $q > 0$ . Then*

$$ZP = \{0\}$$

for every  $P \in \mathcal{P}_0^n$ .

*Proof.* Since  $q > 0$ , Lemma 2 and Lemma 3 imply that  $f$  is continuous on  $S^{n-1}$ . By (7) we have

$$f(x) = \lambda_j^{pq} f(\phi_j^{-1}x) + (1 - \lambda_j)^{pq} f(\psi_j^{-1}x) \quad (39)$$

on  $\mathbb{R}^n \setminus \{0\}$ . The vector  $e_3$  is an eigenvector with eigenvalue 1 of  $\phi_2^{-1}$  and  $\psi_2^{-1}$ . So for  $pq \neq 1$ , (39) and (5) imply  $f(\pm e_k) = 0$  for  $k = 1, 2, \dots, n$ . For  $pq = 1$ , (39) evaluated at  $e_j$  for  $j > 1$  yields

$$f(e_j)\lambda_j^{-p} = f(e_j - (1 - \lambda_j)e_1).$$

Since  $f(e_j - e_1)$  has to be finite and  $f$  is continuous,  $f(e_j)$  has to be zero. Thus also in this case  $f(\pm e_k) = 0$  for  $k = 1, 2, \dots, n$ .

Hence (39) gives

$$f((1 - \lambda_j)e_1 + \lambda_j e_j) = \lambda_j^{pq} f(e_j) + (1 - \lambda_j)^{pq} f(e_1) = 0.$$

Therefore  $f(x_1e_1 + e_j) = 0$  for positive  $x_1$ . Using (39) again shows

$$f(-e_1) = \lambda_j^{pq} f(-e_1) + (1 - \lambda_j)^{pq+p} f(-e_1 + \lambda_j e_j),$$

and so  $f(x_1e_1 + e_j) = 0$  for  $x_1 \leq -1$ . But

$$f(e_j) = \lambda_j^{pq+p} f(-(1 - \lambda_j)e_1 + e_j) + (1 - \lambda_j)^{pq} f(e_j),$$

which proves, together with the observations made before, that  $f(x_1e_1 + e_j) = 0$  for all  $x_1$ . By (5) this implies that  $f(e_1 + x_j e_j) = 0$  for all  $x_j$ . The homogeneity of  $f$  shows  $f(x_1e_1 + x_j e_j) = 0$  for all  $x_1, x_j$ . Thus  $f$  vanishes on all points with at most two coordinates not equal to zero.

We use induction on the number of non-vanishing coordinates. We assume that  $f$  equals zero on points with  $(j - 1)$  non-vanishing coordinates. Set  $x' = x_2e_2 + \dots + x_{j-1}e_{j-1}$ . By (39),

$$f((1 - \lambda_j)e_1 + \lambda_j e_j + x') = \lambda_j^{pq} f(e_j + x') + (1 - \lambda_j)^{pq} f(e_1 + x') = 0,$$

which gives  $f(x_1e_1 + e_j + x'/\lambda_j) = 0$  for  $x_1 > 0$ . Therefore  $f(x_1e_1 + e_j + x') = 0$  for all  $x_1 > 0$  and  $x' = x_2e_2 + \dots + x_{j-1}e_{j-1}$ . But by (39)

$$\begin{aligned} f(-e_1 + x') &= \lambda_j^{pq} f(-e_1 + x') + (1 - \lambda_j)^{pq+p} f(-e_1 + \lambda_j e_j + (1 - \lambda_j)x'), \\ f(e_j + x') &= \lambda_j^{pq+p} f(-(1 - \lambda_j)e_1 + e_j + x') + (1 - \lambda_j)^{pq} f(e_j + \lambda_j x'). \end{aligned}$$

So  $f(x_1e_1 + e_j + x') = 0$  for all  $x_1$  and  $x' = x_2e_2 + \dots + x_{j-1}e_{j-1}$ . By (5),  $f(e_1 + x_j e_j + x') = 0$  for all  $x_j$  and  $x' = x_2e_2 + \dots + x_{j-1}e_{j-1}$ . The homogeneity of  $f$  finally shows that  $f(x_1e_1 + \dots + x_j e_j) = 0$  for all  $x_1, \dots, x_j$ .  $\square$

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