# Asymptotic Approximation of Convex Curves; the Hausdorff Metric Case

Monika Ludwig

## 1 Introduction

Let $C$ and $D$ be closed convex curves in the Euclidean plane. Their Hausdorff distance $\delta^H(C,D)$ is defined by

$$\delta^H(C,D) = \max\{\sup_{p\in C}\inf_{q\in D}\|p-q\|, \sup_{q\in D}\inf_{p\in C}\|p-q\|\},$$

where $\|.\|$ is the Euclidean norm. Let $\mathcal{P}_n^i(C)$ be the set of closed convex polygons which are inscribed into $C$ and have at most $n$ vertices. Beginning with the work of L. Fejes Tóth (cf. the surveys [3], [4]), there are investigations on the asymptotic behavior as $n \to \infty$ of the distance of $C$ to its *best approximating* inscribed polygons with at most $n$ vertices, i.e., of

$$\delta^H(C,\mathcal{P}_n^i) = \inf\{\delta^H(C,P_n) : P_n \in \mathcal{P}_n^i(C)\},$$

and the similarly defined distance $\delta^H(C,\mathcal{P}_n^c)$ for circumscribed polygons.

In the following we assume that $C$ is of class $\mathcal{C}^2$ and has positive curvature. In this case the asymptotic behavior as $n\to\infty$ of $\delta^H(C,\mathcal{P}_n^i)$ and $\delta^H(C,\mathcal{P}_n^c)$ was described by L. Fejes Tóth [1],[2]:

$$\delta^H(C,\mathcal{P}_n^i) \sim \delta^H(C,\mathcal{P}_n^c) \sim \frac{1}{8}\left(\int_0^l \kappa^{1/2}(t)dt\right)^2 \frac{1}{n^2}, \tag{1}$$

where $\kappa(t)$ is the curvature of $C$ given as a function of the arclength $t$ and $l$ the length of $C$. See also McClure and Vitale [7] and for asymptotic formulae for approximation with respect to the Hausdorff metric in higher dimensions R. Schneider [8],[9] and P.M. Gruber [5].

In this article we extend the asymptotic formulae (1) by deriving the second terms in the asymptotic expansions of $\delta^H(C,\mathcal{P}_n^i)$ and $\delta^H(C,\mathcal{P}_n^c)$. This complements results derived for approximation with respect to the symmetric difference metric in [6].

## 2 Some Definitions

Let $P_n$ be a polygon inscribed in $C$ and let $p_1, \ldots, p_n$ be its successive vertices. Denote by $h_i$ the maximal distance that a point lying on $C$ between $p_{i-1}$ and $p_i$ can have from the line segment connecting $p_{i-1}$ and $p_i$. Thus, $\delta^H(C, P_n) = \max_{i=1,\ldots,n} h_i$. For a convex curve $C$ with positive curvature and $n$ large enough it is easy to see that

$$h_1 = h_2 = \ldots = h_n \tag{2}$$

for all best approximating polygons. A similar statement holds for circumscribed polygons.

Because of the asymptotic formulae (1), we introduce a new parameter for $C$ by

$$s(t) = \int_0^t \kappa^{1/2}(\tau)d\tau, \ 0 \le t \le l, \tag{3}$$

where $t$ is the arclength of $C$. Define

$$\lambda = \int_0^l \kappa^{1/2}(\tau)d\tau.$$

In the following $C$ is always of class $\mathcal{C}^4$ with positive curvature and given in a parametrization $x(s)$, $0 \le s \le \lambda$, where $s$ is defined by (3). By $'$ we denote differentiation with respect to $s$.

## 3 Asymptotic Expansion for $\delta^H(C, \mathcal{P}_n^i)$

**Lemma 1** *For $0 \le r \le s \le \lambda$, let $h(r,s)$ be the maximal distance that a point lying on $C$ between $x(r)$ and $x(s)$ can have from the line segment connecting these points. Then*

$$h(r,s) = \frac{1}{8}(s-r)^2 - \frac{1}{384}\left(\kappa(r) + \kappa^{-1}(r)\kappa''(r) - \frac{5}{6}\kappa^{-2}(r)(\kappa'(r))^2\right)(s-r)^4 + o((s-r)^4),$$

*uniformly for all $0 \le r \le s \le \lambda$ as $(s-r) \to 0$.*

*Proof.* The distance of a point $x(\sigma)$, $r \le \sigma \le s$, to the line segment joining $x(r)$ and $x(s)$ is given by

$$|x(\sigma) - x(r), \frac{x(s)-x(r)}{\|x(s)-x(r)\|}|,$$

where $|\,.\,,\,.\,|$ is the determinant. The maximal distance $h(r,s)$ is attained for a point $x(\sigma)$ for which

$$|x'(\sigma), \frac{x(s)-x(r)}{\|x(s)-x(r)\|}| = 0.$$

By this equation a function $\sigma = \phi(r,s)$ is defined with the help of which $h(r,s)$ can be written as

$$h(r,s) = |x(\phi(r,s)) - x(r), \frac{x(s)-x(r)}{\|x(s)-x(r)\|}|.$$

Simple but rather lengthy calculations give the Taylor polynomial of order four of $h(r,s)$ and the statement of the lemma. □

We define

$$k_H^i(s) = \kappa(s) + \kappa^{-1}(s)\kappa''(s) - \frac{5}{6}\kappa^{-2}(s)(\kappa'(s))^2$$

and are able to formulate the first theorem.

**Theorem 1** *Let $C \in \mathcal{C}^4$ be a convex curve with positive curvature. Then*

$$\delta^H(C, \mathcal{P}_n^i) \sim \frac{1}{8}\frac{\lambda^2}{n^2} - \frac{\lambda^3}{384}\int_0^\lambda k_H^i(s)ds\frac{1}{n^4} + o(\frac{1}{n^4})$$

*as $n \to \infty$.*

*Proof:* We consider a sequence of best approximating polygons with vertices at $x(s_{ni})$, $i = 1,\dots,n$. As a first step we derive a simple asymptotic formula for the $\lambda_{ni} = s_{ni} - s_{n,i-1}$.

By (2) we have

$$h_{n1} = h_{n2} = \ldots = h_{nn} = \delta^H(C, \mathcal{P}_n^i).$$

Since $\lambda_{ni} \to 0$ as $n \to \infty$, we see from (1) and the terms of second order in Lemma 1, that for any $\varepsilon > 0$, there is a positive integer $n_0$ such that

$$\frac{\lambda^2}{8n^2}(1-\varepsilon) \le h_{ni} \le \frac{\lambda^2}{8n^2}(1+\varepsilon)$$

and

$$\frac{1}{8}\lambda_{ni}^2(1-\varepsilon) \le h_{ni} \le \frac{1}{8}\lambda_{ni}^2(1+\varepsilon)$$

for all $n \ge n_0$. Combining these inequalities gives

$$\frac{1-\varepsilon}{1+\varepsilon}\frac{\lambda^2}{n^2} \le \lambda_{ni}^2 \le \frac{1+\varepsilon}{1-\varepsilon}\frac{\lambda^2}{n^2}.$$

From this we see that

$$\lambda_{ni} = \frac{\lambda}{n} + o(\frac{1}{n}) \tag{4}$$

uniformly as $n \to \infty$, which is the desired simple asymptotic formula.

For any $i, j, 0 < i, j < n$, we have $h_{ni} = h_{nj}$. Therefore, Lemma 1 and (4) give

$$\frac{1}{8}\lambda_{ni}^2 - \frac{1}{384}k_H^i(s_{n,i-1})\lambda_{ni}^4 + o(\frac{1}{n^4}) = \frac{1}{8}\lambda_{nj}^2 - \frac{1}{384}k_H^i(s_{n,j-1})\lambda_{nj}^4 + o(\frac{1}{n^4}).$$

Extracting the square root of these equations gives

$$\lambda_{ni} - \frac{1}{96}k_H^i(s_{n,i-1})\lambda_{ni}^3 + o(\frac{1}{n^3}) = \lambda_{nj} - \frac{1}{96}k_H^i(s_{n,j-1})\lambda_{nj}^3 + o(\frac{1}{n^3}). \tag{5}$$

We define

$$\lambda_{ni} = \frac{\lambda}{n} + \varepsilon_{ni}$$

and have by (4) $\varepsilon_{ni} = o(\frac{1}{n})$. Combining this with (5) we obtain

$$\varepsilon_{ni} = \frac{1}{96}k_H^i(s_{n,i-1})\frac{\lambda^3}{n^3} + \varepsilon_{nj} - \frac{1}{96}k_H^i(s_{n,j-1})\frac{\lambda^3}{n^3} + o(\frac{1}{n^3}).$$

In these equations we sum on $j$ from 1 to $n$, and since $\sum_{j=0}^{n}\varepsilon_{nj} = 0$, find that

$$n\varepsilon_{ni} = \frac{1}{96}k_H^i(s_{n,i-1})\frac{\lambda^3}{n^2} - \frac{1}{96}\frac{\lambda^2}{n^2}\sum_{j=0}^{n}k_H^i(s_{n,j-1})\lambda_{nj} + o(\frac{1}{n^2}).$$

Therefore,

$$\lim_{n\to\infty} n^3\left(\varepsilon_{ni} - \frac{1}{96}k_H^i(s_{n,i-1})\frac{\lambda^3}{n^3}\right) = -\frac{\lambda^2}{96}\int_0^{\lambda} k_H^i(s)ds$$

and

$$\lambda_{ni} = \frac{\lambda}{n} + \frac{1}{96}k_H^i(s_{n,i-1})\frac{\lambda^3}{n^3} - \frac{\lambda^2}{96}\int_0^{\lambda} k_H^i(s)ds\frac{1}{n^3} + o(\frac{1}{n^3}) \tag{6}$$

uniformly as $n \to \infty$.

Using (6), we obtain the asymptotic expansion of $\delta^H(C, \mathcal{P}_n^i)$. Lemma 1 and (6) give

$$\begin{aligned}
\delta^H(C, \mathcal{P}_n^i) &= \frac{1}{n}\sum_{i=1}^{n} h_{ni} = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{\lambda_{ni}^2}{8} - \frac{1}{384}k_H^i(s_{n,i-1})\lambda_{ni}^4 + o(\frac{1}{n^4})\right) = \\
&= \frac{1}{n}\sum_{i=1}^{n}\Bigg(\frac{1}{8}\Big(\frac{\lambda^2}{n^2} + \frac{1}{48}k_H^i(s_{n,i-1})\frac{\lambda^4}{n^4} - \frac{\lambda^3}{48}\int_0^{\lambda} k_H^i(s)ds\frac{1}{n^4}\Big) - \\
&\qquad - \frac{1}{384}k_H^i(s_{n,i-1})\frac{\lambda^4}{n^4} + o(\frac{1}{n^4})\Bigg).
\end{aligned}$$

Therefore,

$$\delta^H(C, \mathcal{P}_n^i) - \frac{\lambda^2}{8n^2} = -\frac{\lambda^3}{384}\int_0^{\lambda} k_H^i(s)ds\frac{1}{n^4} + o(\frac{1}{n^4})$$

and

$$\lim_{n\to\infty} n^4\left(\delta^H(C, \mathcal{P}_n^i) - \frac{\lambda^2}{8n^2}\right) = -\frac{\lambda^3}{384}\int_0^{\lambda} k_H^i(s)ds. \ \Box$$

Comparing this result with the asymptotic expansion for approximation with respect to the symmetric difference metric

$$\delta^S(C, \mathcal{P}_n^i) \sim \frac{1}{12}\frac{\lambda^3}{n^2} - \frac{1}{2}\frac{\lambda^4}{5!}\int_0^\lambda k(s)ds\frac{1}{n^4} + o(\frac{1}{n^4})$$

(cf. [6]), where $\lambda$ is the affine length of $C$, $s$ the affine arclength and $k(s)$ the affine curvature of $C$, we see that $k_H^i$ plays the same role in the case of the Hausdorff metric as the affine curvature in the case of the symmetric difference metric. But whereas in the case of the symmetric difference metric the second term in the asymptotic expansion for approximation by circumscribed polygons again depends on $k(s)$, for the Hausdorff metric it depends on a function different from $k_H^i$ (see the following section).

# 4 Asymptotic Expansion for $\delta^H(C, \mathcal{P}_n^c)$

**Lemma 2** *For $0 \leq r \leq s \leq \lambda$, let $k(r,s)$ be the maximal distance that a point lying on $C$ between $x(r)$ and $x(s)$ can have from the point where the tangents at $x(r)$ and $x(s)$ meet. Then*

$$\begin{aligned} k(r,s) = &\frac{1}{8}(s-r)^2 + \frac{5}{384}\left(\kappa(r) + \frac{1}{5}\kappa^{-1}(r)\kappa''(r) - \frac{7}{30}\kappa^{-2}(r)(\kappa'(r))^2\right)(s-r)^4 \\ &+ o((s-r)^4), \end{aligned}$$

*uniformly for all $0 \leq r \leq s \leq \lambda$ as $(s-r) \to 0$.*

*Proof.* The point where the tangents at $x(r)$ and $x(s)$ meet is given by

$$y(r,s) = x(r) + \frac{|x(s) - x(r), x'(s)|}{|x'(r), x'(s)|}x'(r).$$

For a point $x(\sigma)$ with maximal distance from that point we have

$$x'(\sigma) \cdot (x(\sigma) - y(r,s)) = 0.$$

By this equation a function $\sigma = \phi(r,s)$ is defined, with the help of which we get

$$k(r,s) = \|x(\phi(r,s)) - y(r,s)\|.$$

As in the proof of Lemma 1 rather lengthy calculations give the Taylor polynomial of order four of $k(r,s)$ and the statement of the lemma. □

We define

$$k_H^c(s) = \kappa(s) + \frac{1}{5}\kappa^{-1}(s)\kappa''(s) - \frac{7}{30}\kappa^{-2}(s)(\kappa'(s))^2$$

and get the following theorem.

**Theorem 2** *Let $C \in \mathcal{C}^4$ be a convex curve with positive curvature. Then*

$$\delta^H(C, \mathcal{P}_n^c) \sim \frac{1}{8}\frac{\lambda^2}{n^2} + \frac{5\lambda^3}{384}\int_0^\lambda k_H^c(s)ds\frac{1}{n^4} + o(\frac{1}{n^4})$$

*as $n \to \infty$.*

The proof of this theorem is analogous to that of Theorem 1.

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