THE HADWIGER THEOREM ON CONVEX FUNCTIONS. I

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ABSTRACT. A complete classification of all continuous, epi-translation and rotation invariant valuations on the space of super-coercive convex functions on $\mathbb{R}^n$ is established. The valuations obtained are functional versions of the classical intrinsic volumes. For their definition, singular Hessian valuations are introduced.

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1. INTRODUCTION AND STATEMENT OF RESULTS

Intrinsic volumes play a fundamental role in Euclidean geometry. For a convex body K (that is, a non-empty, compact convex set) in n-dimensional Euclidean space, $\mathbb{R}^n$, and $j \in \{0, \dots, n\}$, the jth intrinsic volume, $V_j(K)$, measures, in a translation and rotation invariant way, the j-dimensional "size" of K. In particular, $V_n(K)$ is the n-dimensional volume of K and $V_0(K) = 1$ is the Euler characteristic. For $1 \le j \le n-1$, the jth intrinsic volume can be defined in various ways, for example, via the Steiner formula, via integral geometric formulas, or, for a convex body K with smooth boundary, as

$$V_j(K) = \frac{1}{\omega_{n-j}} \int_{\mathbb{S}^{n-1}} [\mathrm{D}^2 h_K(y)]_j \, \mathrm{d}\mathcal{H}^{n-1}(y),$$

where h_K is the support function of K (see Section 2.1 for the definition) and $\mathrm{D}^2 h_K$ the Hessian matrix of h_K. Here, we write $[A]_j$ for the jth elementary symmetric function of the eigenvalues of a symmetric matrix A and use the convention that $[A]_0 = 1$. We write $\mathcal{H}^{n-1}$ for $(n-1)$-dimensional Hausdorff measure and ω_k for the $(k-1)$-dimensional Hausdorff measure of the unit sphere, $\mathbb{S}^{k-1}$, in $\mathbb{R}^k$. The jth intrinsic volume is homogeneous of degree j, that is, $V_j(\lambda K) = \lambda^j V_j(K)$ for all convex bodies K in $\mathbb{R}^n$ and all $\lambda \ge 0$. The intrinsic volumes are continuous functionals on the space of convex bodies, $\mathcal{K}^n$, in $\mathbb{R}^n$ equipped with the standard topology induced by the Hausdorff metric (see [40] for information regarding intrinsic volumes).

One of the most important results in geometry is Hadwiger's characterization theorem [20] for intrinsic volumes. Hadwiger's theorem classifies all continuous, translation and rotation invariant valuations on convex bodies. Here a functional $\mathrm{Z} : \mathcal{K}^n \to \mathbb{R}$ is called a valuation if

$$\mathrm{Z}(K) + \mathrm{Z}(L) = \mathrm{Z}(K \cup L) + \mathrm{Z}(K \cap L) \tag{1.1}$$

for every $K, L \in \mathcal{K}^n$ such that $K \cup L \in \mathcal{K}^n$. Recall that Z is translation invariant if $\mathrm{Z}(\tau K) = \mathrm{Z}(K)$ for all $K \in \mathcal{K}^n$ and translations τ on $\mathbb{R}^n$, and it is rotation invariant if $\mathrm{Z}(\vartheta K) = \mathrm{Z}(K)$ for all $K \in \mathcal{K}^n$ and $\vartheta \in \mathrm{SO}(n)$.

Theorem 1.1 (Hadwiger [20]). *A functional $\mathrm{Z} : \mathcal{K}^n \to \mathbb{R}$ is a continuous, translation and rotation invariant valuation if and only if there are constants $\zeta_0, \dots, \zeta_n \in \mathbb{R}$ such that*

$$\mathrm{Z}(K) = \zeta_0 \, V_0(K) + \dots + \zeta_n \, V_n(K)$$

for every $K \in \mathcal{K}^n$.

Hadwiger's theorem leads to effortless proofs of numerous results in integral geometry and geometric probability (see [20] and [23]). It is the first culmination of a program initiated by Blaschke and the starting point of the geometric theory of valuations, where classification theorems for valuations invariant under various groups are fundamental (see [40, Chapter 6]). Interesting new valuations keep arising (see, for example, [8,21,28]) and see [1,2,4,6,7,18,19,26,29,32,33,36] for some recent classification results.

Currently, a geometric theory of valuations on function spaces is developed. A functional Z defined on a space of (extended) real-valued functions, X, is called a *valuation* if

$$\mathrm{Z}(u) + \mathrm{Z}(v) = \mathrm{Z}(u \vee v) + \mathrm{Z}(u \wedge v)$$

for all $u, v \in X$ such that the pointwise maximum $u \vee v$ and the pointwise minimum $u \wedge v$ belong to X. When X is the space of indicator functions of convex bodies in $\mathbb{R}^n$, we recover the classical notion of valuation on convex bodies. The first classification results were obtained for L_p and Sobolev spaces, for Lipschitz functions and for definable functions (see, for example, [5, 16, 26, 30, 31, 43]).

Spaces of convex functions play a special role because of their close connection to convex bodies. Here classification results were obtained for $\mathrm{SL}(n)$ invariant valuations in [12, 13, 34, 35], the connection to valuations on convex bodies was explored in [3] and first structural results were established in [14,24,25]. The general space of extended real-valued convex functions on $\mathbb{R}^n$ is defined as

$$\mathrm{Conv}(\mathbb{R}^n) := \{u\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \colon u \text{ is convex and lower semicontinuous, } u \not\equiv +\infty\}.$$

It is equipped with the topology induced by epi-convergence (see Section 2.2) and the continuity of valuations defined on $\mathrm{Conv}(\mathbb{R}^n)$ (and its subsets) will always be with respect to this topology.

The aim of this paper is to establish the Hadwiger theorem on convex functions. A valuation Z defined on (a subset of) the space $\mathrm{Conv}(\mathbb{R}^n)$ is said to be *epi-translation invariant* if $\mathrm{Z}(u \circ \tau^{-1} + \alpha) = \mathrm{Z}(u)$ for every translation τ on $\mathbb{R}^n$ and $\alpha \in \mathbb{R}$. It is said to be *rotation invariant* if $\mathrm{Z}(u \circ \vartheta^{-1}) = \mathrm{Z}(u)$ for every $\vartheta \in \mathrm{SO}(n)$. We will identify a convex function with its epi-graph,

$$\mathrm{epi}(u) := \{(x,t) \in \mathbb{R}^{n+1} : t \geq u(x)\}.$$

Note that an epi-translation invariant valuation on $\mathrm{Conv}(\mathbb{R}^n)$ corresponds to a translation invariant valuation on the set of epi-graphs in the sense of (1.1). We say that Z is *epi-homogeneous* of degree j, if the associated valuation on epi-graphs is homogeneous of degree j. It was shown in [14] that a continuous and epi-translation invariant valuation on $\mathrm{Conv}(\mathbb{R}^n)$ is always constant and the corresponding statement was proved on the space of coercive convex functions. Thus, we look at valuations on the smaller space of *super-coercive convex functions*,

$$\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) := \left\{ u \in \mathrm{Conv}(\mathbb{R}^n)\colon \lim_{|x|\to+\infty} \frac{u(x)}{|x|} = +\infty \right\},$$

where $|\cdot|$ is the Euclidean norm. For $n = 1$, a complete classification of continuous and epi-translation invariant valuations was obtained in [14]. So, let $n \geq 2$. For $\zeta \in C_c([0,\infty))$ and $j \in \{0, \dots, n\}$, it follows from results in [15] that a valuation, defined for $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \cap C^2_+(\mathbb{R}^n)$ (that is, with a positive definite Hessian matrix $\mathrm{D}^2 u$) by

$$\mathrm{V}_{j,\zeta}(u) := \int_{\mathbb{R}^n} \zeta(|\nabla u(x)|)[\mathrm{D}^2 u(x)]_{n-j} \,\mathrm{d}x, \tag{1.2}$$

can be extended to a continuous, epi-translation and rotation invariant valuation that is epi-homogeneous of degree j on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$. The valuation defined in (1.2) is a *Hessian valuation*, and the extension of the integral (1.2) uses the notion of Hessian measures. These measures play an important role in the study of a class of fully non-linear elliptic PDEs, the so-called Hessian equations, and in this context were introduced by Trudinger and Wang [41, 42]. In the case of convex functions these measures have been studied, for example, in [11] (see Section 2.3).

As we will show, the functionals $\mathrm{V}_{j,\zeta}$ also extend to continuous valuations on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ for functions $\zeta \in C((0,\infty))$ with a certain type of singularity at the origin. We thus obtain *singular Hessian valuations*. Let $C_b((0,\infty))$ be the set of continuous functions on $(0,\infty)$ with bounded support. For $1 \le j \le n-1$, let

$$D_j^n := \Big\{\zeta \in C_b((0,\infty))\colon \lim_{s\to 0^+} s^{n-j}\zeta(s) = 0, \lim_{s\to 0^+} \int_s^\infty t^{n-j-1}\zeta(t)\,\mathrm{d}t \text{ exists and is finite}\Big\}.$$

In addition, let D_n^n be the set of functions $\zeta \in C_b((0,\infty))$ where $\lim_{s\to 0^+}\zeta(s)$ exists and is finite and set $\zeta(0) = \lim_{s\to 0^+}\zeta(s)$. Let D_0^n be the set of functions $\zeta \in C_b((0,\infty))$ where $\lim_{s\to 0^+}\int_s^\infty t^{n-1}\zeta(t)\,\mathrm{d}t$ exists and is finite. Note that $\mathrm{V}_{0,\zeta}$ is constant on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ for $\zeta \in D_0^n$ (see Section 2.3).

Our first main result establishes the existence of the extension of singular Hessian valuations for functions $\zeta \in D_j^n$.

Theorem 1.2. *For $j \in \{0,\dots,n\}$ and $\zeta \in D_j^n$, there exists a unique, continuous, epi-translation and rotation invariant valuation $\mathrm{V}_{j,\zeta}\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}$ such that*

$$\mathrm{V}_{j,\zeta}(u) = \int_{\mathbb{R}^n} \zeta(|\nabla u(x)|)[\mathrm{D}^2 u(x)]_{n-j}\,\mathrm{d}x$$

for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \cap C_+^2(\mathbb{R}^n)$.

The proof of this result will use analytic tools, developed within the theory of Hessian equations (see Section 3.2 for details). Given a singular function ζ, the integral (1.2) converges absolutely for sufficiently regular functions. For the general extension argument, Hessian valuations, which were introduced in [15], will be used, but singular functions are, in general, not integrable with respect to Hessian measures. For the extension, we will use the Moreau-Yosida approximation and the existence of a homogeneous decomposition for epi-translation invariant valuations that was established in [14].

Our second main result shows that the singular Hessian valuations that were introduced above span the space of continuous, epi-translation and rotation invariant valuations on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$. Let $n \ge 2$.

Theorem 1.3. *A functional $\mathrm{Z} : \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}$ is a continuous, epi-translation and rotation invariant valuation if and only if there exist functions $\zeta_0 \in D_0^n, \dots, \zeta_n \in D_n^n$ such that*

$$\mathrm{Z}(u) = \mathrm{V}_{0,\zeta_0}(u) + \cdots + \mathrm{V}_{n,\zeta_n}(u)$$

for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$.

These theorems show that the singular Hessian valuations $\mathrm{V}_{j,\zeta}$ with $\zeta \in D_j^n$ are functional versions of the intrinsic volumes V_j and that, from the point of view of geometric valuation theory, they are the canonical functional versions of intrinsic volumes.

The space of super-coercive convex functions is related to another subspace of $\mathrm{Conv}(\mathbb{R}^n)$, formed by convex functions with finite values,

$$\mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) := \big\{v \in \mathrm{Conv}(\mathbb{R}^n)\colon v(x) < +\infty \text{ for all } x \in \mathbb{R}^n\big\}.$$

A function v belongs to $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ if and only if its standard conjugate or Legendre transform v^* belongs to $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ (see Section 2.2). It was proved in [15] that Z is a continuous valuation on $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ if and only if $\mathrm{Z}^*\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}$, defined by

$$\mathrm{Z}^*(u) := \mathrm{Z}(u^*),$$

is a continuous valuation on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$. This fact permits us to transfer results valid for valuations on $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ to results for valuations on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ and vice versa. A valuation Z on $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ is called *homogeneous* of degree $j \in \mathbb{R}$ if $\mathrm{Z}(\lambda v) = \lambda^j\,\mathrm{Z}(v)$ for all $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ and $\lambda \geq 0$. It is *dually epi-translation invariant* if

$$\mathrm{Z}(v + \ell + \alpha) = \mathrm{Z}(v)$$

for every $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ and every linear function $\ell\colon \mathbb{R}^n \to \mathbb{R}$ and $\alpha \in \mathbb{R}$. The valuation Z is dually epi-translation invariant and homogeneous of degree j on $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ if and only if Z^* is epi-translation invariant and epi-homogeneous of degree j on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$. Using results on Hessian valuations for conjugate functions, we will obtain the following extension theorem from Theorem 1.2.

Theorem 1.4. *For $j \in \{0,\dots,n\}$ and $\zeta \in D^n_j$, the functional $\mathrm{V}^*_{j,\zeta}\colon \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}$ is a continuous, dually epi-translation and rotation invariant valuation such that*

$$\mathrm{V}^*_{j,\zeta}(v) = \int_{\mathbb{R}^n} \zeta(|x|)[\mathrm{D}^2 v(x)]_j \,\mathrm{d}x \tag{1.3}$$

for every $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \cap C^2_+(\mathbb{R}^n)$.

Theorem 1.3 has the following dual version. Let $n \geq 2$.

Theorem 1.5. *A functional $\mathrm{Z} : \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}$ is a continuous, dually epi-translation and rotation invariant valuation if and only if there exist functions $\zeta_0 \in D^n_0, \dots, \zeta_n \in D^n_n$ such that*

$$\mathrm{Z}(v) = \mathrm{V}^*_{0,\zeta_0}(v) + \cdots + \mathrm{V}^*_{n,\zeta_n}(v)$$

for every $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$.

In the proof of Theorem 1.3 and Theorem 1.5, we will use both the primal setting of epi-translation invariant valuations on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ and the dual setting of dually epi-translation invariant valuations on $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$. We follow the original approach of Hadwiger [20]. We introduce rotational epi-symmetrization to establish a classification of valuations that are epi-homogeneous of degree 1 on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ and induction on the dimension for the general case. The homogeneous decomposition of epi-translation invariant valuations established in [14] will be an important tool in the proof.

An alternate proof of Hadwiger's theorem is due to Klain [22]. We will discuss elements of Klain's proof and how to apply Theorem 1.3 and Theorem 1.5 in integral geometry in a subsequent paper.

2. Preliminaries

We work in n-dimensional Euclidean space $\mathbb{R}^n$, with $n \geq 1$, endowed with the Euclidean norm $|\cdot|$ and the usual scalar product $\langle\cdot,\cdot\rangle$. We write $e_1,\dots,e_n$ for the canonical basis vectors of $\mathbb{R}^n$ and use coordinates, $x = (x_1,\dots,x_n)$, for $x \in \mathbb{R}^n$ with respect to this basis. We write span for linear span. Let $B^n := \{x \in \mathbb{R}^n : |x| \leq 1\}$ be the Euclidean unit ball and $\mathbb{S}^{n-1}$ the unit sphere in $\mathbb{R}^n$. We denote by κ_n the n-dimensional volume of B^n and by ω_n the $(n-1)$-dimensional Hausdorff measure of $\mathbb{S}^{n-1}$.

2.1. Convex Sets.

A basic reference on convex sets is the book by Schneider [40]. Let $K \subset \mathbb{R}^n$ be a closed convex set that is non-empty and such that $K \neq \mathbb{R}^n$. Its *support function* $h_K : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is defined as

$$h_K(y) := \sup\nolimits_{x\in K}\langle x, y\rangle.$$

It is a one-homogeneous and convex function that determines K. If $K \subset \mathbb{R}^n$ is a convex body with boundary of class C^2_+ and $y \in \mathbb{S}^{n-1}$, then the eigenvalues of its Hessian matrix, $\mathrm{D}^2 h_K(y)$, are the principal radii of curvature of K at y. Hence $[\mathrm{D}^2 h_K(y)]_i$ is the ith elementary symmetric function of the principal radii of curvature of K at y (see [40, Section 2.5]).

2.2. **Convex Functions.** We collect some properties of convex functions. Basic references are the books by Rockafellar [38] and Rockafellar & Wets [39].

Let $u \in \text{Conv}(\mathbb{R}^n)$. For $t \in (-\infty, +\infty]$, its *sublevel sets*

$$\{u < t\} := \{x \in \mathbb{R}^n : u(x) < t\}, \qquad \{u \le t\} := \{x \in \mathbb{R}^n : u(x) \le t\}$$

are convex. The latter is a closed convex set by the lower semicontinuity of u and so is its *domain*,

$$\text{dom}(u) := \{x \in \mathbb{R}^n : u(x) < +\infty\}.$$

We equip $\text{Conv}(\mathbb{R}^n)$ and its subspaces with the topology associated to epi-convergence. Here a sequence u_k of functions from $\text{Conv}(\mathbb{R}^n)$ is *epi-convergent* to $u \in \text{Conv}(\mathbb{R}^n)$ if the following conditions hold for all $x \in \mathbb{R}^n$:

(i) For every sequence x_k that converges to x, we have $u(x) \le \liminf_{k\to\infty} u_k(x_k)$.

(ii) There exists a sequence x_k that converges to x such that $u(x) = \lim_{k\to\infty} u_k(x_k)$.

Note that a sequence v_k of functions from $\text{Conv}(\mathbb{R}^n;\mathbb{R})$ epi-converges to $v \in \text{Conv}(\mathbb{R}^n;\mathbb{R})$ if and only if v_k converges pointwise to v and uniformly on compact sets. We will just say that v_k converges to v in this case. We also require the following fact. If u_k is a sequence of functions from $\text{Conv}(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$ that epi-converges to $u \in \text{Conv}(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$, then ∇u_k converges uniformly to ∇u on compact sets (see [38, Theorem 25.7]).

Given $u \in \text{Conv}(\mathbb{R}^n)$, the *convex conjugate* $u^* : \mathbb{R}^n \to (-\infty, \infty]$ is defined by

$$u^*(y) := \sup_{x\in\mathbb{R}^n} \big(\langle y, x\rangle - u(x)\big).$$

As $u \in \text{Conv}(\mathbb{R}^n)$, also $u^* \in \text{Conv}(\mathbb{R}^n)$ and $u^{**} = u$. Moreover, $u \in \text{Conv}_{\text{sc}}(\mathbb{R}^n) \cap C^2_+(\mathbb{R}^n)$ if and only if $u^* \in \text{Conv}(\mathbb{R}^n;\mathbb{R}) \cap C^2_+(\mathbb{R}^n)$. Given a subset $A \subset \mathbb{R}^n$, let $\mathbf{I}_A \colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ denote the (convex) indicatrix function of A,

$$\mathbf{I}_A(x) := \begin{cases} 0 & \text{if } x \in A, \\ +\infty & \text{if } x \notin A. \end{cases}$$

Note that for a convex body $K \subset \mathbb{R}^n$, we have $\mathbf{I}_K \in \text{Conv}_{\text{sc}}(\mathbb{R}^n)$. Let $t \ge 0$. We will often use the following pair of dual functions,

$$u_t(x) := t\,|x| + \mathbf{I}_{B^n}(x) \tag{2.1}$$

and

$$v_t(x) := \begin{cases} 0 & \text{if } |x| \le t, \\ |x| - t & \text{if } |x| > t. \end{cases} \tag{2.2}$$

Note that $u_t \in \text{Conv}_{\text{sc}}(\mathbb{R}^n)$ and $v_t \in \text{Conv}(\mathbb{R}^n;\mathbb{R})$ while $u_t^* = v_t$.

For $u, v \in \text{Conv}(\mathbb{R}^n)$, let

$$(u \,\square\, v)(x) := \inf_{x=y+z} \big(u(y) + v(z)\big)$$

denote the *infimal convolution* of u and v at $x \in \mathbb{R}^n$. Note, that

$$\text{epi}(u \,\square\, v) = \text{epi}\, u + \text{epi}\, v,$$

where on the right we have the Minkowski sum of the sets $\text{epi}\, u$ and $\text{epi}\, v$, and if $u \square v > -\infty$ pointwise, then

$$(u \,\square\, v)^* = u^* + v^*. \tag{2.3}$$

We define the *epi-multiplication* of $u \in \text{Conv}(\mathbb{R}^n)$ and $\lambda > 0$ by setting

$$\lambda \cdot u(x) := \lambda\, u\left(\frac{x}{\lambda}\right)$$

for $x \in \mathbb{R}^n$. Note that the epi-graph of $\lambda \cdot u$ is obtained by rescaling the epi-graph of u by the factor λ. It is easy to see that $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ implies $\lambda \cdot u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ for $\lambda \geq 0$. A functional $\mathrm{Z} : \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}$ is called *epi-homogeneous* of degree $j \in \mathbb{R}$ if

$$\mathrm{Z}(\lambda \cdot u) = \lambda^j \, \mathrm{Z}(u)$$

for all $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ and $\lambda > 0$. Here and in the following, corresponding definitions will be used for $\mathrm{Conv}(\mathbb{R}^n)$ and its subspaces.

Let E and F be orthogonal and complementary subspaces of $\mathbb{R}^n$ with $\dim E, \dim F \geq 1$. We can identify E with $\mathbb{R}^k$ when $k = \dim E$ and write $\mathrm{Conv}(E)$ for $\mathrm{Conv}(\mathbb{R}^k)$ in this case. We use corresponding definitions for subspaces of $\mathrm{Conv}(E)$. For $x \in \mathbb{R}^n$, we write $x = (x_E, x_F)$ with $x_E \in E$ and $x_F \in F$. If $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ is such that $\mathrm{dom}(u) \subset E$, then $v = u^*$ does not depend on x_F. The restriction of v to E, which we denote by $v_E \in \mathrm{Conv}(E; \mathbb{R})$, is equal to the Legendre transform (with respect to the ambient space E) of the restriction of u to E, which we denote by $u_E \in \mathrm{Conv}_{\mathrm{sc}}(E)$. In this case, we will identify u with u_E and v with v_E. Hence, for $u_E \in \mathrm{Conv}_{\mathrm{sc}}(E)$ and $u_F \in \mathrm{Conv}_{\mathrm{sc}}(F)$, we can use this identification and define $u = u_E \,\square\, u_F \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ where we use the usual infimal convolution on $\mathbb{R}^n$. Setting $v_E = u_E^*$ and $v_F = u_F^*$ and using (2.3), we obtain that

$$(u_E \,\square\, u_F)^* = v_E^* + v_F^*. \tag{2.4}$$

For $u \in \mathrm{Conv}(\mathbb{R}^n)$ and $\lambda > 0$, we define the *Moreau-Yosida envelope*, $\mathrm{env}_\lambda u$, of u as

$$\mathrm{env}_\lambda u := u \,\square\, \frac{|\cdot|^2}{2\lambda}.$$

If $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$, then $\mathrm{env}_\lambda u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$, and in particular, $\mathrm{dom}(\mathrm{env}_\lambda u) = \mathbb{R}^n$ (see [39, Theorem 2.26]). If additionally, $u \in C^2(\mathbb{R}^n)$, then so is $\mathrm{env}_\lambda u$ (see [39, Theorem 7.37]). If $u_k \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ and u_k epi-convergences to u, then $\mathrm{env}_\lambda u_k$ epi-convergences to $\mathrm{env}_\lambda u$.

2.3. **Hessian Measures.** In this part we briefly recall the notion of Hessian measures of convex functions. For a more detailed presentation, the reader is referred to [11, 15].

We begin with recalling the definition of the subgradient of a function $u \in \mathrm{Conv}(\mathbb{R}^n)$. For $x \in \mathbb{R}^n$, the subgradient of u at x is defined by

$$\partial u(x) := \{y \in \mathbb{R}^n \colon u(z) \geq u(x) + \langle y, z - x\rangle \ \text{ for } z \in \mathbb{R}^n\}.$$

We set

$$\Gamma_u := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \colon y \in \partial u(x)\},$$

that is, Γ_u is the graph of the subgradient map.

The Hessian measures of u, that we denote by $\Theta_j^n(u, \cdot)$ for $j = 0, \ldots, n$, are non-negative measures defined on the Borel subsets of $\mathbb{R}^n \times \mathbb{R}^n$ that can be introduced in the following way. Given a Borel subset A of $\mathbb{R}^n \times \mathbb{R}^n$ and $s \geq 0$, we consider the set,

$$P_s(u, A) := \{x + sy \colon (x, y) \in \Gamma_u \cap A\}.$$

Its n-dimensional Hausdorff measure is a polynomial in s; in other words, there exist $(n+1)$ non-negative coefficients $\Theta_0^n(u, A), \ldots, \Theta_n^n(u, A)$, such that

$$\mathcal{H}^n(P_s(u, A)) = \sum_{j=0}^{n} s^j \Theta_{n-j}^n(u, A)$$

for every Borel set A in $\mathbb{R}^n \times \mathbb{R}^n$ and $s \geq 0$. The previous relation defines the Hessian measures of u. They are locally finite Borel measures on $\mathbb{R}^n \times \mathbb{R}^n$.

We will frequently use marginals of Hessian measures. Given $j \in \{0, \dots, n\}$ and $u \in \mathrm{Conv}(\mathbb{R}^n)$, we set

$$\Phi_j^n(u, B) := \Theta_{n-j}^n(u, B \times \mathbb{R}^n) \qquad \text{for every Borel subset } B \subseteq \mathbb{R}^n,$$

and

$$\Psi_j^n(u, B) := \Theta_j^n(u, \mathbb{R}^n \times B) \qquad \text{for every Borel subset } B \subseteq \mathbb{R}^n.$$

Note that

$$\mathrm{d}\Phi_j^n(u, x) = [\mathrm{D}^2 u(x)]_j \,\mathrm{d}x \tag{2.5}$$

for $u \in \mathrm{Conv}(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$. If B is a Borel set in $\mathbb{R}^n$ and $u \in \mathrm{Conv}(\mathbb{R}^n)$, then

$$\Phi_0^n(u, B) = \mathcal{H}^n(B) \tag{2.6}$$

and

$$\Phi_j^n(\lambda u, B) = \lambda^j \Phi_j^n(u, B)$$

for $j \in \{0, \dots, n\}$ and $\lambda > 0$.

Let u_k be a sequence in $\mathrm{Conv}(\mathbb{R}^n)$. If u_k epi-converges to $u \in \mathrm{Conv}(\mathbb{R}^n)$, then the sequence of measures $\Theta_j^n(u_k, \cdot)$ converges weakly to the measure $\Theta_j^n(u, \cdot)$ (see [15, Theorem 7.3]). In particular, we obtain the following result, for $\zeta \in C_c(\mathbb{R}^n \times \mathbb{R}^n)$ with compact support in the second variable. If B is a bounded Borel set in $\mathbb{R}^n$ and $\Theta_j^n(u, \partial B) = 0$, then

$$\lim_{k \to \infty} \int_{B \times \mathbb{R}^n} \zeta(x, y) \,\mathrm{d}\Theta_j^n(u_k, (x, y)) = \int_{B \times \mathbb{R}^n} \zeta(x, y) \,\mathrm{d}\Theta_j^n(u, (x, y)). \tag{2.7}$$

The interplay of Hessian measures and convex conjugation is well understood. Let $u \in \mathrm{Conv}(\mathbb{R}^n)$ and $j \in \{0, \dots, n\}$. It was shown in [15, Theorem 8.2] that

$$\Theta_j^n(u, A) = \Theta_{n-j}^n(u^*, \hat{A})$$

for every Borel subset A of $\mathbb{R}^n \times \mathbb{R}^n$ where $\hat{A} = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \colon (y, x) \in A\}$. As an immediate consequence, we obtain

$$\int_B \zeta(y) \,\mathrm{d}\Psi_j^n(u, y) = \int_B \zeta(x) \,\mathrm{d}\Phi_j^n(u^*, x) \tag{2.8}$$

for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ and Borel subset $B \subseteq \mathbb{R}^n$, when $\zeta : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}$ is such that one of the two integrals above and therefore both exist. In particular,

$$\int_{\mathbb{R}^n} \zeta(\nabla u(x)) [\mathrm{D}^2 u(x)]_{n-j} \,\mathrm{d}x = \int_{\mathbb{R}^n} \zeta(x) [\mathrm{D}^2 u^*(x)]_j \,\mathrm{d}x \tag{2.9}$$

for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \cap C_+^2(\mathbb{R}^n)$ and such ζ.

2.4. **Valuations on Convex Functions.**

The following homogeneous decomposition result was established in [14, Theorem 1].

Theorem 2.1. *If* $\mathrm{Z} : \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}$ *is a continuous and epi-translation invariant valuation, then there are continuous, epi-translation invariant valuations* $\mathrm{Z}_i : \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}$ *that are epi-homogeneous of degree* i *such that* $\mathrm{Z} = \mathrm{Z}_0 + \cdots + \mathrm{Z}_n$.

We say that a valuation $\mathrm{Z} : \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}$ is epi-additive if

$$\mathrm{Z}(\alpha \cdot u \,\square\, \beta \cdot v) = \alpha \,\mathrm{Z}(u) + \beta \,\mathrm{Z}(v)$$

for every $\alpha, \beta > 0$ and $u, v \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$. The following result is a consequence of Theorem 2.1.

Lemma 2.2 ([14], Corollary 22). *If* $\mathrm{Z}:\mathrm{Conv_{sc}}(\mathbb{R}^n)\to\mathbb{R}$ *is a continuous and epi-translation invariant valuation that is epi-homogeneous of degree* 1*, then* Z *is epi-additive.*

For $\zeta\in D_0^n$ and $u\in\mathrm{Conv_{sc}}(\mathbb{R}^n)\cap C_+^2(\mathbb{R}^n)$, the substitution $y=\nabla u(x)$ shows that

$$\mathrm{V}_{0,\zeta}(u)=\int_{\mathbb{R}^n}\zeta(|\nabla u(x)|)[\mathrm{D}^2u(x)]_n\,\mathrm{d}x=\int_{\mathbb{R}^n}\zeta(|y|)\,\mathrm{d}y$$

does not depend on u. Hence its extension to $\mathrm{Conv_{sc}}(\mathbb{R}^n)$ is constant. We have the following more general result.

Theorem 2.3 ([14], Theorem 25). *A functional* $\mathrm{Z}:\mathrm{Conv_{sc}}(\mathbb{R}^n)\to\mathbb{R}$ *is a continuous and epi-translation invariant valuation that is epi-homogeneous of degree* 0*, if and only if* Z *is constant.*

For the other extremal degree, we have the following classification result.

Theorem 2.4 ([14], Theorem 2). *A functional* $\mathrm{Z}:\mathrm{Conv_{sc}}(\mathbb{R}^n)\to\mathbb{R}$ *is a continuous and epi-translation invariant valuation that is epi-homogeneous of degree* n*, if and only if there exists* $\zeta\in C_c(\mathbb{R}^n)$ *such that*

$$\mathrm{Z}(u)=\int_{\mathrm{dom}(u)}\zeta(\nabla u(x))\,\mathrm{d}x$$

for every $u\in\mathrm{Conv_{sc}}(\mathbb{R}^n)$.

As a simple consequence, we obtain the following result. Here, we say that Z is *reflection invariant* if $\mathrm{Z}(u)=\mathrm{Z}(u^-)$, where $u^-(x):=u(-x)$ for every $x\in\mathbb{R}^n$.

Corollary 2.5. *For* $n\geq 2$*, a functional* $\mathrm{Z}:\mathrm{Conv_{sc}}(\mathbb{R}^n)\to\mathbb{R}$ *is a continuous, epi-translation and rotation invariant valuation that is epi-homogeneous of degree* n*, if and only if there exists* $\zeta\in C_c([0,\infty))$ *such that*

$$\mathrm{Z}(u)=\int_{\mathrm{dom}(u)}\zeta(|\nabla u(x)|)\,\mathrm{d}x$$

for every $u\in\mathrm{Conv_{sc}}(\mathbb{R}^n)$*. For* $n=1$*, the same representation holds if we replace rotation invariance by reflection invariance.*

Proof. Let Z be given. By Theorem 2.4 there exists $\xi\in C_c(\mathbb{R}^n)$ such that

$$\mathrm{Z}(u)=\int_{\mathrm{dom}(u)}\xi(\nabla u(x))\,\mathrm{d}x$$

for every $u\in\mathrm{Conv_{sc}}(\mathbb{R}^n)$. Fix $y\in\mathbb{R}^n$ and let $u(x)=\mathbf{I}_{B^n}(x)+\langle y,x\rangle$ for $x\in\mathbb{R}^n$. Note, that for $\vartheta\in\mathrm{SO}(n)$

$$u\circ\vartheta^{-1}(x)=\mathbf{I}_{B^n}(x)+\langle\vartheta^{-t}y,x\rangle.$$

Hence, using the $\mathrm{SO}(n)$ invariance of Z, we obtain

$$\kappa_n\,\xi(y)=\mathrm{Z}(u)=\mathrm{Z}(u\circ\vartheta^{-1})=\kappa_n\,\xi(\vartheta^{-t}y)$$

for every $\vartheta\in\mathrm{SO}(n)$. Since $y\in\mathbb{R}^n$ was arbitrary, it follows that there exists $\zeta\in C_c([0,\infty))$ such that $\xi(y)=\zeta(|y|)$ for every $y\in\mathbb{R}^n$. In case $n=1$, we simply need to choose $\vartheta(x)=-x$ in the last argument. ∎

Combining this corollary with Theorem 2.3 gives the following result.

Corollary 2.6. *A functional* $\mathrm{Z}:\mathrm{Conv_{sc}}(\mathbb{R})\to\mathbb{R}$ *is a continuous, epi-translation and reflection invariant valuation, if and only if there exist* $\zeta_0\in D_0^1$ *and* $\zeta_1\in D_1^1$ *such that*

$$\mathrm{Z}(u)=\mathrm{V}_{0,\zeta_0}(u)+\mathrm{V}_{1,\zeta_1}(u)$$

for every $u\in\mathrm{Conv_{sc}}(\mathbb{R}^n)$.

We remark that this implies that Theorem 1.5 is also true in the one-dimensional case if we use the additional assumption of reflection invariance.

All of the previous results have dual versions. The following result is Theorem 4 from [14].

Theorem 2.7. *If* $\mathrm{Z} : \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}$ *is a continuous and dually epi-translation invariant valuation, then there are continuous and dually epi-translation invariant valuations* $\mathrm{Z}_i : \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}$ *that are homogeneous of degree* i *such that* $\mathrm{Z} = \mathrm{Z}_0 + \cdots + \mathrm{Z}_n$.

Theorem 2.8 ([14], Theorem 25)**.** *A functional* $\mathrm{Z} : \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}$ *is a continuous and dually epi-translation invariant valuation that is homogeneous of degree* 0*, if and only if* Z *is constant.*

Theorem 2.9 ([14], Theorem 5)**.** *A functional* $\mathrm{Z} : \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}$ *is a continuous and dually epi-translation invariant valuation that is homogeneous of degree* n*, if and only if there exists* $\zeta \in C_c(\mathbb{R}^n)$ *such that*

$$\mathrm{Z}(v) = \int_{\mathbb{R}^n} \zeta(x)\,\mathrm{d}\Phi_n^n(v,x)$$

for every $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$.

Corollary 2.10. *For* $n \geq 2$*, a functional* $\mathrm{Z} : \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}$ *is a continuous, dually epi-translation and rotation invariant valuation that is homogeneous of degree* n*, if and only if there is* $\zeta \in C_c([0,\infty))$ *such that*

$$\mathrm{Z}(v) = \int_{\mathbb{R}^n} \zeta(|x|)\,\mathrm{d}\Phi_n^n(v,x)$$

for every $v \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$*. For* $n = 1$*, the same representation holds if we replace rotation invariance by reflection invariance.*

The dual to Corollary 2.6 can be written in the following way. The integrals are well-defined by the definition of D_0^1 together with (2.6), as well as, Theorem 2.9.

Corollary 2.11. *If* $\mathrm{Z} : \mathrm{Conv}(\mathbb{R};\mathbb{R}) \to \mathbb{R}$ *is a continuous, dually epi-translation and reflection invariant valuation, then there exist* $\zeta_0 \in D_0^1$ *and* $\zeta_1 \in D_1^1$ *such that*

$$\mathrm{Z}(v) = \int_{\mathbb{R}} \zeta_0(|x|)\,\mathrm{d}\Phi_0^1(v,x) + \int_{\mathbb{R}} \zeta_1(|x|)\,\mathrm{d}\Phi_1^1(v,x)$$

for every $v \in \mathrm{Conv}(\mathbb{R};\mathbb{R})$.

The following was shown in [14, Theorem 17].

Lemma 2.12. *Let* $\zeta \in C(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n)$ *have compact support with respect to the second variable. For* $j \in \{0,1,\dots,n\}$*,*

$$\mathrm{Z}(v) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \zeta(v(x),x,y)\,\mathrm{d}\Theta_j^n(v,(x,y))$$

is well-defined for every $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ *and defines a continuous valuation on* $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$.

We conclude this part with a result which will be repeatedly used throughout this paper. It is a direct consequence of (2.8) and Lemma 2.12.

Lemma 2.13. *For* $\zeta \in C_c([0,\infty))$ *and* $j \in \{0,\dots,n\}$*, the functional* $\mathrm{Z}\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}$*, defined by*

$$\mathrm{Z}(u) = \int_{\mathbb{R}^n} \zeta(|y|)\,\mathrm{d}\Psi_j^n(u,y),$$

is a continuous, epi-translation and rotation invariant valuation that is epi-homogeneous of degree j.

2.5. **The Abel transform.** For $\zeta \in C_b((0,\infty))$, define its *Abel transform* for $t>0$ as

$$\mathcal{A}\,\zeta(t) := \int_t^\infty \frac{s\,\zeta(s)\,\mathrm{d}s}{\sqrt{s^2-t^2}} = \int_{-\infty}^\infty \zeta(s^2+t^2)\,\mathrm{d}s.$$

If $\xi \in C_b^1((0,\infty))$, then the inverse transform is given by

$$\mathcal{A}^{-1}\,\xi(s) = -\frac{1}{\pi}\int_s^\infty \frac{\xi'(t)\,\mathrm{d}t}{\sqrt{t^2-s^2}}.$$

In particular, $\mathcal{A}\,\zeta \equiv 0$ implies that $\zeta \equiv 0$. More generally, we have the following fact.

Lemma 2.14. *Let $\zeta \in C_b((0,\infty))$ and $k \in \mathbb{N}\cup\{0\}$. If*

$$\int_0^\infty \zeta(\sqrt{r^2+t^2})r^k\,\mathrm{d}r = 0 \tag{2.10}$$

for every $t>0$, then $\zeta \equiv 0$.

Proof. First, assume that $k = 2j+1$ with $j \in \mathbb{N}\cup\{0\}$. Then (2.10) can be written in the form

$$\int_{\bar{t}}^\infty \bar{\zeta}(s)(s-\bar{t})^j\,\mathrm{d}s = 0 \qquad \text{for all } \bar{t}>0, \tag{2.11}$$

where

$$\bar{\zeta}(s) := \zeta(\sqrt{s}) \qquad \text{for } s>0 \text{ and } \bar{t} = t^2.$$

Taking $(j+1)$ derivatives in (2.11) leads to $\bar{\zeta} \equiv 0$.

Next, assume that $k = 2j$ with $j \in \mathbb{N}$. Then (2.10) can be reduced to

$$\int_t^\infty \zeta(s)(s^2-t^2)^{j-\frac{1}{2}}s\,\mathrm{d}s = 0 \qquad \text{for all } t>0.$$

Taking j derivatives we obtain that

$$\int_t^\infty \frac{s\,\zeta(s)\,\mathrm{d}s}{\sqrt{s^2-t^2}} = 0 \qquad \text{for all } t>0$$

(which is also (2.10) for $k=0$). This last condition means that the Abel transform of ζ is identically zero, and then the same holds for ζ. □

2.6. **The Classes D_j^n.** Let $j \in \{1,\dots,n-1\}$ and let $\zeta \in D_j^n$. We will associate with ζ two functions $\eta, \rho \in C_c([0,\infty))$ where

$$\eta(t) = \int_t^\infty s^{n-j-1}\zeta(s)\,\mathrm{d}s \tag{2.12}$$

and

$$\rho(t) = t^{n-j}\zeta(t) + (n-j)\eta(t) \tag{2.13}$$

for every $t>0$. Observe that it follows from the definition of D_j^n that $\eta(0) = \lim_{t\to0^+}\eta(t)$ exists and is finite and that $\rho(0) = \lim_{t\to0^+}\rho(t) = (n-j)\eta(0)$.

The following result gives a geometric interpretation to the function ρ using the function u_t and v_t defined in (2.1) and (2.2).

Lemma 2.15. *Let $j \in \{1, \dots, n-1\}$ and $\zeta \in D_j^n$. For every $t \geq 0$,*

$$\int_{\mathbb{R}^n} \zeta(|y|) \,\mathrm{d}\Psi_j^n(u_t, y) = \int_{\mathbb{R}^n} \zeta(|x|) \,\mathrm{d}\Phi_j^n(v_t, x) = \kappa_n \binom{n}{j} \rho(t).$$

Proof. Recall that $v_t = (u_t)^*$. Hence, the first equality follows from (2.8). For $t > 0$, all Hessian measures of v_t, with the exception of the measure $\Theta_n^n(v_t, \cdot)$, vanish in the set $\{x \in \mathbb{R}^n : |x| < t\} \times \mathbb{R}^n$. In particular, we have $\Theta_j^n(v_t, B) = 0$ for every Borel subset B of $\{x \in \mathbb{R}^n : |x| < t\} \times \mathbb{R}^n$. Moreover, the function v_t is of class C^2 in the set $\{x \in \mathbb{R}^n : |x| > t\}$ and the Hessian matrix of v_t at a point x in this set has $(n-1)$ eigenvalues equal to $1/|x|$ and the last eigenvalue equal to zero. Hence

$$[\mathrm{D}^2 v_t(x)]_j = \binom{n-1}{j} \frac{1}{|x|^j}$$

for $|x| > t$. The set $t\,\mathbb{S}^{n-1}$ is the set of singular points for v_t. For $x \in t\,\mathbb{S}^{n-1}$, we have

$$\partial v_t(x) = \Big\{ r\,\frac{x}{|x|} : r \in [0,1] \Big\}.$$

Consequently, $P_s(v_t, t\,\mathbb{S}^{n-1} \times \mathbb{R}^n) = \{(t+r)x : x \in \mathbb{S}^{n-1}, r \in [0,s]\}$ for $s \geq 0$. Thus,

$$\mathcal{H}^n\big(P_s(v_t, t\,\mathbb{S}^{n-1} \times \mathbb{R}^n)\big) = \kappa_n\big((t+s)^n - t^n\big)$$

and therefore

$$\Phi_j^n(v_t, t\,\mathbb{S}^{n-1}) = \Theta_{n-j}^n(v_t, t\,\mathbb{S}^{n-1} \times \mathbb{R}^n) = \kappa_n \binom{n}{j} t^{n-j}.$$

Summing up we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} \zeta(|x|) \,\mathrm{d}\Phi_j^n(v_t, x) &= \kappa_n \binom{n}{j} t^{n-j} \zeta(t) + \binom{n-1}{j} \int_{\{x \in \mathbb{R}^n :\, |x| > t\}} \zeta(|x|) \frac{1}{|x|^j} \,\mathrm{d}x \\ &= \kappa_n \binom{n}{j} t^{n-j} \zeta(t) + n\kappa_n \binom{n-1}{j} \int_t^{+\infty} s^{n-j-1} \zeta(s) \,\mathrm{d}s,\end{aligned}$$

which implies the result for $t > 0$. Note that the set of singular points of v_0 is $\{0\}$ and that $\partial v_0(0) = B^n$. Hence $\Theta_{n-j}^n(v_0, \{0\} \times B^n) = 0$ which implies that the result holds for $t = 0$. $\square$

3. Singular Hessian Valuations

This section is devoted to the proof of Theorem 1.2 and Theorem 1.4. We start with functions in $\mathrm{Conv}(\mathbb{R}^n; \mathbb{R}) \cap C_+^2(\mathbb{R}^n)$ and show that on this class of functions, the valuations from (1.3) are well-defined and finite for $\zeta \in D_j^n$ with $j \in \{0, \dots, n\}$. The dual result shows that (1.2) is well-defined and finite for such ζ on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \cap C_+^2$. The next step is an extension to functions in $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$. For this, we collect results related to the theory of Hessian equations and establish uniform estimates for regular functions. In the last step of the proof, we use Moreau-Yosida envelopes and the polynomiality of epi-translation invariant valuations, which was proved in [14], to extend the valuations to general functions in $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$. In the last part, we apply Theorem 1.2 to establish a representation formula for $\mathrm{V}_{j,\zeta}$ on a certain class of functions.

3.1. **The Smooth Case.** Let $\operatorname{Conv}_{(0)}(\mathbb{R}^n;\mathbb{R})$ denote the set of functions $v\in\operatorname{Conv}(\mathbb{R}^n;\mathbb{R})$ that are of class C^2 in a neighborhood of the origin.

Lemma 3.1. *Let $j\in\{1,\dots,n\}$ and $\zeta\in D^n_j$. If $v\in\operatorname{Conv}_{(0)}(\mathbb{R}^n;\mathbb{R})$, then*

$$\int_{\mathbb{R}^n}\big|\zeta(|x|)\big|\,\mathrm{d}\Phi^n_j(v,x) \tag{3.1}$$

is well-defined and finite.

Proof. Let $\delta>0$ be such that v is of class C^2 on $\{x\in\mathbb{R}^n:|x|<\delta\}$. We write rewrite (3.1) as

$$\int_{\{x\colon |x|<\delta/2\}}\big|\zeta(|x|)\big|\,\mathrm{d}\Phi^n_j(v,x)+\int_{\{x\colon |x|\geq\delta/2\}}\big|\zeta(|x|)\big|\,\mathrm{d}\Phi^n_j(v,x).$$

The second term is bounded, as $\zeta\in C_b((0,\infty))$ and $\Phi^n_j(v,\cdot)$ is locally finite. Concerning the first term, let $\gamma>0$ be such that $[\mathrm{D}^2v(x)]_j\leq\gamma$ for every x such that $|x|\leq\delta/2$. Using (2.5), we get

$$\begin{aligned}\int_{\{x\colon |x|<\delta/2\}}\big|\zeta(|x|)\big|\,\mathrm{d}\Phi^n_j(v,x) &= \int_{\{x\colon |x|<\delta/2\}}\big|\zeta(|x|)\big|[\mathrm{D}^2v(x)]_j\,\mathrm{d}x\\ &\leq \gamma\int_{\{x\colon |x|<\delta/2\}}\big|\zeta(|x|)\big|\,\mathrm{d}x\\ &= \omega_n\gamma\int_0^{\delta/2}r^{n-1}\big|\zeta(r)\big|\,\mathrm{d}r.\end{aligned}$$

As $\zeta\in D^n_j$, the function $r^{n-1}\big|\zeta(r)\big|$ can be extended to $r=0$ as a continuous function. Hence the last integral is finite. ☐

We remark that as a consequence of this Lemma, we obtain that (1.3) in Theorem 1.4 holds for every $v\in\operatorname{Conv}(\mathbb{R}^n;\mathbb{R})\cap C^2(\mathbb{R}^n)$. A special case of the dual result of Lemma 3.1 is the content of the following lemma.

Lemma 3.2. *Let $j\in\{1,\dots,n\}$ and $\zeta\in D^n_j$. If $u\in\operatorname{Conv}_{\mathrm{sc}}(\mathbb{R}^n)\cap C^2_+(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n}\zeta(|\nabla u(x)|)[\mathrm{D}^2u(x)]_{n-j}\,\mathrm{d}x$$

is well-defined and finite.

Proof. Note that $u^*\in\operatorname{Conv}_{(0)}(\mathbb{R}^n;\mathbb{R})$ for $u\in\operatorname{Conv}_{\mathrm{sc}}(\mathbb{R}^n)\cap C^2_+(\mathbb{R}^n)$. Hence, the statement follows from Lemma 3.1 combined with (2.9). ☐

3.2. **Preparatory Results for Theorem 1.2.** We introduce the following class of functions,

$$\operatorname{Conv}_{\mathrm{rg}}(\mathbb{R}^n):=\{u\in\operatorname{Conv}_{\mathrm{sc}}(\mathbb{R}^n): u\ \text{ is regular},\ 0=u(0)<u(x)\ \text{ for every }\ x\neq 0\},$$

where u is *regular* if $u\in C^2(\mathbb{R}^n)$ and the boundary of its sublevel sets $\{u\leq t\}$ is of class C^2 with positive Gaussian curvature for every $t>0$.

Let $u\in\operatorname{Conv}_{\mathrm{rg}}(\mathbb{R}^n)$. Note that $\nabla u(x)\neq 0$ for every $x\neq 0$ and that, for $t>0$, the sublevel set $\{u\leq t\}$ is a convex body with non-empty interior and

$$\partial\{u\leq t\}=\{u=t\}.$$

Given $x\neq 0$ and $j\in\{0,\dots,n-1\}$, we denote by $\tau_j(u,x)$ the jth elementary symmetric function of the principal curvatures at x of $\{u\leq t\}$ for $t=u(x)$. For every $t\geq 0$, we denote by $h(u,\cdot,t)\colon\mathbb{S}^{n-1}\to\mathbb{R}$, the support function of $\{u\leq t\}$. If $t>0$, the regularity properties of the sublevel set $\{u\leq t\}$ imply that $h(u,\cdot,t)\in C^2(\mathbb{S}^{n-1})$. For $y\in\mathbb{S}^{n-1}$, let $\sigma_j(u,y,t)$ denote the jth elementary symmetric function of the principal radii of curvature of $\{u=t\}$ at $\nu_t^{-1}(y)$, where $\nu_t\colon\{u=t\}\to\mathbb{S}^{n-1}$ denotes the Gauss map of $\{u\le t\}$ and $\nu_t^{-1}:\mathbb{S}^{n-1}\to\{u=t\}$ its inverse.

3.2.1. *Hessian Operators and Reilly-Type Lemmas.* Let $A=(a_{ij})$ be a symmetric $n\times n$ matrix and $k\in\{0,\dots,n\}$. We denote by $[A]_k$ the kth elementary symmetric function of the eigenvalues of A, that is,

$$[A]_k := \sum_{1\le i_1<i_2<\cdots<i_k\le n} \lambda_{i_1}\cdots\lambda_{i_k}$$

(with the usual convention $[A]_0\equiv 1$), where $\lambda_1,\dots,\lambda_n$ are the eigenvalues of A. For $l,m\in\{1,\dots,n\}$, we set

$$[A]_k^{lm} := \frac{\partial [A]_k}{\partial a_{lm}}. \tag{3.2}$$

The $n\times n$ matrix formed by the entries $[A]_k^{lm}$ is called the *cofactor matrix* of order k of the matrix A. When $k=n$, this is the usual cofactor matrix. From the homogeneity of the map $A\mapsto[A]_k$, we deduce

$$[A]_k^{lm} a_{lm} = k\,[A]_k \tag{3.3}$$

(here and throughout this section, we adopt the convention of summation over repeated indices). For a $n\times j$ matrix (b^{lm}) with elements of class C^1, we define

$$\operatorname{div}_m((b^{lm}))$$

as the divergence of the vector field whose components are the elements of the lth column of the matrix (b^{lm}).

Let $u\in C^2(\mathbb{R}^n)$. For $k\in\{0,\dots,n\}$, the kth *Hessian operator* applied to u and evaluated at a point $x\in\mathbb{R}^n$ is defined as

$$[\mathrm{D}^2 u(x)]_k.$$

The following result is due to Reilly [37, Proposition 2.1].

Lemma 3.3. *Let $k\in\{0,1,\dots,n\}$ and $l\in\{1,\dots,n\}$. Then*

$$\operatorname{div}_m([\mathrm{D}^2u(x)]_k^{lm})=0$$

for every $u\in C^3(\mathbb{R}^n)$ and $x\in\mathbb{R}^n$.

We will also need a corresponding result for functions defined on the unit sphere. Let $h\in C^2(\mathbb{S}^{n-1})$. Given $x\in\mathbb{S}^{n-1}$, for $i,j\in\{1,\dots,n-1\}$ we denote by h_i and h_{ij} the first and second covariant derivatives of h with respect to a local orthonormal frame defined in a neighborhood of x. Let δ_{ij} be the usual Kronecker symbols. The following result was proved in [10] in the case $m=n-1$, and then extended to the general case in [17, Lemma 3.1].

Lemma 3.4. *Let $k\in\{0,\dots,n-1\}$ and $l\in\{1,\dots,n-1\}$. Then*

$$\operatorname{div}_m\left([h(x)_{lm}+h(x)\delta_{lm}]_k^{lm}\right)=\sum_{m=1}^{n-1}\left([h(x)_{lm}+h(x)\delta_{lm}]_k^{lm}\right)_m=0$$

for every $h\in C^3(\mathbb{S}^{n-1})$ and $x\in\mathbb{S}^{n-1}$, where $\left([h(x)_{lm}+h(x)\delta_{lm}]_k^{lm}\right)_m$ denotes the derivative of $[h(x)_{lm}+h(x)\delta_{lm}]_k^{lm}$ with respect to the mth variable of a local orthonormal frame defined in a neighborhood of x.

As a consequence, we obtain the following facts.

Corollary 3.5. *Let* $k \in \{0, \dots, n-1\}$. *If* $f, g, h \in C^2(\mathbb{S}^{n-1})$, *then*

$$\int_{\mathbb{S}^{n-1}} f\, [h_{lm} + h\delta_{lm}]_k^{lm} g_{lm} \,\mathrm{d}\mathcal{H}^{n-1} = -\int_{\mathbb{S}^{n-1}} [h_{lm} + h\delta_{lm}]_k^{lm} f_l\, g_m \,\mathrm{d}\mathcal{H}^{n-1}.$$

Proof. When $h \in C^3(\mathbb{S}^{n-1})$, the statement follows from the divergence theorem on the sphere and Lemma 3.4:

$$\begin{aligned} 0 &= \int_{\mathbb{S}^{n-1}} \sum_{m=1}^{n-1} \big(f\, [h_{lm} + h\delta_{lm}]_k^{lm} g_l\big)_m \,\mathrm{d}\mathcal{H}^{n-1} \\ &= \int_{\mathbb{S}^{n-1}} f\, [h_{lm} + h\delta_{lm}]_k^{lm} g_{lm} \,\mathrm{d}\mathcal{H}^{n-1} + \int_{\mathbb{S}^{n-1}} [h_{lm} + h\delta_{lm}]_k^{lm} f_l g_m \,\mathrm{d}\mathcal{H}^{n-1}. \end{aligned}$$

The general case is obtained by approximation. □

Corollary 3.6. *Let* $k \in \{0, \dots, n-1\}$. *If* $f, g, h \in C^2(\mathbb{S}^{n-1})$, *then*

$$\int_{\mathbb{S}^{n-1}} f\, [h_{lm} + h\delta_{lm}]_k^{lm} (g_{lm} + g\delta_{lm}) \,\mathrm{d}\mathcal{H}^{n-1} = \int_{\mathbb{S}^{n-1}} g\, [h_{lm} + h\delta_{lm}]_k^{lm} (f_{lm} + f\delta_{lm}) \,\mathrm{d}\mathcal{H}^{n-1}.$$

3.2.2. *Geometric Statements.* The following result is Proposition 2.4 from [27].

Lemma 3.7. *For* $u \in \mathrm{Conv}_{\mathrm{rg}}(\mathbb{R}^n)$ *the function* $h(u, \cdot, \cdot)$ *is of class* $C^2(\mathbb{S}^{n-1} \times (0, \infty))$ *and, for every* $(y, t) \in \mathbb{S}^{n-1} \times (0, \infty)$,

$$\frac{1}{|\nabla u(\nu_t^{-1}(y))|} = \frac{\partial}{\partial t} h(u, y, t).$$

The next result follows from formulas (4.9), (5.55) and (5.56) in [40].

Lemma 3.8. *Let* $1 \le i \le n$ *and* $t > 0$. *For* $u \in \mathrm{Conv}_{\mathrm{rg}}(\mathbb{R}^n)$,

$$\int_{\{u=t\}} \tau_{n-i-1}(u, x) \,\mathrm{d}\mathcal{H}^{n-1}(x) = \alpha\, V_i(\{u \le t\}),$$

and

$$\int_{\mathbb{S}^{n-1}} h(u, y, t) \sigma_{i-1}(u, y, t) \,\mathrm{d}\mathcal{H}^{n-1}(y) = \alpha\, V_i(\{u \le t\}),$$

where α *is a positive constant depending only on* n *and* i.

Lemma 3.9. *Let* $1 \le i \le n$ *and* $0 < t_1 < t_2$. *For* $u \in \mathrm{Conv}_{\mathrm{rg}}(\mathbb{R}^n)$,

$$\int_{\{t_1 < u \le t_2\}} \tau_{n-i}(u, x) \,\mathrm{d}x = \alpha\big(V_i(\{u \le t_2\}) - V_i(\{u \le t_1\})\big),$$

where α *is a positive constant depending only on* n *and* i.

Proof. Note that $\nabla u(x) \neq 0$ for every $x \in \{t_1 < u \le t_2\}$. By the coarea formula we have

$$\int_{\{t_1 < u \le t_2\}} \tau_{n-i}(u, x) \,\mathrm{d}x = \int_{t_1}^{t_2} \int_{\{u=t\}} \frac{1}{|\nabla u(x)|}\, \tau_{n-i}(u, x) \,\mathrm{d}\mathcal{H}^{n-1}(x) \,\mathrm{d}t.$$

Using the change of variable $y = \nu_t(x)$ in the inner integral and Lemma 3.7, we get

$$\begin{aligned} \int_{\{u=t\}} \frac{1}{|\nabla u(x)|}\, \tau_{n-i}(u, x) \,\mathrm{d}\mathcal{H}^{n-1}(x) &= \int_{\mathbb{S}^{n-1}} \frac{1}{|\nabla u(\nu_t^{-1}(y))|} \sigma_{i-1}(u, y, t) \,\mathrm{d}\mathcal{H}^{n-1}(y) \\ &= \int_{\mathbb{S}^{n-1}} \frac{\partial}{\partial t} h(u, y, t)\, \sigma_{i-1}(u, y, t) \,\mathrm{d}\mathcal{H}^{n-1}(y). \end{aligned}$$

Next, we prove that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}^{n-1}} h(u,t,y)\,\sigma_{i-1}(u,y,t)\,\mathrm{d}\mathcal{H}^{n-1}(y) = i\int_{\mathbb{S}^{n-1}} \frac{\partial}{\partial t}h(u,y,t)\,\sigma_{i-1}(u,y,t)\,\mathrm{d}\mathcal{H}^{n-1}(y). \tag{3.4}$$

We have

$$\sigma_{i-1}(u,y,t) = [h(u,y,t)_{lm} + h(u,y,t)\delta_{lm}]_{i-1} \tag{3.5}$$

(see, for example, [40, Section 2.5]). We start from the formula,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}^{n-1}} h(u,y,t)\,\sigma_{i-1}(u,y,t)\,\mathrm{d}\mathcal{H}^{n-1}(y) \;&=\; \int_{\mathbb{S}^{n-1}} \frac{\partial}{\partial t}h(u,y,t)\,\sigma_{i-1}(u,y,t)\,\mathrm{d}\mathcal{H}^{n-1}(y)\\ &\quad+\int_{\mathbb{S}^{n-1}} h(u,y,t)\,\frac{\partial \sigma_{i-1}}{\partial t}(u,y,t)\,\mathrm{d}\mathcal{H}^{n-1}(y).\end{aligned} \tag{3.6}$$

By (3.5), (3.2), Corollary 3.6, and (3.5),

$$\begin{aligned}&\int_{\mathbb{S}^{n-1}} h(u,y,t)\frac{\partial}{\partial t}\,\sigma_{i-1}(u,y,t)\,\mathrm{d}\mathcal{H}^{n-1}(y)\\ &\quad= \int_{\mathbb{S}^{n-1}} h(u,y,t)[h(u,y,t)_{lm}+h(u,y,t)\delta_{lm}]_{i-1}^{lm}\Big(\big(\frac{\partial}{\partial t}h(u,y,t)\big)_{lm} + \frac{\partial}{\partial t}h(u,y,t)\,\delta_{lm}\Big)\,\mathrm{d}\mathcal{H}^{n-1}(y)\\ &\quad= \int_{\mathbb{S}^{n-1}} \frac{\partial}{\partial t}h(u,y,t)\,[h(u,y,t)_{lm}+h(u,y,t)\delta_{lm}]_{i-1}^{lm}\big(h(u,y,t)_{lm}+h(u,y,t)\delta_{lm}\big)\,\mathrm{d}\mathcal{H}^{n-1}(y)\\ &\quad= (i-1)\int_{\mathbb{S}^{n-1}} \frac{\partial}{\partial t}h(u,y,t)[h(u,y,t)_{lm}+h(u,y,t)\delta_{lm}]_{i-1}\,\mathrm{d}\mathcal{H}^{n-1}(y)\\ &\quad= (i-1)\int_{\mathbb{S}^{n-1}} \frac{\partial}{\partial t}h(u,y,t)\sigma_{i-1}(u,y,t)\,\mathrm{d}\mathcal{H}^{n-1}(y).\end{aligned}$$

The last chain of equations and (3.6) imply (3.4). Hence, by the latter relation,

$$\begin{aligned}&\int\limits_{\{t_1<u\le t_2\}} \tau_{n-i}(u,x)\,\mathrm{d}x\\ &\qquad= \int_{t_1}^{t_2}\int_{\mathbb{S}^{n-1}} \frac{\partial}{\partial t}h(u,y,t)\,\sigma_{i-1}(u,y,t)\,\mathrm{d}\mathcal{H}^{n-1}(y)\,\mathrm{d}t\\ &\qquad= \frac{1}{i}\int_{t_1}^{t_2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}^{n-1}} h(u,t,y)\,\sigma_{i-1}(u,y,t)\,\mathrm{d}\mathcal{H}^{n-1}(y)\,\mathrm{d}t\\ &\qquad= \frac{1}{i}\Big(\int_{\mathbb{S}^{n-1}} h(u,y,t_2)\,\sigma_{i-1}(u,y,t_2)\,\mathrm{d}\mathcal{H}^{n-1}(y) - \int_{\mathbb{S}^{n-1}} h(u,y,t_1)\,\sigma_{i-1}(u,y,t_1)\,\mathrm{d}\mathcal{H}^{n-1}(y)\Big)\\ &\qquad= \alpha\,\big(V_i(\{u\le t_2\}) - V_i(\{u\le t_1\})\big),\end{aligned}$$

where we have used Lemma 3.8 in the last equation. □

3.2.3. *Integration by Parts.* We start with a lemma which can be found in [9] (see formula (11)). We write u_k for the partial derivative of u with respect to x_k.

Lemma 3.10 ([9]). *Let $i\in\{1,\dots,n\}$ and $u\in \mathrm{Conv}_{\mathrm{rg}}(\mathbb{R}^n)$. Then*

$$[\mathrm{D}^2u(x)]_{n-i} = |\nabla u(x)|^{n-i}\,\tau_{n-i}(u,x) + \frac{[\mathrm{D}^2u(x)]_{n-i}^{lm}u_k(x)u_l(x)u_{km}(x)}{|\nabla u(x)|^2}$$

for every $x\neq 0$.

We will also need the following lemma for which we refer, for instance, to [37].

Lemma 3.11. *Let* $i \in \{0, \dots, n-1\}$ *and* $u \in \operatorname{Conv}_{\mathrm{rg}}(\mathbb{R}^n)$. *Then*

$$[\mathrm{D}^2 u(x)]_{n-i}^{lm} u_l(x) u_m(x) = |\nabla u(x)|^{n-i+1} \tau_{n-i-1}(u, x)$$

for every $x \neq 0$.

Proposition 3.12. *Let* $i \in \{1, \dots, n\}$ *and* $u \in \operatorname{Conv}_{\mathrm{rg}}(\mathbb{R}^n)$. *If* $\gamma : (0, \infty) \to \mathbb{R}$ *is differentiable, then*

$$\begin{aligned}\operatorname{div}_m \big(\gamma(|\nabla u(x)|)[\mathrm{D}^2 u(x)]_{n-i}^{lm} u_l(x)\big) &= [\mathrm{D}^2 u(x)]_{n-i} \Big((n-i)\gamma(|\nabla u(x)|) + \gamma'(|\nabla u(x)|)|\nabla u(x)|\Big) \\ &\quad - \gamma'(|\nabla u(x)|)|\nabla u(x)|^{n-i+1} \tau_{n-i}(u, x)\end{aligned}$$

for every $x \neq 0$.

Proof. We have

$$\operatorname{div}_m \big(\gamma(|\nabla u|)[\mathrm{D}^2 u]_{n-i}^{lm} u_l\big) = \gamma(|\nabla u|) \operatorname{div}_m([\mathrm{D}^2 u]_{n-i}^{lm} u_l) + \frac{\gamma'(|\nabla u|)}{|\nabla u|} [\mathrm{D}^2 u]_{n-i}^{lm} u_k u_m u_{km}.$$

Note that, by Lemma 3.3 and (3.3),

$$\operatorname{div}_m([\mathrm{D}^2 u]_{n-i}^{lm} u_l) = [\mathrm{D}^2 u]_{n-i}^{lm} u_{lm} = (n-i)[\mathrm{D}^2 u]_{n-i}.$$

The conclusion follows immediately from Lemma 3.10. □

Recall that given $\zeta \in D_i^n$ for $1 \leq i \leq n-1$, we associate two functions $\eta, \rho \in C_c([0, \infty))$ with ζ, defined by

$$\eta(t) = \int_t^\infty s^{n-i-1} \zeta(s) \, \mathrm{d}s, \qquad \rho(t) = t^{n-i} \zeta(t) + (n-i)\eta(t)$$

for $t \in [0, \infty)$.

The main result of this part is the following proposition.

Proposition 3.13. *Let* $1 \leq i \leq n-1$ *and* $\zeta \in D_i^n$. *For every* $u \in \operatorname{Conv}_{\mathrm{rg}}(\mathbb{R}^n)$ *and* t_1, t_2 *with* $0 < t_1 < t_2$,

$$\begin{aligned}\int_{\{t_1 < u \leq t_2\}} \zeta(|\nabla u(x)|)[\mathrm{D}^2 u(x)]_{n-i} \, \mathrm{d}x &= \int_{\{t_1 < u \leq t_2\}} \rho(|\nabla u(x)|) \tau_{n-i}(u, x) \, \mathrm{d}x \\ &\quad - \int_{\{u = t_2\}} \eta(|\nabla u(x)|) \tau_{n-i-1}(u, x) \, \mathrm{d}\mathcal{H}^{n-1}(x) \\ &\quad + \int_{\{u = t_1\}} \eta(|\nabla u(x)|) \tau_{n-i-1}(u, x) \, \mathrm{d}\mathcal{H}^{n-1}(x).\end{aligned}$$

Proof. For $r > 0$, we set

$$\gamma(r) := -\frac{1}{r^{n-i}} \int_r^\infty s^{n-i-1} \zeta(s) \, \mathrm{d}s.$$

Then

$$\gamma'(r) = \frac{n-i}{r^{n-i+1}} \int_r^\infty s^{n-i-1} \zeta(s) \, \mathrm{d}s + \frac{\zeta(r)}{r}$$

and

$$(n-i)\gamma(r) + r\gamma'(r) = \zeta(r)$$

for every $r > 0$. Moreover,

$$r^{n-i+1} \gamma'(r) = (n-i)\eta(r) + r^{n-i} \zeta(r) = \rho(r)$$

for every $r > 0$. By the previous relations, Proposition 3.12 and the divergence theorem, we have

$$\begin{aligned}\int_{\{t_1<u\le t_2\}} \zeta(|\nabla u|)[\mathrm{D}^2 u]_{n-i}\,\mathrm{d}x &= \int_{\{t_1<u\le t_2\}} \big((n-i)\gamma(|\nabla u|)+|\nabla u|\gamma'(|\nabla u|)\big)[\mathrm{D}^2 u]_{n-i}\,\mathrm{d}x\\ &= \int_{\{t_1<u\le t_2\}} |\nabla u|^{n-i+1}\gamma'(|\nabla u|)\tau_{n-i}\,\mathrm{d}x\\ &\quad+\int_{\{t_1<u\le t_2\}} \mathrm{div}_m\left(\gamma(|\nabla u|)[\mathrm{D}^2 u]_{n-i}^{lm}u_l\right)\,\mathrm{d}x\\ &= \int_{\{t_1<u\le t_2\}} \rho(|\nabla u|)\tau_{n-i}\,\mathrm{d}x+\int_{\{u=t_2\}}\frac{\gamma(|\nabla u|)}{|\nabla u|}[\mathrm{D}^2 u]_{n-i}^{lm}u_l u_m\,\mathrm{d}\mathcal{H}^{n-1}\\ &\quad-\int_{\{u=t_1\}}\frac{\gamma(|\nabla u|)}{|\nabla u|}[\mathrm{D}^2 u]_{n-i}^{lm}u_l u_m\,\mathrm{d}\mathcal{H}^{n-1}.\end{aligned}$$

In the last equation we have used the fact that the outer unit normal to the set $\{t_1 < u \le t_2\}$ at a point $x \in \{u = t_2\}$ is given by

$$\frac{\nabla u(x)}{|\nabla u(x)|},$$

while at a point $x \in \{u = t_1\}$ it is given by

$$-\frac{\nabla u(x)}{|\nabla u(x)|}.$$

The conclusion follows from Lemma 3.11. ☐

3.2.4. *An Estimate.* Given a real-valued function u defined in a subset V of $\mathbb{R}^n$, the Lipschitz constant of u in $U \subset V$ is defined as

$$\mathrm{Lip}_U(u) := \sup\left\{\frac{|u(x)-u(y)|}{|x-y|} : x,\, y \in U,\, x \ne y\right\}.$$

For $u \in \mathrm{Conv_{sc}}(\mathbb{R}^n)$ and $t > 0$, we set

$$\mathrm{Lip}(u,t) := \mathrm{Lip}_{\{u\le t\}}(u)$$

and note that for $u \in \mathrm{Conv_{sc}}(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$,

$$\lim_{t\to 0+} \mathrm{Lip}(u,t) = 0 \tag{3.7}$$

if $0 = u(0) \le u(x)$ for $x \in \mathbb{R}^n$.

Lemma 3.14. *Let $1 \le i \le n-1$ and $\zeta \in D_i^n$. If $u \in \mathrm{Conv_{rg}}(\mathbb{R}^n)$, then*

$$\left|\int_{\{t_1<u\le t_2\}} \zeta(|\nabla u(x)|)[\mathrm{D}^2 u(x)]_{n-i}\,\mathrm{d}x\right| \le \alpha\, V_i(\{u\le t_2\})\Big(\max_{[0,\mathrm{Lip}(u,t_2)]}|\rho| + \max_{[0,\mathrm{Lip}(u,t_2)]}|\eta|\Big)$$

for every $0 < t_1 < t_2$ where α is a positive constant depending only on n and i.

Proof. By Lemma 3.9 (and the monotonicity of intrinsic volumes)

$$\begin{aligned}\left|\int_{\{t_1<u\le t_2\}} \rho(|\nabla u(x)|)\tau_{n-i}(u,x)\,\mathrm{d}x\right| &\le \alpha\,(V_i(\{u\le t_2\})-V_i(\{u\le t_1\}))\max_{[0,\mathrm{Lip}(u,t_2)]}|\rho|\\ &\le \alpha\, V_i(\{u\le t_2\})\max_{[0,\mathrm{Lip}(u,t_2)]}|\rho|,\end{aligned}$$

where $\alpha > 0$ only depends on n and i. Similarly, by Lemma 3.8

$$\left| \int_{\{u=t_j\}} \eta(|\nabla u(x)|)\tau_{n-i-1}(u,x)\,\mathrm{d}\mathcal{H}^{n-1}(x) \right| \leq \alpha\, V_i(\{u \leq t_2\}) \max_{[0,\mathrm{Lip}(u,t_2)]} |\eta|$$

for $j \in \{1,2\}$. The conclusion now follows from Proposition 3.13. □

3.3. **Proof of Theorem 1.2.** Throughout this section, fix $j \in \{1,\dots,n-1\}$ and $\zeta \in D_j^n$. We will associate the two functions $\rho, \eta \in C_c([0,\infty))$ defined in (2.12) and (2.13) with ζ.

First, we prove that we may reduce to the case

$$\eta(0) = \rho(0) = 0.$$

Indeed, let

$$\eta_0 := \lim_{r \to 0^+} \eta(r),$$

and choose $\zeta_0 \in C_c([0,\infty))$ such that

$$\int_0^\infty s^{n-j-1}\zeta_0(s)\,\mathrm{d}s = \eta_0.$$

Then $\bar\zeta\colon (0,\infty) \to \mathbb{R}$, defined by

$$\bar\zeta(r) := \zeta(r) - \zeta_0(r),$$

belongs to D_j^n. The function $\bar\eta$, defined by

$$\bar\eta(r) := \int_r^\infty s^{n-j-1}\bar\zeta(s)\,\mathrm{d}s,$$

verifies $\bar\eta(0) = 0$. Moreover, as $\zeta_0 \in C_c([0,\infty))$, the functional $\mathrm{Z}_{\zeta_0}\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}$, given by

$$\mathrm{Z}_{\zeta_0}(u) := \int_{\mathbb{R}^n \times \mathbb{R}^n} \zeta_0(|y|)\,\mathrm{d}\Theta_j^n(u,y),$$

defines, by Lemma 2.13, a continuous, epi-translation and rotation invariant valuation on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$. We have then proved that ζ can be written as

$$\zeta = \bar\zeta + \zeta_0,$$

where ζ_0 gives rise to a continuous valuation on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$, while $\bar\zeta \in D_j^n$, and the function $\bar\eta$ associated to $\bar\zeta$ verifies $\bar\eta(0) = 0$. Finally, note that it follows from the relation

$$\bar\rho(r) = (n-j)\bar\eta(r) + r^{n-j}\bar\zeta(r),$$

that $\bar\rho(0) = 0$.

In the following, let

$$\mathrm{Conv}_{\mathrm{sc},0}(\mathbb{R}^n) := \{u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)\colon u(0) = 0 \leq u(x) \text{ for } x \in \mathbb{R}^n\}$$

and

$$T_j(u) := \{t > 0 : \Theta_j^n(u, \{u = t\} \times \mathbb{R}^n) = 0\}.$$

Since $\Theta_j^n(u,\cdot)$ is a locally finite measure, the complement of $T_j(u)$ in $(0,\infty)$ has at most countably many points.

Lemma 3.15. *Let* $u \in \mathrm{Conv}_{\mathrm{sc},0}(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$*. Then there exists a sequence* u_k *of functions from* $\mathrm{Conv}_{\mathrm{rg}}(\mathbb{R}^n)$ *such that* u_k *epi-converges to* u *and*

$$\lim_{k\to\infty} \int_{\{t_1<u_k\le t_2\}} \zeta(|\nabla u_k(x)|)[\mathrm{D}^2 u_k(x)]_{n-j}\,\mathrm{d}x = \int_{\{(x,y)\colon\, t_1<u(x)\le t_2\}} \zeta(|y|)\,\mathrm{d}\Theta_j^n(u,(x,y))$$

for all $0<t_1<t_2$ *with* $t_1,t_2 \in T_j(u)$.

Proof. The proof is divided into several steps.

(i) There exists a sequence u_k of functions from $\mathrm{Conv}_{\mathrm{rg}}(\mathbb{R}^n)$ that epi-converges to u. Indeed, by a standard mollification procedure, we first find a sequence of convex functions $\bar u_k$ of class $C^2(\mathbb{R}^n)$ such that $0=\bar u_k(0)\le \bar u_k(x)$ for every $x\in\mathbb{R}^n$ and $k\in\mathbb{N}$, and $\bar u_k$ converges to u on compact sets (in particular, $\bar u_k$ epi-converges to u). Then we define, for every $x\in\mathbb{R}^n$ and $k\in\mathbb{N}$,

$$u_k(x) := \bar u_k(x) + \frac{1}{k}|x|^2.$$

Then clearly $u_k\in\mathrm{Conv}_{\mathrm{sc},0}(\mathbb{R}^n)\cap C^2(\mathbb{R}^n)$, and u_k epi-converges to u. Moreover, $\mathrm{D}^2u_k(x)>0$ for every $x\ne 0$ and $k\in\mathbb{N}$, and this implies that the boundary of $\{u_k\le t\}$ is of class C^2 with positive curvature for every $t>0$. Moreover, as $u\in C^1(\mathbb{R}^n)$, the sequence ∇u_k converges to ∇u uniformly on compact sets.

(ii) For $t_1,t_2\in T_j(u)$ such that $0<t_1<t_2$,

$$\lim_{k\to\infty}\int_{\{(x,y)\colon\, t_1<u(x)\le t_2\}} \zeta(|y|)\,\mathrm{d}\Theta_j^n(u_k,(x,y)) = \int_{\{(x,y)\colon\, t_1<u(x)\le t_2\}} \zeta(|y|)\,\mathrm{d}\Theta_j^n(u,(x,y)). \tag{3.8}$$

For $\bar t>0$ fixed, there exists $\alpha>0$ such that

$$|\nabla u(x)|\ge 2\alpha \qquad \text{for every } x \text{ such that } u(x)>\bar t.$$

Since the sequence u_k converges to u in the C^1-norm on compact sets, we may assume that

$$|\nabla u_k(x)|\ge \alpha \qquad \text{for every } x \text{ such that } u_k(x)>\bar t$$

for k sufficiently large (independent of x). Let $\bar\zeta$ be a continuous function in $[0,\infty)$, which coincides with ζ in $[\alpha,\infty)$ and hence has compact support. We have

$$\int_{\{(x,y)\colon\, t_1<u(x)\le t_2\}} \zeta(|y|)\,\mathrm{d}\Theta_j^n(u_k,(x,y)) = \int_{\{(x,y)\colon\, t_1<u(x)\le t_2\}} \bar\zeta(|y|)\,\mathrm{d}\Theta_j^n(u_k,(x,y)),$$

for every $\bar t<t_1<t_2$ and for every k sufficiently large, and the corresponding statement holds for u_k replaced by u. As u_k epi-converges to u and as $t_1,t_2\in T_j(u)$, it follows from (2.7) that

$$\lim_{k\to\infty}\int_{\{t_1<u\le t_2\}\times\mathbb{R}^n} \zeta(|y|)\,\mathrm{d}\Theta_j^n(u_k,(x,y)) = \int_{\{t_1<u\le t_2\}\times\mathbb{R}^n} \zeta(|y|)\,\mathrm{d}\Theta_j^n(u,(x,y)).$$

This proves that (3.8) holds for all $t_1,t_2\in T_j(u)$ such that $t_2>t_1$ and $t_1,t_2>\bar t$. As $\bar t>0$ was arbitrary, the claim is proved.

(iii) For every $t>0$,

$$\lim_{k\to\infty}\Big|\int_{\{u\ge t\}} \zeta(|\nabla u_k(x)|)[\mathrm{D}^2u_k(x)]_{n-j}\,\mathrm{d}x - \int_{\{u_k\ge t\}} \zeta(|\nabla u_k(x)|)[\mathrm{D}^2u_k(x)]_{n-j}\,\mathrm{d}x\Big| = 0. \tag{3.9}$$

We first note that in the previous relation we may replace ζ by $\bar\zeta$ (where $\bar\zeta$ is chosen as in the previous step). Let $\bar\zeta_+$ and $\bar\zeta_-$ denote the positive and negative parts of $\bar\zeta$; these are continuous functions in $[0,\infty)$, with compact support. As $\bar\zeta=\bar\zeta_+-\bar\zeta_-$, it is enough to prove (3.9) for $\bar\zeta_+$ and $\bar\zeta_-$ separately. In other words, we may reduce to prove (3.9) under the assumptions that ζ is non-negative and belongs to $C_c([0,\infty))$.

For every $k\in\mathbb{N}$, let

$$t_{k,1}=\max\{r>0\colon \{u_k\le r\}\subset\{u\le t\}\},$$

and

$$t_{k,2}=\min\{r>0\colon \{u_k\le r\}\supset\{u\le t\}\}.$$

Equivalently, $\{u_k\le t_{k,1}\}$ is the largest sublevel set of u_k contained in $\{u\le t\}$, while $\{u_k\le t_{k,2}\}$ is the smallest sublevel set of u_k containing $\{u\le t\}$. Clearly

$$\{u_k\le t_{k,1}\}\subset\{u\le t\}\subset\{u_k\le t_{k,2}\}$$

for every $k\in\mathbb{N}$. Moreover, by uniform convergence we have

$$\lim_{k\to\infty}\{u_k\le t_{k,1}\}=\lim_{k\to\infty}\{u_k\le t_{k,2}\}=\{u\le t\}$$

(here convergence is in the Hausdorff metric). As ζ is non-negative,

$$\begin{aligned}\int_{\mathbb{R}^n\setminus\{u_k\le t_{k,2}\}}\zeta(|\nabla u_k(x)|)[\mathrm{D}^2u_k(x)]_{n-j}\,\mathrm{d}x &\le \int_{\mathbb{R}^n\setminus\{u\le t\}}\zeta(|\nabla u_k(x)|)[\mathrm{D}^2u_k(x)]_{n-j}\,\mathrm{d}x\\ &\le \int_{\mathbb{R}^n\setminus\{u_k\le t_{k,1}\}}\zeta(|\nabla u_k(x)|)[\mathrm{D}^2u_k(x)]_{n-j}\,\mathrm{d}x.\end{aligned}$$

Therefore, the difference

$$\alpha_k:=\int_{\{u\ge t\}}\zeta(|\nabla u_k(x)|)[\mathrm{D}^2u_k(x)]_{n-j}\,\mathrm{d}x-\int_{\{u_k\ge t\}}\zeta(|\nabla u_k(x)|)[\mathrm{D}^2u_k(x)]_{n-j}\,\mathrm{d}x$$

is bounded from below by

$$\int_{\{a_k\le u_k\le b_k\}}\zeta(|\nabla u_k(x)|)[\mathrm{D}^2u_k(x)]_{n-j}\,\mathrm{d}x, \tag{3.10}$$

where $a_k=\min\{t,t_{k,2}\}$ and $b_k=\max\{t,t_{k,2}\}$. Similarly, α_k is bounded from above by

$$\int_{\{c_k\le u_k\le d_k\}}\zeta(|\nabla u_k(x)|)[\mathrm{D}^2u_k(x)]_{n-j}\,\mathrm{d}x, \tag{3.11}$$

where $c_k=\min\{t,t_{k,1}\}$ and $d_k=\max\{t,t_{k,1}\}$. Note that the sequences a_k, b_k, c_k and d_k converge to t as $k\to\infty$. We prove that the integral in (3.10) converges to zero as $k\to\infty$. The same, with a similar proof, holds for (3.11). This will complete the proof of this step.

By Proposition 3.13, we have

$$\begin{aligned}\int_{\{a_k\le u_k\le b_k\}}\zeta(|\nabla u_k(x)|)[\mathrm{D}^2u_k(x)]_{n-j}\,\mathrm{d}x &= \int_{\{a_k\le u_k\le b_k\}}\rho(|\nabla u_k(x)|)\tau_{n-j}(u_k,x)\,\mathrm{d}x\\ &\quad-\int_{\{u_k=b_k\}}\eta(|\nabla u_k(x)|)\tau_{n-j-1}(u_k,x)\,\mathrm{d}\mathcal{H}^{n-1}(x)\\ &\quad+\int_{\{u_k=a_k\}}\eta(|\nabla u_k(x)|)\tau_{n-j-1}(u_k,x)\,\mathrm{d}\mathcal{H}^{n-1}(x),\end{aligned}$$

where the functions ρ and η are associated to ζ as in Proposition 3.13. For the first term on the right side, we may write, using the continuity of ρ and Lemma 3.9,

$$\left|\int_{\{a_k\le u_k\le b_k\}}\rho(|\nabla u_k(x)|)\tau_{n-j}(u,x)\,\mathrm{d}x\right|\le\alpha\left(V_j(\{u_k\le b_k\})-V_j(\{u_k\le a_k\})\right),$$

where $\alpha > 0$ depends on n, j and ρ. By the uniform convergence and the continuity of intrinsic volumes, the right side of the previous inequality tends to 0 as $k \to \infty$.

Next, we consider

$$\int_{\{u_k=b_k\}} \eta(|\nabla u_k(x)|)\tau_{n-j-1}(u_k,x)\,\mathrm{d}\mathcal{H}^{n-1}(x).$$

This integral can be written as

$$\binom{n-1}{j}\int_{\mathbb{R}^n} \eta(|\nabla u_k(x)|)\,\mathrm{d}C_j(\{u_k\leq b_k\},x),$$

where $C_j(\{u_k \leq b_k\},\cdot)$ is the jth curvature measure of $\{u_k \leq b_k\}$ (see [40, Chapter 4] for the definition of curvature measures). Note that by the convergence of $\{u_k \leq b_k\}$ to $\{u \leq t\}$ and [40, Theorem 4.2.1], the curvature measures $C_j(\{u_k \leq b_k\},\cdot)$ converge weakly to $C_j(\{u \leq t\},\cdot)$. Moreover, the support of the curvature measures of a convex body is contained in the boundary of the convex body. Hence there exists a compact set $C \subset \mathbb{R}^n$, such that the supports of $C_j(\{u_k \leq b_k\},\cdot)$ and of $C_j(\{u \leq t\},\cdot)$ are contained in C for every k. Taking the continuity of η and the uniform convergence of ∇u_k to ∇u on compact sets into account, we deduce that

$$\lim_{k\to\infty}\int_{\{u_k=b_k\}} \eta(|\nabla u_k(x)|)\tau_{n-j-1}(u_k,x)\,\mathrm{d}\mathcal{H}^{n-1}(x) = \binom{n-1}{j}\int_{\mathbb{R}^n}\eta(|\nabla u(x)|)\,\mathrm{d}C_j(\{u\leq t\},x).$$

By a similar argument we may prove that

$$\lim_{k\to\infty}\int_{\{u_k=a_k\}} \eta(|\nabla u_k(x)|)\tau_{n-j-1}(u_k,x)\,\mathrm{d}\mathcal{H}^{n-1}(x) = \binom{n-1}{j}\int_{\mathbb{R}^n}\eta(|\nabla u(x)|)\,\mathrm{d}C_j(\{u\leq t\},x).$$

Thus, the integral in (3.10) converges to zero as $k \to \infty$, which concludes the proof of this step.

(iv) By the previous step we have for every $t_1, t_2 \in T_j(u)$ such that $0 < t_1 < t_2$,

$$\lim_{k\to\infty}\left|\int_{\{t_1<u\leq t_2\}} \zeta(|\nabla u_k(x)|)[\mathrm{D}^2 u_k(x)]_{n-j}\,\mathrm{d}x - \int_{\{t_1<u_k\leq t_2\}} \zeta(|\nabla u_k(x)|)[\mathrm{D}^2 u_k(x)]_{n-j}\,\mathrm{d}x\right| = 0.$$

This, together with (3.8), concludes the proof of this lemma. ☐

Lemma 3.16. *Let $u \in \mathrm{Conv}_{\mathrm{sc},0}(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$. Then for $t_1, t_2 \in T_j(u)$ such that $0 < t_1 < t_2$, we have*

$$\Big|\int_{\{(x,y)\colon\, t_1<u(x)\leq t_2\}} \zeta(|y|)\,\mathrm{d}\Theta_j^n(u,(x,y))\Big| \leq \alpha\, V_j(\{u\leq t_2\})\Big(\max_{[0,\mathrm{Lip}(u,t_2)]}|\rho| + \max_{[0,\mathrm{Lip}(u,t_2)]}|\eta|\Big),$$

where α is a positive constant only depending on n and j.

Proof. The validity of the claimed inequality follows from Lemma 3.14, Lemma 3.15, and the convergence of Lipschitz constants and of sublevel sets under uniform convergence. ☐

Lemma 3.17. *For every $u \in \mathrm{Conv}_{\mathrm{sc},0}(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$, the limit*

$$\lim_{\substack{t\to 0\\ t\in T_j(u)}} \int_{\{(x,y)\colon\, u(x)>t\}} \zeta(|y|)\,\mathrm{d}\Theta_j^n(u,(x,y))$$

exists and is finite.

Proof. From Lemma 3.16, we obtain that for every $t_1, t_2 \in T_j(u)$ such that $0 < t_1 < t_2$,

$$\Big| \int_{\{(x,y)\colon t_1 < u(x) \le t_2\}} \zeta(|y|)\, \mathrm{d}\Theta_j^n(u,(x,y)) \Big| \le \alpha\, V_j(\{u \le t_2\}) \Big(\max_{[0,\mathrm{Lip}(u,t_2)]} |\rho| + \max_{[0,\mathrm{Lip}(u,t_2)]} |\eta| \Big).$$

The conclusion now follows from (3.7) and the fact that both η and ρ are continuous on $[0,\infty)$ and vanish at $t = 0$. □

Lemma 3.17 and the fact that $\mathrm{env}_\lambda u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$ allow us to make the following definition. For $u \in \mathrm{Conv}_{\mathrm{sc},0}(\mathbb{R}^n)$ and $\lambda > 0$, we set

$$\mathrm{V}_{j,\zeta}^{(\lambda)}(u) := \lim_{\substack{t \to 0 \\ t \in T_j(\mathrm{env}_\lambda u)}} \int_{\{(x,y)\colon \mathrm{env}_\lambda u(x) > t\}} \zeta(|y|)\, \mathrm{d}\Theta_j^n(\mathrm{env}_\lambda u,(x,y)). \tag{3.12}$$

An immediate consequence of this definition and Lemma 3.16 is the following result.

Lemma 3.18. *Let $u \in \mathrm{Conv}_{\mathrm{sc},0}(\mathbb{R}^n)$ and $\lambda > 0$. If $t \in T_j(\mathrm{env}_\lambda u)$, then*

$$\Big| \mathrm{V}_{j,\zeta}^{(\lambda)}(u) - \int_{\{(x,y)\colon \mathrm{env}_\lambda u(x) > t\}} \zeta(|y|)\, \mathrm{d}\Theta_j^n(\mathrm{env}_\lambda u,(x,y)) \Big|$$
$$\le \alpha\, V_j(\{\mathrm{env}_\lambda u \le t\}) \big(\max_{[0,\mathrm{Lip}(\mathrm{env}_\lambda u,t)]} |\rho| + \max_{[0,\mathrm{Lip}(\mathrm{env}_\lambda u,t)]} |\eta| \big),$$

where α is a positive constant only depending on n and j.

We introduce the following auxiliary functions. For $r > 0$, we define $\zeta_r\colon [0,\infty) \to \mathbb{R}$ as

$$\zeta_r(t) := \begin{cases} \zeta(t) & \text{for } t \ge r, \\ \zeta(r) & \text{for } 0 \le t < r. \end{cases}$$

Note that $\zeta_r \in C_c([0,\infty))$. We also introduce the corresponding functions $\eta_r, \rho_r\colon [0,\infty) \to \mathbb{R}$, defined as

$$\eta_r(t) := \int_t^\infty s^{n-j+1} \zeta_r(s)\, \mathrm{d}s, \qquad \rho_r(t) := t^{n-1}\zeta_r(t) + (n-j)\eta_r(t).$$

Clearly, $\eta_r, \rho_r \in C_c([0,\infty))$. For $t > r$, we have $\eta_r(t) = \eta(t)$, and, for $0 \le t < r$,

$$\eta_r(t) = \int_t^r s^{n-j-1}\zeta(r)\, \mathrm{d}s + \int_r^\infty s^{n-j-1}\zeta(s)\, \mathrm{d}s = \zeta(r)\, \frac{r^{n-j} - t^{n-j}}{n-j} + \eta(r).$$

Hence

$$\eta_r(t) = \begin{cases} \eta(t) & \text{for } t \ge r, \\ \zeta(r)\, \dfrac{r^{n-j} - t^{n-j}}{n-j} + \eta(r) & \text{for } 0 \le t < r. \end{cases}$$

Concerning the function ρ_r, we have $\rho_r(t) = \rho(t)$ for $t \ge r$ and

$$\rho_r(t) = t^{n-j}\zeta(r) + (n-j)\zeta(r)\, \frac{r^{n-j} - t^{n-j}}{n-j} + (n-j)\eta(r) = r^{n-j}\zeta(r) + (n-j)\eta(r) = \rho(r)$$

for $0 \le t < r$. Hence

$$\rho_r(t) = \begin{cases} \rho(t) & \text{for } t \ge r, \\ \rho(r) & \text{for } 0 \le t < r. \end{cases}$$

For every $\delta > 0$, we obtain

$$\begin{aligned} \max_{[0,\delta]} |\eta_r| &\leq |\eta(r)| + \frac{2}{n-j}\,|r^{n-j}\zeta(r)| + \max_{[0,\delta]} |\eta|, \\ \max_{[0,\delta]} |\rho_r| &\leq |\rho(r)| + \max_{[0,\delta]} |\rho|. \end{aligned} \tag{3.13}$$

These auxiliary functions are used in the following two lemmas.

Lemma 3.19. *If* $u_k \in \mathrm{Conv}_{\mathrm{sc},0}(\mathbb{R}^n)$ *epi-converges to* $u \in \mathrm{Conv}_{\mathrm{sc},0}(\mathbb{R}^n)$*, then*

$$\lim_{k\to\infty} \mathrm{V}_{j,\zeta}^{(\lambda)}(u_k) = \mathrm{V}_{j,\zeta}^{(\lambda)}(u)$$

for every $\lambda > 0$*.*

Proof. Fix $\lambda > 0$ and set $\bar u = \mathrm{env}_\lambda\, u$ and $\bar u_k = \mathrm{env}_\lambda\, u_k$. For any fixed $\bar t > 0$, there exists a constant $\alpha > 0$ (depending on u and $\bar t$) such that

$$|\nabla \bar u(x)| > 2\alpha \qquad \text{for every } x \text{ such that } \bar u(x) \geq \bar t.$$

As u_k epi-converges to u, the sequence $\bar u_k$ converges to $\bar u$ and the sequence $\nabla \bar u_k$ to $\nabla \bar u$. In both cases, the convergence is uniform on compact sets. Therefore, we have

$$|\nabla \bar u_k(x)| > \alpha \qquad \text{for every } x \text{ such that } \bar u_k(x) \geq \bar t$$

for k sufficiently large.

Set $r = \alpha$. We have

$$\Big|\mathrm{V}_{j,\zeta}^{(\lambda)}(u) - \mathrm{V}_{j,\zeta}^{(\lambda)}(u_k)\Big| \leq \beta_0 + \cdots + \beta_4,$$

where

$$\begin{aligned} \beta_0 &= \Big|\mathrm{V}_{j,\zeta_r}^{(\lambda)}(u) - \mathrm{V}_{j,\zeta_r}^{(\lambda)}(u_k)\Big|, \\ \beta_1 &= \Big|\mathrm{V}_{j,\zeta}^{(\lambda)}(u) - \int_{\{(x,y)\colon \bar u(x) > \bar t\}} \zeta(|y|)\,\mathrm{d}\Theta_j^n(\bar u,(x,y))\Big|, \\ \beta_2 &= \Big|\mathrm{V}_{j,\zeta_r}^{(\lambda)}(u) - \int_{\{(x,y)\colon \bar u(x) > \bar t\}} \zeta_r(|y|)\,\mathrm{d}\Theta_j^n(\bar u,(x,y))\Big|, \\ \beta_3 &= \Big|\mathrm{V}_{j,\zeta}^{(\lambda)}(u_k) - \int_{\{(x,y)\colon \bar u_k(x) > \bar t\}} \zeta(|y|)\,\mathrm{d}\Theta_j^n(\bar u_k,(x,y))\Big|, \\ \beta_4 &= \Big|\mathrm{V}_{j,\zeta_r}^{(\lambda)}(u_k) - \int_{\{(x,y)\colon \bar u_k(x) > \bar t\}} \zeta_r(|y|)\,\mathrm{d}\Theta_j^n(\bar u_k,(x,y))\Big|. \end{aligned}$$

These quantities depend, in general, on $\bar t$ and k (note that r depends on $\bar t$). We fix $\varepsilon > 0$ and first consider the terms β_1 and β_3. By Lemma 3.18, the continuity of η and ρ at $t = 0$, the relations $\eta(0) = \rho(0) = 0$, and the uniform convergence of $\bar u_k$ to $\bar u$ on compact sets, we may choose $\bar t_1 > 0$ so that $\beta_1,\ \beta_3 \leq \varepsilon$ for every

$$\bar t \in (0, \bar t_1) \cap T_j(\bar u) \cap \bigcap\nolimits_{k \geq k_1} T_j(\bar u_k)$$

with k_1 sufficiently large. Note that this set has full measure in $(0, \bar t_1)$.

Next, we deal with the term β_2. For $t > 0$, set

$$r(t) := \inf\big\{|\nabla \bar u(x)| : x \ \text{ such that } \bar u(x) > t\big\}.$$

Note that $r(t)$ tends to zero as $t \to 0$. By Lemma 3.18, we have for $\bar t \in T_j(\bar u)$,

$$\beta_2 \le \alpha\, V_j(\{\bar u \le \bar t\})\big(\max_{[0,\mathrm{Lip}(\bar u,\bar t)]} |\rho_{r(\bar t)}| + \max_{[0,\mathrm{Lip}(\bar u,\bar t)]} |\eta_{r(\bar t)}|\big).$$

By (3.7) and (3.13) combined with the conditions on ρ and η, we deduce that there exists $\bar t_2 > 0$ so that $\beta_2 \le \varepsilon$ for every $\bar t \in (0, \bar t_2) \cap T_j(\bar u)$.

We proceed in a similar way for β_4. For $k \in \mathbb{N}$ and $t > 0$, let

$$r_k(t) := \inf\big\{ |\nabla \bar u_k(x)| : x \text{ such that } \bar u_k(x) > t\big\}.$$

As before, $r_k(t)$ tends to zero as $t \to 0$. Moreover, for $\bar t \in T_j(\bar u_k)$,

$$\beta_4 \le \alpha\, V_j(\{\bar u_k \le \bar t\})\big(\max_{[0,\mathrm{Lip}(\bar u_k,\bar t]} |\rho_{r_k(\bar t)}| + \max_{[0,\mathrm{Lip}(\bar u_k,\bar t)]} |\eta_{r_k(\bar t)}|\big).$$

By (3.7) and (3.13) combined with the uniform convergence of $\bar u_k$ to $\bar u$ on compact sets, there exists $\bar t_3 > 0$ such that $\beta_4 \le \varepsilon$ for every $\bar t \in (0, \bar t_3) \cap \bigcap_{k \ge k_3} T_j(\bar u_k)$ with k_3 sufficiently large.

Finally, for every fixed $r > 0$, and hence for every fixed $\bar t > 0$, the term $\beta_0 \to 0$ as $k \to \infty$, since $\zeta_r \in C_c([0,\infty))$ and by Lemma 2.12. This concludes the proof. ☐

Lemma 3.20. *For every* $u \in \mathrm{Conv}_{\mathrm{sc},0}(\mathbb{R}^n)$ *and* $\lambda > 0$,

$$\lim_{r \to 0^+} \mathrm{V}_{j,\zeta_r}^{(\lambda)}(u) = \mathrm{V}_{j,\zeta}^{(\lambda)}(u).$$

Proof. For $r > 0$, let

$$t(r) := \inf\big\{ t > 0 : |\nabla \operatorname{env}_\lambda u(x)| \ge r \text{ for all } x \text{ such that } \operatorname{env}_\lambda u(x) > t\big\}.$$

This defines a monotone function of r, and

$$\lim_{r \to 0} t(r) = 0$$

since $\operatorname{env}_\lambda u$ is of class C^1.

For every $r > 0$, let $\bar t(r) \in T_j(\operatorname{env}_\lambda u)$ be such that $t(r) \le \bar t(r) \le t(r) + r$. We have

$$\begin{aligned} \Big| \mathrm{V}_{j,\zeta}^{(\lambda)}(u) - \mathrm{V}_{j,\zeta_r}^{(\lambda)}(u)\Big| \;\le\; & \Big| \mathrm{V}_{j,\zeta}^{(\lambda)}(u) - \int_{\{(x,y):\ \operatorname{env}_\lambda u(x) > \bar t(r)\}} \zeta(|y|)\, \mathrm{d}\Theta_j^n(\operatorname{env}_\lambda u, (x,y))\Big| \\ & + \Big| \mathrm{V}_{j,\zeta_r}^{(\lambda)}(u) - \int_{\{(x,y):\ \operatorname{env}_\lambda u(x) > \bar t(r)\}} \zeta_r(|y|)\, \mathrm{d}\Theta_j^n(\operatorname{env}_\lambda u, (x,y))\Big|. \end{aligned}$$

Let $r \to 0^+$. The conclusion follows from Lemma 3.18, (3.7), and (3.13). ☐

We extend $\mathrm{V}_{j,\zeta}^{(\lambda)}$ from a functional defined on $\mathrm{Conv}_{\mathrm{sc},0}(\mathbb{R}^n)$ to a functional defined on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ in the following way. For every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$, there exists $u_0 \in \mathrm{Conv}_{\mathrm{sc},0}(\mathbb{R}^n)$ such that $\mathrm{epi}(u_0)$ is a translate of $\mathrm{epi}(u)$. Indeed, if $x_0 \in \mathbb{R}^n$ is such that u attains its absolute minimum at x_0, it is sufficient to define u_0 for $x \in \mathbb{R}^n$ as

$$u_0(x) := u(x + x_0) - \min\nolimits_{\mathbb{R}^n} u.$$

We note that u_0 is not uniquely determined (as x_0 is not), but any two functions of this form can be obtained from each other by a translation of the epi-graph. Therefore,

$$\mathrm{V}_{j,\zeta}^{(\lambda)}(u) := \mathrm{V}_{j,\zeta}^{(\lambda)}(u_0)$$

is independent of the choice of the particular u_0. Clearly, $\mathrm{V}_{j,\zeta}^{(\lambda)}$ extended in this way is epi-translation invariant and, as $\mathrm{V}_{j,\zeta}^{(\lambda)}$ is rotation invariant on $\mathrm{Conv}_{\mathrm{sc},0}(\mathbb{R}^n)$, the same holds for its extension to $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$. By Lemma 3.19, this extension is also continuous.

For every $r > 0$, we have $\zeta_r \in C_c([0,\infty))$. Hence, it follows from Lemma 3.20, that the functional $\mathrm{V}_{j,\zeta_r}^{(\lambda)}$, defined in (3.12), is

$$\mathrm{V}_{j,\zeta_r}^{(\lambda)}(u) = \int_{\mathbb{R}^n\times\mathbb{R}^n} \zeta_r(|y|)\,\mathrm{d}\Theta_j^n(\mathrm{env}_\lambda u,(x,y))$$

for $u \in \mathrm{Conv}_{\mathrm{sc},0}(\mathbb{R}^n)$, extends by this representation to $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ and defines there an epi-translation invariant valuation. Hence, using the epi-translation invariance of both $\mathrm{V}_{j,\zeta}^{(\lambda)}$ and $\mathrm{V}_{j,\zeta_r}^{(\lambda)}$ on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$, we deduce from Lemma 3.20 that also $\mathrm{V}_{j,\zeta}^{(\lambda)}$ is a valuation on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$. We conclude that $\mathrm{V}_{j,\zeta}^{(\lambda)}$ is a continuous, epi-translation invariant and rotation invariant valuation on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$.

The inclusion $D_j^n \subset D_i^n$ for $i \in \{1,\dots,j\}$ ensures that we may repeat this construction replacing j by any $i \in \{1,\dots,j\}$.

Lemma 3.21. *For every $i \in \{1,\dots,j\}$ and $\lambda > 0$, the function $\mathrm{V}_{i,\zeta}^{(\lambda)}$ is a continuous, epi-translation and rotation invariant valuation on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$.*

For $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \cap C_+^2(\mathbb{R}^n)$, Proposition 3.2 shows that

$$\mathrm{V}_{j,\zeta}(u) := \int_{\mathbb{R}^n} \zeta(|\nabla u(x)|)[\mathrm{D}^2 u(x)]_{n-j}\,\mathrm{d}x$$

is well-defined. Moreover, as $D_j^n \subset D_i^n$ for $i \in \{1,\dots,j\}$, we may also set

$$\mathrm{V}_{i,\zeta}(u) := \int_{\mathbb{R}^n} \zeta(|\nabla u(x)|)[\mathrm{D}^2 u(x)]_{n-i}\,\mathrm{d}x$$

for every $i = 1,\dots,j$ and $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \cap C_+^2(\mathbb{R}^n)$. Since $\mathrm{env}_\lambda u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \cap C_+^2(\mathbb{R}^n)$ for such u and $\lambda > 0$, we have

$$\mathrm{V}_{i,\zeta}(\mathrm{env}_\lambda u) = \mathrm{V}_{i,\zeta}^{(\lambda)}(u) \tag{3.14}$$

by Proposition 3.2.

For the final step of the proof of Theorem 1.2, we require the following statement.

Lemma 3.22. *There exist $\alpha_1,\dots,\alpha_{j+1} \in \mathbb{R}$ such that*

$$\mathrm{V}_{j,\zeta}(u) = \sum_{i=1}^{j+1} \alpha_i\,\mathrm{V}_{j,\zeta}^{(i)}(u) \tag{3.15}$$

for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \cap C_+^2(\mathbb{R}^n)$.

Proof. Let $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \cap C_+^2(\mathbb{R}^n)$ and let $\lambda > 0$. As usual, we denote by u^* the conjugate of u. Note that

$$(\mathrm{env}_\lambda u)^*(y) = u^*(y) + \lambda\frac{|y|^2}{2}$$

and that

$$[\mathrm{D}^2(\mathrm{env}_\lambda u)^*(y)]_j = [\mathrm{D}^2(u^*(y) + \lambda\frac{|y|^2}{2})]_j = \sum_{i=0}^{j}\binom{n-i}{j-i}\lambda^{j-i}[\mathrm{D}^2 u^*(y)]_i$$

for every $y \in \mathbb{R}^n$. Hence, it follows from (3.14) and (2.9) that

$$\begin{aligned}
\mathrm{V}_{j,\zeta}^{(\lambda)}(u) &= \int_{\mathbb{R}^n} \zeta(|\nabla(\operatorname{env}_\lambda u)(x)|)[\mathrm{D}^2(\operatorname{env}_\lambda u)(x)]_{n-j}\,\mathrm{d}x \\
&= \int_{\mathbb{R}^n} \zeta(|y|)[\mathrm{D}^2(\operatorname{env}_\lambda u)^*(y)]_j\,\mathrm{d}y \\
&= \sum_{i=0}^{j} \binom{n-i}{j-i} \lambda^{j-i} \int_{\mathbb{R}^n} \zeta(|y|)[\mathrm{D}^2 u^*(y)]_i\,\mathrm{d}x \\
&= \sum_{i=0}^{j} \binom{n-i}{j-i} \lambda^{j-i} \int_{\mathbb{R}^n} \zeta(|\nabla u(x)|)[\mathrm{D}^2 u(x)]_{n-i}\,\mathrm{d}x \\
&= \sum_{i=0}^{j} \binom{n-i}{j-i} \lambda^{j-i}\, \mathrm{V}_{i,\zeta}(u).
\end{aligned}$$

Inverting the system that we obtain by writing the previous equation for $\lambda = 1, \dots, n+1$, we deduce (3.15). We use that the matrix is a Vandermonde matrix and therefore invertible. ☐

Combined with Lemma 3.21, Lemma 3.22 implies that $\mathrm{V}_{j,\zeta}$ continuously extends to $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$. This completes the proof of Theorem 1.2.

3.4. **Representation formulas.** As an application of Theorem 1.2, we derive representation formulas for $\mathrm{V}_{j,\zeta}$ and $\mathrm{V}_{j,\zeta}^*$ on convex functions with certain regularity properties.

Lemma 3.23. *Let $j \in \{0, \dots, n\}$ and $\zeta \in D_j^n$. We have*

$$\mathrm{V}_{j,\zeta}^*(v) = \int_{\mathbb{R}^n} \zeta(|x|)\,\mathrm{d}\Phi_j^n(v,x)$$

for $v \in \mathrm{Conv}_{(0)}(\mathbb{R}^n;\mathbb{R})$.

Proof. By Theorem 1.2 and (2.5), we know that for every $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \cap C_+^2(\mathbb{R}^n)$,

$$\mathrm{V}_{j,\zeta}^*(v) = \int_{\mathbb{R}^n} \zeta(|x|)\,\mathrm{d}\Phi_j^n(v,x) = \int_{\mathbb{R}^n} \zeta(|x|)[\mathrm{D}^2 v(x)]_j\,\mathrm{d}x.$$

Now, let $v \in \mathrm{Conv}_{(0)}(\mathbb{R}^n;\mathbb{R})$. There exists a sequence v_k of functions from $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \cap C_+^2(\mathbb{R}^n)$ that converges to v. Indeed, by a standard mollification procedure, we first find a sequence of convex functions $\bar{v}_k$ of class $C^2(\mathbb{R}^n)$ that converges to v on compact sets. Then we define, for every $x \in \mathbb{R}^n$ and $k \in \mathbb{N}$,

$$v_k(x) := \bar{v}_k(x) + \frac{1}{k}|x|^2.$$

Then clearly $v_k \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \cap C_+^2(\mathbb{R}^n)$, and v_k converges to v. Moreover, if $\bar{r} > 0$ is such that v is of class C^2 in the set $\bar{r}B^n$, then the convergence is in C^2-norm in this set.

Since $\mathrm{V}_{j,\zeta}^*$ is continuous, it is now sufficient to prove that

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} \zeta(|x|)\,\mathrm{d}\Phi_j^n(v_k,x) = \int_{\mathbb{R}^n} \zeta(|x|)\,\mathrm{d}\Phi_j^n(v,x).$$

By (2.7) and since ζ is continuous on $(0,\infty)$, we have

$$\lim_{k\to\infty} \int_{\{x\colon |x|\geq r\}} \zeta(|x|)\,\mathrm{d}\Phi_j^n(v_k,x) = \int_{\{x\colon |x|\geq r\}} \zeta(|x|)\,\mathrm{d}\Phi_j^n(v,x) \tag{3.16}$$

for $r > 0$ with $\Phi_j^n(v, \{v = r\}) = 0$. Since $\Phi_j^n(v, \cdot)$ is locally finite, (3.16) holds for a.e. $r > 0$. On the other hand, for every $r < \bar{r}$ and for every k we have

$$\int_{\{x\colon |x|<r\}} \zeta(|x|) \,\mathrm{d}\Phi_j^n(v_k, x) = \int_{\{x\colon |x|<r\}} \zeta(|x|)[\mathrm{D}^2 v_k(x)]_j \,\mathrm{d}x$$

and a corresponding relation for v. Using the convergence of v_k to v in $\bar{r}B^n$ in C^2-norm, polar coordinates and the fact that for $\delta > 0$,

$$\int_0^\delta r^{n-1}\zeta(r)\,\mathrm{d}r$$

is finite, we obtain that for every $\varepsilon > 0$ there exists $0 < \delta < \bar{r}$ such that

$$\Big|\int_{\{x\colon |x|<\delta\}} \zeta(|x|)[\mathrm{D}^2 v(x)]_j \,\mathrm{d}x\Big|, \Big|\int_{\{x\colon |x|<\delta\}} \zeta(|x|)[\mathrm{D}^2 v_m(x)]_j \,\mathrm{d}x\Big| \le \varepsilon \tag{3.17}$$

for every $k \in \mathbb{N}$. The conclusion follows from (3.16) and (3.17). □

An immediate consequence is the following result.

Lemma 3.24. *Let $j \in \{0, \dots, n\}$ and $\zeta \in D_j^n$. We have*

$$\mathrm{V}_{j,\zeta}(u) = \int_{\mathbb{R}^n} \zeta(|y|)\,\mathrm{d}\Psi_j^n(u, y)$$

for $u^* \in \mathrm{Conv}_{(0)}(\mathbb{R}^n; \mathbb{R})$.

4. The Classification Result

In the proof of Theorem 1.3 and Theorem 1.5, we follow the original approach by Hadwiger [20]. First, we establish a classification of valuations on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ that are epi-homogeneous of degree 1. For this, we introduce rotational epi-symmetrization and reduce the classification problem to the solution of an integral equation. We immediately obtain, by duality, a classification of valuations on $\mathrm{Conv}(\mathbb{R}^n; \mathbb{R})$ that are homogeneous of degree 1. Next, we introduce orthogonal cylinder functions on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ and show that if a valuation vanishes on this class of functions, then it has to be epi-homogeneous of degree 1. By duality, we also obtain that if a valuation vanishes on dual orthogonal cylinder functions on $\mathrm{Conv}(\mathbb{R}^n; \mathbb{R})$, then it has to be homogeneous of degree 1. In the final step, we work in the dual setting and use induction on the dimension to establish a representation formula of valuations on dual orthogonal cylinder functions. Combined with the results established in the first part and the classification of one-homogeneous valuations, it completes the proof in the dual setting, that is, the proof of Theorem 1.5. The result in the primal setting, that is, Theorem 1.3, follows by duality.

4.1. **The Case of Epi-Homogeneity of Degree 1.** Let $n \ge 2$. A classical result by Hadwiger [20, §4.5.3] (or see, [40, Theorem 3.3.5]) states that for every convex body $K \subset \mathbb{R}^n$, that is at least one-dimensional, there exists a sequence of rotation means of K that converges to a ball. Here a *rotation mean* of K is given, for $k \ge 1$ and rotations $\vartheta_1, \dots, \vartheta_k \in \mathrm{SO}(n)$, as

$$\frac{1}{k}\left(\vartheta_1 K + \dots + \vartheta_k K\right).$$

For $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ and rotations $\vartheta_1, \dots, \vartheta_k \in \mathrm{SO}(n)$, we call

$$\frac{1}{k} \cdot \left(u \circ \vartheta_1^{-1} \,\square \cdots \square\, u \circ \vartheta_k^{-1}\right)$$

a *rotation epi-mean* of u. A function $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ is called *radial* if $u = u \circ \vartheta^{-1}$ for every $\vartheta \in \mathrm{SO}(n)$.

For $u \in \text{Conv}_{\text{sc}}(\mathbb{R}^n)$, define its *rotational epi-symmetrization* u° via the support function of its epigraph for $y \in \mathbb{R}^n$ and $t \in \mathbb{R}$ by

$$h_{\operatorname{epi} u^\circ}(y,t) := \int_{\mathrm{SO}(n)} h_{\operatorname{epi} u}(\vartheta^{-1} y, t)\,\mathrm{d}\vartheta,$$

where we integrate with respect to the Haar probability measure on $\mathrm{SO}(n)$. The following result is obtained by suitably approximating the above integral.

Lemma 4.1. *For every $u \in \text{Conv}_{\text{sc}}(\mathbb{R}^n)$ with at least one-dimensional domain, there is a sequence of rotation epi-means of u epi-converging to its rotation epi-mean $u^\circ \in \text{Conv}_{\text{sc}}(\mathbb{R}^n)$.*

Recall that for given $t \geq 0$, we have $u_t(x) := t|x| + \mathbf{I}_{B^n}(x)$ for $x \in \mathbb{R}^n$.

Lemma 4.2. *If $u \in \text{Conv}_{\text{sc}}(\mathbb{R}^n)$ is radial, non-negative, and such that $u(0) = 0$, then there exists a sequence u_k which epi-converges to u and such that each u_k is the finite epi-sum of functions of the form $r \cdot u_t$ with $r > 0$ and $t \geq 0$.*

Proof. Since the function u is radial, this is a one-dimensional problem. Any piecewise affine function $\bar{u} : [0,\infty) \to [0,\infty)$ with positive derivative at 0 can be written as an epi-sum of finitely many functions of the form $s \mapsto t\,s + \mathbf{I}_{[0,r]}(s)$ with suitable $r > 0$ and $t \geq 0$. Hence the statement follows from the fact that such piecewise affine functions are dense. $\square$

The main result of this section is the following result.

Proposition 4.3. *Let $n \geq 2$. If $\mathrm{Z}\colon \text{Conv}_{\text{sc}}(\mathbb{R}^n) \to \mathbb{R}$ is a continuous, epi-translation and rotation invariant valuation that is epi-homogeneous of degree 1, then there exists $\zeta \in D_1^n$ such that*

$$\mathrm{Z}(u) = \mathrm{V}_{1,\zeta}(u)$$

for every $u \in \text{Conv}_{\text{sc}}(\mathbb{R}^n)$.

Proof. Since Z is a continuous valuation that is epi-homogeneous of degree 1, Lemma 2.2 implies that it is epi-additive. Let $u \in \text{Conv}_{\text{sc}}(\mathbb{R}^n)$ and let u_k be any rotation epi-mean of u, that is, there are rotations $\vartheta_1, \dots, \vartheta_{m_k}$ such that

$$u_k = \frac{1}{m_k} \cdot \big(u \circ \vartheta_1^{-1} \,\square \cdots \square\, u \circ \vartheta_{m_k}^{-1}\big).$$

Since Z is epi-additive, rotation invariant and epi-homogeneous of degree 1, we see that

$$\mathrm{Z}(u) = \mathrm{Z}(u_k).$$

By Lemma 4.1, there is a sequence of rotation epi-means of u that converges to the rotation epi-symmetral u°. Since Z is continuous, it follows that

$$\mathrm{Z}(u) = \mathrm{Z}(u^\circ).$$

Hence we obtain that Z is determined by its values on radial functions.

Since Z is epi-translation invariant, Z is even determined by radial functions u that are non-negative with $u(0) = 0$. Combined with the epi-additivity and the epi-homogeneity of degree 1 of Z, Lemma 4.2 implies that Z is already determined by its values on the one-parameter family u_t for $t \geq 0$. Thus, the function $t \mapsto \mathrm{Z}(u_t)$ on $[0,\infty)$ uniquely determines Z and it suffices to show that there is $\zeta \in D_1^n$ such that

$$\mathrm{Z}(u_t) = \mathrm{V}_{1,\zeta}(u_t) \tag{4.1}$$

for $t \geq 0$.

Note that the continuity of Z implies that $t \mapsto \mathrm{Z}(u_t)$ is continuous in $[0,\infty)$. Let us prove that it has compact support. Assume that this is not true. Hence, there exists a sequence t_k such that $t_k \to \infty$ as $k \to \infty$, and

$$\alpha_k := \mathrm{Z}(u_{t_k}) > 0$$

for all $k \in \mathbb{N}$. Define $\bar{u}_k \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ by

$$\bar{u}_k(x) := \frac{1}{\alpha_k} \cdot u_{t_k}(x) = t_k|x| + \mathbf{I}_{\frac{1}{\alpha_k}B^n}(x).$$

By the epi-homogeneity of Z, we have $\mathrm{Z}(\bar{u}_k) = 1$ for every $k \in \mathbb{N}$. On the other hand the sequence $\bar{u}_k$ epi-converges to $\mathbf{I}_{\{0\}}$, and it is easy to prove, using once more the fact that Z is epi-homogeneous of degree 1, that $\mathrm{Z}(\mathbf{I}_{\{0\}}) = 0$. Hence we have a contradiction to the continuity of Z.

To prove (4.1), we solve an integral equation. Let u_t^* be the dual function of u_t and note that we have $u_t^* \in \mathrm{Conv}_{(0)}(\mathbb{R}^n;\mathbb{R})$. By the definition of $\mathrm{V}_{1,\bar{\zeta}}$ combined with Lemma 3.24, we can calculate $\mathrm{V}_{1,\bar{\zeta}}(u_t)$ for $\bar{\zeta} \in D_1^n$ by Lemma 2.15 and obtain

$$\mathrm{V}_{1,\bar{\zeta}}(u_t) = \omega_n\Big(\bar{\zeta}(t)\,t^{n-1} + (n-1)\int_t^\infty r^{n-2}\bar{\zeta}(r)\,\mathrm{d}r\Big).$$

Hence, we need to solve the integral equation,

$$\zeta(t) + \frac{n-1}{t^{n-1}}\int_t^\infty r^{n-2}\zeta(r)\,\mathrm{d}r = \frac{\mathrm{Z}(u_t)}{\omega_n t^{n-1}},$$

to determine ζ when $\mathrm{Z}(u_t)$ is given. We claim that

$$\zeta(t) := \frac{\mathrm{Z}(u_t)}{\omega_n t^{n-1}} - \frac{(n-1)}{\omega_n}\int_t^\infty \frac{\mathrm{Z}(u_r)}{r^n}\,\mathrm{d}r \tag{4.2}$$

is a solution. Indeed, using integration by parts and that $t \mapsto \mathrm{Z}(u_t)$ has compact support, we obtain that

$$(n-1)\int_t^\infty r^{n-2}\int_r^\infty \frac{\mathrm{Z}(u_s)}{s^n}\,\mathrm{d}s\,\mathrm{d}r = -t^{n-1}\int_t^\infty \frac{\mathrm{Z}(u_r)}{r^n}\,\mathrm{d}r + \int_t^\infty \frac{\mathrm{Z}(u_r)}{r}\,\mathrm{d}r \tag{4.3}$$

and the result follows. Since $t \mapsto \mathrm{Z}(u_t)$ is continuous with compact support, also ζ is continuous on $(0,\infty)$ and it has bounded support.

Finally, we show that the solution ζ defined in (4.2) is in D_1^n. First, we show that

$$\lim_{t\to 0^+}\int_t^\infty r^{n-2}\zeta(r)\,\mathrm{d}r \qquad \text{exists and is finite.} \tag{4.4}$$

Note, that for $t > 0$,

$$\omega_n\int_t^\infty r^{n-2}\zeta(r)\,\mathrm{d}r = \int_t^\infty\left(\frac{\mathrm{Z}(u_r)}{r} - (n-1)r^{n-2}\int_r^\infty \frac{\mathrm{Z}(u_s)}{s^n}\,\mathrm{d}s\right)\mathrm{d}r$$

and therefore, by (4.3),

$$\omega_n\int_t^\infty r^{n-2}\zeta(r)\,\mathrm{d}r = \int_t^\infty \frac{\mathrm{Z}(u_r)}{r}\,\mathrm{d}r + t^{n-1}\int_t^\infty \frac{\mathrm{Z}(u_r)}{r^n}\,\mathrm{d}r - \int_t^\infty \frac{\mathrm{Z}(u_r)}{r}\,\mathrm{d}r = t^{n-1}\int_t^\infty \frac{\mathrm{Z}(u_r)}{r^n}\,\mathrm{d}r.$$

Since $r \mapsto \mathrm{Z}(u_r)$ is in $C_c([0,\infty))$ it follows from L'Hospital's rule that

$$\lim_{t\to 0^+} t^{n-1}\int_t^\infty \frac{\mathrm{Z}(u_r)}{r^n}\,\mathrm{d}r = \frac{\mathrm{Z}(u_0)}{n-1} \tag{4.5}$$

and we have shown (4.4).

Second, since

$$\omega_n t^{n-1}\zeta(t) = \omega_n t^{n-1}\left(\frac{\mathrm{Z}(u_t)}{\omega_n t^{n-1}} - \frac{n-1}{\omega_n}\int_t^\infty \frac{\mathrm{Z}(u_r)}{r^n}\,\mathrm{d}r\right) = \mathrm{Z}(u_t) - (n-1)t^{n-1}\int_t^\infty \frac{\mathrm{Z}(u_r)}{r^n}\,\mathrm{d}r,$$

it follows from (4.5) that

$$\lim_{t\to 0^+} t^{n-1}\zeta(t) = 0.$$

Thus, $\zeta \in D_1^n$ and we have proved (4.1). $\square$

We immediately obtain the following dual result.

Proposition 4.4. *Let $n \geq 2$. If $\mathrm{Z}\colon \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}$ is a continuous, dually epi-translation and rotation invariant valuation that is homogeneous of degree 1, then there exists $\zeta \in D_1^n$ such that*

$$\mathrm{Z}(v) = \mathrm{V}_{1,\zeta}^*(v)$$

for every $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$.

4.2. **Orthogonal Cylinder Functions.** Let $P \subset \mathbb{R}^n$ be an n-dimensional convex polytope. We call the n-dimensional convex polytopes $P_1, \ldots, P_m \subset \mathbb{R}^n$ a *dissection* of P, if

$$P = \bigcup_{i=1}^m P_i$$

and the interiors of the polytopes P_i and P_j are disjoint for $i \neq j$. In this case we write

$$P = \bigsqcup_{i=1}^m P_i.$$

Two n-dimensional convex polytopes P and Q are *translatively equi-dissectable*, written $P \sim Q$, if there are dissections $P = \bigsqcup_{i=1}^m P_i$ and $Q = \bigsqcup_{i=1}^m Q_i$ such that P_i is a translate of Q_i for $i = 1, \ldots, m$.

A valuation $\mathrm{Z} : \mathcal{P}^n \to \mathbb{R}$ is *simple* if it vanishes on lower dimensional polytopes. For a simple valuation $\mathrm{Z} : \mathcal{P}^n \to \mathbb{R}$, we have

$$\mathrm{Z}\Big(\bigsqcup_{i=1}^m P_i\Big) = \sum_{i=1}^m \mathrm{Z}(P_i) \tag{4.6}$$

(see, for example, [40, Section 6.2]).

An n-dimensional simplex S is the convex hull of $(n+1)$ affinely independent points $p_0, \ldots, p_n$. Set $x_i = p_i - p_{i-1}$. We say that S is orthogonal if the vectors $x_1, \ldots, x_n$ are pairwise orthogonal. We set $x_0 = p_0$ and write $S = \langle x_0; x_1, \ldots, x_n\rangle$. For $0 < t < 1$, the *canonical simplex dissection* is

$$S = \bigsqcup_{k=0}^n \big((1-t)\,\underline{S}_k + t\,\overline{S}_{n-k}\big), \tag{4.7}$$

where

$$\underline{S}_k := \langle x_0; x_1, \ldots, x_k\rangle, \qquad \overline{S}_{n-k} := \langle x_0 + \sum_{i=1}^k x_i; x_{k+1}, \ldots, x_n\rangle$$

while $+$ in (4.7) denotes Minkowski addition (see Hadwiger [20]).

Let $\mathcal{P}^n$ denote the set of convex polytopes in $\mathbb{R}^n$. We say that $P \in \mathcal{P}^n$ is a *proper orthogonal cylinder* if there are orthogonal and complementary subspaces E and F with $\dim E, \dim F \geq 1$ and polytopes $P_E \subset E$ and $P_F \subset F$ such that $P = P_E + P_F$. If S is an orthogonal simplex, then the terms in (4.7) for $1 < k < n$ are proper orthogonal cylinders.

The following result is due to Hadwiger [20, Section 1.3.4, Theorem VII].

Lemma 4.5. *For an n-dimensional polytope $P \in \mathcal{P}^n$, there are orthogonal simplices $S_1, \dots, S_m$, $S'_1, \dots, S'_{m'}$ such that*

$$P \sqcup \bigsqcup_{i=1}^{m} S_i \sim \bigsqcup_{j=1}^{m'} S'_j.$$

Let $\mathrm{Z} : \mathrm{Conv_{sc}}(\mathbb{R}^n) \to \mathbb{R}$ be a valuation. We say that it is *simple* if Z vanishes on functions with lower dimensional domain. The following analogue of (4.6) is a consequence of the inclusion-exclusion principle established in [14]. Let $u_1, \dots, u_m \in \mathrm{Conv_{sc}}(\mathbb{R}^n)$ be such that the domains of $u_i \vee u_j$ are lower dimensional for $i \neq j$. If $\mathrm{Z} : \mathrm{Conv_{sc}}(\mathbb{R}^n) \to \mathbb{R}$ is a simple valuation, then

$$\mathrm{Z}\,\Big(\bigwedge_{i=1}^{m} u_i\Big) = \sum_{i=1}^{m} \mathrm{Z}(u_i). \tag{4.8}$$

A function $u \in \mathrm{Conv_{sc}}(\mathbb{R}^n)$ is called *piecewise affine* if its domain is a convex polytope and u is the pointwise maximum of finitely many affine functions on its domain. Hence there is a dissection of $\mathrm{dom}(u)$ into polytopes $P_1, \dots, P_m$ such that

$$u = \bigwedge_{i=1}^{m} (\ell_i + \alpha_i + \mathbf{I}_{P_i})$$

with linear functions $\ell_i : \mathbb{R}^n \to \mathbb{R}$ and $\alpha_i \in \mathbb{R}$ for $i = 1, \dots, m$. If $\mathrm{Z} : \mathrm{Conv_{sc}}(\mathbb{R}^n) \to \mathbb{R}$ is an epi-translation invariant and simple valuation, we obtain by (4.8) that

$$\mathrm{Z}(u) = \sum_{i=1}^{m} \mathrm{Z}(\ell_i + \mathbf{I}_{P_i}). \tag{4.9}$$

Note that piecewise affine functions are dense in $\mathrm{Conv_{sc}}(\mathbb{R}^n)$.

We say that the function $u \in \mathrm{Conv_{sc}}(\mathbb{R}^n)$ is an *orthogonal cylinder function* if there are orthogonal and complementary subspaces E and F with $\dim E, \dim F \geq 1$ and functions $u_E \in \mathrm{Conv_{sc}}(\mathbb{R}^n)$ with $\mathrm{dom}(u_E) \subset E$ and $u_F \in \mathrm{Conv_{sc}}(\mathbb{R}^n)$ with $\mathrm{dom}(u_F) \subset F$ such that $u = u_E \,\square\, u_F$.

Proposition 4.6. *If $\mathrm{Z}\colon \mathrm{Conv_{sc}}(\mathbb{R}^n) \to \mathbb{R}$ is a continuous and epi-translation invariant valuation that vanishes on orthogonal cylinder functions, then it is epi-homogeneous of degree 1.*

Proof. For $y \in \mathbb{R}^n$, set $\ell_y(x) = \langle y, x\rangle$. Define $\tilde{\mathrm{Z}} : \mathcal{P}^n \to \mathbb{R}$ by

$$\tilde{\mathrm{Z}}(P) := \mathrm{Z}(\ell_y + \mathbf{I}_P).$$

Since Z is continuous, so is $\tilde{\mathrm{Z}}$.

Let S be an n-dimensional orthogonal simplex in $\mathbb{R}^n$ and $0 < t < 1$. In the canonical simplex dissection

$$S = \bigsqcup_{k=0}^{n} ((1-t)\underline{S}_k + t\,\overline{S}_{n-k}),$$

the simplices $\underline{S}_k$ and $\overline{S}_{n-k}$ are orthogonal and lie in orthogonal subspaces. Since Z vanishes on orthogonal cylinder functions, it is simple. Consequently, $\tilde{\operatorname{Z}}_y$ vanishes on orthogonal cylinders and is simple. By (4.6), this implies

$$\tilde{\operatorname{Z}}(S) = \tilde{\operatorname{Z}}((1-t)S) + \tilde{\operatorname{Z}}(t\,S).$$

Let $r, s > 0$. Setting $\alpha(r) = \tilde{\operatorname{Z}}(r\,S)$ and $t = r/(r+s)$, we obtain

$$\alpha(r+s) = \alpha(r) + \alpha(s)$$

for $r, s > 0$. Since $\tilde{\operatorname{Z}}$ is continuous, so is $\alpha : (0,\infty) \to \mathbb{R}$. Hence, α is a continuous solution of Cauchy's functional equation and we obtain that

$$\tilde{\operatorname{Z}}(t\,S) = t\,\tilde{\operatorname{Z}}(S)$$

for any $t > 0$ and any orthogonal simplex S.

For $P \in \mathcal{P}^n$, by Lemma 4.5 there are orthogonal simplices such that

$$P \sqcup \bigsqcup_{i=1}^{m} S_i \sim \bigsqcup_{j=1}^{m'} S_j'.$$

Therefore, for every $t > 0$,

$$\tilde{\operatorname{Z}}(t\,P) = \tilde{\operatorname{Z}}\Big(\bigsqcup_{j=1}^{m'} t\,S_j'\Big) - \tilde{\operatorname{Z}}_y\Big(\bigsqcup_{i=1}^{m} t\,S_i\Big) = t\,\tilde{\operatorname{Z}}\Big(\bigsqcup_{j=1}^{m'} S_j'\Big) - t\,\tilde{\operatorname{Z}}_y\Big(\bigsqcup_{i=1}^{m} S_i\Big) = t\,\tilde{\operatorname{Z}}(P).$$

Hence, $\tilde{\operatorname{Z}}$ is homogeneous of degree 1 on polytopes. Consequently, Z is epi-homogeneous of degree 1 on functions of the form

$$u = \ell_y + \mathbf{I}_P$$

for every $y \in \mathbb{R}^n$ and $P \in \mathcal{P}^n$. Since Z is epi-translation invariant and simple, (4.9) now implies that Z is epi-homogeneous of degree 1 on piecewise affine functions. Since piecewise affine functions are dense in $\operatorname{Conv}_{\rm sc}(\mathbb{R}^n)$, the continuity of Z now implies that Z is epi-homogeneous of degree 1 on $\operatorname{Conv}_{\rm sc}(\mathbb{R}^n)$. $\square$

We say that $v \in \operatorname{Conv}(\mathbb{R}^n;\mathbb{R})$ is a *dual orthogonal cylinder function* if there are orthogonal and complementary subspaces E and F with $\dim E, \dim F \geq 1$ as well as functions $v_E \in \operatorname{Conv}(E;\mathbb{R})$ and $v_F \in \operatorname{Conv}(F;\mathbb{R})$ such that $v = v_E + v_F$. We use (2.4) and obtain the following result as an immediate consequence of Proposition 4.6.

Proposition 4.7. *If* $\operatorname{Z}\colon \operatorname{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}$ *is a continuous and dually epi-translation invariant valuation that vanishes on dual orthogonal cylinder functions, then it is homogeneous of degree 1.*

4.3. **Additional Results on Hessian Measures and Valuations.**

Lemma 4.8. *Let* E *and* F *be orthogonal and complementary subspaces of* $\mathbb{R}^n$ *such that* $1 \leq k < n$ *with* $k = \dim E$*. If* $v_E \in \operatorname{Conv}(E;\mathbb{R})$ *and* $v_F \in \operatorname{Conv}(F;\mathbb{R})$*, then*

$$\Phi_l^n(v_E + v_F, B) = \sum_{i=0\vee(l+k-n)}^{k\wedge l} \Phi_i^k(v_E, B\cap E)\Phi_{l-i}^{n-k}(v_F, B\cap F)$$

for every $0 \leq l \leq n$ *and every Borel subset* $B \subseteq \mathbb{R}^n$*.*

Proof. Let $v = v_E + v_F$. Observe that $y \in \partial v(x)$ if and only if $y_E \in \partial v_E(x_E)$ and $y_F \in \partial v_F(x_F)$. Therefore, for any $s \geq 0$ and any Borel subset $B \subseteq \mathbb{R}$,

$$\begin{aligned} P_s(v, B \times \mathbb{R}^n) &= \{x + sy \colon x \in B, y \in \partial v(x)\} \\ &= \{(x_E, x_F) + s(y_E, y_F) \colon x_E \in B \cap E, x_F \in B \cap F, y_E \in \partial v_E(x_E), y_F \in \partial v_F(x_F)\} \\ &= P_s(v_E, (B \cap E) \times E) \times P_s(v_F, (B \cap F) \times F). \end{aligned}$$

Thus,

$$\begin{aligned} \sum_{l=0}^{n} s^l \Phi_l^n(v, B) &= \mathcal{H}^n(P_s(v, B \times \mathbb{R}^n)) \\ &= \mathcal{H}^k(P_s(v_E, (B \cap E) \times E))\, \mathcal{H}^{n-k}(P_s(v_F, (B \cap F) \times F)) \\ &= \sum_{i=0}^{k} s^i \Phi_i^k(v_E, B \cap E) \sum_{j=0}^{n-k} s^j \Phi_j^{n-k}(v_F, B \cap F) \end{aligned}$$

and consequently,

$$\Phi_l^n(v, B) = \sum_{i=0 \vee (l-(n-k))}^{k \wedge l} \Phi_i^k(v_E, B \cap E)\, \Phi_{l-i}^{n-k}(v_F, B \cap F),$$

which completes the proof. □

We repeatedly evaluate Hessian measures at piecewise affine functions. Fix $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n) \in \mathbb{R}^n$ and define $\bar{v} \in \text{Conv}(\mathbb{R}^n; \mathbb{R})$ as

$$\bar{v}(x_1, \dots, x_n) := \frac{1}{2} \sum_{i=1}^{n} |x_i - \bar{x}_i|. \tag{4.10}$$

Note, that if $\bar{x}_1, \dots, \bar{x}_n \neq 0$, then $\bar{v} \in \text{Conv}_{(0)}(\mathbb{R}^n; \mathbb{R})$. If $\bar{u} = \bar{v}^* \in \text{Conv}_{\text{sc}}(\mathbb{R}^n)$, then

$$\bar{u}(x) = \mathbf{I}_{[-\frac{1}{2}, \frac{1}{2}]^n}(x) + \langle \bar{x}, x \rangle$$

for $x \in \mathbb{R}^n$. It is now easy to see that

$$\int_{\mathbb{R}^n} \zeta(x)\, \mathrm{d}\Phi_n^n(\bar{v}, x) = \int_{\mathbb{R}^n} \zeta(y)\, \mathrm{d}\Psi_n^n(\bar{u}, y) = \int_{[-\frac{1}{2}, \frac{1}{2}]^n} \zeta(\bar{x})\, \mathrm{d}x = \zeta(\bar{x})$$

for every function $\zeta : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}$. Hence, we conclude that

$$\Phi_n^n(\bar{v}, \cdot) = \delta_{\bar{x}}, \tag{4.11}$$

where $\delta_{\bar{x}}$ denotes the Dirac point measure concentrated at $\bar{x}$.

Next, for $1 \leq j < n$, let

$$v(x_1, \dots, x_n) := \frac{1}{2} \sum_{i=1}^{j} |x_i - \bar{x}_i|$$

with $\bar{x}_1, \dots, \bar{x}_j \in \mathbb{R}$. Observe that v is of the form

$$v(x_1, \dots, x_n) = v_1(x_1, \dots, x_j) + v_2(x_{j+1}, \dots, x_n)$$

where v_1 is of the same form as (4.10) and $v_2 \equiv 0$. In particular, $\Phi_i^{n-j}(v_2, A) = 0$ for every $1 \leq i \leq n-j$ and Borel subset $A \subseteq \mathbb{R}^{n-j}$. Hence, it follows from Lemma 4.8, (2.6) and (4.11) that

$$\begin{aligned} \mathrm{d}\Phi_j^n(v,(x_1,\dots,x_n)) &= \mathrm{d}\Phi_j^j(v_1,(x_1,\dots,x_j))\,\mathrm{d}\Phi_0^{n-j}(v_2,(x_{j+1},\dots,x_n)) \\ &= \mathrm{d}\delta_{(\bar{x}_1,\dots,\bar{x}_j)}((x_1,\dots,x_j))\,\mathrm{d}x_{j+1}\cdots\mathrm{d}x_n \end{aligned} \tag{4.12}$$

and

$$\Phi_i^n(v,B) = 0 \tag{4.13}$$

for every $j < i \leq n$ and Borel subset $B \subseteq \mathbb{R}^n$.

Lemma 4.9. *Let $\zeta \in C_b((0,\infty))$ and $j \in \{1,\dots,n\}$. If*

$$\int_{\mathbb{R}^n} \zeta(|x|)\,\mathrm{d}\Phi_j^n(v,x) = 0$$

for every $v \in \mathrm{Conv}_{(0)}(\mathbb{R}^n;\mathbb{R})$, then $\zeta \equiv 0$.

Proof. Fix $\bar{x}_1,\dots,\bar{x}_j \in \mathbb{R}\backslash\{0\}$, set $t = \sqrt{\bar{x}_1^2+\cdots+\bar{x}_j^2} \neq 0$ and $v(x_1,\dots,x_n) = \frac{1}{2}\sum_{i=1}^j |x_i - \bar{x}_i|$. For $j = n$, the measure $\Phi_n^n(v,\cdot)$ is the Dirac point mass concentrated at $(\bar{x}_1,\dots,\bar{x}_n)$ by (4.11). Hence

$$0 = \int_{\mathbb{R}^n} \zeta(|x|)\,\mathrm{d}\Phi_j^n(v,x) = \zeta(t)$$

and the result follows. For $1 \leq j < n$, we use (4.12) and obtain

$$\begin{aligned} 0 &= \int_{\mathbb{R}^n} \zeta(|x|)\,\mathrm{d}\Phi_j^n(v,x) \\ &= \int_{\mathbb{R}^{n-j}} \zeta(|(\bar{x}_1,\dots,\bar{x}_j,x_{j+1},\dots,x_n)|)\,\mathrm{d}x_{j+1}\cdots\mathrm{d}x_n \\ &= \omega_{n-j}\int_0^\infty \zeta(\sqrt{r^2+t^2})r^{n-j-1}\,\mathrm{d}r. \end{aligned}$$

Since $\bar{x}_1,\dots,\bar{x}_j \in \mathbb{R}\backslash\{0\}$ and hence $t > 0$ are arbitrary, the result now follows from Lemma 2.14. □

Lemma 4.10. *Let $j \in \{1,\dots,n\}$ and $\zeta_m, \zeta \in D_j^n$. If*

$$\lim_{m\to\infty}\int_{\mathbb{R}^n} \zeta_m(|x|)\,\mathrm{d}\Phi_j^n(v_m,x) = \int_{\mathbb{R}^n} \zeta(|x|)\,\mathrm{d}\Phi_j^n(v,x) \tag{4.14}$$

for every $v_m, v \in \mathrm{Conv}_{(0)}(\mathbb{R}^n;\mathbb{R})$ with v_m epi-convergent to v, then $\lim_{m\to\infty}\zeta_m(t) = \zeta(t)$ for every $t > 0$.

Proof. In case $j = n$, we may simply choose $v_m(x) = \bar{v}(x) = \frac{1}{2}\sum_{j=1}^n |x_j - \bar{x}_j|$ with $\bar{x}_j \in \mathbb{R}\backslash\{0\}$. It now follows from (4.11) and (4.14) that $\lim_{m\to\infty}\zeta_m(t) = \zeta(t)$ for $t = \sqrt{\bar{x}_1^2+\cdots+\bar{x}_n^2} > 0$.

So let $1 \leq j < n$. For $t > 0$ we will consider the functions v_t defined in (2.2). Note, that $v_t \equiv 0$ on tB^n and hence $v_t \in \mathrm{Conv}_{(0)}(\mathbb{R}^n;\mathbb{R})$. It follows from Lemma 2.15 that

$$\int_{\mathbb{R}^n} \zeta_m(|x|)\,\mathrm{d}\Phi_j^n(v_t,x) = \kappa_n\binom{n}{j}\rho_m(t),$$

where $\rho_m \in C_c([0,\infty))$ is defined by

$$\rho_m(t) := (n-j)\eta_m(t) - t\eta_m'(t) \tag{4.15}$$

and $\eta_m \in C_c([0,\infty))$ is in turn defined by

$$\eta_m(t) := \int_t^\infty r^{n-j-1}\zeta_m(r)\,\mathrm{d}r.$$

We will denote by ρ and η the corresponding functions associated to ζ according to the same relations. For every $t>0$ and for every sequence t_m, with $t_m>0$, converging to t, we have $v_{t_m}\to v_t$ and hence, by the assumptions of the present lemma,

$$\lim_{m\to\infty}\kappa_n\binom{n}{j}\rho_m(t_m)=\lim_{m\to\infty}\int_{\mathbb{R}^n}\zeta_m(|x|)\,\mathrm{d}\Phi_j^n(v_{t_m},x)=\int_{\mathbb{R}^n}\zeta(|x|)\,\mathrm{d}\Phi_j^n(v_t,x)=\kappa_n\binom{n}{j}\rho(t).$$

In other words, the sequence of functions ρ_m converges uniformly to ρ on compact subsets of $(0,\infty)$.

Next, we will show that there exists $\bar{t}>0$ such that

$$\rho_m(t)=0 \qquad \text{for every } t>\bar{t} \text{ and } m\in\mathbb{N}, \tag{4.16}$$

which means that the supports of the functions ρ_m are uniformly bounded. Assume that there exists a sequence $t_m\to\infty$ such that

$$\rho_m(t_m)>0 \qquad \text{for every } m\in\mathbb{N}.$$

For $m\in\mathbb{N}$, set

$$r_m := \rho_m(t_m)^{-\frac{1}{j}}$$

and

$$v_m := r_m v_{t_m}.$$

Since $v_{t_m}(x)=0$ whenever $|x|\le t_m$, it follows that v_m converges to $v\equiv 0\in \mathrm{Conv}_{(0)}(\mathbb{R}^n;\mathbb{R})$ as $m\to\infty$. On the other hand, by homogeneity,

$$\int_{\mathbb{R}^n}\zeta_m(|x|)\,\mathrm{d}\Phi_j^n(v_m,x)=\kappa_n\binom{n}{j}r_m^j\rho_m(t_m)=\kappa_n\binom{n}{j} \qquad \text{for } m\in\mathbb{N},$$

which contradicts the convergence to $v\equiv 0$. Thus (4.16) holds.

Let $\bar{t}>0$ be such that $\rho_m(t)=0$ for every $t\ge\bar{t}$ and $m\in\mathbb{N}$. Since η_m has compact support, there exists $t_m>\bar{t}$ such that $\eta_m(t_m)=0$. Moreover, η_m solves the Cauchy problem,

$$\begin{aligned}&(n-j)\eta_m(t)-t\eta_m'(t)=0 \qquad \text{in } (\bar{t},\infty),\\ &\eta_m(t_m)=0.\end{aligned}$$

This implies that $\eta_m\equiv 0$ in $(\bar{t},\infty)$. As $\bar{t}$ is independent of m, we deduce that the supports of the functions η_m are uniformly bounded as well.

Equation (4.15) implies

$$-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\eta_m(t)}{t^{n-j}}\right)=\frac{\rho_m(t)}{t^{n-j+1}} \qquad \text{for } m\in\mathbb{N} \text{ and } t>0.$$

Therefore,

$$-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\eta_m(t)}{t^{n-j}}\right)$$

converges uniformly to

$$\frac{\rho(t)}{t^{n-j+1}}$$

on compact subsets of $(0,\infty)$. Set

$$\bar{\eta}_m(t) := \frac{\eta_m(t)}{t^{n-j}}$$

for $t > 0$ and $m \in \mathbb{N}$. As the supports of the functions η_m are uniformly bounded, there exists $t_1 > 0$ such that $\bar{\eta}_m(t_1) = 0$ and $\rho_m(t_1) = 0$ for every $m \in \mathbb{N}$. We deduce that $\bar{\eta}_m$ converges to a primitive of

$$-\frac{\rho(t)}{t^{n-j+1}}$$

which vanishes at t_1. But since

$$\frac{\eta(t)}{t^{n-j}}$$

has the same properties, it follows that η_m converges to η uniformly on compact subsets of $(0, \infty)$. We also have, from the definition of η_m and ρ_m,

$$t^{n-j}\zeta_m(t) = \rho_m(t) - (n-j)\eta_m(t) \tag{4.17}$$

for every $t > 0$. Analogously,

$$t^{n-j}\zeta(t) = \rho(t) - (n-j)\eta(t)$$

for every $t > 0$. Hence, passing to the limit as $m \to \infty$ in (4.17), we obtain the statement of the lemma. □

4.4. **Proof of Theorem 1.5.** For $\zeta_0 \in D_0^n, \ldots, \zeta_n \in D_n^n$, we obtain from Theorem 1.4 that

$$v \mapsto \mathrm{V}^*_{0,\zeta_0}(v) + \cdots + \mathrm{V}^*_{n,\zeta_n}(v)$$

defines a continuous, dually epi-translation and rotation invariant valuation on $\mathrm{Conv}(\mathbb{R}^n; \mathbb{R})$.

Conversely, let $\mathrm{Z} : \mathrm{Conv}(\mathbb{R}^n; \mathbb{R}) \to \mathbb{R}$ be a continuous, dually epi-translation and rotation invariant valuation. We want to show that there are $\zeta_0 \in D_0^n, \ldots, \zeta_n \in D_n^n$ such that

$$\mathrm{Z}(v) = \mathrm{V}^*_{0,\zeta_0}(v) + \cdots + \mathrm{V}^*_{n,\zeta_n}(v)$$

for every $v \in \mathrm{Conv}(\mathbb{R}^n; \mathbb{R})$. By Theorem 2.7, there is a homogeneous decomposition of Z. Therefore, we may assume that Z is homogeneous of degree l with $l \in \{0, \ldots, n\}$. Since the cases $l = 0, 1, n$ are settled in Theorem 2.8, Proposition 4.4 and Theorem 2.9, respectively, we may assume that $2 \le l \le n-1$. In particular, we already have a full classification for the case $n = 2$. Therefore, we assume that $n \ge 3$.

Recall that $v \in \mathrm{Conv}_{(0)}(\mathbb{R}^n; \mathbb{R})$ if $v \in \mathrm{Conv}(\mathbb{R}^n; \mathbb{R})$ and v is of class C^2 in a neighborhood of the origin. This space is dense in $\mathrm{Conv}(\mathbb{R}^n; \mathbb{R})$ and by Lemma 3.23, we have for $\zeta \in D_l^n$,

$$\mathrm{V}^*_{l,\zeta}(v) = \int_{\mathbb{R}^n} \zeta(|x|)\,\mathrm{d}\Phi_l^n(v, x) \tag{4.18}$$

for every $v \in \mathrm{Conv}_{(0)}(\mathbb{R}^n; \mathbb{R})$.

We proceed by induction on the dimension n. By (4.18), we may use the induction hypothesis in the following form. Recall that the statement is true for $n = 2$.

Induction Hypothesis. *Let $2 \le k \le n-1$. If $\mathrm{Z} : \mathrm{Conv}(\mathbb{R}^k; \mathbb{R}) \to \mathbb{R}$ is a continuous, dually epi-translation and rotation invariant valuation, then there exist $\zeta_i \in D_i^k$, for $0 \le i \le k$, such that*

$$\mathrm{Z}(v) = \sum_{i=0}^{k} \int_{\mathbb{R}^n} \zeta_i(|x|)\,\mathrm{d}\Phi_i^k(v, x)$$

for every $v \in \mathrm{Conv}_{(0)}(\mathbb{R}^k; \mathbb{R})$.

The main ingredient of our proof is the following result (which we prove using the induction hypothesis).

Proposition 4.11. *If* $\mathrm{Z}:\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})\to\mathbb{R}$ *is a continuous, dually epi-translation and rotation invariant valuation that is homogeneous of degree* l *with* $2\le l\le n-1$*, then there exists* $\zeta\in D_l^n$ *such that*

$$\mathrm{Z}(v_E+v_F)=\int_{\mathbb{R}^n}\zeta(|x|)\,\mathrm{d}\Phi_l^n(v_E+v_F,x)$$

for every pair of orthogonal and complementary subspaces E,F *with* $1\le\dim E<n$ *and every pair of functions* $v_E\in\mathrm{Conv}_{(0)}(E;\mathbb{R})$ *and* $v_F\in\mathrm{Conv}_{(0)}(F;\mathbb{R})$.

Before proving this result, we complete the proof of Theorem 1.5. Define

$$\tilde{\mathrm{Z}}(v):=\mathrm{Z}(v)-\mathrm{V}^*_{l,\zeta}(v)$$

for $v\in\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$. Note that $\tilde{\mathrm{Z}}$ is a continuous, dually epi-translation and rotation invariant valuation. From Proposition 4.11 and the continuity of $\tilde{\mathrm{Z}}$, we obtain that

$$\tilde{\mathrm{Z}}(v)=0$$

for all dual orthogonal cylinder functions $v\in\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$. Thus, it follows from Proposition 4.7 that $\tilde{\mathrm{Z}}$ is homogeneous of degree 1. Since by construction $\tilde{\mathrm{Z}}$ is also homogeneous of degree $l>1$, this implies that $\tilde{\mathrm{Z}}\equiv 0$ and thus,

$$\mathrm{Z}(v)=\mathrm{V}^*_{l,\zeta}(v)$$

for every $v\in\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$. This completes the proof of Theorem 1.5.

4.5. **Proof of Proposition 4.11.** Since Z is rotation invariant and the roles of E and F can be interchanged, we may assume that $E=\mathrm{span}\{e_1,\dots,e_k\}$, $F=\mathrm{span}\{e_{k+1},\dots,e_n\}$, and $\lceil\frac{n}{2}\rceil\le k\le n-1$. Since $n\ge 3$, this also implies that $k\ge 2$.

Lemma 4.12. *Let* $v_F\in\mathrm{Conv}(F;\mathbb{R})$. *There exist* $\zeta_{i,v_F}\in D_i^k$ *for* $1\le i\le k$ *such that*

$$\mathrm{Z}(v_E+v_F)=\sum_{i=0}^k\int_E\zeta_{i,v_F}(|x_E|)\,\mathrm{d}\Phi_i^k(v_E,x_E) \tag{4.19}$$

for every $v_E\in\mathrm{Conv}_{(0)}(E;\mathbb{R})$. *Moreover, the map*

$$v_F\mapsto\int_E\zeta_{0,v_F}(|x_E|)\,\mathrm{d}\Phi_0^k(v_E,x_E)$$

and the maps $v_F\mapsto\zeta_{i,v_F}(s)$ *for* $1\le i\le k$ *and* $s>0$ *are continuous, dually epi-translation and* $\mathrm{O}(n-k)$ *invariant valuations for every* $v_E\in\mathrm{Conv}_{(0)}(E;\mathbb{R})$.

Proof. For $v_F\in\mathrm{Conv}(F;\mathbb{R})$, the map

$$v_E\mapsto\mathrm{Z}(v_E+v_F)$$

is a continuous, dually epi-translation and $\mathrm{SO}(k)$ invariant valuation on $\mathrm{Conv}(E;\mathbb{R})$. Since $k\ge 2$, it follows from the induction hypothesis that there exist functions $\zeta_{i,v_F}\in D_i^k$ for $0\le i\le k$, depending on $v_F\in\mathrm{Conv}(F;\mathbb{R})$, such that (4.19) holds for every $v_E\in\mathrm{Conv}_{(0)}(E;\mathbb{R})$.

Note, that the map $v_E\mapsto\int_E\zeta_{i,v_F}(|x_E|)\,\mathrm{d}\Phi_i^k(v_E,x_E)$ is homogeneous of degree i. Since the map $v_F\mapsto\mathrm{Z}(v_E+v_F)$ is a continuous, dually epi-translation and $\mathrm{SO}(n-k)$ invariant valuation on $\mathrm{Conv}(F;\mathbb{R})$ for every $v_E\in\mathrm{Conv}_{(0)}(E;\mathbb{R})$, it follows from (4.19) and homogeneity that also

$$v_F\mapsto\int_E\zeta_{i,v_F}(|x_E|)\,\mathrm{d}\Phi_i^k(v_E,x_E)$$

is a valuation with the same properties on $\operatorname{Conv}(F;\mathbb{R})$ for every $v_E \in \operatorname{Conv}_{(0)}(E;\mathbb{R})$ and $0 \le i \le k$. This map is actually $\mathrm{O}(n-k)$ invariant, since Z is rotation invariant and since (4.19) shows that the map $v_E \mapsto \mathrm{Z}(v_E + v_F)$ is $\mathrm{O}(k)$ invariant.

Fix i with $1 \le i \le k$ and let $v_{F,1}, v_{F,2} \in \operatorname{Conv}(F;\mathbb{R})$ be such that also $v_{F,1} \wedge v_{F,2} \in \operatorname{Conv}(F;\mathbb{R})$. By the valuation property we have

$$\int_E \big(\zeta_{i,v_{F,1}\vee v_{F,2}}(|x_E|) + \zeta_{i,v_{F,1}\wedge v_{F,2}}(|x_E|) - \zeta_{i,v_{F,1}}(|x_E|) - \zeta_{i,v_{F,2}}(|x_E|)\big)\, \mathrm{d}\Phi_i^k(v_E, x_E) = 0$$

for every $v_E \in \operatorname{Conv}_{(0)}(E;\mathbb{R})$. Hence, it follows from Lemma 4.9 that $v_F \mapsto \zeta_{i,v_F}(s)$ defines a valuation on $\operatorname{Conv}(F;\mathbb{R})$ for every $s > 0$. Similarly, it can be seen that this valuation is dually epi-translation and $\mathrm{O}(n-k)$ invariant. To prove that it is also continuous, note that the map

$$(v_E, v_F) \mapsto \int_E \zeta_{i,v_F}(|x_E|)\, \mathrm{d}\Phi_i^k(v_E, x_E)$$

is jointly continuous in the two variables $v_E \in \operatorname{Conv}_{(0)}(E;\mathbb{R})$ and $v_F \in \operatorname{Conv}(F;\mathbb{R})$. Hence continuity follows from Lemma 4.10. □

In order to avoid unnecessary distinction of cases, we set $\Phi_j^m \equiv 0$ if $j > m$ or $j < 0$.

Lemma 4.13. *For $0 \le i \le k$ with $(l+k-n) \le i \le l$, there exist functions $\zeta_{i,l-i} : (0,\infty)^2 \to \mathbb{R}$ such that*

$$\begin{aligned}\mathrm{Z}(v_E + v_F) = &\int_F \int_E \zeta_{0,l}(|x_E|, |x_F|)\, \mathrm{d}\Phi_0^k(v_E, x_E)\, \mathrm{d}\Phi_l^{n-k}(v_F, x_F)\\ &+ \sum_{i=1\vee(l+k-n)}^{k\wedge l} \int_E \int_F \zeta_{i,l-i}(|x_E|, |x_F|)\, \mathrm{d}\Phi_{l-i}^{n-k}(v_F, x_F)\, \mathrm{d}\Phi_i^k(v_E, x_E)\end{aligned}$$

for every $v_E \in \operatorname{Conv}_{(0)}(E;\mathbb{R})$ and $v_F \in \operatorname{Conv}_{(0)}(F;\mathbb{R})$. Moreover, $\zeta_{0,l}(\cdot, t) \in D_0^k$ for $t > 0$ and

$$\int_E \zeta_{0,l}(|x_E|, \cdot)\, \mathrm{d}\Phi_0^k(v_E, x_E) \in D_l^{n-k}$$

for every $v_E \in \operatorname{Conv}_{(0)}(E;\mathbb{R})$, while, for $i \ge 1$, we have $\zeta_{i,l-i}(s, \cdot) \in D_{l-i}^{n-k}$ for $s > 0$ and

$$\int_F \zeta_{i,l-i}(\cdot, |x_F|)\, \mathrm{d}\Phi_{l-i}^{n-k}(v_F, x_F) \in D_i^k$$

for every $v_F \in \operatorname{Conv}_{(0)}(F;\mathbb{R})$. For $i = l+k-n$ with $i \ge 1$, the statements even hold for every $v_F \in \operatorname{Conv}(F;\mathbb{R})$.

Proof. By Lemma 4.12 combined with the induction hypothesis and Corollary 2.11, there exist functions $\tilde{\zeta}_{i,j,s} \in D_j^{n-k}$ for every $1 \le i \le k$ and $0 \le j \le n-k$, depending on $s > 0$, such that

$$\zeta_{i,v_F}(s) = \sum_{j=0}^{n-k} \int_F \tilde{\zeta}_{i,j,s}(|x_F|)\, \mathrm{d}\Phi_j^{n-k}(v_F, x_F)$$

for every $v_F \in \operatorname{Conv}_{(0)}(F;\mathbb{R})$. Since the function $s \mapsto \zeta_{i,v_F}(s)$ belongs to D_i^k and each of the summands on the right-hand side is homogeneous of a different degree in v_F, it follows that also the functions

$$s \mapsto \int_F \tilde{\zeta}_{i,j,s}(|x_F|)\, \mathrm{d}\Phi_j^{n-k}(v_F, x_F)$$

belong to D_i^k for every $v_F \in \operatorname{Conv}_{(0)}(F;\mathbb{R})$ and $0 \le j \le n-k$. Note, that since both ζ_{i,v_F} and the last integral for the case $j = n-k$ are well-defined for every $v_F \in \operatorname{Conv}(F;\mathbb{R})$, the last statement also holds for every $v_F \in \operatorname{Conv}(\mathbb{R}^n;\mathbb{R})$ in this case. We remark that this also follows from Theorem 2.7 and Corollary 2.10. Hence, there exist functions $\zeta_{i,j} : (0,\infty)^2 \to \mathbb{R}$ with $\zeta_{i,j}(s,\cdot) \in D_j^{n-k}$ for every $s > 0$ and $\int_F \zeta_{i,j}(\cdot, |x_F|) \,\mathrm{d}\Phi_j^{n-k}(v_F, x_F) \in D_i^k$ for every $v \in \operatorname{Conv}_{(0)}(F;\mathbb{R})$ such that

$$\tilde{\zeta}_{i,j,s}(t) = \zeta_{i,j}(s,t)$$

for every $s, t > 0$. Moreover, $\int_F \zeta_{i,n-k}(\cdot, |x_F|) \,\mathrm{d}\Phi_{n-k}^{n-k}(v_F, x_F) \in D_i^k$ for every $v \in \operatorname{Conv}(F;\mathbb{R})$.

For the case $i = 0$, note that

$$\int_E \zeta_{0,v_F}(|x_E|) \,\mathrm{d}\Phi_0^k(v_E, x_E) = \int_E \zeta_{0,v_F}(|x_E|) \,\mathrm{d}x_E = \tilde{\zeta}_{0,v_F}$$

is a constant that is independent of v_E. By Lemma 4.12 the map $v_F \mapsto \tilde{\zeta}_{0,v_F}$ is a continuous, dually epi-translation and $\mathrm{O}(n-k)$ invariant valuation on $\operatorname{Conv}(F;\mathbb{R})$. Therefore, by the induction hypothesis, there exist functions $\tilde{\zeta}_{0,j} \in D_j^{n-k}$ for every $0 \le j \le n-k$ such that

$$\tilde{\zeta}_{0,v_F} = \sum_{j=0}^{n-k} \int_F \tilde{\zeta}_{0,j}(|x_F|) \,\mathrm{d}\Phi_j^{n-k}(v_F, x_F)$$

for every $v_F \in \operatorname{Conv}_{(0)}(F;\mathbb{R})$. It is now possible to find (non-unique) functions $\zeta_{0,j} : (0,\infty)^2 \to \mathbb{R}$ such that $\zeta_{0,j}(\cdot, t) \in D_0^k$ for every $t > 0$ and

$$\tilde{\zeta}_{0,j}(t) = \int_E \zeta_{0,j}(|x_E|, t) \,\mathrm{d}x_E$$

for every $t > 0$ and $0 \le j \le n-k$.

Combining the cases $i > 0$ and $i = 0$, we obtain that

$$\begin{aligned}
\mathrm{Z}(v_E + v_F) = &\sum_{j=0}^{n-k} \int_F \int_E \zeta_{0,j}(|x_E|, |x_F|) \,\mathrm{d}\Phi_0^k(v_E, x_E) \,\mathrm{d}\Phi_j^{n-k}(v_F, x_F) \\
&+ \sum_{i=1}^{k} \sum_{j=0}^{n-k} \int_E \int_F \zeta_{i,j}(|x_E|, |x_F|) \,\mathrm{d}\Phi_j^{n-k}(v_F, x_F) \,\mathrm{d}\Phi_i^k(v_E, x_E)
\end{aligned}$$

for every $v_E \in \operatorname{Conv}_{(0)}(E;\mathbb{R})$ and $v_F \in \operatorname{Conv}_{(0)}(F;\mathbb{R})$. As Z is homogeneous of degree l, and $\Phi_i^k(v_E, \cdot)$ and $\Phi_j^{n-k}(v_F, \cdot)$ in the sum on the right side of the previous equation are homogeneous of degree i and j, respectively, only those terms for which $i + j = l$ may appear. Moreover we must have

$$i \le l, \quad i \le k, \quad l - i \le n-k$$

which gives the desired representation of Z. □

Recall that $\lceil \frac{n}{2} \rceil \le k \le n-1$ and that $E = \operatorname{span}\{e_1, \dots, e_k\}$ while $F = \operatorname{span}\{e_{k+1}, \dots, e_n\}$. Set $E' = \operatorname{span}\{e_1, \dots, e_{k-1}\}$ and $F' = \operatorname{span}\{e_{k+2}, \dots, e_n\}$. In particular, for $k = n-1$, we have $F' = \{0\}$ and $\operatorname{Conv}_{(0)}(F';\mathbb{R})$ can be identified with $\mathbb{R}$. In order to simplify notation, we write $\int_{F'} \mathrm{d}\Phi_0^0(v, x) = 1$ for $v \in \operatorname{Conv}_{(0)}(F';\mathbb{R})$.

Lemma 4.14. *Let $1 \le i < k$ and $a \in \mathbb{R}\backslash\{0\}$. For $v_{E'} \in \operatorname{Conv}_{(0)}(E';\mathbb{R})$ and $v_{F'} \in \operatorname{Conv}_{(0)}(F';\mathbb{R})$,*

$$\begin{aligned}
&\int_{F'} \int_E \zeta_{0,l}(|x_E|, |(a, x_{F'})|) \,\mathrm{d}x_E \,\mathrm{d}\Phi_{l-1}^{n-k-1}(v_{F'}, x_{F'}) \\
&\quad = \int_{E'} \int_{\mathbb{R}} \int_{F'} \zeta_{1,l-1}(|(x_{E'}, a)|, |(t, x_{F'})|) \,\mathrm{d}\Phi_{l-1}^{n-k-1}(v_{F'}, x_{F'}) \,\mathrm{d}t \,\mathrm{d}x_{E'}
\end{aligned}$$

and

$$\begin{aligned}\int_{E'}\int_{\mathbb{R}}\int_{F'}&\zeta_{i,l-i}(|(x_{E'},s)|,|(a,x_{F'})|)\,\mathrm{d}\Phi_{l-i-1}^{n-k-1}(v_{F'},x_{F'})\,\mathrm{d}s\,\mathrm{d}\Phi_i^{k-1}(v_{E'},x_{E'})\\&=\int_{E'}\int_{\mathbb{R}}\int_{F'}\zeta_{i+1,l-i-1}(|(x_{E'},a)|,|(t,x_{F'})|)\,\mathrm{d}\Phi_{l-i-1}^{n-k-1}(v_{F'},x_{F'})\,\mathrm{d}t\,\mathrm{d}\Phi_i^{k-1}(v_{E'},x_{E'})\end{aligned}$$

where $(l+k-n)\leq i<l$.

Proof. Let $\alpha,\beta\in\mathbb{R}$ and $t_\alpha,t_\beta\in\mathbb{R}\backslash\{0\}$. Set

$$v_E(x_E)=v_{E'}(x_{E'})+\frac{\alpha}{2}|x_k-t_\alpha|$$

for $v_{E'}\in\mathrm{Conv}_{(0)}(E';\mathbb{R})$ and

$$v_F(x_F)=\frac{\beta}{2}|x_{k+1}-t_\beta|+v_{F'}(x_{F'})$$

for $v_{F'}\in\mathrm{Conv}_{(0)}(F';\mathbb{R})$. Using Lemma 4.8, (2.6) and (4.11), we obtain

$$\begin{aligned}\mathrm{d}\Phi_i^k(v_E,x_E)&=\mathrm{d}\Phi_i^{k-1}(v_{E'},x_{E'})\,\mathrm{d}x_k+\alpha\,\mathrm{d}\Phi_{i-1}^{k-1}(v_{E'},x_{E'})\,\mathrm{d}\delta_{t_\alpha}(x_k),\\ \mathrm{d}\Phi_{l-i}^{n-k}(v_F,x_F)&=\mathrm{d}x_{k+1}\,\mathrm{d}\Phi_{l-i}^{n-k-1}(v_{F'},x_{F'})+\beta\,\mathrm{d}\delta_{t_\beta}(x_{k+1})\,\mathrm{d}\Phi_{l-i-1}^{n-k-1}(v_{F'},x_{F'})\end{aligned}$$

for $(l+k-n)\leq i\leq l$. Thus, by Lemma 4.13,

$$\begin{aligned}&\mathrm{Z}(v_E+v_F)\\&=\int_F\int_E\zeta_{0,l}(|x_E|,|x_F|)\,\mathrm{d}\Phi_0^k(v_E,x_E)\,\mathrm{d}\Phi_l^{n-k}(v_F,x_F)\\&\quad+\sum_{i=1\vee(l+k-n)}^{k\wedge l}\int_E\int_F\zeta_{i,l-i}(|x_E|,|x_F|)\,\mathrm{d}\Phi_{l-i}^{n-k}(v_F,x_F)\,\mathrm{d}\Phi_i^k(v_E,x_E)\\&=\int_{F'}\int_{\mathbb{R}}\int_E\zeta_{0,l}(|x_E|,|(x_{k+1},x_{F'})|)\,\mathrm{d}x_E\,\mathrm{d}x_{k+1}\,\mathrm{d}\Phi_l^{n-k-1}(v_{F'},x_{F'})\\&\quad+\beta\int_{F'}\int_E\zeta_{0,l}(|x_E|,|(t_\beta,x_{F'})|)\,\mathrm{d}x_E\,\mathrm{d}\Phi_{l-1}^{n-k-1}(v_{F'},x_{F'})\\&\quad+\sum_{i=1\vee(l+k-n)}^{k\wedge l}\Big(\int_{E'}\int_{\mathbb{R}}\int_{F'}\int_{\mathbb{R}}\zeta_{i,l-i}(|(x_{E'},s)|,|(t,x_{F'})|)\,\mathrm{d}t\,\mathrm{d}\Phi_{l-i}^{n-k-1}(v_{F'},x_{F'})\,\mathrm{d}s\,\mathrm{d}\Phi_i^{k-1}(v_{E'},x_{E'})\\&\qquad+\alpha\int_{E'}\int_{\mathbb{R}}\int_{F'}\zeta_{i,l-i}(|(x_{E'},t_\alpha)|,|(t,x_{F'})|)\,\mathrm{d}\Phi_{l-i}^{n-k-1}(v_{F'},x_{F'})\,\mathrm{d}t\,\mathrm{d}\Phi_{i-1}^{k-1}(v_{E'},x_{E'})\\&\qquad+\beta\int_{E'}\int_{\mathbb{R}}\int_{F'}\zeta_{i,l-i}(|(x_{E'},t)|,|(t_\beta,x_{F'})|)\,\mathrm{d}\Phi_{l-i-1}^{n-k-1}(v_{F'},x_{F'})\,\mathrm{d}t\,\mathrm{d}\Phi_i^{k-1}(v_{E'},x_{E'})\\&\qquad+\alpha\beta\int_{E'}\int_{F'}\zeta_{i,l-i}(|(x_{E'},t_\alpha)|,|(t_\beta,x_{F'})|)\,\mathrm{d}\Phi_{l-i-1}^{n-k-1}(v_{F'},x_{F'})\,\mathrm{d}\Phi_{i-1}^{k-1}(v_{E'},x_{E'})\Big).\end{aligned}$$

Here we have split the integral over E with respect to $\mathrm{d}\Phi_i^k(v_E,x_E)$ and the integral over F with respect to $\mathrm{d}\Phi_{l-i}^{n-k}(v_F,x_F)$ into multiple integrals. We are allowed to do so, since for given i each of the integrals so obtained is homogeneous of different degrees in $v_{E'}$ and $v_{F'}$. In particular, each of the integrals in the last expression is well-defined and finite. Since Z is rotation invariant and this expression only depends on the absolute values of t_α and t_β, respectively, we may exchange (α,t_α) and (β,t_β) while preserving equality. Making this exchange, setting $t_\alpha = t_\beta$, comparing with the original expression and considering those parts which are i-homogeneous in $v_{E'}$ and $(l-i-1)$-homogeneous $v_{F'}$ gives

$$\begin{aligned}
&\beta \int_{F'} \int_E \zeta_{0,l}(|x_E|, |(t_\alpha, x_{F'})|) \,\mathrm{d}x_E \,\mathrm{d}\Phi_{l-1}^{n-k-1}(v_{F'}, x_{F'}) \\
&\qquad + \alpha \int_{E'} \int_{\mathbb{R}} \int_{F'} \zeta_{1,l-1}(|(x_{E'}, t_\alpha)|, |(t, x_{F'})|) \,\mathrm{d}\Phi_{l-1}^{n-k-1}(v_{F'}, x_{F'}) \,\mathrm{d}t \,\mathrm{d}x_{E'} \\
&= \alpha \int_{F'} \int_E \zeta_{0,l}(|x_E|, |(t_\alpha, x_{F'})|) \,\mathrm{d}x_E \,\mathrm{d}\Phi_{l-1}^{n-k-1}(v_{F'}, x_{F'}) \\
&\qquad + \beta \int_{E'} \int_{\mathbb{R}} \int_{F'} \zeta_{1,l-1}(|(x_{E'}, t_\alpha)|, |(t, x_{F'})|) \,\mathrm{d}\Phi_{l-1}^{n-k-1}(v_{F'}, x_{F'}) \,\mathrm{d}t \,\mathrm{d}x_{E'}
\end{aligned}$$

for $i = 0$ and

$$\begin{aligned}
&\beta \int_{E'} \int_{\mathbb{R}} \int_{F'} \zeta_{i,l-i}(|(x_{E'}, t)|, |(t_\alpha, x_{F'})|) \,\mathrm{d}\Phi_{l-i-1}^{n-k-1}(v_{F'}, x_{F'}) \,\mathrm{d}t \,\mathrm{d}\Phi_i^{k-1}(v_{E'}, x_{E'}) \\
&\qquad + \alpha \int_{E'} \int_{\mathbb{R}} \int_{F'} \zeta_{i+1,l-i-1}(|(x_{E'}, t_\alpha)|, |(t, x_{F'})|) \,\mathrm{d}\Phi_{l-i-1}^{n-k-1}(v_{F'}, x_{F'}) \,\mathrm{d}t \,\mathrm{d}\Phi_i^{k-1}(v_{E'}, x_{E'}) \\
&= \alpha \int_{E'} \int_{\mathbb{R}} \int_{F'} \zeta_{i,l-i}(|(x_{E'}, t)|, |(t_\alpha, x_{F'})|) \,\mathrm{d}\Phi_{l-i-1}^{n-k-1}(v_{F'}, x_{F'}) \,\mathrm{d}t \,\mathrm{d}\Phi_i^{k-1}(v_{E'}, x_{E'}) \\
&\qquad + \beta \int_{E'} \int_{\mathbb{R}} \int_{F'} \zeta_{i+1,l-i-1}(|(x_{E'}, t_\alpha)|, |(t, x_{F'})|) \,\mathrm{d}\Phi_{l-i-1}^{n-k-1}(v_{F'}, x_{F'}) \,\mathrm{d}t \,\mathrm{d}\Phi_i^{k-1}(v_{E'}, x_{E'})
\end{aligned}$$

for $i \geq 1$. The desired equality now follows after rearranging, using the fact that α and β are arbitrary, and setting $a = t_\alpha$. ☐

Lemma 4.15. *For $1 \leq i < k$ with $(l+k-n) \leq i < l$, there exists $\zeta_i \in C_b((0,\infty))$ such that*

$$\zeta_{i,l-i}(a,b) = \zeta_i(\sqrt{a^2+b^2})$$

for every $a, b > 0$, and by continuity this extends to all $a, b \geq 0$ with $(a,b) \neq (0,0)$. Moreover, if $k = n-1$, then $\zeta_{l-1} \in D_l^n$.

Proof. Since $i \geq 1$, it follows from Lemma 4.9 and Lemma 4.14 that

$$\int_{\mathbb{R}} \int_{F'} \big(\zeta_{i+1,l-i-1}(|(a,b)|, |(t, x_{F'})|) - \zeta_{i,l-i}(|(a,t)|, |(b, x_{F'})|)\big) \,\mathrm{d}\Phi_{l-i-1}^{n-k-1}(v_{F'}, x_{F'}) \,\mathrm{d}t = 0$$

for every $a, b > 0$ and $v_{F'} \in \mathrm{Conv}_{(0)}(F'; \mathbb{R})$. Writing the same equation again but this time replacing b by ε with $0 < \varepsilon < b$ and a by $\sqrt{a^2+b^2-\varepsilon^2}$, we obtain

$$\int_{\mathbb{R}} \int_{F'} \big(\zeta_{i+1,l-i-1}(|(a,b)|, |(t, x_{F'})|) - \zeta_{i,l-i}(|(\sqrt{a^2+b^2-\varepsilon^2}, t)|, |(\varepsilon, x_{F'})|)\big) \,\mathrm{d}\Phi_{l-i-1}^{n-k-1}(v_{F'}, x_{F'}) \,\mathrm{d}t = 0$$

for every $a > 0$ and $b > \varepsilon > 0$. By subtracting the last two equations, we obtain

$$\int_{\mathbb{R}} \int_{F'} \big(\zeta_{i,l-i}(|(\sqrt{a^2+b^2-\varepsilon^2}, t)|, |(\varepsilon, x_{F'})|) - \zeta_{i,l-i}(|(a,t)|, |(b, x_{F'})|)\big) \,\mathrm{d}\Phi_{l-i-1}^{n-k-1}(v_{F'}, x_{F'}) \,\mathrm{d}t = 0$$

for every $a > 0$ and $b > \varepsilon > 0$. By the properties of $\zeta_{i,l-i}$ we may apply Lemma 2.14 for fixed $b > \varepsilon > 0$ and obtain

$$\int_{F'} \big(\zeta_{i,l-i}(\sqrt{a^2+b^2-\varepsilon^2}, |(\varepsilon, x_{F'})|) - \zeta_{i,l-i}(a, |(b, x_{F'})|)\big) \,\mathrm{d}\Phi_{l-i-1}^{n-k-1}(v_{F'}, x_{F'}) = 0$$

for every $a > 0$. Similarly as before we write the last equation again but this time, we replace a by δ with $0 < \delta < a$. Furthermore, we may replace b by $\sqrt{a^2+b^2-\delta^2}$ since by our choice of δ we have $\sqrt{a^2+b^2-\delta^2} > b > \varepsilon$. Thus, we obtain

$$\int_{F'} \big(\zeta_{i,l-i}(\sqrt{a^2+b^2-\varepsilon^2}, |(\varepsilon, x_{F'})|) - \zeta_{i,l-i}(\delta, |(\sqrt{a^2+b^2-\delta^2}, x_{F'})|)\big) \,\mathrm{d}\Phi_{l-i-1}^{n-k-1}(v_{F'}, x_{F'}) = 0$$

for every $a > \delta > 0$ and $b > \varepsilon > 0$. Subtracting the last two equations yields

$$\int_{F'} \big(\zeta_{i,l-i}(\delta, |(\sqrt{a^2+b^2-\delta^2}, x_{F'})|) - \zeta_{i,l-i}(a, |(b, x_{F'})|)\big) \,\mathrm{d}\Phi_{l-i-1}^{n-k-1}(v_{F'}, x_{F'}) = 0 \tag{4.20}$$

for every $a > \delta > 0$ and $b > \varepsilon > 0$. Since this expression does not depend on ε anymore and $\varepsilon > 0$ was arbitrary, we conclude that (4.20) holds for every $b > 0$.

We now claim that

$$\zeta_{i,l-i}(\sqrt{a^2+b^2-\delta^2}, \delta) = \zeta_{i,l-i}(a,b) \tag{4.21}$$

for every $a > \delta > 0$ and $b > 0$. In case $l-i-1 > 0$, this is a direct consequence of Lemma 4.9. In case $l-i-1 = 0$ and $k = n-1$, we have $F' = \{0\}$ and $\int_{F'} \mathrm{d}\Phi_{l-i-1}^{n-k-1}(v_{F'}, x_{F'}) = 1$ and the claim is immediate. In the remaining case $l-i-1 = 0$ and $k < n-1$, we want to emphasize that

$$\mathrm{d}\Phi_{l-i-1}^{n-k-1}(v_{F'}, x_{F'}) = \mathrm{d}x_{F'}$$

and therefore, (4.21) follows from Lemma 2.14 after switching to spherical coordinates in (4.20) and considering that $b > 0$ is arbitrary.

Note, that the right side of (4.21) does not depend on the choice of $\delta < a$. Hence, there exists a function $\zeta_i : (0,\infty) \to \mathbb{R}$ such that

$$\zeta_{i,l-i}(a,b) = \lim_{\delta\to 0^+} \zeta_{i,l-i}(\sqrt{a^2+b^2-\delta^2}, \delta) =: \zeta_i(\sqrt{a^2+b^2}) \tag{4.22}$$

for every $a, b > 0$. Furthermore, it follows from the properties of $\zeta_{i,l-i}$ that ζ_i is continuous with bounded support. Moreover, for fixed $a > 0$ we obtain

$$\lim_{b\to 0^+} \zeta_{i,l-i}(a,b) = \lim_{b\to 0^+} \zeta_i(\sqrt{a^2+b^2}) = \zeta_i(\sqrt{a^2/2 + a^2/2}) = \zeta_{i,l-i}(a/\sqrt{2}, a/\sqrt{2}).$$

Thus, (4.22) continuously extends to all $a, b \geq 0$ with $(a,b) \neq (0,0)$.

Note, that in case $k = n-1$ and $i = l-1$, we have $\zeta_{l-1,1}(s,\cdot) \in D_1^1$ for every $s > 0$ and that $\int_F \zeta_{l-1,1}(\cdot, |x_F|) \,\mathrm{d}\Phi_1^1(v_F, x_F) \in D_{l-1}^{n-1}$ for every $v_F \in \operatorname{Conv}(F;\mathbb{R})$. Thus, choosing $v_F(x_F) = \frac{1}{2}\sum_{i=k+1}^{n} |x_i|$ we obtain by (4.11)

$$\zeta_{l-1}(s) = \zeta_{l-1,1}(s,0) = \int_F \zeta_{l-1,1}(s, |x_F|) \,\mathrm{d}\Phi_1^1(v_F, x_F)$$

for every $s > 0$. Hence $\zeta_{l-1} \in D_{l-1}^{n-1} = D_l^n$. □

Lemma 4.16. *For $\lceil \frac{n}{2} \rceil \leq k \leq n-1$, there exists $\zeta_{(k)} \in C_b((0,\infty))$ such that*

$$\begin{aligned}\mathrm{Z}(v_E + v_F) = &\int_F \int_E \zeta_{(k)}(\sqrt{|x_E|^2 + |x_F|^2}) \,\mathrm{d}\Phi_0^k(v_E, x_E) \,\mathrm{d}\Phi_l^{n-k}(v_F, x_F) \\ &+ \sum_{i=1\vee(l+k-n)}^{k\wedge l} \int_E \int_F \zeta_{(k)}(\sqrt{|x_E|^2 + |x_F|^2}) \,\mathrm{d}\Phi_{l-i}^{n-k}(v_F, x_F) \,\mathrm{d}\Phi_i^k(v_E, x_E)\end{aligned}$$

for every $v_E \in \operatorname{Conv}_{(0)}(E;\mathbb{R})$ and $v_F \in \operatorname{Conv}_{(0)}(F;\mathbb{R})$. Moreover, $\zeta_{(n-1)} \in D_l^n$.

Proof. By Lemma 4.13 combined with Lemma 4.15, we have

$$\begin{aligned}\mathrm{Z}(v_E+v_F) &= \int_F\int_E \zeta_{0,l}(|x_E|,|x_F|)\,\mathrm{d}\Phi_0^k(v_E,x_E)\,\mathrm{d}\Phi_l^{n-k}(v_F,x_F)\\ &\quad+\sum_{i=1\vee(l+k-n)}^{(k\wedge l)-1}\int_E\int_F \zeta_i\big(\sqrt{|x_E|^2+|x_F|^2}\big)\,\mathrm{d}\Phi_{l-i}^{n-k}(v_F,x_F)\,\mathrm{d}\Phi_i^k(v_E,x_E)\\ &\quad+\int_E\int_F \zeta_{k\wedge l,l-(k\wedge l)}(|x_E|,|x_F|)\,\mathrm{d}\Phi_{l-(k\wedge l)}^{n-k}(v_F,x_F)\,\mathrm{d}\Phi_{k\wedge l}^k(v_E,x_E)\end{aligned}$$

for every $v_E\in\mathrm{Conv}_{(0)}(E;\mathbb{R})$ and $v_F\in\mathrm{Conv}_{(0)}(F;\mathbb{R})$ where $\zeta_i\in C_b((0,\infty))$ for every i and, if $k=n-1$, in addition, $\zeta_{l-1}\in D_l^n$. We need to show that we can choose $\zeta_{1\vee(l+k-n)}=\cdots=\zeta_{(k\wedge l)-1}=:\zeta_{(k)}$ and that we can also write the first integral (in case it does not vanish) using $\zeta_{(k)}$. Moreover, in case $k<l$, our proof will show that $\zeta_{k,l-k}=\zeta_{(k)}$ and in case $l\le k$, we will rewrite the last integral using $\zeta_{(k)}$.

Fix $0\vee(l+k-n)\le i<k\wedge l$. We will first consider the case $i>0$ and $l-i-1>0$ and we will treat the cases $i=0$ and $i=l-1$ separately. Since $i>0$ it follows from Lemma 4.9, Lemma 4.14 and Lemma 4.15 that

$$\int_{\mathbb{R}}\int_{F'}\Big(\zeta_{i+1,l-i-1}(a,|(t,x_{F'})|)-\zeta_i(|(a,t,x_{F'})|)\Big)\,\mathrm{d}\Phi_{l-i-1}^{n-k-1}(v_{F'},x_{F'})\,\mathrm{d}t=0 \tag{4.23}$$

for every $a>0$ and $v_{F'}\in\mathrm{Conv}_{(0)}(F';\mathbb{R})$. By (4.12), choosing $a>0$ and $v_{F'}(x_{F'})=\frac{1}{2}(|x_{k+2}-\bar{x}_{k+2}|+\cdots+|x_{k+l-i}-\bar{x}_{k+l-i}|)$ with arbitrary $\bar{x}_{k+2},\ldots,\bar{x}_{k+l-i}\in\mathbb{R}\backslash\{0\}$ gives

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}^{n+i-k-l}}\Big(&\zeta_{i+1,l-i-1}(a,|(t,b,x_{k+l-i+1},\ldots,x_n)|)\\ &-\zeta_i(|(a,t,b,x_{k+l-i+1},\ldots,x_n)|)\Big)\,\mathrm{d}x_{k+l-i+1}\cdots\mathrm{d}x_n\,\mathrm{d}t=0\end{aligned}$$

for every $b>0$. Hence, by Lemma 2.14,

$$\zeta_{i+1,l-i-1}(a,b)=\zeta_i(\sqrt{a^2+b^2})$$

for every $a,b>0$ and similarly as in the proof of Lemma 4.15, for every $a,b\ge0$ with $(a,b)\ne(0,0)$. Note, that in case $i+1<k\wedge l$ it follows from Lemma 4.15 that also $\zeta_{i+1,l-i-1}(a,b)=\zeta_{i+1}(\sqrt{a^2+b^2})$ for every $a,b\ge0$ with $(a,b)\ne(0,0)$. Thus, we have shown that there exists $\zeta_{(k)}\in C_b((0,\infty))$ such that

$$\zeta_{(k)}(\sqrt{a^2+b^2}):=\zeta_{j,l-j}(a,b)$$

for every $a,b\ge0$ with $(a,b)\ne(0,0)$ and every $1\vee(l+k-n)\le j\le k\wedge(l-1)$, except for the cases $l=2$ or $k=n-1$ (for which only $i=0$ and/or $i=l-1$ satisfy the conditions on i above).

Observe that the case $i=l-1$ can only occur if $l\le k$. Since $l\ge2$ we have $i>0$ and therefore equation (4.23) still holds. We now claim that we may replace $\zeta_i=\zeta_{l-1}$ by $\zeta_{(k)}$ in (4.23). If $k<n-1$ and $l>2$ we have $1\vee(l+k-n)\le l-2$ and therefore this follows from the previous considerations for the case $i=l-2$. If $k=n-1$ or $l=2$ we simply set $\zeta_{(k)}:=\zeta_{l-1}$ by Lemma 4.15 which also shows that $\zeta_{(n-1)}\in D_l^n$. Thus, we obtain

$$\begin{aligned}0&=\int_{\mathbb{R}}\int_{F'}\big(\zeta_{l,0}(a,|(t,x_{F'})|)-\zeta_{(k)}(|(a,t,x_{F'})|)\big)\,\mathrm{d}x_{F'}\,\mathrm{d}t\\ &=\int_F\big(\zeta_{l,0}(a,|x_F|)-\zeta_{(k)}(|(a,x_F)|)\big)\,\mathrm{d}\Phi_0^{n-k}(v_F,x_F)\end{aligned}$$

for every $a>0$ and $v_F \in \operatorname{Conv}_{(0)}(F;\mathbb{R})$. In particular, this implies that

$$\int_E \int_F \zeta_{l,0}(|x_E|,|x_F|) \,\mathrm{d}\Phi_0^{n-k}(v_F,x_F)\,\mathrm{d}\Phi_l^k(v_E,x_E) \\ = \int_E \int_F \zeta_{(k)}(\sqrt{|x_E|^2+|x_F|^2})\,\mathrm{d}\Phi_0^{n-k}(v_F,x_F)\,\mathrm{d}\Phi_l^k(v_E,x_E)$$

for every $v_E \in \operatorname{Conv}_{(0)}(E;\mathbb{R})$ and $v_F \in \operatorname{Conv}_{(0)}(F;\mathbb{R})$.

Note that the remaining case $i=0$ can only occur if $l \leq n-k$. Since by our assumptions $l \geq 2$ and $k \geq 2$ it follows from the previous considerations for the case $i=1$ that $\zeta_1 = \zeta_{(k)}$. Thus, by Lemma 4.14 and Lemma 4.15,

$$\int_{E'}\int_{\mathbb{R}}\int_{F'} \zeta_{(k)}(|(x_{E'},a,t,x_{F'})|)\,\mathrm{d}\Phi_{l-1}^{n-k-1}(v_{F'},x_{F'})\,\mathrm{d}t\,\mathrm{d}x_{E'} \\ = \int_{F'}\int_{\mathbb{R}}\int_{E'} \zeta_{0,l}(|(x_{E'},t)|,|(a,x_{F'})|)\,\mathrm{d}x_{E'}\,\mathrm{d}t\,\mathrm{d}\Phi_{l-1}^{n-k-1}(v_{F'},x_{F'})$$

for every $a>0$ and $v_{F'} \in \operatorname{Conv}_{(0)}(F';\mathbb{R})$. If we set $v_{F'}(x_{F'}) = \frac{1}{2}(|x_{k+2}-\bar{x}_{k+2}| + \cdots + |x_{k+l}-\bar{x}_{k+l}|)$ with arbitrary $\bar{x}_{k+2},\dots,\bar{x}_{k+l} \in \mathbb{R}\backslash\{0\}$ and consider (4.12) this becomes

$$\int_{E'}\int_{\mathbb{R}}\int_{\mathbb{R}^{n-k-l}} \zeta_{(k)}(|(x_{E'},a,t,b,x_{k+l+1},\dots,x_n)|)\,\mathrm{d}x_{k+l+1}\cdots\,\mathrm{d}x_n\,\mathrm{d}t\,\mathrm{d}x_{E'} \\ = \int_{\mathbb{R}^{n-k-l}}\int_{\mathbb{R}}\int_{E'} \zeta_{0,l}(|(x_{E'},t)|,|(a,b,x_{k+l+1},\dots,x_n)|)\,\mathrm{d}x_{E'}\,\mathrm{d}t\,\mathrm{d}x_{k+l+1}\cdots\,\mathrm{d}x_n$$

for every $a,b>0$. Thus, by renaming the integration variables on the left side and rearranging, we obtain

$$\int_{\mathbb{R}^{n-k-l}}\int_{\mathbb{R}}\int_{E'} \Big(\zeta_{(k)}(|(x_{E'},a,t,b,x_{k+l+1},\dots,x_n)|) \\ - \zeta_{0,l}(|(x_{E'},t)|,|(a,b,x_{k+l+1},\dots,x_n)|)\Big)\,\mathrm{d}x_{E'}\,\mathrm{d}t\,\mathrm{d}x_{k+l+1}\cdots\,\mathrm{d}x_n = 0$$

for every $a,b>0$ and thus, by Lemma 2.14 if $l<n-k$ and trivially if $l=n-k$,

$$0 = \int_{\mathbb{R}}\int_{E'} \Big(\zeta_{(k)}(|x_{E'},t,a)|) - \zeta_{0,l}(|x_{E'},t)|,a)\Big)\,\mathrm{d}x_{E'}\,\mathrm{d}t \\ = \int_E \Big(\zeta_{(k)}(|(x_E,a)|) - \zeta_{0,l}(|x_E|,a)\Big)\,\mathrm{d}\Phi_0^k(v_E,x_E)$$

for every $a>0$ and $v_E \in \operatorname{Conv}_{(0)}(E;\mathbb{R})$. In particular, as in the case $i=l-1$, this implies that

$$\int_F\int_E \zeta_{0,l}(|x_E|,|x_F|)\,\mathrm{d}\Phi_0^k(v_E,x_E)\,\mathrm{d}\Phi_l^{n-k}(v_F,x_F) \\ = \int_F\int_E \zeta_{(k)}(\sqrt{|x_E|^2+|x_F|^2})\,\mathrm{d}\Phi_0^k(v_E,x_E)\,\mathrm{d}\Phi_l^{n-k}(v_F,x_F)$$

for every $v_E \in \operatorname{Conv}_{(0)}(E;\mathbb{R})$ and $v_F \in \operatorname{Conv}_{(0)}(F;\mathbb{R})$. $\square$

We can now complete the proof of Proposition 4.11. Let

$$v(x_1,\dots,x_n)=\frac{1}{2}\left(|x_1-\bar{x}_1|+\cdots+|x_l-\bar{x}_l|\right)$$

with $\bar{x}_1,\dots,\bar{x}_l\in\mathbb{R}\backslash\{0\}$ and let $a=\sqrt{\bar{x}_1^2+\cdots+\bar{x}_l^2}>0$. Note, that for every $\lceil\frac{n}{2}\rceil\le k\le n-1$ we can write v in the form $v=v_E+v_F$. Furthermore, by (4.13), $\mathrm{d}\Phi_i^k(v_E,x_E)=0$ if $i>l$ and $\mathrm{d}\Phi_j^{n-k}(v_F,x_F)=0$ if $j>(l-k)\vee 0$. Thus, for any k, it follows from Lemma 4.16, (4.11) and (4.12) that

$$\begin{aligned}\operatorname{Z}(v)&=\int_{\mathbb{R}^{n-l}}\zeta_{(\lceil\frac{n}{2}\rceil)}(|(a,x_{l+1},\dots,x_n)|)\,\mathrm{d}x_{l+1}\cdots\mathrm{d}x_n\\&=\cdots\\&=\int_{\mathbb{R}^{n-l}}\zeta_{(n-1)}(|(a,x_{l+1},\dots,x_n)|)\,\mathrm{d}x_{l+1}\cdots\mathrm{d}x_n.\end{aligned}$$

Hence, by Lemma 2.14, $\zeta_{(\lceil\frac{n}{2}\rceil)}\equiv\cdots\equiv\zeta_{(n-1)}=:\zeta$ and $\zeta\in D_l^n$. Thus, by Lemma 4.16,

$$\begin{aligned}\operatorname{Z}(v_E+v_F)&=\int_F\int_E\zeta(|(x_E,x_F)|)\,\mathrm{d}\Phi_0^k(v_E,x_E)\,\mathrm{d}\Phi_l^{n-k}(v_F,x_F)\\&\quad+\sum_{i=1\vee(l+k-n)}^{k\wedge l}\int_E\int_F\zeta(|(x_E,x_F)|)\,\mathrm{d}\Phi_{l-i}^{n-k}(v_F,x_F)\,\mathrm{d}\Phi_i^k(v_E,x_E)\end{aligned}\tag{4.24}$$

for every $v_E\in\operatorname{Conv}_{(0)}(E;\mathbb{R})$ and $v_F\in\operatorname{Conv}_{(0)}(F;\mathbb{R})$ and every $\lceil\frac{n}{2}\rceil\le k\le n-1$. Since Hessian measures are non-negative and since $v_E+v_F\in\operatorname{Conv}_{(0)}(\mathbb{R}^n;\mathbb{R})$ for every $v_E\in\operatorname{Conv}_{(0)}(E;\mathbb{R})$ and $v_F\in\operatorname{Conv}_{(0)}(F;\mathbb{R})$, it follows from Lemma 3.1 and Lemma 4.8 that

$$\int_F\int_E\big|\zeta(|(x_E,x_F)|)\big|\,\mathrm{d}\Phi_0^k(v_E,x_E)\,\mathrm{d}\Phi_l^{n-k}(v_F,x_F)\le\int_{\mathbb{R}^n}\big|\zeta(|x|)\big|\,\mathrm{d}\Phi_l^n(v_E+v_F,x)<+\infty$$

for every $v_E\in\operatorname{Conv}_{(0)}(E;\mathbb{R})$ and $v_F\in\operatorname{Conv}_{(0)}(F;\mathbb{R})$. Furthermore, since $\zeta\in D_l^n$ is measurable and has bounded support and since Hessian measures are locally finite, it now follows from the Fubini-Tonelli theorem that we may exchange the order of integration of the first term on the right side in (4.24). Together with Lemma 4.8 we now obtain

$$\begin{aligned}\operatorname{Z}(v_E+v_F)&=\sum_{i=0\vee(l+k-n)}^{k\wedge l}\int_E\int_F\zeta(|(x_E,x_F)|)\,\mathrm{d}\Phi_{l-i}^{n-k}(v_F,x_F)\,\mathrm{d}\Phi_i^k(v_E,x_E)\\&=\int_{\mathbb{R}^n}\zeta(|x|)\,\mathrm{d}\Phi_l^n(v_E+v_F,x)\end{aligned}$$

for every $v_E\in\operatorname{Conv}_{(0)}(E;\mathbb{R})$ and $v_F\in\operatorname{Conv}_{(0)}(F;\mathbb{R})$ and every $\lceil\frac{n}{2}\rceil\le k\le n-1$, which completes the proof.

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