General Affine Surface Areas

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Abstract

Two families of general affine surface areas are introduced. Basic properties and affine isoperimetric inequalities for these new affine surface areas as well as for L_ϕ affine surface areas are established.

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Finding the *right* notion of affine surface area was one of the first questions asked within affine differential geometry. At the beginning of the last century, Blaschke [5] and his School studied this question and introduced equi-affine surface area – a notion of surface area that is equi-affine invariant, that is, $\mathrm{SL}(n)$ and translation invariant. The first fundamental result regarding equi-affine surface area was the classical affine isoperimetric inequality of differential geometry [5]. Numerous important results regarding equi-affine surface area were obtained in recent years (see, for example, [1,2,45,48–51]). Using valuations on convex bodies, the author and Reitzner [27] were able to characterize a much richer family of affine surface areas (see Theorem 2). Classical equi-affine and centro-affine surface area as well as all L_p affine surface areas for $p > 0$ belong to this family of L_ϕ affine surface areas.

The present paper has two aims. The first is to establish affine isoperimetric inequalities and basic duality relations for all L_ϕ affine surface areas. The second aim is to define new general notions of affine surface area that complement L_ϕ affine surface areas and include L_p affine surface areas for $p < -n$ and $-n < p < 0$. Let $\mathcal{K}_0^n$ denote the space of convex bodies, that is, compact convex sets, in $\mathbb{R}^n$ that contain the origin in their interiors. Whereas L_ϕ affine surface areas are always finite and are upper semicontinuous functionals on $\mathcal{K}_0^n$, the affine surface areas of the new families are infinite for certain convex bodies including polytopes and are lower semicontinuous functionals on $\mathcal{K}_0^n$. Basic properties and affine isoperimetric inequalities for these new affine surface areas are established. In Section 6,

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it is conjectured that together with L_ϕ affine surface areas, these new affine surface areas constitute – in a certain sense – *all* affine surface areas.

For a smooth convex body $K \subset \mathbb{R}^n$, equi-affine surface area is defined by

$$\Omega(K) = \int_{\partial K} \kappa_0(K,x)^{\frac{1}{n+1}} \, d\mu_K(x). \tag{1}$$

Here $d\mu_K(x) = x \cdot u(K,x) \, d\mathcal{H}(x)$ is the cone measure on ∂K, $x \cdot u$ is the standard inner product of $x, u \in \mathbb{R}^n$, $u(K,x)$ is the exterior unit normal vector to K at $x \in \partial K$, $\mathcal{H}$ is the $(n-1)$-dimensional Hausdorff measure,

$$\kappa_0(K,x) = \frac{\kappa(K,x)}{(x \cdot u(K,x))^{n+1}},$$

and $\kappa(K,x)$ is the Gaussian curvature of K at x. Note that $\kappa_0(K,x)$ is (up to a constant) just a power of the volume of the origin-centered ellipsoid osculating K at x and thus is an $\mathrm{SL}(n)$ covariant notion. Also μ_K is an $\mathrm{SL}(n)$ covariant notion. Thus Ω is easily seen to be $\mathrm{SL}(n)$ invariant and it is also easily seen to be translation invariant. The notion of equi-affine surface area is fundamental in affine differential and convex geometry. Since many basic problems in discrete and stochastic geometry are equi-affine invariant, equi-affine surface area has found numerous applications in these fields (see, for example, [3, 4, 12, 40]).

The extension of the definition of equi-affine surface area to general convex bodies was obtained much more recently in a series of papers [21, 29, 43]. Since $\kappa_0(K, \cdot)$ exists μ_K a.e. on ∂K by Aleksandrov's differentiability theorem, definition (1) still can be used. The long conjectured upper semicontinuity of equi-affine surface area (for smooth surfaces as well as for general convex surfaces) was proved by Lutwak [29] in 1991, that is,

$$\limsup_{j \to \infty} \Omega(K_j) \le \Omega(K)$$

for any sequence of convex bodies K_j converging to K (in the Hausdorff metric). Let $\mathcal{K}^n$ denote the space of convex bodies in $\mathbb{R}^n$. Schütt [42] showed that Ω is a valuation on $\mathcal{K}^n$, that is,

$$\Omega(K) + \Omega(L) = \Omega(K \cup L) + \Omega(K \cap L)$$

for all $K, L \in \mathcal{K}^n$ with $K \cup L \in \mathcal{K}^n$. An equi-affine version of Hadwiger's celebrated classification theorem [18] was established in [26]: (up to multiplication with a positive constant) equi-affine surface area is the unique upper semicontinuous, $\mathrm{SL}(n)$ and translation invariant valuation on $\mathcal{K}^n$ that vanishes on polytopes.

During the past decade and a half, there has been an explosive growth of an L_p extension of the classical Brunn Minkowski theory (see, for example, [6–8,15–17,24,25,31,34–38,46,47]). Within this theory, L_p affine surface area is the notion corresponding to equi-affine surface area in the classical Brunn Minkowski theory. For $p > 1$, L_p affine surface area, Ω_p, was introduced by Lutwak [32] and shown to be $\mathrm{SL}(n)$ invariant, homogeneous of degree $q = p(n-p)/(n+p)$ (that is, $\Omega_p(t\,K) = t^q\,\Omega_p(K)$ for $t > 0$), and upper semicontinuous on $\mathcal{K}_0^n$. Hug [19] defined L_p affine surface area for every $p > 0$ and obtained the following representation for $K \in \mathcal{K}_0^n$:

$$\Omega_p(K) = \int_{\partial K} \kappa_0(K,x)^{\frac{p}{n+p}}\, d\mu_K(x). \tag{2}$$

Note that $\Omega_1 = \Omega$ and that Ω_n is the classical (and $\mathrm{GL}(n)$ invariant) centro-affine surface area. Geometric interpretations of L_p affine surface areas were obtained in [11, 39, 44, 52], and an application of L_p affine surface areas to partial differential equations is given in [33].

The L_p affine surface areas for $p > 0$ are special cases of the following family of affine surface areas introduced in [27]. Let $\mathrm{Conc}(0,\infty)$ be the set of functions $\phi : (0,\infty) \to (0,\infty)$ such that ϕ is concave, $\lim_{t\to 0}\phi(t) = 0$, and $\lim_{t\to\infty}\phi(t)/t = 0$. Set $\phi(0) = 0$. For $\phi \in \mathrm{Conc}(0,\infty)$, we define the L_ϕ affine surface area of K by

$$\Omega_\phi(K) = \int_{\partial K} \phi(\kappa_0(K,x))\, d\mu_K(x). \tag{3}$$

The following basic properties of L_ϕ affine surface areas were established in [27]. Let $\mathcal{P}_0^n$ denote the set of convex polytopes containing the origin in their interiors.

Theorem 1 ([27]). *If $\phi \in \mathrm{Conc}(0,\infty)$, then $\Omega_\phi(K)$ is finite for every $K \in \mathcal{K}_0^n$ and $\Omega_\phi(P) = 0$ for every $P \in \mathcal{P}_0^n$. In addition, $\Omega_\phi : \mathcal{K}_0^n \to [0,\infty)$ is both upper semicontinuous and an $\mathrm{SL}(n)$ invariant valuation.*

The family of L_ϕ affine surface areas for $\phi \in \mathrm{Conc}(0,\infty)$ is distinguished by the following basic properties (see [23] and [27], for characterizations of functionals that do not necessarily vanish on polytopes).

Theorem 2 ([27]). *If $\Phi : \mathcal{K}_0^n \to \mathbb{R}$ is an upper semicontinuous and $\mathrm{SL}(n)$ invariant valuation that vanishes on $\mathcal{P}_0^n$, then there exists $\phi \in \mathrm{Conc}(0,\infty)$ such that*

$$\Phi(K) = \Omega_\phi(K)$$

for every $K \in \mathcal{K}_0^n$.

One of the most important inequalities of affine geometry is the classical affine isoperimetric inequality. The following theorem establishes affine isoperimetric inequalities for all L_ϕ affine surface areas. Let $\mathcal{K}_c^n$ denote the space of $K \in \mathcal{K}_0^n$ that have their centroids at the origin and let $|K|$ denote the n-dimensional volume of K.

Theorem 3. *Let $K \in \mathcal{K}_c^n$ and $B_K \in \mathcal{K}_c^n$ be the ball such that $|B_K| = |K|$. If $\phi \in \mathrm{Conc}(0,\infty)$, then*

$$\Omega_\phi(K) \le \Omega_\phi(B_K)$$

and there is equality for strictly increasing ϕ if and only if K is an ellipsoid.

For $\phi(t) = t^{1/(n+1)}$ and smooth convex bodies, Theorem 3 is the classical affine isoperimetric inequality of differential geometry. For general convex bodies, proofs of the classical affine isoperimetric inequality were given by Leichtweiß [21], Lutwak [29], and Hug [19]. For L_p affine surface areas, the affine isoperimetric inequality was established by Lutwak [32] for $p > 1$ and by Werner and Ye [53] for $p > 0$.

Polarity on convex bodies induces the following duality on L_ϕ affine surface areas. Let $K^* = \{x \in \mathbb{R}^n : x \cdot y \le 1 \text{ for } y \in K\}$ denote the polar body of $K \in \mathcal{K}_0^n$. For $\phi \in \mathrm{Conc}(0,\infty)$, define $\phi_* : (0,\infty) \to (0,\infty)$ by $\phi_*(s) = s\,\phi(1/s)$.

Theorem 4. *If $\phi \in \mathrm{Conc}(0,\infty)$, then $\Omega_\phi(K^*) = \Omega_{\phi_*}(K)$ holds for every $K \in \mathcal{K}_0^n$.*

For L_p affine surface areas and $p > 0$, Theorem 4 is due to Hug [20]: $\Omega_p(K^*) = \Omega_{n^2/p}(K)$ for every $K \in \mathcal{K}_0^n$.

An alternative definition of L_p affine surface area uses integrals of the curvature function $f(K,\cdot)$ over the unit sphere $\mathbb{S}^{n-1}$ (see [32]). This approach can also be used for L_ϕ affine surface areas.

Theorem 5. *If $\phi \in \mathrm{Conc}(0,\infty)$, then*

$$\Omega_\phi(K) = \int_{\mathbb{S}^{n-1}} \phi_*(a_0(K,u))\, d\nu_K(u)$$

for every $K \in \mathcal{K}_0^n$.

Here $a_0(K,u) = f_{-n}(K,u) = h(K,u)^{n+1} f(K,u)$ is the L_p curvature function of K (see [32]) for $p = -n$, while $h(K,u)$ is the support function of K, and $d\nu_K(u) = d\mathcal{H}(u)/h(K,u)^n$ (see Section 1 for precise definitions). For L_p affine surface areas and $p > 0$, Theorem 5 is due to Hug [19].

The family of L_ϕ affine surface areas for $\phi \in \mathrm{Conc}(0,\infty)$ includes all $\mathrm{SL}(n)$ invariant and upper semicontinuous valuations on $\mathcal{K}_0^n$ that vanish on polytopes and, in particular, all L_p affine surface areas for $p > 0$. However, L_p affine surface areas for $p < 0$ do not belong to the family of L_ϕ affine surface areas. Recent results by Meyer and Werner [39], Schütt and Werner [44], Werner [52], and Werner and Ye [53] underline the importance of L_p affine surface area also for $p < 0$.

A new family of affine surface areas generalizes L_p affine surface area for $-n < p < 0$. Let $\mathrm{Conv}(0,\infty)$ be the set of functions $\psi : (0,\infty) \to (0,\infty)$ such that ψ is convex, $\lim_{t\to 0} \psi(t) = \infty$, and $\lim_{t\to\infty} \psi(t) = 0$. Set $\psi(0) = \infty$. For $\psi \in \mathrm{Conv}(0,\infty)$, we define the L_ψ affine surface area of K by

$$\Omega_\psi(K) = \int_{\partial K} \psi(\kappa_0(K,x))\, d\mu_K(x). \tag{4}$$

The following theorem establishes basic properties of L_ψ affine surface areas.

Theorem 6. *If $\psi \in \mathrm{Conv}(0,\infty)$, then $\Omega_\psi(K)$ is positive for every $K \in \mathcal{K}_0^n$ and $\Omega_\psi(P) = \infty$ for every $P \in \mathcal{P}_0^n$. In addition, $\Omega_\psi : \mathcal{K}_0^n \to (0,\infty]$ is both lower semicontinuous and an $\mathrm{SL}(n)$ invariant valuation.*

An immediate consequence of Theorem 6 is the following result for L_p affine surface area.

Corollary 7. *If $-n < p < 0$, then $\Omega_p(K)$ is positive for every $K \in \mathcal{K}_0^n$ and $\Omega_p(P) = \infty$ for every $P \in \mathcal{P}_0^n$. In addition, $\Omega_p : \mathcal{K}_0^n \to (0,\infty]$ is both lower semicontinuous and an $\mathrm{SL}(n)$ invariant valuation.*

Affine isoperimetric inequalities for L_ψ affine surface areas are established in

Theorem 8. *Let $K \in \mathcal{K}_c^n$ and $B_K \in \mathcal{K}_c^n$ be the ball such that $|B_K| = |K|$. If $\psi \in \mathrm{Conv}(0,\infty)$, then*

$$\Omega_\psi(K) \geq \Omega_\psi(B_K)$$

and there is equality for strictly decreasing ψ if and only if K is an ellipsoid.

For $\psi(t) = t^{p/(n+p)}$ and $-n < p < 0$, this result was proved (in a different way) by Werner and Ye [53].

For $\psi \in \mathrm{Conv}(0,\infty)$, define $\Omega_\psi^* : \mathcal{K}_0^n \to (0,\infty]$ by $\Omega_\psi^*(K) := \Omega_\psi(K^*)$. The following theorem establishes basic properties of these affine surface areas.

Theorem 9. *If* $\psi \in \mathrm{Conv}(0,\infty)$*, then* $\Omega^*_\psi(K)$ *is positive for every* $K \in \mathcal{K}^n_0$ *and* $\Omega^*_\psi(P) = \infty$ *for every* $P \in \mathcal{P}^n_0$*. In addition,* $\Omega^*_\psi : \mathcal{K}^n_0 \to (0,\infty]$ *is both lower semicontinuous and an* $\mathrm{SL}(n)$ *invariant valuation.*

The family of affine surface areas Ω^*_ψ for $\psi \in \mathrm{Conv}(0,\infty)$ complements L_ϕ affine surface areas for $\phi \in \mathrm{Conc}(0,\infty)$ and L_ψ affine surface areas for $\psi \in \mathrm{Conv}(0,\infty)$. Whereas L_ϕ affine surface areas for $\phi \in \mathrm{Conc}(0,\infty)$ include affine surface areas homogeneous of degree q for all $|q| < n$ and L_ψ affine surface areas for $\psi \in \mathrm{Conv}(0,\infty)$ include affine surface areas homogeneous of degree q for all $q > n$, the new family includes affine surface areas homogeneous of degree q for all $q < -n$.

The next theorem gives a representation of Ω^*_ψ corresponding to that of Theorem 5.

Theorem 10. *If* $\psi \in \mathrm{Conv}(0,\infty)$*, then*

$$\Omega^*_\psi(K) = \int_{\mathbb{S}^{n-1}} \psi(a_0(K,u))\, d\nu_K(u)$$

for every $K \in \mathcal{K}^n_0$*.*

For $p < -n$, L_p affine surface area was defined by Schütt and Werner [44] using (2). Here a different approach is used and a different definition of L_p affine surface areas for $p < -n$ is given:

$$\Omega_p(K) := \int_{\mathbb{S}^{n-1}} a_0(K,u)^{\frac{n}{n+p}}\, d\nu_K(u). \tag{5}$$

By Theorem 10, $\Omega_p(K) = \Omega^*_{n^2/p}(K) = \Omega^*_\psi(K)$ with $\psi(t) = t^{n/(n+p)}$ and $p < -n$.

An immediate consequence of Theorem 9 is the following result for L_p affine surface area as defined by (5).

Corollary 11. *If* $p < -n$*, then* $\Omega_p(K)$ *is positive for every* $K \in \mathcal{K}^n_0$ *and* $\Omega_p(P) = \infty$ *for every* $P \in \mathcal{P}^n_0$*. In addition,* $\Omega_p : \mathcal{K}^n_0 \to (0,\infty]$ *is both lower semicontinuous and an* $\mathrm{SL}(n)$ *invariant valuation.*

1 Tools

Basic notions on convex bodies and their curvature measures are collected. For detailed information, see [10,13,41]. Let $K \in \mathcal{K}^n_0$. The support function of K is defined for $x \in \mathbb{R}^n$ by

$$h(K,x) = \max\{x \cdot y : y \in K\}.$$

The radial function of K is defined for $x \in \mathbb{R}^n$ and $x \neq 0$ by

$$\rho(K, x) = \max\{t > 0 : t\,x \in K\}.$$

Note that these definitions immediately imply that

$$\rho(K, x) = 1 \quad \text{for} \quad x \in \partial K, \tag{6}$$

$$\rho(K, t\,u) = \frac{1}{t}\,\rho(K, x) \quad \text{for} \quad t > 0, \tag{7}$$

and

$$h(K, u) = \frac{1}{\rho(K^*, u)}, \tag{8}$$

where K^* is the polar body of K.

Let $\mathcal{B}(\mathbb{R}^n)$ denote the family of Borel sets in $\mathbb{R}^n$ and $\sigma(K, \beta)$ the spherical image of $\beta \in \mathcal{B}(\mathbb{R}^n)$, that is, the set of all exterior unit normal vectors of K at points of β. Note that $\sigma(K, \beta)$ is Lebesgue measurable for each $\beta \in \mathcal{B}(\mathbb{R}^n)$. For a sequence of convex bodies $K_j \in \mathcal{K}_0^n$ converging to $K \in \mathcal{K}_0^n$ and a closed set $\beta \subset \mathbb{R}^n$, we have

$$\limsup_{j \to \infty} \sigma(K_j, \beta) \subset \sigma(K, \beta). \tag{9}$$

For $\beta \in \mathcal{B}(\mathbb{R}^n)$, set

$$C(K, \beta) = \int_{\sigma(K,\beta)} \frac{d\mathcal{H}(u)}{h(K, u)^n},$$

where $\mathcal{H}$ denotes the $(n-1)$-dimensional Hausdorff measure. Hence $C(K, \cdot)$ is a Borel measure on $\mathbb{R}^n$ that is concentrated on ∂K. By (8), we obtain

$$C(K, \partial K) = n\,|K^*|. \tag{10}$$

It follows from (9) that for every closed set $\beta \subset \mathbb{R}^n$,

$$\limsup_{j \to \infty}\, C(K_j, \beta) \le C(K, \beta). \tag{11}$$

Let $C_0(K, \cdot) : \mathcal{B}(\mathbb{R}^n) \to [0, \infty)$ be the 0-th curvature measure of the convex body K (see [41], Section 4.2). For $\beta \in \mathcal{B}(\mathbb{R}^n)$, we have

$$C_0(K, \beta) = \mathcal{H}(\sigma(K, \beta)). \tag{12}$$

We decompose the measure $C_0(K, \cdot)$ into measures absolutely continuous and singular with respect to $\mathcal{H}$, say, $C_0(K, \cdot) = C_0^a(K, \cdot) + C_0^s(K, \cdot)$. Note that

$$\frac{dC_0^a(K, \cdot)}{d\mathcal{H}} = \kappa(K, \cdot). \tag{13}$$

Let $\operatorname{reg} K$ denote the set of regular boundary points of K, that is, boundary points with a unique exterior unit normal vector. From (12), we obtain for $\omega \subset \operatorname{reg} K$ and $\omega \in \mathcal{B}(\mathbb{R}^n)$,

$$C(K,\omega) = \int_{\sigma(K,\omega)} \frac{d\mathcal{H}(u)}{h(K,u)^n} = \int_\omega \frac{dC_0(K,x)}{(x \cdot u(K,x))^n}. \tag{14}$$

We decompose the measure $C(K,\cdot)$ into measures absolutely continuous and singular with respect to the measure μ_K, say, $C(K,\cdot) = C^a(K,\cdot) + C^s(K,\cdot)$. The singular part is concentrated on a μ_K null set $\omega_0 \subset \partial K$, that is, for $\beta \in \mathcal{B}(\mathbb{R}^n)$

$$C^s(K, \beta \backslash \omega_0) = 0. \tag{15}$$

Since $C^a(K,\cdot)$ is concentrated on $\operatorname{reg} K$, (13) and (14) imply for $\omega \subset \partial K$ and $\omega \in \mathcal{B}(\mathbb{R}^n)$,

$$C^a(K,\omega) = \int_\omega \frac{\kappa(K,x)}{(x \cdot u(K,x))^n}\, d\mathcal{H}(x) = \int_\omega \kappa_0(K,x)\, d\mu_K(x). \tag{16}$$

Combined with (10), this implies

$$\int_{\partial K} \kappa_0(K,x)\, d\mu_K(x) \le n\,|K^*|. \tag{17}$$

Hug [20] proved that for almost all $x \in \partial K$,

$$\kappa(K,x) = \big(\frac{x}{|x|} \cdot u_K(x)\big)^{n+1} f(K^*, \frac{x}{|x|}).$$

Hence we have for almost all $y \in \partial K^*$,

$$\kappa_0(K^*, y) = a_0(K, \frac{y}{|y|}). \tag{18}$$

Here $|x|$ denotes the length of x.

2 Proof of Theorems 3 and 8

Let $\phi \in \text{Conc}(0,\infty)$ and $K \in \mathcal{K}_c^n$. By definition (3), Jensen's inequality, (17), and the monotonicity of ϕ, we obtain

$$\begin{aligned}\Omega_\phi(K) &= \int_{\partial K} \phi(\kappa_0(K,x))\, d\mu_K(x)\\ &\leq n\,|K|\ \phi\Big(\frac{1}{n\,|K|}\int_{\partial K} \kappa_0(K,x)\, d\mu_K(x)\Big)\\ &\leq n\,|K|\ \phi\Big(\frac{|K^*|}{|K|}\Big).\end{aligned}$$

For origin-centered ellipsoids, $\kappa_0(K,\cdot)$ is constant and there is equality in the above inequalities. Now we use the Blaschke-Santaló inequality: for $K \in \mathcal{K}_c^n$

$$|K|\,|K^*| \leq |B^n|^2$$

with equality precisely for origin-centered ellipsoids (see, for example, [28]). Here B^n is the unit ball in $\mathbb{R}^n$. We obtain

$$\Omega_\phi(K) \leq n\,|K|\ \phi\Big(\frac{|K^*|}{|K|}\Big) \leq n\,|K|\ \phi\Big(\frac{|B^n|^2}{|K|^2}\Big) = \Omega_\phi(B_K). \tag{19}$$

For ϕ strictly increasing, equality in the second inequality of (19) holds if and only if there is equality in the Blaschke-Santaló inequality, that is, precisely for ellipsoids. This completes the proof of Theorem 3 and the proof of Theorem 8 follows along similar lines.

3 Proof of Theorems 4 and 9

Define Ω_ϕ^* on $\mathcal{K}_0^n$ by $\Omega_\phi^*(K) := \Omega_\phi(K^*)$. Since Ω_ϕ is upper semicontinuous, so is Ω_ϕ^*. For $K, L, K\cup L \in \mathcal{K}_0^n$, we have

$$(K\cup L)^* = K^* \cap L^* \quad \text{and} \quad (K\cap L)^* = K^* \cup L^*.$$

Since Ω_ϕ is a valuation, this implies that

$$\begin{aligned}\Omega_\phi^*(K) + \Omega_\phi^*(L) &= \Omega_\phi(K^*) + \Omega_\phi(L^*)\\ &= \Omega_\phi(K^*\cup L^*) + \Omega_\phi(K^*\cap L^*)\\ &= \Omega_\phi((K\cap L)^*) + \Omega_\phi((K\cup L)^*)\\ &= \Omega_\phi^*(K\cap L) + \Omega_\phi^*(K\cup L),\end{aligned}$$

that is, Ω_ϕ^* is a valuation on $\mathcal{K}_0^n$. For $A \in \mathrm{SL}(n)$ and $K \in \mathcal{K}_0^n$, we have $(A\,K)^* = A^{-t}\,K^*$, where A^{-t} denotes the inverse of the transpose of A. Since Ω_ϕ is $\mathrm{SL}(n)$ invariant, this implies $\Omega_\phi^*(AK) = \Omega_\phi^*(K)$, that is, $\Omega_\phi^* : \mathcal{K}_0^n \to \mathbb{R}$ is $\mathrm{SL}(n)$ invariant. Since Ω_ϕ vanishes on polytopes, so does Ω_ϕ^*. Therefore Ω_ϕ^* satisfies the assumptions of Theorem 2. Thus there exists $\alpha \in \mathrm{Conc}(0,\infty)$ such that $\Omega_\phi^* = \Omega_\alpha$. Let B^n denote the unit ball in $\mathbb{R}^n$. For $r > 0$, we obtain from (3) that

$$\Omega_\alpha(rB^n) = n\,|B^n|\,r^n\,\alpha(\frac{1}{r^{2n}})$$

and

$$\Omega_\phi^*(rB^n) = \Omega_\phi(\frac{1}{r}B^n) = \frac{n\,|B^n|}{r^n}\,\phi(r^{2n}).$$

This shows that $\alpha = \phi_*$ and completes the proof of Theorem 4. The proof of Theorem 9 follows along the lines of the proof that Ω_ϕ^* satisfies the assumptions of Theorem 2.

4 Proof of Theorems 5 and 10

Define $y : \mathbb{S}^{n-1} \to \partial K^*$ by $u \mapsto \rho(K^*, u)\,u$. Note that this is a Lipschitz function. For the Jacobian Jy of y, we have a.e. on $\mathbb{S}^{n-1}$,

$$Jy(u) = \frac{\rho(K^*, u)^{n-1}}{u \cdot u_{K^*}(\rho(K^*, u)\,u)} \tag{20}$$

(see, for example, [20]). By the area formula (see, for example, [9]), we have for every a.e. defined function $g : \mathbb{S}^{n-1} \to [0,\infty]$,

$$\int_{\mathbb{S}^{n-1}} g(u)\,Jy(u)\,d\mathcal{H}(u) = \int_{\partial K^*} g(\frac{y}{|y|})\,d\mathcal{H}(y).$$

Setting

$$g(u) = \frac{\tau(a_0(K,u))}{h(K,u)^n\,Jy(u)}$$

for $\tau : [0,\infty] \to [0,\infty]$, we get by (6), (7), (8), and (18),

$$\begin{aligned}
\int\limits_{\mathbb{S}^{n-1}} \tau(a_0(K,u))\,d\nu_K(u) &= \int\limits_{\mathbb{S}^{n-1}} \tau(a_0(K,u))\,\frac{d\mathcal{H}(u)}{h(K,u)^n} \\
&= \int\limits_{\partial K^*} \tau(\kappa_0(K^*, y))\,\frac{\frac{y}{|y|} \cdot u_{K^*}(y)}{\rho(K^*, \frac{y}{|y|})^{n-1}}\,\rho(K^*, \frac{y}{|y|})^n\,d\mathcal{H}(y) \\
&= \int\limits_{\partial K^*} \tau(\kappa_0(K^*, y))\,d\mu_{K^*}(y).
\end{aligned}$$

For $\tau \in \mathrm{Conv}(0,\infty)$, this implies Theorem 10. To obtain Theorem 5, we set $\tau = \phi_* \in \mathrm{Conc}(0,\infty)$ and apply Theorem 4.

5 Proof of Theorem 6

Let $\psi \in \mathrm{Conv}(0,\infty)$ and $K \in \mathcal{K}_0^n$. Note that ψ is strictly decreasing and positive. By definition (4), the Jensen inequality, (17), and the monotonicity of ψ, we obtain

$$\begin{aligned}\Omega_\psi(K) &= \int_{\partial K} \psi(\kappa_0(K,x))\, d\mu_K(x)\\ &\geq n\,|K|\;\psi\Big(\frac{1}{n\,|K|}\int_{\partial K}\kappa_0(K,x)\,d\mu_K(x)\Big)\\ &\geq n\,|K|\;\psi\Big(\frac{|K^*|}{|K|}\Big).\end{aligned}$$

This shows that $\Omega_\psi(K) > 0$. The $\mathrm{SL}(n)$ invariance of Ω_ψ follows immediately from the definition. So does the fact that $\Omega_\psi(P) = \infty$ for $P \in \mathcal{P}_0^n$.

Next, we show that Ω_ψ is a valuation on $\mathcal{K}_0^n$, that is, for $K, L \in \mathcal{K}_0^n$ such that $K \cup L \in \mathcal{K}_0^n$,

$$\Omega_\psi(K\cup L) + \Omega_\psi(K\cap L) = \Omega_\psi(K) + \Omega_\psi(L). \tag{21}$$

Let $K^c = \{x \in \mathbb{R}^n : x \notin K\}$ and let $\mathrm{int}\, K$ denote the interior of K. We follow Schütt [42] (see also [14]) and work with the decompositions

$$\begin{aligned}\partial(K\cup L) &= \big(\partial K\cap\partial L\big)\cup\big(\partial K\cap L^c\big)\cup\big(\partial L\cap K^c\big),\\ \partial(K\cap L) &= \big(\partial K\cap\partial L\big)\cup\big(\partial K\cap \mathrm{int}\,L\big)\cup\big(\partial L\cap \mathrm{int}\,K\big),\\ \partial K &= \big(\partial K\cap\partial L\big)\cup\big(\partial K\cap L^c\big)\cup\big(\partial K\cap \mathrm{int}\,L\big),\\ \partial L &= \big(\partial K\cap\partial L\big)\cup\big(\partial L\cap K^c\big)\cup\big(\partial L\cap \mathrm{int}\,K\big),\end{aligned}$$

where all unions on the right hand side are disjoint. Note that for x such that the curvatures $\kappa_0(K,x)$, $\kappa_0(L,x)$, $\kappa_0(K\cup L,x)$, and $\kappa_0(K\cap L,x)$ exist,

$$u(K,x) = u(L,x) = u(K\cup L,x) = u(K\cap L,x) \tag{22}$$

and

$$\begin{aligned}\kappa_0(K\cup L,x) &= \min\{\kappa_0(K,x),\kappa_0(L,x)\},\\ \kappa_0(K\cap L,x) &= \max\{\kappa_0(K,x),\kappa_0(L,x)\}.\end{aligned} \tag{23}$$

To prove (21), we use (4), split the involved integrals using the above decompositions, and use (22) and (23).

Finally, we show that Ω_ψ is lower semicontinuous on $\mathcal{K}_0^n$. The proof complements the proofs in [22] and [30]. Let $K \in \mathcal{K}_0^n$ and $\varepsilon > 0$ be chosen. Since $\kappa_0(K, \cdot)$ is measurable a.e. on ∂K and since the set ω_0, where the singular part of $C(K, \cdot)$ is concentrated, is a μ_K null set, we can choose by Lusin's theorem (see, for example, [9]) pairwise disjoint closed sets $\omega_l \subset \partial K$, $l \in \mathbb{N}$, such that $\kappa_0(K, \cdot)$ is continuous as a function restricted to ω_l, such that for every $l \in \mathbb{N}$,

$$\omega_l \cap \omega_0 = \emptyset \tag{24}$$

and such that

$$\mu_K(\bigcup_{l=1}^{\infty} \omega_l) = \mu_K(\partial K). \tag{25}$$

For $\omega \subset \mathbb{R}^n$, let $\bar\omega$ be the cone generated by ω, that is, $\bar\omega = \{t\,x \in \mathbb{R}^n : t \geq 0, x \in \omega\}$. Note that $\bar\omega_l$ is closed and that $\partial K \cap \bar\omega_l = \omega_l$.

Let K_j be a sequence of convex bodies converging to K. First, we show that for $l \in \mathbb{N}$,

$$\liminf_{j\to\infty} \int_{\partial K_j \cap \bar\omega_l} \psi(\kappa_0(K_j, x))\, d\mu_{K_j}(x) \geq \int_{\partial K \cap \bar\omega_l} \psi(\kappa_0(K, x))\, d\mu_K(x). \tag{26}$$

Let $\eta > 0$ be chosen. We choose a monotone sequence $t_i \in (0, \infty)$, $i = \mathbb{Z}$, $\lim_{i\to-\infty} t_i = 0$, $\lim_{i\to\infty} t_i = \infty$, such that

$$\max_{i\in\mathbb{Z}} |\psi(t_{i+1}) - \psi(t_i)| \leq \eta \tag{27}$$

and such that for $i \in \mathbb{Z}$, $j \geq 0$,

$$\mu_{K_j}(\{x \in \partial K_j : \kappa_0(K_j, x) = t_i\}) = 0, \tag{28}$$

where $K_0 = K$. This is possible, since $\mu_{K_j}(\{x \in K_j : \kappa_0(K_j, x) = t\}) > 0$ holds only for countably many t. Set

$$\omega_{li} = \{x \in \omega_l : t_i \leq \kappa_0(K, x) \leq t_{i+1}\}.$$

Since $\kappa_0(K, \cdot)$ is continuous on ω_l and ω_l is closed, the sets $\bar\omega_{li}$ are closed for $i \in \mathbb{Z}$. This implies by (11) that

$$\limsup_{j\to\infty} C(K_j, \bar\omega_{li}) \leq C(K, \bar\omega_{li}). \tag{29}$$

By (24), (15), and the definition of ω_{li},

$$C(K, \bar\omega_{li}) = C^a(K, \bar\omega_{li}) \leq t_{i+1}\, \mu_K(\partial K \cap \bar\omega_{li}). \tag{30}$$

By (16),

$$\int_{\partial K_j \cap \bar{\omega}_{li}} \kappa_0(K_j, x)\, d\mu_{K_j}(x) \le C(K_j, \bar{\omega}_{li}). \tag{31}$$

Using the monotonicity of ψ, we obtain

$$\begin{aligned} \int_{\omega_l} \psi(\kappa_0(K,x))\, d\mu_K(x) &\le \sum_{i\in\mathbb{Z}} \int_{\omega_{li}} \psi(\kappa_0(K,x))\, d\mu_K(x) \\ &\le \sum_{i\in\mathbb{Z}} \psi(t_i)\, \mu_K(\omega_{li}). \end{aligned} \tag{32}$$

Using (28), the Jensen inequality, (31), and the monotonicity of ψ, we obtain

$$\begin{aligned} \int\limits_{\partial K_j \cap \bar{\omega}_l} \psi(\kappa_0(K_j,x))\, d\mu_{K_j}(x) &= \sum_{i\in\mathbb{Z}} \int\limits_{\partial K_j \cap \bar{\omega}_{li}} \psi(\kappa_0(K_j,x))\, d\mu_{K_j}(x) \\ &= \sum_{i\in\mathbb{Z}}{}' \int\limits_{\partial K_j \cap \bar{\omega}_{li}} \psi(\kappa_0(K_j,x))\, d\mu_{K_j}(x) \\ &\ge \sum_{i\in\mathbb{Z}}{}' \psi\left(\frac{C(K_j,\bar{\omega}_{li})}{\mu_{K_j}(\partial K_j \cap \bar{\omega}_{li})}\right) \mu_{K_j}(\partial K_j \cap \bar{\omega}_{li}) \end{aligned}$$

where the $'$ indicates that we sum only over $\bar{\omega}_{li}$ with $\mu_{K_j}(\partial K_j \cap \bar{\omega}_{li}) \neq 0$. Since

$$\begin{aligned} &\liminf_{j\to\infty} \sum_{i\in\mathbb{Z}}{}' \psi\left(\frac{C(K_j,\bar{\omega}_{li})}{\mu_{K_j}(\partial K_j \cap \bar{\omega}_{li})}\right) \mu_{K_j}(\partial K_j \cap \bar{\omega}_{li}) \\ &\qquad \ge \sum_{i\in\mathbb{Z}}{}' \psi\left(\limsup_{j\to\infty}\left(\frac{C(K_j,\bar{\omega}_{li})}{\mu_{K_j}(\partial K_j \cap \bar{\omega}_{li})}\right)\right) \liminf_{j\to\infty} \mu_{K_j}(\partial K_j \cap \bar{\omega}_{li}), \end{aligned}$$

we obtain by (29), (30),(32),(27), and (28) that

$$\begin{aligned} &\liminf_{j\to\infty} \int_{\partial K_j \cap \bar{\omega}_l} \psi(\kappa_0(K_j,x))\, d\mu_{K_j}(x) \\ &\qquad \ge \sum_{i\in\mathbb{Z}}{}' \psi\left(\frac{C(K,\bar{\omega}_{li})}{\mu_K(\partial K \cap \bar{\omega}_{li})}\right) \mu_K(\partial K \cap \bar{\omega}_{li}) \\ &\qquad \ge \sum_{i\in\mathbb{Z}} \psi(t_{i+1})\, \mu_K(\partial K \cap \bar{\omega}_{li}) \\ &\qquad = \sum_{i\in\mathbb{Z}} \psi(t_i)\, \mu_K(\partial K \cap \bar{\omega}_{li}) - \sum_{i\in\mathbb{Z}} (\psi(t_i) - \psi(t_{i+1}))\ \mu_K(\partial K \cap \bar{\omega}_{li}) \\ &\qquad \ge \int_{\partial K \cap \bar{\omega}_l} \psi(\kappa_0(K,x))\, d\mu_K(x) - \eta\, \mu_K(\partial K \cap \bar{\omega}_l). \end{aligned}$$

Since $\eta > 0$ is arbitrary, this proves (26).

Finally, (28) and (26) imply

$$\begin{aligned}\liminf_{j\to\infty} \int\limits_{\partial K_j} \psi(\kappa_0(K_j,x))\,d\mu_{K_j}(x) &= \liminf_{j\to\infty} \sum_{l=1}^{\infty} \int\limits_{\partial K_j \cap \bar{\omega}_l} \psi(\kappa_0(K_j,x))\,d\mu_{K_j}(x)\\ &\geq \sum_{l=1}^{\infty} \liminf_{j\to\infty} \int\limits_{\partial K_j \cap \bar{\omega}_l} \psi(\kappa_0(K_j,x))\,d\mu_{K_j}(x)\\ &\geq \int\limits_{\partial K} \psi(\kappa_0(K,x))\,d\mu_K(x).\end{aligned}$$

This completes the proof of the theorem.

6 Open problems

The affine surface areas Ω_ψ and Ω^*_ψ for $\psi \in \operatorname{Conv}(0,\infty)$ are lower semi-continuous and $\operatorname{SL}(n)$ invariant valuations. More general examples of such functionals are

$$\Psi = \Omega_{\psi_1} + \Omega^*_{\psi_2} - \Omega_\phi$$

for $\psi_1, \psi_2 \in \operatorname{Conv}(0,\infty)$ and $\phi \in \operatorname{Conc}(0,\infty)$. Additional examples are the following continuous functionals

$$K \mapsto c_0 + c_1\,|K| + c_2\,|K^*|$$

for $c_0, c_1, c_2 \in \mathbb{R}$. In view of Theorem 2, this gives rise to the following

Conjecture 1. *If $\Psi : \mathcal{K}^n_0 \to (-\infty,\infty]$ is a lower semicontinuous and $\operatorname{SL}(n)$ invariant valuation, then there exist $\psi_1, \psi_2 \in \operatorname{Conv}(0,\infty)$, $\phi \in \operatorname{Conc}(0,\infty)$, and $c_0, c_1, c_2 \in \mathbb{R}$ such that*

$$\Psi(K) = c_0 + c_1\,|K| + c_2\,|K^*| + \Omega_{\psi_1}(K) + \Omega^*_{\psi_2}(K) - \Omega_\phi(K)$$

for every $K \in \mathcal{K}^n_0$.

The following special case of the above conjecture is of particular interest.

Conjecture 2. *If $\Psi : \mathcal{K}^n_0 \to (-\infty,\infty]$ is a lower semicontinuous and $\operatorname{SL}(n)$ invariant valuation that is homogeneous of degree $q < -n$ or $q > n$, then there exists $c \geq 0$ such that*

$$\Psi(K) = c\,\Omega_p(K)$$

for every $K \in \mathcal{K}^n_0$, where $p = n\,(n-q)/(n+q)$.

References

[1] B. Andrews, *Contraction of convex hypersurfaces by their affine normal*, J. Differential Geom. **43** (1996), 207–230.

[2] B. Andrews, *The affine curve-lengthening flow*, J. Reine Angew. Math. **506** (1999), 43–83.

[3] I. Bárány, *Affine perimeter and limit shape*, J. Reine Angew. Math. **484** (1997), 71–84.

[4] I. Bárány, *Sylvester's question: the probability that n points are in convex position*, Ann. Probab. **27** (1999), 2020–2034.

[5] W. Blaschke, *Differentialgeometrie II*, Springer, Berlin, 1923.

[6] S. Campi and P. Gronchi, *The L^p-Busemann-Petty centroid inequality*, Adv. Math. **167** (2002), 128–141.

[7] K. Chou and X.-J. Wang, *The L_p-Minkowski problem and the Minkowski problem in centroaffine geometry*, Adv. Math. **205** (2006), 33–83.

[8] A. Cianchi, E. Lutwak, D. Yang, and G. Zhang, *Affine Moser-Trudinger and Morrey-Sobolev inequalities*, Calc. Var. Partial Differential Equations **36** (2009), 419–436.

[9] H. Federer, *Geometric measure theory*, Springer, Berlin, 1969.

[10] R. Gardner, *Geometric tomography*, second ed., Encyclopedia of Mathematics and its Applications, vol. 58, Cambridge University Press, Cambridge, 2006.

[11] P. M. Gruber, *Asymptotic estimates for best and stepwise approximation of convex bodies I*, Forum Math. **5** (1993), 281–297.

[12] P. M. Gruber, *Asymptotic estimates for best and stepwise approximation of convex bodies II*, Forum Math. **5** (1993), 521–538.

[13] P. M. Gruber, *Convex and discrete geometry*, Grundlehren der Mathematischen Wissenschaften, vol. 336, Springer, Berlin, 2007.

[14] C. Haberl, *Star body valued valuations*, Indiana Univ. Math. J. **2009** (58), 2253 – 2276.

[15] C. Haberl and M. Ludwig, *A characterization of L_p intersection bodies*, Int. Math. Res. Not. **10548** (2006), 1–29.

[16] C. Haberl and F. Schuster, *Asymmetric affine L_p Sobolev inequalities*, J. Funct. Anal. **257** (2009), 641–658.

[17] C. Haberl and F. Schuster, *General L_p affine isoperimetric inequalities*, J. Differential Geom. **83** (2009), 1–26.

[18] H. Hadwiger, *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie*, Springer, Berlin, 1957.

[19] D. Hug, *Contributions to affine surface area*, Manuscripta Math. **91** (1996), 283–301.

[20] D. Hug, *Curvature relations and affine surface area for a general convex body and its polar*, Result. Math. **29** (1996), 233–248.

[21] K. Leichtweiß, *Über einige Eigenschaften der Affinoberfläche beliebiger konvexer Körper*, Result. Math. **13** (1988), 255–282.

[22] M. Ludwig, *On the semicontinuity of curvature integrals*, Math. Nachr. **227** (2001), 99–108.

[23] M. Ludwig, *Valuations on polytopes containing the origin in their interiors*, Adv. Math. **170** (2002), 239–256.

[24] M. Ludwig, *Ellipsoids and matrix valued valuations*, Duke Math. J. **119** (2003), 159–188.

[25] M. Ludwig, *Minkowski valuations*, Trans. Amer. Math. Soc. **357** (2005), 4191–4213.

[26] M. Ludwig and M. Reitzner, *A characterization of affine surface area*, Adv. Math. **147** (1999), 138–172.

[27] M. Ludwig and M. Reitzner, *A classification of SL(n) invariant valuations*, Ann. of Math., in press.

[28] E. Lutwak, *On the Blaschke-Santaló inequality*, Ann. New York Acad. Sci. **440** (1985), 106–112.

[29] E. Lutwak, *Extended affine surface area*, Adv. Math. **85** (1991), 39–68.

[30] E. Lutwak, *On the semicontinuity of curvatures*, Comment. Math. Helv. **67** (1992), 664–669.

[31] E. Lutwak, *The Brunn-Minkowski-Firey theory. I. Mixed volumes and the Minkowski problem*, J. Differential Geom. **38** (1993), 131–150.

[32] E. Lutwak, *The Brunn-Minkowski-Firey theory. II: Affine and geominimal surface areas*, Adv. Math. **118** (1996), 244–294.

[33] E. Lutwak and V. Oliker, *On the regularity of solutions to a generalization of the Minkowski problem*, J. Differential Geom. **41** (1995), 227–246.

[34] E. Lutwak, D. Yang, and G. Zhang, *L_p affine isoperimetric inequalities*, J. Differential Geom. **56** (2000), 111–132.

[35] E. Lutwak, D. Yang, and G. Zhang, *Sharp affine L_p Sobolev inequalities*, J. Differential Geom. **62** (2002), 17–38.

[36] E. Lutwak, D. Yang, and G. Zhang, *Volume inequalities for subspaces of L_p*, J. Differential Geom. **68** (2004), 159–184.

[37] E. Lutwak, D. Yang, and G. Zhang, *Optimal Sobolev norms and the L^p Minkowski problem*, Int. Math. Res. Not. **62987** (2006), 1–10.

[38] E. Lutwak and G. Zhang, *Blaschke-Santaló inequalities*, J. Differential Geom. **47** (1997), 1–16.

[39] M. Meyer and E. Werner, *On the p-affine surface area*, Adv. Math. **152** (2000), 288–313.

[40] M. Reitzner, *The combinatorial structure of random polytopes*, Adv. Math. **191** (2005), 178–208.

[41] R. Schneider, *Convex bodies: the Brunn-Minkowski theory*, Encyclopedia of Mathematics and its Applications, vol. 44, Cambridge University Press, Cambridge, 1993.

[42] C. Schütt, *On the affine surface area*, Proc. Amer. Math. Soc. **118** (1993), 1213–1218.

[43] C. Schütt and E. Werner, *The convex floating body*, Math. Scand. **66** (1990), 275–290.

[44] C. Schütt and E. Werner, *Surface bodies and p-affine surface area*, Adv. Math. **187** (2004), 98–145.

[45] W.M. Sheng, N. S. Trudinger, and X.-J. Wang, *Enclosed convex hypersurfaces with maximal affine area*, Math. Z. **252** (2006), 497–510.

[46] A. Stancu, *The discrete planar L_0-Minkowski problem*, Adv. Math. **167** (2002), 160–174.

[47] A. Stancu, *On the number of solutions to the discrete two-dimensional L_0-Minkowski problem*, Adv. Math. **180** (2003), 290–323.

[48] N. S. Trudinger and X.-J. Wang, *The Bernstein problem for affine maximal hypersurfaces*, Invent. Math. **140** (2000), 399–422.

[49] N. S. Trudinger and X.-J. Wang, *Affine complete locally convex hypersurfaces*, Invent. Math. **150** (2002), 45–60.

[50] N. S. Trudinger and X.-J. Wang, *The affine Plateau problem*, J. Amer. Math. Soc. **18** (2005), 253–289.

[51] N. S. Trudinger and X.-J. Wang, *Boundary regularity for the Monge-Ampère and affine maximal surface equations*, Ann. of Math. **167** (2008), 993–1028.

[52] E. Werner, *On L_p-affine surface areas*, Indiana Univ. Math. J. **56** (2007), 2305–2323.

[53] E. Werner and D. Ye, *New L_p affine isoperimetric inequalities*, Adv. Math. **218** (2008), 762–780.

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