Anisotropic Fractional Sobolev Norms

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Abstract

Bourgain, Brezis & Mironescu showed that (with suitable scaling) the fractional Sobolev s-seminorm of a function $f \in W^{1,p}(\mathbb{R}^n)$ converges to the Sobolev seminorm of f as $s \to 1^-$. The anisotropic s-seminorms of f defined by a norm on $\mathbb{R}^n$ with unit ball K are shown to converge to the anisotropic Sobolev seminorm of f defined by the norm with unit ball $\operatorname{Z}_p^* K$, the polar L_p moment body of K. The limiting behavior for $s \to 0^+$ is also determined (extending results by Maz′ya & Shaposhnikova).

For $p \geq 1$ and $0 < s < 1$, Gagliardo introduced the fractional Sobolev s-seminorm of a function $f \in L^p(\Omega)$ as

$$\|f\|^p_{W^{s,p}(\Omega)} = \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{n+ps}}\, dx\, dy, \tag{1}$$

where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^n$ and $\Omega \subset \mathbb{R}^n$. This seminorm turned out to be critical in the study of traces of Sobolev functions in the Sobolev space $W^{1,p}(\Omega)$ (cf. [11]). Fractional Sobolev norms have found numerous applications within mathematics and applied mathematics (cf. [3, 7, 27]).

The limiting behavior of fractional Sobolov s-seminorms as $s \to 1^-$ and $s \to 0^+$ turns out to be very interesting. Bourgain, Brezis & Mironescu [2] showed that

$$\lim_{s \to 1^-} (1-s) \|f\|^p_{W^{s,p}(\Omega)} = \alpha_{n,p}\, \|f\|^p_{W^{1,p}(\Omega)} \tag{2}$$

for $f \in W^{1,p}(\Omega)$ and $\Omega \subset \mathbb{R}^n$ a smooth and bounded domain, where $\alpha_{n,p}$ is a constant depending on n and p, and

$$\|f\|_{W^{1,p}(\Omega)} = \Big(\int_\Omega |\nabla f(x)|^p dx \Big)^{1/p} \tag{3}$$

is the Sobolev seminorm of f.

Maz′ya & Shaposhnikova [28] showed that if $f \in W^{s,p}(\mathbb{R}^n)$ for all $s \in (0,1)$, where $W^{s,p}(\mathbb{R}^n)$ are the functions in $L^p(\mathbb{R}^n)$ with finite Gagliardo seminorm (1) with $\Omega = \mathbb{R}^n$, then

$$\lim_{s \to 0+} s\, \|f\|^p_{W^{s,p}(\mathbb{R}^n)} = \frac{2\,n}{p}\, |B|\, |f|^p_p, \tag{4}$$

where $B \subset \mathbb{R}^n$ is n-dimensional Euclidean unit ball, $|B|$ its n-dimensional volume and $|f|_p$ the L^p norm of f on $\mathbb{R}^n$.

An anisotropic Sobolev seminorm is obtained by replacing the Euclidean norm $|\cdot|$ in (3) by an arbitrary norm $\|\cdot\|_L$ with unit ball L. We set

$$\|f\|_{W^{1,p},K} = \Big(\int_{\mathbb{R}^n} \|\nabla f(x)\|_{K^*}^p \, dx\Big)^{1/p},$$

where $K^* = \{v \in \mathbb{R}^n : v \cdot x \le 1 \text{ for all } x \in K\}$ is the polar body of K. Anisotropic Sobolev seminorms have attracted increased interest in recent years (cf. [1, 5, 9, 13]).

A natural question is to study the limiting behavior of anisotropic s-seminorms as $s \to 1^-$ and $s \to 0^+$. While one might suspect that the limit as $s \to 1^-$ of the anisotropic s-seminorms defined using a norm with unit ball K is the Sobolev seminorm with the same unit ball, this turns out not to be true in general.

Theorem 1. *If $f \in W^{1,p}(\mathbb{R}^n)$ has compact support, then*

$$\lim_{s\to 1^-} (1-s) \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{\|x-y\|_K^{n+sp}}\, dx\, dy = \int_{\mathbb{R}^n} \|\nabla f(x)\|_{\mathrm{Z}_p^* K}^p \, dx \tag{5}$$

where $\mathrm{Z}_p^* K$ is the polar L_p moment body of K.

For the Euclidean s-seminorms and the Euclidean unit ball B, the convex body $\mathrm{Z}_p^* B$ is just a multiple of B. Hence Theorem 1 recovers the result by Bourgain, Brezis & Mironescu (2) including the value of the constant $\alpha_{n,p}$. For a convex body $K \subset \mathbb{R}^n$, the polar L_p moment body is the unit ball of the norm defined by

$$\|v\|_{\mathrm{Z}_p^* K}^p = \frac{n+p}{2} \int_K |v \cdot x|^p \, dx$$

for $v \in \mathbb{R}^n$.

The polar body of $\mathrm{Z}_1^* K$, the convex body $\mathrm{Z}_1 K$, is the moment body of K. The convex body

$$\frac{2}{(n+1)|K|}\, \mathrm{Z}_1 K$$

is the centroid body of K, a classical concept that goes back at least to Dupin (cf. [12]). If we intersect the origin-symmetric convex body K by halfspaces orthogonal to $u \in S^{n-1}$, then the centroids of these intersections trace out the boundary of twice the centroid body of K, which explains the name centroid body. The name moment body comes from the fact that the corresponding moment vectors trace out the boundary (of a constant multiple) of $\mathrm{Z}_1 K$. Centroid bodies play an important role within the affine geometry of convex bodies (cf. [12, 20]) and moment bodies within the theory of valuations on convex bodies (see [14, 17, 18]).

The polar body of $\mathrm{Z}_p^* K$, the convex body $\mathrm{Z}_p K$, is the L_p moment body of K and

$$\frac{2}{(n+p)|K|}\,\mathrm{Z}_p K$$

is the L_p centroid body of K, a concept introduced by Lutwak & Zhang [26]. L_p centroid bodies and L_p moment bodies have found important applications within convex geometry, probability theory, and the local theory of Banach spaces (cf. [10, 15–17, 21–25, 29–32]).

For $p > 1$, it follows from Bourgain, Brezis & Mironescu [2, Theorem 2] that (5) also holds for $f \in L^p(\Omega)$ in the sense that if $f \notin W^{1,p}(\Omega)$, then both sides of (5) are infinite. For $p = 1$, it follows from [2, Theorem 3'] that a corresponding result holds for $f \notin BV(\mathbb{R}^n)$ (see also Dávila [6]). In [19], the limiting behavior of fractional anisotropic Sobolev seminorms on $BV(\mathbb{R}^n)$ is discussed using fractional anisotropic perimeters. Ponce [33] obtained several extensions of the results in [2], from which Theorem 1 can also be deduced if anisotropic s-seminorms are used. The proof given in this paper is independent of Ponce's results. It makes use of the one-dimensional case of the Bourgain, Brezis & Mironescu Theorem (2) and the Blaschke-Petkanschin Formula from integral geometry.

Corresponding to the result of Maz′ya & Shaposhnikova (4), we obtain the following result.

Theorem 2. *If $f \in W^{s,p}(\mathbb{R}^n)$ for all $s \in (0,1)$ and f has compact support, then*

$$\lim_{s\to 0^+} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{\|x-y\|_K^{n+sp}}\,dx\,dy = \frac{2\,n}{p}\,|K|\,|f|_p^p.$$

The proof of Theorem 2 is based on the one-dimensional case of (4) and the Blaschke-Petkantschin Formula.

1 Preliminaries

We state the Blaschke-Petkantschin Formula (cf. [34, Theorem 7.2.7]) in the case in which it will be used. Let H^k denote the k-dimensional Hausdorff measure on $\mathbb{R}^n$ and let $\mathrm{Aff}(n,1)$ denote the affine Grassmannian of lines in $\mathbb{R}^n$. If $g : \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$ is Lebesgue measurable, then

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} g(x,y)\,dH^n(x)\,dH^n(y) = \int_{\mathrm{Aff}(n,1)}\int_L\int_L g(x,y)\,|x-y|^{n-1}\,dH^1(x)\,dH^1(y)\,dL, \quad (6)$$

where dL denotes integration with respect to a suitably normalized rigid motion invariant Haar measure on $\mathrm{Aff}(n,1)$. This measure can be described in the following way.

Any line $L \in \mathrm{Aff}(n,1)$ can be parameterized using one of its direction unit vectors $u \in S^{n-1}$ and its base point $x \in u^\perp$, where $u^\perp$ is the hyperplane orthogonal to u, as $L = \{x + \lambda\, u : \lambda \in \mathbb{R}\}$. Hence, for $h : \mathrm{Aff}(n,1) \to [0,\infty)$ measurable,

$$\int_{\mathrm{Aff}(n,1)} h(L)\, dL = \frac{1}{2} \int_{S^{n-1}} \int_{u^\perp} h(x + L_u)\, dH^{n-1}(x)\, dH^{n-1}(u), \tag{7}$$

where $L_u = \{\lambda u : \lambda \in \mathbb{R}\}$.

For $f \in W^{1,p}(\mathbb{R}^n)$, we denote by $\bar f$ its precise representative (cf. [8, Section 1.7.1]). We require the following result. For every $u \in S^{n-1}$, the precise representative $\bar f$ is absolutely continuous on the lines $L = \{x + \lambda\, u : \lambda \in \mathbb{R}\}$ for H^{n-1}- a.e. $x \in u^\perp$ and its first-order (classical) partial derivatives belong to $L^p(\mathbb{R}^n)$ (cf. [8, Section 4.9.2, Theorem 2]). Hence we have for the restriction of $\bar f$ to L,

$$\bar f|_L \in W^{1,p}(L) \tag{8}$$

for a.e. line L parallel to u.

We require the following one-dimensional case of (2).

Proposition 1 ([2])**.** *If $g \in W^{1,p}(\mathbb{R})$ has compact support, then*

$$\lim_{s\to 1^-} (1-s) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|g(x)-g(y)|^p}{|x-y|^{1+ps}}\, dx\, dy = \frac{2}{p}\, \|g\|_{W^{1,p}(\mathbb{R})}^p.$$

We require the following one-dimensional case of (4).

Proposition 2 ([28])**.** *If $g \in W^{s,p}(\mathbb{R})$ for all $s \in (0,1)$, then*

$$\lim_{s\to 0^+} s \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|g(x)-g(y)|^p}{|x-y|^{1+ps}}\, dx\, dy = \frac{4}{p}\, |g|_p^p.$$

We also need the following result. The proof is based on the one-dimensional case of some estimates from [2]. Let $\mathrm{diam}(C) = \sup\{|x-y| : x \in C, y \in C\}$ denote the diameter of $C \subset \mathbb{R}$.

Lemma 1. *If $g \in W^{1,p}(\mathbb{R})$ has compact support C, then there exists a constant γ_p depending only on p such that*

$$(1-s) \int_{-\infty}^{\infty} \frac{|g(x)-g(y)|^p}{|x-y|^{1+ps}}\, dx\, dy \le \gamma_p\, \max(1, \mathrm{diam}(C))^p\, \|g\|_{W^{1,p}(\mathbb{R})}^p$$

for all $1/2 \le s < 1$.

Proof. If $g \in W^{1,p}(\mathbb{R})$ is smooth, then for $h \in \mathbb{R}$

$$g(x+h) - g(x) = h \int_0^1 g'(x+th)\, dt.$$

Hence for $h \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} |g(x+h) - g(x)|^p \, dx \leq |h|^p \, \|g\|^p_{W^{1,p}(\mathbb{R})}. \tag{9}$$

The same estimate is obtained for $g \in W^{1,p}(\mathbb{R})$ by approximation (cf. [4, Proposition 9.3]). Let the support of g be contained in $[-r, r]$, where $r \geq 1$. By (9) we get

$$\begin{aligned}
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|g(x) - g(y)|^p}{|x-y|^{1+ps}} \, dx \, dy &= \int_{-2r}^{-2r} \int_{-\infty}^{\infty} \frac{|g(x+h) - g(x)|^p}{|h|^{1+ps}} \, dx \, dh \\
&\leq \int_{-2r}^{2r} |h|^{-(1-p(1-s))} dh \, \|g\|^p_{W^{1,p}(\mathbb{R})} \\
&\leq \frac{2\,(2r)^{p(1-s)}}{p(1-s)} \, \|g\|^p_{W^{1,p}(\mathbb{R})}.
\end{aligned}$$

This completes the proof of the lemma. $\square$

The following estimate is used in the proof of Theorem 2.

Lemma 2. *If $g \in W^{s,p}(\mathbb{R})$ for all $s \in (0,1)$, then*

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|g(x) - g(y)|^p}{|x-y|^{1+ps}} \, dx \, dy \leq \frac{2^{p+1}}{ps} \, |g|_p^p + \|g\|^p_{W^{s',p}(\mathbb{R})}$$

for all $0 < s \leq s' < 1$.

Proof. Note that

$$\int_{\mathbb{R}} \int_{\{|x-y| \geq 1\}} \frac{|g(x)|^p}{|x-y|^{1+ps}} \, dx \, dy \leq \int_{\{|z| \geq 1\}} \frac{dz}{|z|^{1+ps}} \, |g|_p^p = \frac{2}{ps} \, |g|_p^p.$$

Hence, by Jensen's inequality,

$$\int\int_{\{|x-y| \geq 1\}} \frac{|g(x) - g(y)|^p}{|x-y|^{1+ps}} \, dx \, dy \leq 2^{p-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|g(x)|^p + |g(y)|^p}{|x-y|^{1+ps}} \, dx \, dy \leq \frac{2^{p+1}}{ps} \, |g|_p^p.$$

On the other hand,

$$\iint_{\{|x-y| < 1\}} \frac{|g(x) - g(y)|^p}{|x-y|^{1+sp}} \, dx \, dy \quad \leq \quad \iint_{\{|x-y| < 1\}} \frac{|g(x) - g(y)|^p}{|x-y|^{1+s'p}} \, dx \, dy$$

for $0 < s < s'$. $\square$

2 Proof of Theorem 1

By the Blaschke-Petkantschin Formula (6), we have

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{\|x-y\|_K^{n+ps}}\,dx\,dy = \int_{\mathrm{Aff}(n,1)} \|u(L)\|_K^{-(n+ps)} \int_L\int_L \frac{|f(x)-f(y)|^p}{|x-y|^{1+ps}}\,dH^1(x)\,dH^1(y)\,dL. \tag{10}$$

By Proposition 1 and (8), we have

$$\lim_{s\to 1^-}(1-s)\int_L\int_L \frac{|f(x)-f(y)|^p}{|x-y|^{1+ps}}dx\,dy = \frac{2}{p}\int_L |\nabla f(x)\cdot u|^p\,dH^1(x) \tag{11}$$

for a.e. line L parallel to $u \in S^{n-1}$.

By Fubini's Theorem, the definition of the measure on the affine Grassmannian (7) and the polar coordinate formula, we get

$$\begin{aligned}
&\frac{2}{p}\int_{\mathrm{Aff}(n,1)} \|u(L)\|_K^{-(n+p)} \int_L |\nabla f(x)\cdot u|^p\,dH^1(x)\,dL\\
&\quad= \frac{1}{p}\int_{S^{n-1}}\int_{u^\perp} \|u\|_K^{-(n+p)} \int_{y+L_u} |\nabla f(x)\cdot u|^p\,dH^1(x)\,dH^{n-1}(y)\,dH^{n-1}(u)\\
&\quad= \frac{1}{p}\int_{S^{n-1}}\int_{\mathbb{R}^n} \|u\|_K^{-(n+p)}\, |\nabla f(x)\cdot u|^p\,dH^n(x)\,dH^{n-1}(u)\\
&\quad= \frac{n+p}{p}\int_K\int_{\mathbb{R}^n} |\nabla f(x)\cdot y|^p\,dH^n(x)\,dH^n(y).
\end{aligned}$$

Using Fubini's Theorem and the definition of the L_p moment body of K, we obtain

$$\int_{\mathrm{Aff}(n,1)} \|u(L)\|_K^{-(n+ps)} \int_L |\nabla f(x)\cdot u|^p\,dH^1(x)\,dL = \int_{\mathbb{R}^n} \|\nabla f(x)\|_{\mathrm{Z}_p^* K}^p\,dx. \tag{12}$$

So, in particular, we have

$$\int_{\mathrm{Aff}(n,1)} \int_L |\nabla f(x)\cdot u|^p\,dH^1(x)\,dL = \alpha_{n,p}\int_{\mathbb{R}^n} |\nabla f(x)|^p\,dx < \infty, \tag{13}$$

where $\alpha_{n,p}$ is a constant.

Using the Dominated Convergence Theorem combined with Lemma 1 and (13), we obtain from (10), (11) and (12) that

$$\lim_{s\to 1^-}(1-s)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{\|x-y\|_K^{n+ps}}\,dx\,dy = \int_{\mathbb{R}^n} \|\nabla f(x)\|_{\mathrm{Z}_p^* K}^p\,dx.$$

This concludes the proof of the theorem.

3 Proof of Theorem 2

By the Blaschke-Petkantschin Formula (6), we obtain

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{\|x-y\|_K^{n+ps}}\,dx\,dy = \int_{\mathrm{Aff}(n,1)} \|u(L)\|_K^{-(n+ps)} \int_L\int_L \frac{|f(x)-f(y)|^p}{|x-y|^{s+1}}\,dx\,dy\,dL.$$

Thus we obtain by the Dominated Convergence Theorem, Lemma 2 and Proposition 2 that

$$\lim_{s\to 0^+} s \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{\|x-y\|_K^{n+ps}} dx\,dy = \frac{4}{p}\int_{\mathrm{Aff}(n,1)} \|u(L)\|_K^{-n} \int_L |f(x)|^p\,dH^1(x)\,dL.$$

By Fubini's Theorem, the definition of the measure on the affine Grassmannian (7) and the polar coordinate formula for volume, we get

$$\begin{aligned}
&\frac{4}{p}\int_{\mathrm{Aff}(n,1)} \|u(L)\|_K^{-n} \int_L |f(x)|^p\,dH^1(x)\,dL \\
&\quad= \frac{2}{p}\int_{S^{n-1}}\int_{u^\perp} \|u\|_K^{-n} \int_{y+L_u} |f(x)|^p\,dH^1(x)\,dH^{n-1}(y)\,dH^{n-1}(u) \\
&\quad= \frac{2}{p}\int_{S^{n-1}}\int_{\mathbb{R}^n} \|u\|_K^{-n}\,|f(x)|^p\,dH^n(x)\,dH^{n-1}(u) \\
&\quad= \frac{2n}{p}\,|K|\;|f|_p^p.
\end{aligned}$$

This concludes the proof of the theorem.

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