On the Geometric Classification of Functions

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Sobolev Inequality

$$\frac{1}{n}\int_{\mathbb{R}^n} |\nabla f(x)|\, dx \geqslant v_n^{1/n}\, |f|_{\frac{n}{n-1}}$$

- $f \in W^{1,1}(\mathbb{R}^n) = \{f \in L^1(\mathbb{R}^n) : |\nabla f| \in L^1(\mathbb{R}^n)\}$
- $|x|$ Euclidean norm of $x \in \mathbb{R}^n$
- $|f|_p = \left(\int_{\mathbb{R}^n} |f(x)|^p\, dx\right)^{1/p}$
- v_n volume of n-dimensional unit ball

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- v_n volume of n-dimensional unit ball
- Equality for indicator functions of balls
- Equivalent to Euclidean isoperimetric inequality
- Federer & Fleming 1960, Maz′ya 1960

General Sobolev Inequality

$$\frac{1}{n}\int_{\mathbb{R}^n} \|\nabla f(x)\|_{K^*}\, dx \geqslant v_n^{1/n}\, |f|_{\frac{n}{n-1}}$$

- $f \in W^{1,1}(\mathbb{R}^n)$
- $\|\cdot\|_L$ norm with unit ball L
- $K \subset \mathbb{R}^n$ origin-symmetric convex body with $V_n(K) = v_n$
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- Gromov 1986
- Cordero-Erausquin, Nazaret & Villani 2004

Optimal Sobolev Inequality

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Question (Lutwak, Yang & Zhang 2006)

For given $f \in W^{1,1}(\mathbb{R}^n)$, which convex body K of volume v_n minimizes

$$\frac{1}{n}\int_{\mathbb{R}^n} \|\nabla f(x)\|_{K^*}\,dx?$$

Which norm is optimal?

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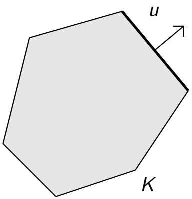
- Surface area measure $S(K,\cdot) : \mathcal{B}(\mathbb{S}^{n-1}) \to [0,\infty)$



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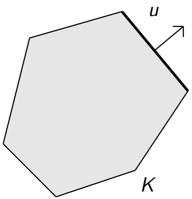

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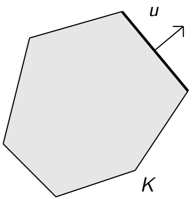

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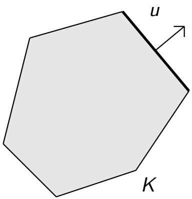

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- **Minkowski problem.** Find necessary and sufficient conditions on a finite Borel measure μ on $\mathbb{S}^{n-1}$ so that μ is the surface area measure of a convex body in $\mathbb{R}^n$.
  Solution. Minkowski, Alexandrov, Fenchel and Jessen;
  Lewy (1938), Nirenberg (1953), Cheng and Yau (1976), Pogorelov (1978), Caffarelli (1990), . . .

Optimal Sobolev Body

Definition (Lutwak, Yang & Zhang 2006)

For $f \in W^{1,1}(\mathbb{R}^n)$, the measure $\nu_f : \mathcal{B}(\mathbb{S}^{n-1}) \to [0, \infty)$ is defined as the unique even measure such that

$$\int_{\mathbb{S}^{n-1}} g(u)\, d\nu_f(u) = \int_{\mathbb{R}^n} g(\nabla f(x))\, dx$$

for all even and positively 1-homogeneous functions $g \in C(\mathbb{R}^n)$.

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- $f \mapsto \langle f \rangle$ LYZ operator

Optimal Sobolev Body

Definition (Functional Minkowski Problem; LYZ 2006)

For $f \in W^{1,1}(\mathbb{R}^n)$, the optimal Sobolev body, $\langle f \rangle$, of f is defined as the unique origin-symmetric convex body such that

$$\int_{\mathbb{S}^{n-1}} g(u)\, dS(\langle f \rangle, u) = \int_{\mathbb{R}^n} g(\nabla f(x))\, dx$$

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Theorem (LYZ 2006)

For $f \in W^{1,1}(\mathbb{R}^n)$, the infimum over all origin-symmetric convex bodies K of volume $V_n(K) = v_n$ over

$$\int_{\mathbb{R}^n} \|\nabla f(x)\|_{K^*}\, dx$$

is attained if and only if K is a dilate of $\langle f \rangle$.

LYZ Operator

- $\langle \cdot \rangle : \begin{cases} W^{1,1}(\mathbb{R}^n) & \to & \mathcal{K}_c^n \\ f & \mapsto & \langle f \rangle \end{cases}$
- $\langle \cdot \rangle$ is the solution of the functional Minkowski problem.
- $\langle \cdot \rangle$ associates with f its optimal Sobolev body.

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- $\langle\cdot\rangle : W^{1,1}(\mathbb{R}^n) \to \mathcal{K}_c^n$ is $\mathrm{SL}(n)$ covariant (LYZ 2006):

$$\langle f \circ A^{-1} \rangle = A\langle f \rangle \quad \text{for all } f \in W^{1,1}(\mathbb{R}^n),\ A \in \mathrm{SL}(n)$$

- $\langle\cdot\rangle : W^{1,1}(\mathbb{R}^n) \to \mathcal{K}_c^n$ is **affinely covariant** (LYZ 2006):
  - $\langle\cdot\rangle$ is $\mathrm{SL}(n)$ covariant
  - $\langle\cdot\rangle$ is translation invariant
  - $\langle\cdot\rangle$ is scaling covariant, that is, $\langle f \circ (s\,\mathrm{id})\rangle = |s|^p \langle f \rangle$ for some $p \in \mathbb{R}$
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- $\langle\cdot\rangle : W^{1,1}(\mathbb{R}^n) \to \mathcal{K}_c^n$ is continuous.

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- Group of rigid motions: $x \mapsto Ux + b$
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- Special linear group $\mathrm{SL}(n)$: $x \mapsto Ax$
  where A is an $n \times n$ matrix of determinant 1

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$$\mathrm{z}(K) + \mathrm{z}(L) = \mathrm{z}(K \cup L) + \mathrm{z}(K \cap L)$$

for all $K, L \in \mathcal{K}^n$ such that $K \cup L \in \mathcal{K}^n$.

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- Classification of valuations:

Blaschke 1937, **Hadwiger** 1949, Schneider 1971, Groemer 1972, McMullen 1977, Betke & Kneser 1985, Klain 1995, Ludwig 1999, Reitzner 1999, Alesker 1999, Fu 2006, Haberl 2006, Schuster 2006, Tsang 2010, Wannerer 2010, Abardia 2011, Bernig & Fu 2011, Parapatits 2011, Faifman 2014, ...

Hadwiger's Classification Theorem 1952

Theorem

A functional $\mathrm{z} : \mathcal{K}^n \to \langle \mathbb{R}, + \rangle$ *is a continuous, rigid motion invariant valuation*

$$\iff$$

$\exists \; c_0, c_1, \dots, c_n \in \mathbb{R}$ *such that*

$$\mathrm{z}(K) = c_0 \, V_0(K) + \cdots + c_n \, V_n(K)$$

for every $K \in \mathcal{K}^n$.

$V_0(K), \dots, V_n(K)$	intrinsic volumes of K
$V_n(K)$	n-dimensional volume
$2\,V_{n-1}(K) = S(K)$	surface area
$V_0(K) = 1$	Euler characteristic

Intrinsic Volumes

- K convex body with smooth boundary

$$V_i(K) = \frac{\binom{n}{i}}{n v_{n-i}} \int_{\mathbb{S}^{n-1}} s_i(K,u)\,du = \frac{\binom{n}{i}}{n v_{n-i}} \int_{\partial K} H_{n-i-1}(K,x)\,dx$$

- Steiner formula

$$V_n(K+\rho\,B) = \sum_{j=0}^{n} \rho^{n-j} v_{n-j} V_j(K)$$

- Crofton Formula

$$V_i(K) = \int_{Graff(n,i)} V_0(K\cap E)\,d\mu_i(E) = \int_{Gr(n,i)} V_i(K|E)\,d\nu_i(E)$$

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"Introduction to Geometric Probability" by Klain and Rota 1997

Abstract Hadwiger Theorem 2007

Theorem (Alesker: Annals 1999, GAFA 2007)

For a compact subgroup G of $\mathrm{SO}(n)$, the space of continuous, G invariant and translation invariant valuations on $\mathcal{K}^n$ is finite dimensional

$$\Longleftrightarrow$$

G acts transitively on $\mathbb{S}^{n-1}$.

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- $\mathrm{U}(n)$ invariance (Alesker: GAFA 2001, Fu: JDG 2006, Bernig & Fu: Annals 2011, Wannerer: JDG 2014)
- $\mathrm{SU}(n)$ invariance (Bernig: GAFA 2009)
- G_2, $\mathrm{Spin}(7)$, $\mathrm{Spin}(9)$ invariance (Bernig: Israel J. 2011)
- $\mathrm{Sp}(n)$, $\mathrm{Sp}(n)\cdot\mathrm{U}(1)$, $\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)$ invariance (Bernig & Solanes: JFA 2014)

Affine Classification Theorems

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- Upper semicontinuous valuations on $\mathcal{K}_0^n$
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- Tensor valued valuations
  - L. (DMJ 2003)
  - Haberl & Parapatits (2014+)

Brunn Minkowski Theory

Rolf Schneider (*Convex Bodies: The Brunn Minkowski Theory*, 1993; 2014)
"Merging two elementary notions for point sets in Euclidean space: vector addition and volume"

- Minkowski sum (or vector sum) of $K, L \in \mathcal{K}^n$

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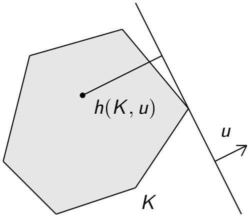
Convex Bodies

- Support function $h(K, \cdot) : \mathbb{R}^n \to \mathbb{R}$



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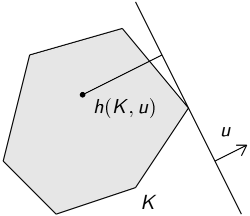

- $h(K,u) = \max\{u \cdot x : x \in K\}$
- $h(K,u+v) \leqslant h(K,u) + h(K,v)$
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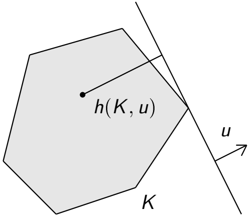

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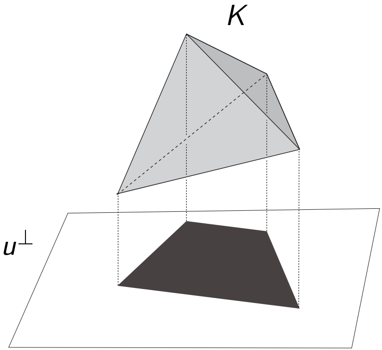
- Every sublinear function is the support function of a unique convex body.

Projection Body, ΠK, of K



- $u \in \mathbb{S}^{n-1}$
- $u^\perp$ hyperplane orthogonal to u
- $K|u^\perp$ projection of K to $u^\perp$

Definition (Minkowski 1901)

$$h(\Pi K, u) = V_{n-1}(K|u^\perp) = \frac{1}{2}\int_{\mathbb{S}^{n-1}} |u \cdot v| \, dS(K, v)$$

Classification of Minkowski Valuations

Theorem (Ludwig: AIM 2002)

$\mathrm{Z} : \mathcal{K}^n \to \langle \mathcal{K}^n, + \rangle$ *is a continuous,* $\mathrm{SL}(n)$ *contravariant and translation invariant Minkowski valuation*

$$\Longleftrightarrow$$

$\exists\, c \geqslant 0$*:*

$$\mathrm{Z}(K) = c\, \Pi\, K$$

for every $K \in \mathcal{K}^n$.

- $\mathrm{Z} : \mathcal{K}^n \to \mathcal{K}^n$ is $\mathrm{SL}(n)$ contravariant $\Longleftrightarrow$

$$\mathrm{Z}(AK) = A^{-t}\, \mathrm{Z}(K) \qquad \forall\, A \in \mathrm{SL}(n)$$

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- Ludwig (TAMS 2005, JDG 2010), Schuster (TAMS 2007), Schuster & Wannerer (TAMS 2009), Wannerer (IUMJ 2009), Haberl (JEMS 2012), Abardia (JFA 2012, IMRN 2014+), Abardia & Bernig (AIM 2011), Parapatits (TAMS 2014, JLMS 2014)

Valuations on Function Spaces

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- Valuations on convex functions, log-concave functions, ...
- L. (AIM 2011, AJM 2012), Andy Tsang (IMRN 2010, TAMS 2012), Tuo Wang (IUMJ 2014), Baryshnikov, Ghrist, Wright (AIM 2013), ...

Valuations on Sobolev Spaces

- $W^{1,1}(\mathbb{R}^n) = \{f \in L^1(\mathbb{R}^n) : |\nabla f| \in L^1(\mathbb{R}^n)\}$
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Theorem (Ludwig: AJM 2012)

An operator $\mathrm{z} : W^{1,1}(\mathbb{R}^n) \to \langle \mathcal{K}_c^n, + \rangle$ *is a continuous and affinely contravariant Minkowski valuation*

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$\exists\, c \geqslant 0$ *such that*

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for every $f \in W^{1,1}(\mathbb{R}^n)$.

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Sketch of the Proof

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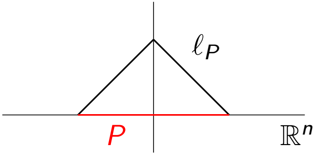
- $L^{1,1}(\mathbb{R}^n) \subset W^{1,1}(\mathbb{R}^n)$ piecewise linear continuous functions

- $P^{1,1}(\mathbb{R}^n) \subset L^{1,1}(\mathbb{R}^n)$ 'linear elements'



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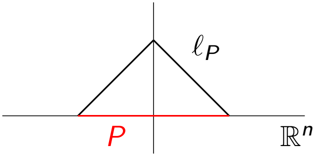

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- $\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{K}_c^n, + \rangle, \quad \mathrm{Z}(P) = \mathrm{z}(\ell_P)$
  GL(n) contravariant valuation on $\mathcal{P}_0^n$
- Ludwig: JDG 2010 $\Rightarrow$ $\mathrm{z}(\ell_P) = c\,\Pi\,\langle \ell_P \rangle$
- $\Rightarrow$ $\mathrm{z}(f) = c\,\Pi\,\langle f \rangle$

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Affine Sobolev inequality

Theorem (Gaoyong Zhang: JDG 1999)

For $f \in W^{1,1}(\mathbb{R}^n)$,

$$\frac{1}{n}\int_{\mathbb{S}^{n-1}}\Big(\int_{\mathbb{R}^n}|u\cdot\nabla f(x)|\,dx\Big)^{-n}du \leqslant \Big(\frac{v_n}{2\,v_{n-1}}\Big)^n |f|^{-n}_{\frac{n}{n-1}}.$$

- Affine isoperimetric inequality
- Hölder's inequality $\Rightarrow$

$$\Big(\frac{1}{nv_n}\int_{\mathbb{S}^{n-1}}\Big(\int_{\mathbb{R}^n}|u\cdot\nabla f(x)|\,dx\Big)^{-n}\Big)^{-\frac{1}{n}} \leqslant \frac{1}{nv_n}\int_{\mathbb{S}^{n-1}}\int_{\mathbb{R}^n}|u\cdot\nabla f(x)|\,dx\,du$$

$\Rightarrow$ Sobolev inequality

- Left hand side is multiple of $V_n(\Pi^*\langle f\rangle)$
- Extended to $\mathrm{BV}(\mathbb{R}^n)$ by Tuo Wang (AIM 2012)

Valuations on Sobolev Spaces

- $\langle \cdot \rangle : W^{1,1}(\mathbb{R}^n) \to \mathcal{K}_c^n$ is affinely covariant (LYZ 2006).

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A Characterization of the Fisher Information Matrix

Theorem (Ludwig: AIM 2011)

An operator $\mathrm{Z} : W^{1,2}(\mathbb{R}^n) \to \langle \mathbb{M}^n, + \rangle$ *is a continuous and affinely contravariant valuation*

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- Connection between Fisher information matrix and LYZ ellipsoid (Lutwak, Yang & Zhang: DMJ 2000)
- Characterization of matrix-valued valuations on convex bodies (Ludwig: DMJ 2003)

Thank you !!!