

UNIMODULAR VALUATIONS BEYOND EHRHART

ANSGAR FREYER, MONIKA LUDWIG, AND MARTIN RUBEY

ABSTRACT. A complete classification of unimodular valuations on the set of lattice polygons with values in the spaces of polynomials and formal power series, respectively, is established. The valuations are classified in terms of their behaviour with respect to dilation using extensions to unbounded polyhedra and basic invariant theory.

2020 AMS subject classification: 52B20, 52B45

1. INTRODUCTION

For $n \geq 2$, let $\mathcal{P}(\mathbb{Z}^n)$ denote the set of lattice polytopes in $\mathbb{R}^n$, that is, convex polytopes with vertices in the integer lattice $\mathbb{Z}^n$. A function Z on $\mathcal{P}(\mathbb{Z}^n)$ with values in a vector space V is a *valuation*, if

$$Z(P) + Z(Q) = Z(P \cup Q) + Z(P \cap Q)$$

for all $P, Q \in \mathcal{P}(\mathbb{Z}^n)$ such that both $P \cup Q$ and $P \cap Q$ are also in $\mathcal{P}(\mathbb{Z}^n)$. The lattice point enumerator $L^0: \mathcal{P}(\mathbb{Z}^n) \rightarrow \mathbb{R}$, which counts the number of lattice points in $P \in \mathcal{P}(\mathbb{Z}^n)$, and n -dimensional volume are important valuations on $\mathcal{P}(\mathbb{Z}^n)$.

A fundamental result on valuations on lattice polytopes is the Betke–Kneser Theorem [5]. It provides a complete classification of real-valued valuations on lattice polytopes in $\mathbb{R}^n$ that are invariant with respect to the *affine unimodular group*, obtained by the general linear group over the integers $\mathrm{GL}_n(\mathbb{Z})$ and translations by vectors from $\mathbb{Z}^n$. It also provides a characterization of the so-called Ehrhart coefficients [10]. We state the two-dimensional version.

Theorem 1 (Betke–Kneser). *A function $Z: \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{R}$ is a translation and $\mathrm{GL}_2(\mathbb{Z})$ invariant valuation if and only if there are $c_0, c_1, c_2 \in \mathbb{R}$ such that*

$$Z(P) = c_0 L_0^0(P) + c_1 L_1^0(P) + c_2 L_2^0(P)$$

for every $P \in \mathcal{P}(\mathbb{Z}^2)$.

In the planar case, the Ehrhart coefficients have a simple description: $L_0^0(P) := 1$ for a non-empty lattice polygon P , while $L_1^0(P)$ is half the number of lattice points in the boundary of P , and $L_2^0(P)$ is the area of P . We refer to [2, 3, 11, 12, 22] for general information on valuations on lattice polytopes and to [1, 4, 6, 13, 14, 19, 22] for some recent results.

We generalize the above theorem and establish a complete classification of polynomial-valued, unimodular valuations on $\mathcal{P}(\mathbb{Z}^2)$. Here, we call a valuation *unimodular* if it is translationally polynomial and $\mathrm{GL}_2(\mathbb{Z})$ equivariant, and we recall both definitions below.

Let $\mathbb{R}[x, y]$ denote the vector space of bivariate polynomials and consider the (left) action of $\mathrm{GL}_2(\mathbb{Z})$ on $\mathbb{R}[x, y]$ defined by $\phi f := f \circ \phi^*$ for $f \in \mathbb{R}[x, y]$ and $\phi \in \mathrm{GL}_2(\mathbb{Z})$, where ϕ^* is the transpose of ϕ . A function $Z: \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{R}[x, y]$ is $\mathrm{GL}_2(\mathbb{Z})$ *equivariant* if $Z(\phi P) = \phi Z(P)$ for every $\phi \in \mathrm{GL}_2(\mathbb{Z})$ and $P \in \mathcal{P}(\mathbb{Z}^2)$.

Since $\mathbb{R}[x, y]$ is the direct sum of the spaces of r -homogeneous polynomials $\mathbb{R}[x, y]_r$ for $r \in \mathbb{Z}_{\geq 0}$, it suffices to classify valuations with values in $\mathbb{R}[x, y]_r$. This corresponds to a classification of valuations from $\mathcal{P}(\mathbb{Z}^2)$ to the space of symmetric r -tensors. In this setting, the question was first considered in [19] (see also [22, Chapter 21]), where a complete classification was established for $r \leq 8$. We remark that $\mathrm{SL}_n(\mathbb{Z})$ equivariance was considered in [19], but there is a mistake in a calculation in the planar case. However, the results are correct for $\mathrm{GL}_2(\mathbb{Z})$ equivariance in the planar case and for $\mathrm{SL}_n(\mathbb{Z})$ equivariance for $n \geq 3$.

All tensor valuations that appear for $r \leq 8$ are tensor-valued Ehrhart coefficients (introduced in [7, 8] for $r \geq 1$) obtained through a homogeneous decomposition of the r th *discrete moment tensor*. This tensor corresponds to the r th discrete moment polynomial,

$$L^r(P) := \frac{1}{r!} \sum_{v \in P \cap \mathbb{Z}^2} v^r,$$

where $v^r(x, y) = (ax + by)^r$ for $v = (a, b)$, that is, we consider (x, y) as element of the dual of $\mathbb{R}^2$ and v as an element of its bidual. In particular, L^0 is the classical lattice point enumerator. Similar to Ehrhart's celebrated result, there is a homogeneous decomposition of L^r (see [7]): for $r \geq 1$ and $1 \leq i \leq r+2$, there exist i -homogeneous valuations $L_i^r: \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{R}[x, y]_r$ such that

$$(1) \quad L^r = \sum_{i=1}^{r+2} L_i^r.$$

Here, a function Z is called *i -homogeneous* if $Z(mP) = m^i Z(P)$ for every $m \in \mathbb{Z}_{\geq 0}$ and $P \in \mathcal{P}(\mathbb{Z}^2)$.

A valuation $Z: \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{R}[x, y]$ is *translatively polynomial* (of degree at most r) if there exist *associated valuations* $Z^{r-j}: \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{R}[x, y]$ for $0 \leq j \leq r$ such that

$$(2) \quad Z(P + v) = \sum_{j=0}^r Z^{r-j}(P) \frac{v^j}{j!}$$

for every $P \in \mathcal{P}(\mathbb{Z}^2)$ and $v \in \mathbb{Z}^2$. The associated valuations are uniquely determined, and by setting $v = 0$, we see that $Z^r = Z$. Khovanskii and Pukhlikov [15] showed that every translatively polynomial valuation $Z: \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{R}[x, y]$ of degree at most r can be written as

$$Z = \sum_{i=0}^{r+2} Z_i^r,$$

where Z_i^r are i -homogeneous valuations.

Let Val denote the vector space of translatively polynomial and $\mathrm{GL}_2(\mathbb{Z})$ equivariant valuations $Z: \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{R}[x, y]$. Let Val^r be the subspace of valuations $Z: \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{R}[x, y]_r$. We call r the *rank* of the valuation. It was proved in [19] that the space

$$\mathrm{Val}_i^r := \{Z \in \mathrm{Val}^r : Z \text{ is } i\text{-homogeneous}\}$$

is spanned by the Ehrhart coefficient L_i^r for $1 \leq r \leq 8$ and $1 \leq i \leq r+2$, and that it is zero-dimensional for $i = 0$ and $r \geq 1$, as well as for $i > r+2$. Moreover, it was proved that $\dim \text{Val}_1^9 \geq 2$.

Our first main result determines the dimensions of the vector spaces Val_i^r .

Theorem 2. *For $i, r \in \mathbb{Z}_{\geq 0}$,*

$$\dim \text{Val}_i^r = \begin{cases} 1 & \text{for } 1 \leq i < r \text{ and } r-i \text{ odd, and for } r \leq i \leq r+2, \\ p_{2,3}(r) & \text{for } i = 1 < r \text{ and } r-1 \text{ even,} \\ p_{2,3}\left(\frac{r-i}{2} + 1\right) & \text{for } 1 < i < r \text{ and } r-i \text{ even,} \\ 0 & \text{otherwise,} \end{cases}$$

where $p_{2,3}(r) := |\{(k, \ell) \in \mathbb{Z}_{\geq 0}^2 : 2k + 3\ell = r\}|$ is the number of integer partitions of r into parts 2 and 3.

The vector space Val_i^r is spanned by L_i^r if $\dim \text{Val}_i^r = 1$, and $L_i^r \neq 0$ (that is, L_i^r is not identically zero) for every $1 \leq i \leq r+2$. Since $p_{2,3}(r) = 1$ for $r \in \{3, 5, 7\}$, Theorem 2 gives a basis of Val_1^r for every even r and for $r \leq 7$.

The following result provides bases of the vector spaces Val_1^r also for the remaining r . Let $\mathbf{T}$ be the standard triangle with vertices at the origin, $e_1 := (1, 0)$, and $e_2 := (0, 1)$. Recall that every $P \in \mathcal{P}(\mathbb{Z}^2)$ has a (generally non-unique) unimodular triangulation $\mathcal{T}$; that is, P can be written as finite union of non-overlapping triangles $\phi\mathbf{T} + v \in \mathcal{T}$ with $\phi \in \text{GL}_2(\mathbb{Z})$ and $v \in \mathbb{Z}^2$.

Theorem 3. *For $r > 1$ odd, a basis of Val_1^r is given by $L_1^{2k, 3\ell} : \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{R}[x, y]_r$ with $k, \ell \in \mathbb{Z}_{\geq 0}$ and $2k + 3\ell = r$, defined for $P \in \mathcal{P}(\mathbb{Z}^2)$ and $\mathcal{T}$ any unimodular triangulation of P , by*

$$L_1^{2k, 3\ell}(P) := \sum_{\phi\mathbf{T}+v \in \mathcal{T}} (L_1^2(\mathbf{T})^k L_1^3(\mathbf{T})^\ell) \circ \phi^*$$

Here, $L_1^{2k, 3\ell}(P)$ does not depend on the triangulation $\mathcal{T}$ of P .

Note that it follows that the Ehrhart coefficient L_1^7 is a multiple of $L_1^{4, 3}$.

Our approach also yields a structure theorem for valuations with values in the space of bivariate formal power series $\mathbb{R}[[x, y]]$. A valuation $\bar{Z} : \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{R}[[x, y]]$ is called *translatively exponential* if

$$\bar{Z}(P + v) = e^v \bar{Z}(P)$$

for every $P \in \mathcal{P}(\mathbb{Z}^2)$ and $v \in \mathbb{Z}^2$, where we again consider v as a linear function on the dual of $\mathbb{R}^2$. Let $\overline{\text{Val}}$ denote the vector space of translatively exponential, $\text{GL}_2(\mathbb{Z})$ equivariant valuations $\bar{Z} : \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{R}[[x, y]]$. Note that the functions

$$(3) \quad \bar{L}(P) := \sum_{v \in P \cap \mathbb{Z}^2} e^v, \quad \bar{E}(P) := \int_P e^v dv$$

are both in $\overline{\text{Val}}$ (see [2, 3] for more information and applications).

For $d \in \mathbb{Z}$, we say that $\bar{Z} : \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{R}[[x, y]]$ is *d-dilative* if

$$\bar{Z}(mP)(x, y) = m^{-d} \bar{Z}(P)(mx, my)$$

for every $m \in \mathbb{Z}_{>0}$, $P \in \mathcal{P}(\mathbb{Z}^2)$, and $x, y \in \mathbb{R}$. For instance, the exponential integral $\bar{E}$ defined in (3) is (-2) -dilative. Set

$$\overline{\text{Val}}_d := \{\bar{Z} \in \overline{\text{Val}} : \bar{Z} \text{ is } d\text{-dilative}\}$$

for $d \in \mathbb{Z}$. Similar to the translatively polynomial case, dilative valuations decompose the space $\overline{\text{Val}}$, and we determine the dimensions of the vector spaces $\overline{\text{Val}}_d$.

Theorem 4. *For every $\overline{Z} \in \overline{\text{Val}}$, there are unique $\overline{Z}_d \in \overline{\text{Val}}_d$ with $d \in \mathbb{Z}$ such that $\overline{Z} = \sum_{d \geq -2} \overline{Z}_d$. Moreover,*

$$\dim \overline{\text{Val}}_d = \begin{cases} 1 & \text{for } -2 \leq d \leq 0 \text{ and for } d > 0 \text{ odd,} \\ p_{2,3} \left(\frac{d}{2} + 1 \right) & \text{for } d > 0 \text{ even,} \\ 0 & \text{otherwise.} \end{cases}$$

The r -summand (that is, the sum of the monomials of degree r) of a valuation in $\overline{\text{Val}}_d$ is in Val_i^r for $d = r - i$. We will see that all valuations in Val_i^r for $i > 1$ occur as $(r - i)$ -summand of a translatively exponential valuation in $\overline{\text{Val}}_{r-i}$. For $d \leq 0$ and for $d > 0$ odd, the space $\overline{\text{Val}}_d$ is spanned by the d -dilative part $\overline{L}_d$ of $\overline{L}$ (see Section 7). For $d > 0$ even, we will present in (25) an explicit representation for all valuations in $\overline{\text{Val}}_d$. Thus, we obtain a triangulation-free description of the valuations in Val_i^r for $i > 1$ and $r - i$ even as summands of valuations in $\overline{\text{Val}}_{r-i}$. In Section 10, we describe some consequences of our results.

2. OVERVIEW OF THE PROOFS

In the following sections, we simultaneously prove Theorem 2 and Theorem 4. When the vector spaces in question are zero- or one-dimensional, the results essentially follow from the arguments in [19]. We repeat these arguments in Section 8 to keep the proof as self-contained as possible. The majority of the work thus goes in the second and third cases of Theorem 2 and the second case of Theorem 4.

In the second and third cases of Theorem 2, the difference $d = r - i$ of rank and homogeneity degree is a positive even integer. If we fix a positive even integer d , it turns out that the vector spaces in question are connected by the diagram in Fig. 1. In this diagram, the horizontal maps assign to a valuation in Val_i^{d+i} its first associated valuation, that is, the linear coefficient of v in (2). In Section 3, we recapitulate how it follows from [19] that the horizontal maps are well-defined. The vertical maps in the diagram simply map a valuation $\overline{Z} \in \overline{\text{Val}}$ to its $(d + i)$ -summand Z^{d+i} , where $Z^{d+i}(P)$ is the $(d + i)$ -summand of the power series $\overline{Z}(P)$ for $P \in \mathcal{P}(\mathbb{Z}^2)$. The well-definedness of this map is the subject of Lemma 25.

The proofs of Theorem 2 and Theorem 4 for even $d > 0$ can be summarized as follows: In Section 4, we show that $\dim \text{Val}_1^{d+1} = p_{2,3}(d + 1)$ by relating the respective valuations to invariants of a finite group (see Theorem 15). In Section 5, we show that the horizontal maps in the diagram in Fig. 1 are injective (except for $\text{Val}_1^{d+1} \rightarrow 0$, see Lemma 18) and compute the dimension of the image of $\text{Val}_2^{d+2} \rightarrow \text{Val}_1^{d+1}$. From this, we will obtain that $\dim \text{Val}_2^{d+2} = p_{2,3}(\frac{d}{2} + 1)$ (see Proposition 21).

At this point, it remains to show that the dimensions of $\overline{\text{Val}}_d$ and Val_i^{d+i} for $i > 2$ are equal to $\dim \text{Val}_2^{d+2}$. Our strategy for this is as follows: First, in Section 6, we show that the diagram in Fig. 1 commutes. Then, in Section 7, we construct an inverse map to $\overline{\text{Val}}_d \rightarrow \text{Val}_2^{d+2}$, that is, for an arbitrary $Z_2^{d+2} \in \text{Val}_2^{d+2}$, we construct a valuation $\overline{Z} \in \overline{\text{Val}}_d$ such that the $(d + 2)$ -summand of $\overline{Z}(P)$ is $Z_2^{d+2}(P)$. The construction is based on an extension to rational, possibly unbounded polyhedra. Using Lawrence's application of the Brion identity [16], we construct a valuation

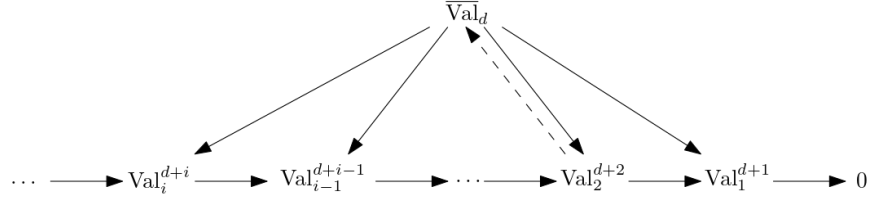


FIGURE 1. A commutative diagram connecting the spaces of valuations.

on rational polyhedra into the space of meromorphic functions whose restriction to lattice polygons takes analytic values.

In Section 9, we complete the proofs of Theorem 2 and Theorem 4. This section should be regarded as a summary of the previous sections. Finally, we derive two corollaries from our classification in Section 10.

3. PRELIMINARIES

We collect results and definitions that are used in our arguments.

3.1. Associated valuations. For $r \in \mathbb{Z}_{\geq 0}$, let $Z: \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{R}[x, y]_r$ be a translately polynomial valuation, that is, for any $P \in \mathcal{P}(\mathbb{Z}^2)$ and $v \in \mathbb{Z}^2$, we have

$$Z(P + v) = \sum_{j=0}^r Z^{r-j}(P) \frac{v^j}{j!}$$

with suitable functions $Z^{r-j}: \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{R}[x, y]_{r-j}$. By comparing the coefficients of v^j , we see that each Z^{r-j} is uniquely determined and a valuation. Moreover, setting $v = 0$, we obtain $Z^r = Z$. We refer to the valuations Z^{r-j} as the *associated valuations of Z* . It was shown in [19] that associated valuations are, in the following way, hereditary.

Lemma 5 (Proposition 6 in [19]). *If $Z: \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{R}[x, y]_r$ is a translately polynomial valuation with associated valuation $Z^{r-j}: \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{R}[x, y]_{r-j}$ for $0 \leq j \leq r$, then each Z^{r-j} is translately polynomial and its associated valuations are Z^{r-k} for $j \leq k \leq r$.*

It follows from a result of Khovanskii and Pukhlikov [15] that Z can be decomposed into *homogeneous components*.

Theorem 6 (Theorem 14 in [19]). *If $Z: \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{R}[x, y]_r$ is a translately polynomial valuation, then there exist i -homogeneous valuations $Z_i: \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{R}[x, y]_r$ for $0 \leq i \leq r+2$ such that $Z = \sum_{i=0}^{r+2} Z_i$.*

The next lemma follows from this decomposition and Propositions 6 and 7 in [19].

Lemma 7. *Let $Z: \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{R}[x, y]_r$ be a translately polynomial valuation. For $0 \leq i \leq r+2$, each homogeneous component Z_i is translately polynomial. The associated functions of Z_i are given by the homogeneous decomposition of the associated functions of Z , that is, $(Z_i)^{r-j} = (Z^{r-j})_{i-j}$ for $j \leq \min\{i, r\}$ and $(Z_i)^{r-j} = 0$ otherwise.*

We will use the abbreviation $Z_{i-j}^{r-j} := (Z^{r-j})_{i-j}$.

3.2. Invariant and equivariant valuations. If the valuation Z in Lemma 7 is $\mathrm{GL}_2(\mathbb{Z})$ invariant, then its associated valuations inherit the $\mathrm{GL}_2(\mathbb{Z})$ invariance. For a subgroup G of $\mathrm{GL}_2(\mathbb{R})$, let $\mathbb{R}[x, y]^G$ be the ring of G invariant polynomials. It is easy to see that

$$(4) \quad \mathbb{R}[x, y]^{\mathrm{GL}_2(\mathbb{Z})} \cong \mathbb{R}.$$

As observed in [19], this implies that the constant term in the homogeneous decomposition in Theorem 6 vanishes if Z is $\mathrm{GL}_2(\mathbb{Z})$ invariant.

Lemma 8. *For $r \geq 0$ and $Z \in \mathrm{Val}^r$, there exists $c \in \mathbb{R}$ such that $Z_0(P) = c$ for every $P \in \mathcal{P}(\mathbb{Z}^2)$. If $r > 0$, then this constant is zero.*

It was also observed in [19] that this implies the following result.

Lemma 9. *If $r \geq 2$ and $Z \in \mathrm{Val}_1^r$, then Z is translation invariant.*

A valuation $Z: \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{R}[x, y]_r$ is called *odd* if it is equivariant with respect to reflection at the origin, that is, we have $Z(-P) = -Z(P)$. Note that since the reflection at the origin is in $\mathrm{GL}_2(\mathbb{Z})$, we have $Z(-P) = (-1)^r Z(P)$ for any $Z \in \mathrm{Val}^r$. So $Z \in \mathrm{Val}^r$ is odd if and only if r is an odd number.

In the following, we will also deal with valuations taking values in the spaces of formal power series or meromorphic functions on $(\mathbb{R}^2)^\star$. For such valuations, the term $\mathrm{GL}_2(\mathbb{Z})$ equivariance will also refer to the action $\phi f = f \circ \phi^\star$, where $\phi \in \mathrm{GL}_2(\mathbb{Z})$ and f is a (formal) function.

3.3. Planar lattice triangulations. A *decomposition* of a two-dimensional lattice polygon P is a set of lattice polygons $\mathcal{T}$ such that the following properties hold:

- (1) $P = \bigcup_{T \in \mathcal{T}} T$.
- (2) For all $S, T \in \mathcal{T}$, the set $S \cap T$ is a (possibly empty) face of S and T .

If all $S \in \mathcal{T}$ are triangles, we call $\mathcal{T}$ a *triangulation*.

Recall that a triangle $S = \mathrm{conv}\{v_0, v_1, v_2\}$ with $v_i \in \mathbb{Z}^2$ is called *unimodular* if it can be expressed as $S = \phi \mathbf{T} + v$, where $\mathbf{T} = \mathrm{conv}\{0, e_1, e_2\}$ and $\phi \in \mathrm{GL}_2(\mathbb{Z})$, $v \in \mathbb{Z}^2$. Here, conv stands for convex hull. The triangle S is unimodular if and only if the edge vectors $v_i - v_0$ for $i \in \{1, 2\}$ generate $\mathbb{Z}^2$. Note that v_0 might be any of the three vertices of S in this condition. In fact, it follows that S is unimodular if and only if $\mathrm{vol}(S) = \frac{1}{2}$, where vol stands for (two-dimensional) volume. Note that one has $\mathrm{vol}(S) > \frac{1}{2}$ for a non-unimodular lattice triangle since its edge vectors generate a proper two-dimensional sublattice of $\mathbb{Z}^2$. A triangulation is called unimodular if all of its simplices are unimodular.

Moreover, a lattice triangle is unimodular if and only if it is *empty*; that is, it contains no lattice points other than the vertices. It follows that every lattice polygon $P \in \mathcal{P}(\mathbb{Z}^2)$ admits an unimodular triangulation.

For triangulations $\mathcal{T}$ and $\mathcal{T}'$ of a lattice polygon P one says that $\mathcal{T}$ and $\mathcal{T}'$ are *connected by a flip*, if $\mathcal{T} \setminus \mathcal{T}' = \{S_1, S_2\}$ and $\mathcal{T}' \setminus \mathcal{T} = \{S'_1, S'_2\}$ and, moreover, $S_1 \sqcup S_2 = S'_1 \sqcup S'_2$ is a convex quadrilateral. In this case, $\{S_1, S_2\}$ and $\{S'_1, S'_2\}$ correspond to the two possible triangulations of the quadrilateral.

Theorem 10 (Lawson [17]). *Let $P \in \mathcal{P}(\mathbb{Z}^2)$ and let $\mathcal{T}$ and $\mathcal{T}'$ be triangulations of P . Then there exists a finite sequence of triangulations $\mathcal{T} = \mathcal{T}_0, \mathcal{T}_1, \dots, \mathcal{T}_m = \mathcal{T}'$ such that $\mathcal{T}_{i-1}$ and $\mathcal{T}_i$ are connected by a flip for all $1 \leq i \leq m$.*

Lawson's theorem holds in more generality for triangulations of arbitrary (non-lattice) point sets.

A valuation Z on $\mathcal{P}(\mathbb{Z}^2)$ is called *simple* if it vanishes on points and segments. Using the valuation property inductively, it is easy to see that valuations on $\mathcal{P}(\mathbb{Z}^2)$ satisfy the inclusion-exclusion principle. In particular if Z is a simple valuation and $\mathcal{T}$ is a triangulation of a polygon $P \in \mathcal{P}(\mathbb{Z}^2)$, we have

$$(5) \quad Z(P) = \sum_{S \in \mathcal{T}} Z(S).$$

The inclusion-exclusion principle also holds for valuations on lattice polytopes in higher dimensions, but its proof is no longer trivial (see [21]).

The following corollary of Theorem 10 helps to construct a valuation “from scratch”. Let ρ_i denote the reflection on the i th coordinate axis and $\mathcal{S}(\mathbb{Z}^2)$ the set of unimodular lattice triangles.

Corollary 11. *If $Z' : \mathcal{S}(\mathbb{Z}^2) \rightarrow \mathbb{R}[x, y]$ is $\mathrm{GL}_2(\mathbb{Z})$ equivariant and such that*

$$(6) \quad Z'(v + \mathbf{T}) + Z'(v + e_1 + e_2 - \mathbf{T}) = Z'(v + e_1 + \rho_2 \mathbf{T}) + Z'(v + e_2 + \rho_1 \mathbf{T})$$

for every $v \in \mathbb{Z}^2$, then there is a unique simple, $\mathrm{GL}_2(\mathbb{Z})$ equivariant valuation $Z : \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{R}[x, y]$ with $Z(T) = Z'(T)$ for every $T \in \mathcal{S}(\mathbb{Z}^2)$. Conversely, if $Z : \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{R}[x, y]$ is a simple, $\mathrm{GL}_2(\mathbb{Z})$ equivariant valuation, then its restriction Z' to $\mathcal{S}(\mathbb{Z}^2)$ satisfies (6).

Proof. Note that

$$(7) \quad \mathbf{T} \sqcup (e_1 + e_2 - \mathbf{T}) = (e_1 + \rho_2 \mathbf{T}) \sqcup (e_2 + \rho_1 \mathbf{T}) = [0, 1]^2.$$

Let Z' be as in the statement of the corollary. For $P \in \mathcal{P}(\mathbb{Z}^2)$ two-dimensional, we define $Z(P; \mathcal{T}) := \sum_{T \in \mathcal{T}} Z'(T)$, where $\mathcal{T}$ is a unimodular triangulation of P . We claim that $Z(P; \mathcal{T})$ is independent of $\mathcal{T}$. By Theorem 10, it is enough to consider a triangulation $\mathcal{T}'$, which is connected to $\mathcal{T}$ by a flip. Let Q be the convex quadrilateral in which $\mathcal{T}$ and $\mathcal{T}'$ differ. It is easy to see that $Q = v + \phi[0, 1]^2$ for some $\phi \in \mathrm{GL}_2(\mathbb{Z})$ and $v \in \mathbb{Z}^2$. Since there are exactly two lattice triangulations of $[0, 1]^2$, the ones given in (7), it follows that, say, $\mathcal{T}$ contains $v + \phi(e_1 + e_2 - \mathbf{T})$ and $v + \phi \mathbf{T}$. Consequently, $\mathcal{T}'$ contains $v + \phi(e_1 + \rho_2 \mathbf{T})$ and $v + \phi(e_2 + \rho_1 \mathbf{T})$. Since Z' is $\mathrm{GL}_2(\mathbb{Z})$ equivariant, we obtain $Z'(v + \phi T) = Z'(\phi^{-1}v + T)$ for any unimodular triangle T . Thus, $Z(P; \mathcal{T}) = Z(P; \mathcal{T}')$ follows from (6), and we can define $Z(P) := \sum_{T \in \mathcal{T}} Z'(T)$ for $P \in \mathcal{P}(\mathbb{Z}^2)$ two-dimensional. Setting $Z(P) := 0$ for $\dim P < 2$, we obtain a simple, $\mathrm{GL}_2(\mathbb{Z})$ equivariant valuation with $Z(T) = Z'(T)$ for every $T \in \mathcal{S}(\mathbb{Z}^2)$. The uniqueness of Z follows from (5). The converse statement is immediate since both sides of (6) are equal to $Z(v + [0, 1]^2)$. $\square$

The following lemma will be used frequently in the following sections.

Lemma 12. *Let $Z \in \mathrm{Val}^r$ be simple with associated valuations Z^{r-j} for $0 \leq j \leq r$. If $P \in \mathcal{P}(\mathbb{Z}^2)$ is two-dimensional with unimodular triangulation $\mathcal{T} = \{\phi_i \mathbf{T} + v_i : \phi_i \in \mathrm{GL}_2(\mathbb{Z}), v_i \in \mathbb{Z}^2, 1 \leq i \leq m\}$, then*

$$(8) \quad Z(P) = \sum_{i=1}^m \sum_{j=0}^r (Z^{r-j}(\mathbf{T}) \circ \phi_i^*) \frac{v_i^j}{j!}.$$

In particular, Z is uniquely determined by the values $Z^{r-j}(\mathbf{T})$ for $1 \leq j \leq r$.

Proof. By using the equivariance properties of Z to (5), we immediately obtain (8). For the last statement, we recall that every two-dimensional lattice polygon has a unimodular triangulation, so $Z(P)$ can always be evaluated via (8). $\square$

3.4. Integer partitions. In Theorem 2, we defined

$$(9) \quad p_{2,3}(r) := |\{(k, \ell) \in \mathbb{Z}_{\geq 0}^2 : 2k + 3\ell = r\}|,$$

the number of integer partitions of r into parts 2 and 3. In explicit terms, one has

$$(10) \quad p_{2,3}(r) = \left\lfloor \frac{r+2}{2} \right\rfloor - \left\lfloor \frac{r+2}{3} \right\rfloor.$$

The sequence is A103221 in the Online Encyclopedia of Integer Sequences.

4. ONE-HOMOGENEOUS VALUATIONS OF ODD RANK

In this section, we prove Theorem 2 for $i = 1$ and $r > 1$ odd. We require the following result.

Lemma 13. *Let $r - i > 0$ be even and $i \geq 1$. If $Z \in \text{Val}_i^r$, then Z is simple.*

Proof. First, let $i = 1$. It follows from the assumption that r is odd and from Lemma 9, that Z is translation invariant. Let $Q \in \mathcal{P}(\mathbb{Z}^2)$ be a segment or a point. In either case, a vector $v \in \mathbb{Z}^2$ exists, such as $-Q = Q + v$. Since r is odd and Z is $\text{GL}_2(\mathbb{Z})$ equivariant, the valuation Z is also odd. Hence, by translation invariance,

$$-Z(Q) = Z(-Q) = Z(Q + v) = Z(Q)$$

and, thus, $Z(Q) = 0$.

Now, let $i > 1$. By induction and Lemma 8, the associated functions Z_{i-j}^{r-j} of Z are simple for $0 < j < i$. This implies $Z(Q + v) = Z(Q)$ for any $v \in \mathbb{Z}^2$ and any lower-dimensional $Q \in \mathcal{P}(\mathbb{Z}^2)$.

It follows from (4) that $Z(\{0\}) = 0$. For any $p \in \mathbb{Z}^2$ with $p \neq 0$, the translation invariance on lower-dimensional polygons implies that $Z(\{p\}) = Z(p + \{0\}) = 0$. So, let $S \in \mathcal{P}(\mathbb{Z}^2)$ be a segment. For $m \geq 1$, we can decompose the dilate mS into m translates S_i with $1 \leq i \leq m$ of S that intersect at lattice points. We just saw that our valuation vanishes on points. Hence, we obtain from the homogeneity of Z that

$$m^i Z(S) = Z(mS) = \sum_{i=1}^m Z(S_i) = m Z(S).$$

Since $i > 1$, we conclude that $Z(S) = 0$. □

Let $Z \in \text{Val}_1^r$ with $r > 1$ odd. As we saw earlier, Z is translation invariant by Lemma 9, simple by Lemma 13, and odd. The converse statement is also true.

Lemma 14. *If r is odd and $Z: \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{R}[x, y]_r$ is a simple, translation invariant, $\text{GL}_2(\mathbb{Z})$ equivariant valuation, then Z is one-homogeneous, that is, $Z \in \text{Val}_1^r$.*

Proof. Since the valuation Z is translation invariant, all its associated valuations $Z^{r-j}: \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{R}[x, y]_{r-j}$ vanish except for $Z^r = Z$. Thus, Lemma 12 implies that it is enough to check $Z(m\mathbf{T}) = m Z(\mathbf{T})$ for $m \in \mathbb{Z}_{>1}$.

We observe that

$$m\mathbf{T} = \left(\bigsqcup_{v \in (m-2)\mathbf{T} \cap \mathbb{Z}^2} (v + [0, 1]^2) \right) \sqcup \left(\bigsqcup_{a, b \in \mathbb{Z}_{\geq 0}, a+b=m-1} (ae_1 + be_2 + \mathbf{T}) \right).$$

Since Z is odd and translation invariant and $[0, 1]^2$ is centrally symmetric, it follows that $Z(v + [0, 1]^2) = 0$ for all $v \in \mathbb{Z}^2$. Thus, by (5),

$$Z(m\mathbf{T}) = \sum_{a, b \in \mathbb{Z}_{\geq 0}, a+b=m-1} Z(ae_1 + be_2 + \mathbf{T}) = m Z(\mathbf{T}),$$

where we used again that Z is simple and translation invariant. $\square$

Let $\mathbf{G}$ denote the subgroup of unimodular transformations $\phi \in \mathrm{GL}_2(\mathbb{Z})$ for which there exists $v \in \mathbb{Z}^2$ such that $\mathbf{T} = \phi\mathbf{T} + v$.

Theorem 15. *Let $r > 1$ be odd. The evaluation map*

$$\Theta^r : \mathrm{Val}_1^r \rightarrow \mathbb{R}[x, y]_r^{\mathbf{G}}, \quad \Theta^r(Z) := Z(\mathbf{T})$$

is a well-defined linear isomorphism. The valuation $Z_f := (\Theta^r)^{-1}(f)$, obtained by the inverse map, is simple for $f \in \mathbb{R}[x, y]_r^{\mathbf{G}}$ and given by

$$(11) \quad Z_f(P) = \sum_{i=1}^m f \circ \phi_i^*.$$

for any two-dimensional $P \in \mathcal{P}(\mathbb{Z}^2)$ with unimodular triangulation $\mathcal{T} = \{\phi_i\mathbf{T} + v_i : \phi_i \in \mathrm{GL}_2(\mathbb{Z}), v_i \in \mathbb{Z}^2, 1 \leq i \leq m\}$.

Proof. First, we show that $\Theta^r(Z) \in \mathbb{R}[x, y]_r^{\mathbf{G}}$ for $Z \in \mathrm{Val}_1^r$. If $\phi \in \mathbf{G}$, then there exists $v \in \mathbb{Z}^2$ such that $\mathbf{T} = \phi\mathbf{T} + v$. Since Z is translation invariant by Lemma 9 and $\mathrm{GL}_2(\mathbb{Z})$ equivariant, it follows that $Z(\mathbf{T}) = Z(\mathbf{T}) \circ \phi^*$ for $\phi \in \mathbf{G}$, which shows that $Z(\mathbf{T}) \in \mathbb{R}[x, y]_r^{\mathbf{G}}$ as required.

The linearity of Θ^r is obvious. The injectivity follows directly from Lemma 12. To show that Θ^r is surjective, we show that for any $f \in \mathbb{R}[x, y]_r^{\mathbf{G}}$ the expression (11) can serve as a definition of a simple valuation $Z_f \in \mathrm{Val}_1^r$, which is then necessarily $(\Theta^r)^{-1}(f)$.

To this end, consider a unimodular triangle $T \in \mathcal{P}(\mathbb{Z}^2)$ represented in two ways as an affine image of the standard triangle $\mathbf{T}$, that is, $T = \phi\mathbf{T} + v = \phi'\mathbf{T} + v'$ for certain $\phi, \phi' \in \mathrm{GL}_2(\mathbb{Z})$ and $v, v' \in \mathbb{Z}^2$. Rearranging yields

$$\mathbf{T} = \phi^{-1}\phi'\mathbf{T} + \phi^{-1}(v' - v).$$

Hence, $\phi^{-1}\phi' \in \mathbf{G}$. Since f is $\mathbf{G}$ invariant, it follows that $f \circ (\phi')^* = f \circ \phi^*$, that is, $Z'(T) := f \circ \phi^*$ is a well-defined translation invariant, $\mathrm{GL}_2(\mathbb{Z})$ equivariant function on the set of unimodular triangles.

Next, we note that the triangles $\rho_1\mathbf{T}$ and $\rho_2\mathbf{T}$ occurring in (6) are opposite in the sense that $\rho_1\mathbf{T} = -\rho_2\mathbf{T}$. Thus, since the degree of f is odd, we have $Z'(-\mathbf{T}) = -Z'(\mathbf{T})$ and $Z'(\rho_1\mathbf{T}) = -Z'(\rho_2\mathbf{T})$. Since Z' is translation invariant, (6) is satisfied. Hence, Corollary 11 yields a simple $\mathrm{GL}_2(\mathbb{Z})$ equivariant valuation $Z : \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{R}[x, y]_r$ with $Z(T) = Z'(T)$ for all unimodular triangles T . Since Z' is translation invariant, it follows from (5) that Z is translation invariant as well. In particular, $Z \in \mathrm{Val}^r$ and Lemma 12 shows that $Z(P) = Z_f(P)$ as given by (11).

Finally, Lemma 14 yields that Z_f is 1-homogeneous, that is, we have $Z_f \in \mathrm{Val}_1^r$ as desired. $\square$

For $r = 1$, the valuations in Val_1^1 are not necessarily translation invariant. Thus, $Z(\mathbf{T})$ is not $\mathbf{G}$ invariant in this case. For $r > 1$ even, the valuations in Val_1^r are translation invariant by Lemma 9, and so the evaluation map Θ^r is a well-defined linear map to $\mathbb{R}[x, y]_r^{\mathbf{G}}$. However, it is, in general, not an isomorphism.

To compute the dimension of Val_1^r for $r > 1$ odd, it is now enough to compute the Molien series of the invariant ring $\mathbb{R}[x, y]^{\mathbf{G}}$. To this end, we observe that $\mathbf{G}$ has order 6. Any unimodular transformation $\phi \in \text{GL}_2(\mathbb{Z})$ maps $\mathbf{T}$ to a triangle with a vertex at the origin. So if $\phi\mathbf{T} = \mathbf{T} + v$, we have $v \in \{0, -e_1, -e_2\}$. The two non-zero vertices of $\mathbf{T}$ must then be mapped by ϕ to the non-zero vertices of $\mathbf{T} + v$. Hence, we have

$$\mathbf{G} = \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix} \right\rangle,$$

since the group on the right-hand side has order 6 and is contained in $\mathbf{G}$. From this, we see that $\mathbf{G}$ is a finite group generated by *pseudo-reflections*, that is, linear transformations that fix a space of codimension 1. The Chevalley–Shephard–Todd Theorem (see, for example, [23, Theorem 7.1.4]) now implies that the ring of $\mathbf{G}$ invariant polynomials, $\mathbb{R}[x, y]^{\mathbf{G}}$, is finitely generated and that its generators are algebraically independent. Molien’s formula (for example, [23, Theorem 3.1.3]) yields that the dimension of $\mathbb{R}[x, y]_r^{\mathbf{G}}$ is $p_{2,3}(r)$ (defined in (9)). In particular, $\mathbb{R}[x, y]^{\mathbf{G}}$ is generated by suitable polynomials of degree 2 and 3.

Lemma 16. *The algebraically independent polynomials $\mathbf{p}_2(x, y) := x^2 - xy + y^2$ and $\mathbf{p}_3(x, y) := x^3 - \frac{3}{2}(x^2y + xy^2) + y^3$ generate the invariant ring $\mathbb{R}[x, y]^{\mathbf{G}}$.*

Proof. It is enough to apply the Reynolds operator $\frac{1}{|\mathbf{G}|} \sum_{\phi \in \mathbf{G}} \phi$, which is a projection onto $\mathbb{R}[x, y]^{\mathbf{G}}$, to the polynomials x^2 and x^3 and thus compute the generators $\mathbf{p}_2$ and $\mathbf{p}_3$. $\square$

Combining Lemma 16 with Theorem 15 gives the second case of Theorem 2.

Corollary 17. *For $r > 1$ odd, we have $\dim \text{Val}_1^r = p_{2,3}(r)$.*

Note that $r = 9$ is the first odd number such that $p_{2,3}(r) > 1$. Since $L^9(m\mathbf{T})$ is a polynomial of degree 11 in m with coefficients $L_i^9(\mathbf{T})$, we can compute $L_1^9(\mathbf{T})$ via interpolation. Using **SageMath** [25], we obtain $L_1^9 = Z_f$ (defined in (11)), where

$$f = \frac{1}{990 \cdot 9!} (4\mathbf{p}_3^3 - 79\mathbf{p}_2^3\mathbf{p}_3).$$

The valuation constructed in [19, Section 8.2] is a unimodular valuation that is not a linear combination of Ehrhart tensor coefficients. It corresponds (up to a constant factor) to Z_g with $g = \mathbf{p}_3^3$.

5. TWO-HOMOGENEOUS VALUATIONS OF EVEN RANK

Throughout the section, we assume that $r > 1$ is odd and consider a valuation $Z_1^r \in \text{Val}_1^r$. When does there exist $Z_2^{r+1} \in \text{Val}^{r+1}$ such that

$$(12) \quad Z_2^{r+1}(P + v) = Z_2^{r+1}(P) + Z_1^r(P)v$$

for every $v \in \mathbb{Z}^2$ and $P \in \mathcal{P}(\mathbb{Z}^2)$?

To answer this question, we first prove the following lemma on associated valuations, which we will also use in the classification for higher homogeneity degrees.

Lemma 18. *Let $Y, Z \in \text{Val}_i^r$ with $r > i > 1$ and $r - i$ even. If the associated valuations satisfy $Y^{r-j} = Z^{r-j}$ for all $0 < j < i$, then $Y = Z$.*

Proof of Lemma 18. Let $X := Z - Y$. By assumption, X is translation invariant, as well as $\text{GL}_2(\mathbb{Z})$ equivariant and i -homogeneous, and by Lemma 13, it is simple. If $i = 2$, then [19, Proposition 23] yields $X = 0$ and thus the claim. If $i > 2$, we have $X = 0$ by [20, Theorem 5]. $\square$

By Lemma 18, there is at most one solution Z_2^{r+1} of (12). We will now determine for which Z_1^r a solution Z_2^{r+1} exists. We write $\mathbf{D}$ for the group generated by the reflections at the coordinate axes and permutation of the coordinates, that is,

$$\mathbf{D} = \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle.$$

Note that $\mathbf{D}$ is a finite subgroup of $\text{GL}_2(\mathbb{Z})$.

Proposition 19. *Let $r > 1$ be odd and $Z_1^r \in \text{Val}_1^r$. The following are equivalent:*

- (1) *There exists $Z_2^{r+1} \in \text{Val}_2^{r+1}$ such that (12) holds for every $P \in \mathcal{P}(\mathbb{Z}^2)$ and $v \in \mathbb{Z}^2$.*
- (2) *There exists $h \in \mathbb{R}[x, y]_{r+1}^{\mathbf{G}}$ such that $h + (x + y) Z_1^r(\mathbf{T}) \in \mathbb{R}[x, y]_{r+1}^{\mathbf{D}}$.*

Moreover, the polynomial h is uniquely determined by $Z_1^r(\mathbf{T})$ if the conditions (1) and (2) are satisfied.

Proof. First, we observe that condition (2) is by linearity equivalent to

$$(2') \text{ There exists } h \in \mathbb{R}[x, y]_{r+1}^{\mathbf{G}} \text{ such that } 2h - \left(\frac{x}{3} + \frac{y}{3}\right) Z_1^r(\mathbf{T}) \in \mathbb{R}[x, y]_{r+1}^{\mathbf{D}}.$$

In the proof, we will work with (2') rather than (2).

We start by showing that (1) implies (2'). By Lemma 13, the valuation Z_2^{r+1} is simple. So, it is determined by

$$f_1 := Z_1^r(\mathbf{T}) \text{ and } f_2 := Z_2^{r+1}(\mathbf{T})$$

by Lemma 12. Consider a unimodular transformation $\phi \in \mathbf{G}$ such that $\phi\mathbf{T} + v = \mathbf{T}$ for some $v \in \mathbb{Z}^2$. Then, by (12) and since $f_1 \in \mathbb{R}[x, y]_r^{\mathbf{G}}$,

$$(13) \quad f_2 = Z_2^{r+1}(\phi\mathbf{T} + v) = f_2 \circ \phi^* + f_1 v.$$

Suppose $\tilde{f}_2$ is another solution of (13). Then $f_2 - \tilde{f}_2 \in \mathbb{R}[x, y]_{r+1}^{\mathbf{G}}$. Next, we make the guess that $(\frac{x}{3} + \frac{y}{3})f_1$ is a particular solution to (13). To see that the guess is correct, note that $\text{cen}(\mathbf{T}) := \frac{x}{3} + \frac{y}{3}$ corresponds to the centroid of the standard triangle.

$$(\text{cen}(\mathbf{T})f_1) \circ \phi^* + v f_1 = \text{cen}(\phi\mathbf{T})(f_1 \circ \phi^*) + v f_1 = \text{cen}(\phi\mathbf{T} + v)f_1 = \text{cen}(\mathbf{T})f_1.$$

So $(\frac{x}{3} + \frac{y}{3})f_1$ is indeed a solution to (13) and $f_2 = h + (\frac{x}{3} + \frac{y}{3})f_1$ for a suitable $h \in \mathbb{R}[x, y]_{r+1}^{\mathbf{G}}$.

Now consider the square $[0, 1]^2$. This is a centrally symmetric polygon, so we have $Z_1^r([0, 1]^2) = 0$ because Z_1^r is odd and translation invariant. From (12), it follows that $Z_2^{r+1}([0, 1]^2 + v) = Z_2^{r+1}([0, 1]^2)$ for any $v \in \mathbb{Z}^2$, which implies that

$$(14) \quad Z_2^{r+1}([0, 1]^2) \in \mathbb{R}[x, y]_{r+1}^{\mathbf{D}},$$

since $\phi[0, 1]^2$ is a translate of $[0, 1]^2$ for $\phi \in \mathbf{D}$. Using the triangulation $[0, 1]^2 = \mathbf{T} \sqcup (-\mathbf{T} + e_1 + e_2)$, we obtain

$$\begin{aligned} Z_2^{r+1}([0, 1]^2) &= Z_2^{r+1}(\mathbf{T}) + Z_2^{r+1}(-\mathbf{T} + e_1 + e_2) \\ &= f_2(x, y) + f_2(-x, -y) + (x + y)f_1(-x, -y) \\ &= 2f_2(x, y) - (x + y)f_1(x, y) \\ &= 2h - \left(\frac{x}{3} + \frac{y}{3}\right)f_1. \end{aligned}$$

Combined with (14), this proves the implication.

Second, we show that (2') implies (1). We define $f_2 := (\frac{x}{3} + \frac{y}{3})f_1 + h$. For a unimodular triangle $T = \phi\mathbf{T} + v$ with $\phi \in \mathrm{GL}_2(\mathbb{Z})$ and $v \in \mathbb{Z}^2$, we set

$$Z'(T) := f_2 \circ \phi^* + v \cdot f_1 \circ \phi^*.$$

We claim that this is independent of the representation of T as an affine image of $\mathbf{T}$. As in the proof of Theorem 15, consider $\phi\mathbf{T} + v = \psi\mathbf{T} + w$ for suitable $\phi, \psi \in \mathrm{GL}_2(\mathbb{Z})$ and $v, w \in \mathbb{Z}^2$. Then, $\mathbf{T} = \phi^{-1}\psi\mathbf{T} + \phi^{-1}(w - v)$. This means $\phi^{-1}\psi \in \mathbf{G}$. Set $f_0 := (\frac{x}{3} + \frac{y}{3})f_1$. We already saw in the previous step that f_0 satisfies

$$f_0 = f_0 \circ (\phi^{-1}\psi)^* + \phi^{-1}(w - v) \cdot f_1 = f_0 \circ \psi^* \circ \phi^{-*} + \phi^{-1}(w - v) \cdot f_1.$$

Adding h to the equation, applying ϕ and rearranging gives

$$\begin{aligned} f_2 \circ \phi^* + v \cdot f_1 \circ \phi^* &= h \circ \phi^* + f_0 \circ \phi^* + v \cdot f_1 \circ \phi^* \\ &= h \circ \phi^* + f_0 \circ \psi^* + w \cdot (f_1 \circ \phi^*) \\ &= h \circ (\phi^{-1}\psi) \circ \phi^* + f_0 \circ \psi^* + w \cdot (f_1 \circ (\phi^{-1}\psi) \circ \phi^*) \\ &= h \circ \psi^* + f_0 \circ \psi^* + w \cdot (f_1 \circ \psi^*) \\ &= f_2 \circ \psi^* + w \cdot (f_1 \circ \psi^*) \end{aligned}$$

Thus, $Z'(T)$ is, in fact, well-defined. Moreover, it is easy to verify that Z' is $\mathrm{GL}_2(\mathbb{Z})$ equivariant.

For $v \in \mathbb{Z}^2$, consider the triangles $T_1 = v + \mathbf{T}$ and $T_2 = v + e_1 + e_2 - \mathbf{T}$, as well as $T_3 = v + e_1 + \rho_2\mathbf{T}$ and $T_4 = v + e_2 + \rho_1\mathbf{T}$, where again $\rho_i \in \mathbf{D}$ denotes the reflection at the i th coordinate axis. Using that r is odd, we have

$$\begin{aligned} Z'(T_1) + Z'(T_2) &= f_2 + vf_1 + f_2 - (x + y)f_1 - vf_1 \\ &= 2(h + (\frac{x}{3} + \frac{y}{3})f_1) - (x + y)f_1 \\ &= 2h - (\frac{x}{3} + \frac{y}{3})f_1. \end{aligned}$$

Using that r is odd, we also obtain

$$\begin{aligned} Z'(T_3) + Z'(T_4) &= f_2(-x, y) + (x + v)f_1(-x, y) + f_2(x, -y) + (y + v)f_1(x, -y) \\ &= 2f_2(-x, y) + (x - y)f_1(-x, y) \\ &= 2h(-x, y) - (-\frac{x}{3} + \frac{y}{3})f_1(-x, y) \\ &= 2h - (\frac{x}{3} + \frac{y}{3})f_1. \end{aligned}$$

We used the $\mathbf{D}$ invariance of $2h - (\frac{x}{3} + \frac{y}{3})f_1$ to obtain the last line. So the function Z' satisfies (6). By Corollary 11, we find a simple, $\mathrm{GL}_2(\mathbb{Z})$ equivariant valuation $Z: \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{R}[x, y]_{r+1}$ with $Z(T) = Z'(T)$ for all unimodular triangles T . In fact, Z

satisfies (12): Since Z and Z_1^r are simple, it suffices to check (12) for a unimodular triangle $T = \phi\mathbf{T} + w$ and a translation vector $v \in \mathbb{Z}^2$. We have

$$\begin{aligned} Z(T + v) &= Z'(T + v) = f_2 \circ \phi^\star + (v + w) \cdot f_1 \circ \phi^\star \\ &= Z'(T) + v \cdot Z_1^r(\mathbf{T}) \circ \phi^\star = Z(T) + v Z_1^r(T), \end{aligned}$$

where in the last step we used that Z_1^r is translation invariant by Lemma 9. It remains to show that Z is 2-homogeneous. First, we show that $Z(2^k T) = 2^{2k} Z(T)$ holds all unimodular simplices T and for all $k \in \mathbb{Z}_{\geq 0}$ by induction on k . For $k = 0$, there is nothing to prove, so let $k > 0$. Multiplying the triangulation

$$2\mathbf{T} = \mathbf{T} \sqcup (\mathbf{T} + e_1) \sqcup (\mathbf{T} + e_2) \sqcup (-\mathbf{T} + e_1 + e_2).$$

with 2^{k-1} , we obtain a triangulation of $2^k \mathbf{T}$ into four copies of $\pm 2^{k-1} \mathbf{T}$:

$$2^k \mathbf{T} = 2^{k-1} \mathbf{T} \sqcup 2^{k-1}(\mathbf{T} + e_1) \sqcup 2^{k-1}(\mathbf{T} + e_2) \sqcup 2^{k-1}(-\mathbf{T} + e_1 + e_2).$$

Using induction, we find

$$\begin{aligned} Z(2^k \mathbf{T}) &= Z(2^{k-1} \mathbf{T}) + Z(2^{k-1}(\mathbf{T} + e_1)) + Z(2^{k-1}(\mathbf{T} + e_2)) + Z(2^{k-1}(-\mathbf{T} + e_1 + e_2)) \\ &= 2^{2k-2} (Z(\mathbf{T}) + Z(\mathbf{T} + e_1) + Z(\mathbf{T} + e_2) + Z(-\mathbf{T} + e_1 + e_2)) \\ &= 2^{2k} Z(\mathbf{T}), \end{aligned}$$

where we used (12) to obtain the last line. For any $\phi \in \text{GL}_2(\mathbb{Z})$, we then also have $Z(2^k \phi \mathbf{T}) = 2^{2k} Z(\phi \mathbf{T})$. For an arbitrary unimodular triangle $\phi \mathbf{T} + v$, it follows from (12) that

$$\begin{aligned} Z(2^k(\phi \mathbf{T} + v)) &= Z(2^k \phi \mathbf{T}) + Z_1^r(2^k \phi \mathbf{T}) 2^k v \\ &= 2^{2k} (Z(\phi \mathbf{T}) + Z_1^r(\phi \mathbf{T}) v) = 2^{2k} Z(\phi \mathbf{T} + v), \end{aligned}$$

where we used that Z_1^r is one-homogeneous. So we have $Z(2T) = 2^{2k} Z(T)$ for all unimodular triangles T and all $k \in \mathbb{Z}_{\geq 0}$. By Theorem 6, this means that Z is 2-homogeneous on unimodular triangles. It follows from (5) that Z is 2-homogeneous on arbitrary $P \in \mathcal{P}(\mathbb{Z}^2)$.

Finally, we establish the uniqueness of h . Note that in the above construction we have

$$Z([0, 1]^2) = 2h - \left(\frac{x}{3} + \frac{y}{3}\right) f_1.$$

Suppose that a second $\mathbf{D}$ invariant polynomial $\tilde{h}$ exists such that

$$2\tilde{h} - \left(\frac{x}{3} + \frac{y}{3}\right) f_1 \in \mathbb{R}[x, y]_{r+1}^{\mathbf{D}}.$$

Then there exists a valuation $\tilde{Z} \in \text{Val}_2^{r+1}$ that satisfies (12) and has

$$\tilde{Z}([0, 1]^2) = 2\tilde{h} - \left(\frac{x}{3} + \frac{y}{3}\right) f_1 \neq Z([0, 1]^2).$$

However, by Lemma 18, we have $Z = \tilde{Z}$, a contradiction. $\square$

Proposition 19 allows us to determine the dimension of Val_2^{r+1} as follows. By Lemma 18 and Theorem 15, the linear map

$$\text{Val}_2^{r+1} \rightarrow \mathbb{R}[x, y]_r^{\mathbf{G}}, \quad Z_2^{r+1} \mapsto Z_1^r(\mathbf{T})$$

is injective. Proposition 19 tells us that its image consists of those $f \in \mathbb{R}[x, y]_r^{\mathbf{G}}$ for which we can find $h \in \mathbb{R}[x, y]_{r+1}^{\mathbf{G}}$ such that $h + (x + y)f \in \mathbb{R}[x, y]_r^{\mathbf{D}}$.

We consider the linear maps

$$\rho: \mathbb{R}[x, y]_r^{\mathbf{G}} \times \mathbb{R}[x, y]_{r+1}^{\mathbf{G}} \rightarrow \mathbb{R}[x, y]_{r+1}, \quad (f, h) \mapsto h + (x + y)f$$

and

$$\delta: \mathbb{R}[x, y]_{r+1} \rightarrow \mathbb{R}[x, y]_{r+1}, \quad g \mapsto \frac{1}{2}(g(x, y) - g(x, -y)).$$

Since the permutation of the coordinates x and y is in both $\mathbf{G}$ and $\mathbf{D}$, it follows for $f, h \in \mathbb{R}[x, y]_r^{\mathbf{G}}$ that $h + (x + y)f \in \mathbb{R}[x, y]_r^{\mathbf{D}}$ if and only if $h + (x + y)f \in \ker(\delta)$, the kernel of δ . We have

$$(15) \quad \dim \text{Val}_2^{r+1} = \dim \ker(\delta \circ \rho),$$

since the natural projection $\ker(\delta \circ \rho) \rightarrow \mathbb{R}[x, y]_r^{\mathbf{G}}$ is injective by Proposition 19.

In the remainder of the section, we prove the following result.

Lemma 20. *For $r > 1$ odd, the dimension of the image of $\delta \circ \rho$ is $\lfloor \frac{r+3}{4} \rfloor$.*

Using (10), it can be checked that $p_{2,3}(r) + p_{2,3}(r+1) - \lfloor \frac{r+3}{4} \rfloor = p_{2,3}(\frac{r+1}{2})$ for r odd. Since $p_{2,3}(r) + p_{2,3}(r+1) = \dim(\mathbb{R}[x, y]_r^{\mathbf{G}} \times \mathbb{R}[x, y]_{r+1}^{\mathbf{G}})$, we obtain the following result from Lemma 20 and (15).

Proposition 21. *For $r > 1$ odd, $\dim \text{Val}_2^{r+1} = p_{2,3}(\frac{r+1}{2})$.*

For the proof of Lemma 20, we use a third matrix group,

$$\mathbf{H} := \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle,$$

a subgroup of $\mathbf{D}$. The algebraically independent polynomials

$$\mathbf{q} := xy \quad \text{and} \quad \tilde{\mathbf{q}} := x^2 + y^2$$

generate the invariant ring $\mathbb{R}[x, y]^{\mathbf{H}}$.

Lemma 22. *For $r > 1$ odd, the image of ρ is in $\mathbb{R}[x, y]_{r+1}^{\mathbf{H}}$. As a vector space, it is spanned by the polynomials*

$$\left\{ (\tilde{\mathbf{q}} - \mathbf{q})^k (2\tilde{\mathbf{q}} - 5\mathbf{q})^\ell (\tilde{\mathbf{q}} + 2\mathbf{q})^{\lfloor \frac{\ell+1}{2} \rfloor} : k = \frac{r+1}{2} - \ell - \left\lfloor \frac{\ell+1}{2} \right\rfloor, 0 \leq \ell \leq \left\lfloor \frac{r+1}{3} \right\rfloor \right\}.$$

Proof. We observe that

$$\begin{aligned} \mathbf{p}_2 &= \tilde{\mathbf{q}} - \mathbf{q}, \\ (x + y)\mathbf{p}_3 &= \frac{1}{2}(x + y)^2(2x^2 - 5xy + 2y^2) \\ &= \frac{1}{2}(\tilde{\mathbf{q}} + 2\mathbf{q})(2\tilde{\mathbf{q}} - 5\mathbf{q}), \\ \mathbf{p}_3^2 &= \frac{1}{4}(\tilde{\mathbf{q}} + 2\mathbf{q})(2\tilde{\mathbf{q}} - 5\mathbf{q})^2. \end{aligned}$$

Therefore, any element in the image of ρ is a linear combination of terms of the form $(\tilde{\mathbf{q}} - \mathbf{q})^k (2\tilde{\mathbf{q}} - 5\mathbf{q})^\ell (\tilde{\mathbf{q}} + 2\mathbf{q})^{\lfloor \frac{\ell+1}{2} \rfloor}$ such that $2(k + \ell + \lfloor \frac{\ell+1}{2} \rfloor) = r + 1$.

To determine the range of ℓ , note that k must be non-negative, and therefore

$$\ell + \left\lfloor \frac{\ell+1}{2} \right\rfloor \leq \frac{r+1}{2}.$$

By distinguishing cases, we see that this is equivalent to

$$\ell \leq \left\lfloor \frac{r+1}{3} \right\rfloor,$$

provided that r is odd. $\square$

Lemma 23. *For $r > 1$ odd, we have*

$$\delta(\mathbf{q}^i \tilde{\mathbf{q}}^j) = \begin{cases} \mathbf{q}^i \tilde{\mathbf{q}}^j & \text{for } i \text{ odd,} \\ 0 & \text{otherwise,} \end{cases}$$

for $0 \leq i \leq \frac{r+1}{2}$ and $j := \frac{r+1}{2} - i$. In particular, $\delta(\mathbb{R}[x, y]_{r+1}^{\mathbf{H}}) \subseteq \mathbb{R}[x, y]_{r+1}^{\mathbf{H}}$.

Proof. It suffices to observe that $\tilde{\mathbf{q}}$ is even in y and $\mathbf{q}$ is odd in y . $\square$

Proof of Lemma 20. For $r > 1$ odd, let $d(r)$ be the dimension of the image of $\delta \circ \rho$. Set

$$\mathbf{g}_\ell := (\tilde{\mathbf{q}} - \mathbf{q})^k (2\tilde{\mathbf{q}} - 5\mathbf{q})^\ell (\tilde{\mathbf{q}} + 2\mathbf{q})^{\lfloor \frac{\ell+1}{2} \rfloor},$$

with $k := \frac{r+1}{2} - \ell - \lfloor \frac{\ell+1}{2} \rfloor$ for $0 \leq \ell \leq \lfloor \frac{r+1}{3} \rfloor$. It follows from Lemma 22 that $d(r)$ is the number of linearly independent polynomials in

$$\left\{ \delta(\mathbf{g}_\ell) : 0 \leq \ell \leq \left\lfloor \frac{r+1}{3} \right\rfloor \right\}.$$

Let R be the matrix of coefficients of monomials of these polynomials, that is, $R_{\ell,i}$ is the coefficient of $\mathbf{q}^i \tilde{\mathbf{q}}^{\frac{r+1}{2}-i}$ in $\delta(\mathbf{g}_\ell)$. Since $\delta(\mathbf{g}_\ell)$ has no monomials of even degree in $\mathbf{q}$ by Lemma 23, every second column of R vanishes. We will show that the remaining

$$\left\lfloor \frac{r+3}{4} \right\rfloor = \left\lfloor \frac{\frac{r+1}{2} + 1}{2} \right\rfloor$$

columns of R are linearly independent. This determines the dimension $d(r)$.

By Lemma 23, the entries of R are integers. Therefore, it suffices to show that the remaining columns of R are linearly independent modulo 2. This is easier, since we have

$$\mathbf{g}_\ell = (\tilde{\mathbf{q}} + \mathbf{q})^k \mathbf{q}^\ell \tilde{\mathbf{q}}^{\lfloor \frac{\ell+1}{2} \rfloor}$$

in $\mathbb{Z}_2[\tilde{\mathbf{q}}, \mathbf{q}]$. By the binomial theorem, we obtain that $R_{\ell,2i+1}$ equals, modulo 2,

$$\binom{k}{2i+1-\ell} = \binom{\frac{r+1}{2} - \ell - \lfloor \frac{\ell+1}{2} \rfloor}{2i+1-\ell}.$$

We now show that the indices of the last odd entries in the remaining columns of R are all distinct. More precisely, we show that these indices are

$$\ell_i := \begin{cases} 2i+1 & \text{for } 0 \leq i < \left\lfloor \frac{r+3}{6} \right\rfloor, \\ r-1-4i & \text{for } \left\lfloor \frac{r+3}{6} \right\rfloor \leq i < \left\lfloor \frac{r+3}{4} \right\rfloor. \end{cases}$$

In both cases, the binomial coefficient equals 1, which follows in the second case from ℓ_i being even. Since r is odd, the indices ℓ_i are distinct. It remains to show that $0 \leq \ell_i \leq \left\lfloor \frac{r+1}{3} \right\rfloor$ and that subsequent entries in the same column vanish.

For the first case, we observe that $2i + 1 \leq \lfloor \frac{r+1}{3} \rfloor$ if and only if $i < \lfloor \frac{r+3}{6} \rfloor$, and that the binomial coefficient vanishes when replacing ℓ_i with any larger value.

For the second case, we check the extremal values of i . On the one hand, for $i = \lfloor \frac{r+3}{6} \rfloor$, we require

$$\ell_i = r - 1 - 4 \left\lfloor \frac{r+3}{6} \right\rfloor \leq \left\lfloor \frac{r+1}{3} \right\rfloor.$$

Indeed, this is the case for all three relevant congruence classes modulo 6, namely, for $r + 1 = 6s$, $r + 1 = 6s + 2$, and $r + 1 = 6s + 4$. On the other hand, for $i = \lfloor \frac{r+3}{6} \rfloor - 1 = \lfloor \frac{r-1}{4} \rfloor$, we require

$$\ell_i = r - 1 - 4 \left\lfloor \frac{r-1}{4} \right\rfloor \geq 0,$$

which is obviously true. It remains to show that all subsequent entries in the same column vanish. This is certainly the case if, for $j > 0$,

$$\frac{r+1}{2} - (\ell_i + j) - \left\lfloor \frac{\ell_i + j + 1}{2} \right\rfloor < 2i + 1 - (\ell_i + j).$$

Using $\ell_i = r - 1 - 4i$, we obtain the condition

$$\frac{r+1}{2} - \left\lfloor \frac{r-1-4i+j+1}{2} \right\rfloor < 2i + 1,$$

which simplifies to

$$- \left\lfloor \frac{j-1}{2} \right\rfloor < 1.$$

This concludes the proof. $\square$

6. EXPONENTIAL VALUATIONS

This section contains the proof of the second case of Theorem 4 and the third case of Theorem 2. Let $\bar{Z} \in \overline{\text{Val}}$. For $P \in \mathcal{P}(\mathbb{Z}^2)$, we can write

$$(16) \quad \bar{Z}(P) = \sum_{r \geq 0} Z^r(P),$$

where $Z^r : \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{R}[x, y]_r$ is a $\text{GL}_2(\mathbb{Z})$ equivariant valuation for each r . We call Z^r the r -summand of $\bar{Z}$. We start with two simple observations, which use the notation from (16).

Lemma 24. *The valuation $\bar{Z} \in \overline{\text{Val}}$ is translatively exponential if and only if every Z^r is translatively polynomial with associated functions Z^{r-j} for $0 \leq j \leq r$.*

Proof. If $\bar{Z}$ is translatively exponential, then

$$\sum_{r \geq 0} Z^r(P + v) = \bar{Z}(P + v) = e^v \bar{Z}(P) = \sum_{k \geq 0} \frac{v^k}{k!} \sum_{r \geq 0} Z^r(P) = \sum_{r \geq 0} \sum_{j=0}^r Z^{r-j}(P) \frac{v^j}{j!}$$

for every $P \in \mathcal{P}(\mathbb{Z}^2)$ and $v \in \mathbb{Z}^2$. Comparing the r -summands on both sides shows that Z^r is translatively polynomial with associated valuations Z^{r-j} for $0 \leq j \leq r$. The reverse implication can be seen in the same way. $\square$

Lemma 25. *The valuation $\bar{Z} \in \overline{\text{Val}}$ is d -dilative if and only if the valuation Z^r is $(r-d)$ -homogeneous for every $r \geq 0$. In particular, we have*

- (1) $\overline{\text{Val}}_d = 0$ for $d < -2$, and
- (2) if $\bar{Z}$ is d -dilative, then $Z^r = 0$ for every $r < d$.

Proof. If $\bar{Z}$ is d -dilative, then

$$\begin{aligned} \sum_{r \geq 0} Z^r(mP)(x, y) &= \bar{Z}(mP)(x, y) \\ &= m^{-d} \sum_{r \geq 0} Z^r(P)(mx, my) = \sum_{r \geq 0} m^{r-d} Z^r(P)(x, y) \end{aligned}$$

for $m \in \mathbb{Z}_{>0}$. Comparing the r -summands on both sides shows that Z^r is $(r-d)$ -homogeneous. The reverse implication can be seen in the same way. The last two statements immediately follow from Theorem 6. $\square$

We deduce the decomposition of $\overline{\text{Val}}$ claimed in Theorem 4.

Proposition 26. *The map*

$$\prod_{d \geq -2} \overline{\text{Val}}_d \rightarrow \overline{\text{Val}}, \quad (\bar{Z}_d)_{d \geq -2} \mapsto \sum_{d \geq -2} \bar{Z}_d$$

is a linear isomorphism.

Proof. The linearity is clear. To see that the map is injective, suppose that

$$\sum_{d \geq -2} \bar{Z}_d = 0.$$

We write $\bar{Z}_d = \sum_{r \geq 0} Z_{r-d}^r$. Note that the difference of upper and lower indices is d , so there is no conflict of notation for different d . Extracting the r -summand of the sum, we see that $\sum_{d \geq -2} Z_{r-d}^r = 0$. It follows from Lemma 25 that Z_{r-d}^r is $(r-d)$ -homogeneous. This implies $Z_{r-d}^r = 0$ for all $d \geq -2$. Since r was arbitrary, we obtain $\bar{Z}_d = 0$ for all $d \geq -2$.

For the surjectivity, let $\bar{Z} \in \overline{\text{Val}}$ be written as in (16). By Theorem 6, each Z^r has a homogeneous decomposition $Z^r = \sum_{i=0}^{r+2} Z_i^r$ with $Z_i^r \in \text{Val}_i^r$. Since Z is translative exponential, Lemma 24 implies that Z^r is translative polynomial with associated valuation Z^{r-j} for $0 \leq j \leq r$. Thus, it follows from Lemma 7 that Z_i^r is translative polynomial with associated valuations Z_{i-j}^{r-j} for $0 \leq j \leq r$. Since the difference between rank and homogeneity degree is the same for every j , the reverse implication of Lemma 24 shows that the valuation

$$\bar{Z}_d := \sum_{r \geq 0} Z_{r-d}^r$$

is translative exponential for any $d \geq -2$. By Lemma 25, it is d -dilative. Since the r -summands of $\bar{Z}_d$ are zero for $r < d$ by Lemma 25, in the sum $\sum_{d \geq -2} \bar{Z}_d(P)$, we see only finitely many polynomial terms of each degree of homogeneity. Therefore, $\sum_{d \geq -2} \bar{Z}_d(P)$ defines a formal power series which is equal to $\bar{Z}(P)$. $\square$

For the results in this section, we did not use that our valuations are defined on polygons. The corresponding results can be proven similarly in the n -dimensional setting.

7. UNBOUNDED POLYHEDRA

Next, we classify d -dilative, translatively exponential valuations. Here, we are only concerned with the second case of Theorem 4 as the remaining cases will be treated in the next section. So, for the remainder of the section, let $d > 0$ be an even integer.

It follows from Lemma 24 and Lemma 25 that we have well-defined linear maps

$$(17) \quad \Lambda^r : \overline{\text{Val}}_d \rightarrow \text{Val}_{r-d}^r, \quad \bar{Z} = \sum_{s \geq 0} Z^s \mapsto Z^r.$$

We require the following result.

Lemma 27. *Let $d > 0$ be even. The map Λ^r defined in (17) is injective for $r > d$ and identically zero for $r \leq d$.*

Proof. The fact that $\Lambda^r = 0$ for $r \leq d$ follows from Theorem 6 and Lemma 8. To see that Λ^r is injective for $r > d$, consider $\bar{Z} = \sum_{s \geq 0} Z^s \in \overline{\text{Val}}_d$ with $0 = \Lambda^r \bar{Z} = Z^r$. It follows from Lemma 24 that $Z^s = 0$ for all $s \leq r$, since these are the associated valuations of Z^r . Again by Lemma 24, the associated valuations of $Z^{r+1} \in \text{Val}_{r-d+1}^{r+1}$ (other than Z^{r+1} itself) are all zero. Since by assumption $r - d + 1 \geq 2$, Lemma 18 applies and we obtain $Z^{r+1} = 0$. Using induction on j , we see that $Z^{r+j} = 0$ for all $j > 0$. Thus, $\bar{Z} = 0$, as desired. $\square$

In the following, we want to prove that Λ^{d+2} is a linear isomorphism. To this end, we construct a valuation on the set of rational polyhedra $\mathcal{Q}(\mathbb{Q}^2)$ with values in the vector space

$$\mathcal{V} := \text{span}\{e^v \cdot f : v \in \mathbb{R}^2, f \in \mathbb{R}(x, y)\},$$

a subspace of the space of meromorphic functions. Here, $\mathbb{R}(x, y)$ denotes the space of rational functions in x and y . The restriction of this valuation to lattice polygons will take values in $\mathbb{R}[[x, y]]$. Here, a polyhedron $Q \subseteq \mathbb{R}^2$ is called *rational* if it can be represented as $Q = \{v \in \mathbb{R}^2 : z_i(v) \leq b_i, 1 \leq i \leq m\}$ for certain $z_1, \dots, z_m \in (\mathbb{Q}^2)^*$ and $b_1, \dots, b_m \in \mathbb{Q}$. For $X \subseteq \mathbb{R}^2$, let $\text{pos}(X)$ denote the positive hull of X . We call a rational polyhedron $Q \in \mathcal{Q}(\mathbb{Q}^2)$ a *rational cone* if $\text{pos}(Q) = Q$, and we denote the set of rational cones by $\mathcal{C}(\mathbb{Q}^2)$.

We will use the following special case of [16, Theorem 1]. Let $\text{vert}(Q)$ be the set of vertices of $Q \in \mathcal{Q}(\mathbb{Q}^2)$.

Theorem 28 (Lawrence). *A valuation $\zeta_0 : \mathcal{C}(\mathbb{Q}^2) \rightarrow \mathcal{V}$ that vanishes on cones containing a line can be extended to a valuation ζ on $\mathcal{Q}(\mathbb{Q}^2)$ by*

$$\zeta(Q) := \sum_{v \in \text{vert}(Q)} e^v \zeta_0(\text{fcone}(v; Q)),$$

where $\text{fcone}(v; Q) := \text{pos}(Q - v)$.

To construct our particular ζ_0 , we start with a valuation $Z_2^{d+2} \in \text{Val}_2^{d+2}$ and consider the rational function

$$(18) \quad R(x, y) := \frac{Z_2^{d+2}([0, 1]^2)}{xy}.$$

Note that $R \in \mathbb{R}(x, y)_d$, the space of d -homogeneous rational functions in x and y .

We require some results and notions related to cones (see, for example, [9]). A pointed two-dimensional cone $C \in \mathcal{C}(\mathbb{Q}^2)$ is called *unimodular* if $C = \phi_C(\mathbb{R}_{\geq 0}^2)$ with $\phi_C \in \text{GL}_2(\mathbb{Z})$. Here $\mathbb{R}_{\geq 0}^2 := \{(x, y) \in \mathbb{R}^2 : x, y \geq 0\}$. Note that ϕ_C is unique up to

a permutation of the coordinates. Since R is symmetric in x and y , the expression $R \circ \phi_C^*$ is independent of the choice of ϕ_C . For a general two-dimensional pointed cone $K \in \mathcal{C}(\mathbb{Q}^2)$, let $u_1, \dots, u_m$ be the primitive lattice vectors on the boundary of the convex hull of $K \cap \mathbb{Z}^2 \setminus \{0\}$ (labeled in any direction along the boundary), where a lattice vector is *primitive* if its coordinates are co-prime. For a two-dimensional cone, the *Hilbert decomposition* $\mathcal{H}(K)$ can be defined as

$$\mathcal{H}(K) := \{\text{pos}\{u_{i-1}, u_i\} : 1 \leq i \leq m\}.$$

It is easy to see that the decomposition $\mathcal{H}(K)$ is *unimodular*; that is, the cones of $\mathcal{H}(K)$ are unimodular. Moreover, any unimodular decomposition of $\mathcal{H}(K)$ is a refinement of $\mathcal{H}(K)$ (cf. [24, Proposition 1.19]). We can describe the refinements in a structured way. For a unimodular cone $C = \text{pos}\{u, v\}$, where $u, v \in \mathbb{Z}^2$ are primitive vectors, we define the *balanced decomposition* of C as $\{\text{pos}\{u, u+v\}, \text{pos}\{u+v, v\}\}$. For two unimodular decompositions $\mathcal{S}$ and $\mathcal{T}$ of the cone $K \in \mathcal{C}(\mathbb{Q}^2)$, we say that $\mathcal{T}$ is an *elementary refinement* of $\mathcal{S}$ if $\mathcal{T}$ can be obtained from $\mathcal{S}$ by replacing a unimodular cone of $\mathcal{S}$ by its balanced decomposition.

Lemma 29. *Let $K \in \mathcal{C}(\mathbb{Q}^2)$ be two-dimensional and pointed. For every unimodular decomposition $\mathcal{T}$ of K , there exist unimodular decompositions $\mathcal{T}_0, \dots, \mathcal{T}_k$ (for some $k \in \mathbb{Z}_{\geq 0}$) of K such that $\mathcal{T}_0 = \mathcal{H}(K)$, $\mathcal{T}_k = \mathcal{T}$, and $\mathcal{T}_i$ is an elementary refinement of $\mathcal{T}_{i-1}$ for all $1 \leq i \leq k$.*

Proof. Since any unimodular triangulation of K is a refinement of $\mathcal{H}(K)$, it suffices to prove the lemma for K being itself a unimodular cone, that is, $\mathcal{H}(K) = \{K\}$ and $K = \text{pos}\{u, v\}$, where u and v generate $\mathbb{Z}^2$. We use induction on $m := |\mathcal{T}|$. For $m = 1$, there is nothing to prove. So, let $m > 1$ and suppose that the statement of the lemma is true for any unimodular cone K and any decomposition $\mathcal{T}$ of size less than m .

First, we show that the ray $\text{pos}\{u+v\}$ is contained in the boundary of two cones of $\mathcal{T}$. To see this, we apply a unimodular transformation so that $K = \mathbb{R}_{\geq 0}^2$ and, thus, $u+v = (1, 1)$. If $\text{pos}\{(1, 1)\}$ is not a boundary ray of a cone in $\mathcal{T}$, there exists a unimodular cone $C \in \mathcal{C}(\mathbb{Q}^2)$ with primitive generators $(a, b), (c, d) \in \mathbb{Z}^2$ such that $a > b \geq 0$ and $d > c \geq 0$ and such that $(1, 1)$ is contained in the interior C . Since C is unimodular, there exist $k, \ell \in \mathbb{Z}_{>0}$ with $(1, 1) = k(a, b) + \ell(c, d)$. Using that the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in $\text{SL}_2(\mathbb{Z})$ and solving for k and ℓ gives $k = d - c \geq 1$ and $\ell = a - b \geq 1$. This implies $(a, b) = (1, 0)$ and $(c, d) = (0, 1)$. Hence, $C = K$, which contradicts $m > 1$. Thus, the ray $\text{pos}\{u+v\}$ must be contained in the boundary of a cone of $\mathcal{T}$.

Consequently, we obtain that $\mathcal{T}$ is the disjoint union of the two unimodular decompositions $\{T \in \mathcal{T} : T \subseteq \text{pos}\{u, u+v\}\}$ and $\{T \in \mathcal{T} : T \subseteq \text{pos}\{u+v, v\}\}$ of the unimodular cones $\text{pos}\{u, u+v\}$ and $\text{pos}\{u+v, v\}$, respectively. Since each of the two decompositions contains fewer cones than $\mathcal{T}$, the induction hypothesis applies, and we see that $\mathcal{T}$ is obtained by taking the balanced decomposition of K first and then refining the two halves inductively. $\square$

For a pointed two-dimensional cone $K \in \mathcal{C}(\mathbb{Q}^2)$, set

$$(19) \quad \zeta_0(K) := \sum_{C \in \mathcal{H}(K)} R \circ \phi_C^*.$$

If K is not pointed or if K is lower-dimensional, we set $\zeta_0(K) := 0$.

Lemma 30. *For $d > 0$ even, the function $\zeta_0: \mathcal{C}(\mathbb{Q}^2) \rightarrow \mathbb{R}(x, y)_d$, defined in (19), is a simple, $\mathrm{GL}_2(\mathbb{Z})$ equivariant valuation that vanishes on cones with lines.*

Proof. We only have to show that ζ_0 is a valuation. Let $K \in \mathcal{C}(\mathbb{Q}^2)$ and $L \subset \mathbb{R}^2$ be a rational one-dimensional subspace (L is generated by a rational vector) that intersects K . Let K_+ and K_- denote the intersections of K with the two closed halfspaces determined by L . By [2, Lemma 19.5], it is sufficient to check that

$$(20) \quad \zeta_0(K) = \zeta_0(K_+) + \zeta_0(K_-)$$

for all $K \in \mathcal{C}(\mathbb{Q}^2)$ and rational one-dimensional subspaces L .

If $\dim K < 2$, this is trivial. So, let K be two-dimensional. First, we assume that K is pointed. Let $\mathcal{H} := \mathcal{H}(K)$ and $\mathcal{H}_\sigma := \mathcal{H}(K_\sigma)$ for $\sigma \in \{-, +\}$. Clearly, $\mathcal{T} := \mathcal{H}_- \cup \mathcal{H}_+$ is a unimodular decomposition of K . As such, it is obtained from $\mathcal{H}$ by a finite sequence of elementary refinements using Lemma 29. Consider a unimodular cone $C \in \mathcal{T}$. By $\mathrm{GL}_2(\mathbb{Z})$ equivariance, we may assume that $C = \mathbb{R}_{\geq 0}^2$. If we show that

$$\zeta_0(\mathbb{R}_{\geq 0}^2) = \zeta_0(\mathrm{pos}\{e_1, e_1 + e_2\}) + \zeta_0(\mathrm{pos}\{e_2 + e_1, e_2\}),$$

then we can use the $\mathrm{GL}_2(\mathbb{Z})$ equivariance and apply Lemma 29 inductively to obtain (20). Hence, by definition of ζ_0 , we need to verify that

$$(21) \quad R(x, y) = R(x, x + y) + R(x + y, y)$$

for $x, y \in \mathbb{R}$. Writing $P_0 := [0, 1]^2$, $P_1 := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} P_0$, and $P_2 := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} P_0$, we have

$$R(x, x + y) = \frac{Z_2^{d+2}(P_0)(x, x + y)}{x(x + y)} = \frac{Z_2^{d+2}(P_1)}{x(x + y)}.$$

Similarly, we obtain

$$R(x + y, y) = \frac{Z_2^{d+2}(P_2)}{(x + y)y}.$$

Thus,

$$R(x, x + y) + R(x + y, y) = \frac{x Z_2^{d+2}(P_2) + y Z_2^{d+2}(P_1)}{xy(x + y)}.$$

Let $T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mathbf{T}$. Using the decompositions

$$P_0 = T \sqcup (-T + e_1 + e_2), \quad P_1 = T \sqcup (-T + 2e_1 + e_2), \quad P_2 = (-T + e_1 + e_2) \sqcup (e_2 + T),$$

we compute

$$\begin{aligned} x(Z_2^{d+2}(P_2) - Z_2^{d+2}(P_0)) &= x(Z_2^{d+2}(T + e_2) - Z_2^{d+2}(T)) \\ &= xy Z_1^{d+1}(T) \\ &= y(-x Z_1^{d+1}(-T + e_1 + e_2)) \\ &= y(Z_2^{d+2}(-T + e_1 + e_2) - Z_2^{d+2}(-T + 2e_1 + e_2)) \\ &= y(Z_2^{d+2}(P_0) - Z_2^{d+2}(P_1)). \end{aligned}$$

We have used that Z_2^{d+2} is simple by Lemma 13 and that Z_1^{d+1} is odd and translation invariant by Lemma 9. Hence,

$$x Z_2^{d+2}(P_2) + y Z_2^{d+2}(P_1) = (x + y) Z_2^{d+2}(P_0)$$

and (21) follows. So we obtain (20) if K is a pointed unimodular cone.

Next, let K be a halfspace. We may assume that L intersects the interior of K . Then, K_+ is unimodular if and only if K_- is unimodular. First, assume that the cones K_+ and K_- are unimodular. By $\mathrm{GL}_2(\mathbb{Z})$ equivariance, it suffices to consider the case where K is the upper halfplane and L is the y -axis. In this case, (20) is equivalent to the functional equation

$$(22) \quad 0 = R(x, y) + R(-x, y)$$

for $x, y \in \mathbb{R}$. Since $Z_2^{d+2}([0, 1]^2)$ is $\mathbf{D}$ invariant, (22) is fulfilled.

Next, we assume that K_+ is a non-unimodular cone. We can choose a one-dimensional rational subspace L' that meets the interior of K_+ such that the cone K'_+ bounded by the boundary of K and L' is unimodular. Let K'_- be the closure of $K \setminus K'_+$, which is a unimodular cone. Thus,

$$\zeta_0(K) = \zeta_0(K'_+) + \zeta_0(K'_-).$$

Moreover, K'_- is decomposed by the subspace L into the cones K_- and K''_+ , where K''_+ is the closure of $K_+ \setminus K'_+$. Since we already proved that (20) holds for the pointed cone K'_- we deduce

$$\zeta_0(K) = \zeta_0(K'_+) + \zeta_0(K''_+) + \zeta_0(K_-) = \zeta_0(K_+) + \zeta_0(K_-),$$

which verifies (20) for halfspaces. Since the statement is trivial for $K = \mathbb{R}^2$, the proof is complete. $\square$

Using Theorem 28, we obtain a simple, $\mathrm{GL}_2(\mathbb{Z})$ equivariant, and translatively exponential valuation

$$(23) \quad \zeta: \mathcal{Q}(\mathbb{Q}^2) \rightarrow \mathcal{V}, \quad \zeta(P) := \sum_{v \in \mathrm{vert}(P)} e^v \zeta_0(\mathrm{fcone}(v; P)).$$

Moreover, it follows from $\zeta_0(\mathrm{fcone}(v; P)) \in \mathbb{R}(x, y)_d$ that ζ is d -dilative. Let us evaluate ζ at the standard triangle $\mathbf{T}$.

Lemma 31. *For $Z_2^{d+2} \in \mathrm{Val}_2^{d+2}$ with $d > 0$ even, the valuation ζ defined in (23) satisfies*

$$\begin{aligned} \zeta(\mathbf{T}) &= 2 Z_2^{d+2}(\mathbf{T}) \cdot \left(\frac{e^y - 1}{y(y-x)} - \frac{e^x - 1}{x(y-x)} \right) - Z_1^{d+1}(\mathbf{T}) \cdot \left(\frac{x(e^y - 1)}{y(y-x)} - \frac{y(e^x - 1)}{x(y-x)} \right) \\ &= 2 Z_2^{d+2}(\mathbf{T}) \sum_{m>0} \frac{[m]_{x,y}}{(m+1)!} - Z_1^{d+1}(\mathbf{T}) \sum_{m>0} \frac{[m]_{x,y} - x^{m-1} - y^{m-1}}{m!}, \end{aligned}$$

where $[m]_{x,y} := \sum_{j=0}^{m-1} x^j y^{m-j}$ and $Z_1^{d+1} \in \mathrm{Val}_1^{d+1}$ is the associated valuation of Z_2^{d+2} . In particular, the function $\zeta(\mathbf{T})$ is analytic.

Proof. We set $f_2 := Z_2^{d+2}(\mathbf{T})$ and $f_1 := Z_1^{d+1}(\mathbf{T})$. Since Z_i^{d+i} is simple for $i \in \{1, 2\}$ by Lemma 13, considering the triangulation $[0, 1]^2 = \mathbf{T} \sqcup (-\mathbf{T} + e_1 + e_2)$, we obtain

$$(24) \quad R(x, y) = \frac{2f_2(x, y) - (x+y)f_1(x, y)}{xy}.$$

For the standard triangle, we have

$$\mathrm{fcone}(0; \mathbf{T}) = \mathbb{R}_{\geq 0}^2, \quad \mathrm{fcone}(e_1; \mathbf{T}) = \phi_1 \mathbb{R}_{\geq 0}^2, \quad \text{and} \quad \mathrm{fcone}(e_2; \mathbf{T}) = \phi_2 \mathbb{R}_{\geq 0}^2,$$

where $\phi_1 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$ and $\phi_2 = \begin{pmatrix} 0 & -1 \\ -1 & -1 \end{pmatrix}$. Note that $\phi_i \mathbf{T} = \mathbf{T} - e_i$ for $i \in \{1, 2\}$. So, in particular, we have $\phi_1, \phi_2 \in \mathbf{G}$.

Using the representation (24) of R and the $\mathbf{G}$ invariance of f_1 , we compute

$$\begin{aligned} R \circ \phi_2^* &= \frac{1}{(x-y)(-y)} (2f_2 \circ \phi_2^* - (x-y+(-y))f_1 \circ \phi_2^*) \\ &= \frac{1}{y(y-x)} \left(2Z_2^{d+2}(\mathbf{T} - e_2) - (x-2y)f_1 \right) \\ &= \frac{1}{y(y-x)} (2(f_2 - yf_1) - (x-2y)f_1) \\ &= \frac{1}{y(y-x)} (2f_2 - xf_1). \end{aligned}$$

Following the same lines, we compute

$$R \circ \phi_1^* = \frac{1}{x(x-y)} (2f_2 - yf_1).$$

So we obtain

$$\begin{aligned} \zeta(\mathbf{T}) &= e^0 \zeta_0(\mathbb{R}_{\geq 0}^2) + e^x \zeta_0(\phi_1 \mathbb{R}_{\geq 0}^2) + e^y \zeta_0(\phi_2 \mathbb{R}_{\geq 0}^2) \\ &= R + e^x R \circ \phi_1^* + e^y R \circ \phi_2^* \\ &= \frac{2f_2 - (x+y)f_1}{xy} + e^x \frac{2f_2 - yf_1}{x(x-y)} + e^y \frac{2f_2 - xf_1}{y(y-x)} \\ &= 2f_2 \cdot \left(\frac{e^y - 1}{y(y-x)} - \frac{e^x - 1}{x(y-x)} \right) - f_1 \cdot \left(\frac{x(e^y - 1)}{y(y-x)} - \frac{y(e^x - 1)}{x(y-x)} \right). \end{aligned}$$

This proves the first identity.

For the second identity, note that $\frac{e^x - 1}{x} = \sum_{m \geq 0} \frac{x^m}{(m+1)!}$ is an analytic function and that $y^m - x^m = (y-x)[m]_{x,y}$. For the first bracket term, we thus obtain

$$\frac{e^y - 1}{y(y-x)} - \frac{e^x - 1}{x(y-x)} = \sum_{m \geq 0} \frac{y^m - x^m}{(m+1)!(y-x)} = \sum_{m \geq 0} \frac{[m]_{x,y}}{(m+1)!}.$$

In particular, the left-hand side is an analytic function on all $\mathbb{R}^2$. Similarly, we have for the second bracket term:

$$\begin{aligned} \frac{x(e^y - 1)}{y(y-x)} - \frac{y(e^x - 1)}{x(y-x)} &= \sum_{m \geq 0} \frac{xy^m - yx^m}{(m+1)!(y-x)} \\ &= -1 + \sum_{m > 0} \frac{xy(y^{m-1} - x^{m-1})}{(m+1)!(y-x)} = -1 + \sum_{m > 0} \frac{xy[m-1]_{x,y}}{(m+1)!} \\ &= \sum_{m \geq 0} \frac{[m+1]_{x,y} - x^m - y^m}{(m+1)!} = \sum_{m > 0} \frac{[m]_{x,y} - x^{m-1} - y^{m-1}}{m!}. \end{aligned}$$

Again, the left-hand side is an analytic function on all $\mathbb{R}^2$. $\square$

Let $\bar{Z}$ be the restriction of ζ to $\mathcal{P}(\mathbb{Z}^2)$. By Lemma 31, we have $\bar{Z}(\mathbf{T}) \in \mathbb{R}[[x, y]]$. Since ζ is simple, it follows that $\bar{Z} \in \overline{\text{Val}}_d$. From the series expansion in Lemma 31, we see that the $(d+1)$ -summand of $\bar{Z}$ is Z_1^{d+1} and the $(d+2)$ -summand of $\bar{Z}$ is Z_2^{d+2} . This shows that the map Λ^r is surjective. Combined with Lemma 27, we obtain the following result.

Lemma 32. *For $d > 0$ even, the map $\Lambda^{d+2}: \overline{\text{Val}}_{d+2} \rightarrow \text{Val}_2^{d+2}$ defined in (17) is a linear isomorphism.*

This allows us to compute the dimensions of several spaces of equivariant valuations.

Corollary 33. *Let $d > 0$ be even. Then we have*

$$\dim \overline{\text{Val}}_d = \dim \text{Val}_i^{d+i} = p_{2,3} \left(\frac{d}{2} + 1 \right)$$

for all $i \geq 2$.

Proof. By Proposition 21, we have $\dim \text{Val}_2^{d+2} = p_{2,3} \left(\frac{d}{2} + 1 \right)$. By Lemma 32, we have $\dim \overline{\text{Val}}_d = p_{2,3} \left(\frac{d}{2} + 1 \right)$. Now let $i \geq 2$ and let $A_i^{d+i}: \text{Val}_i^{d+i} \rightarrow \text{Val}_{i-1}^{d+i-1}$ be the linear operator that maps a valuation Z_i^{d+i} to its associated valuation of rank $d+i-1$. Note that this associated valuation is $(i-1)$ -homogeneous by Lemma 7. Since $i \geq 2$, applying Lemma 18 inductively shows that each A_i^{d+i} with $i \geq 2$ is injective. So, the linear map $A_i^{d+i} \circ \dots \circ A_3^{d+3}$ from Val_i^{d+i} to Val_2^{d+2} is injective. At the same time, the map $\Lambda^{d+i}: \overline{\text{Val}}_d \rightarrow \text{Val}_i^{d+i}$ is injective by Lemma 27. This shows the claim for all $i \geq 2$ (cf. Fig. 1). $\square$

Let us have a closer look at the valuations we obtain. For a lattice polygon $P \in \mathcal{P}(\mathbb{Z}^2)$ and a vertex v of P , let $u_1(v), \dots, u_{m_v}(v)$ be the lattice points on the boundary of the convex hull of $P \cap \mathbb{Z}^2 \setminus \{v\}$ that are visible from v (labeled in any direction along the boundary). Let $\phi_i[v]$ denote the linear transformation with $\phi_i[v](e_j) = u_{i+j-2}(v) - v$ for $1 \leq i \leq m_v$ and $1 \leq j \leq 2$. Then we have $\phi_i[v] \in \text{GL}_2(\mathbb{Z})$ and it follows from (23) that

$$(25) \quad \overline{Z}(P) = \sum_{v \in \text{vert}(P)} e^v \sum_{i=1}^{m_v} R \circ \phi_i[v]^\star.$$

By (18), the function R is determined by Z_2^{d+2} . As we saw in Proposition 19, there is a one-to-one correspondence between valuations $Z_2^{d+2} \in \text{Val}_2^{d+2}$ and $\mathbf{G}$ invariant polynomials $f \in \mathbb{R}[x, y]_{d+1}^{\mathbf{G}}$ for which there exists an $h \in \mathbb{R}[x, y]_{d+2}^{\mathbf{G}}$ such that $h + (x+y)f$ is $\mathbf{D}$ invariant. If it exists, this h is uniquely determined by f by Proposition 19. Hence, (25) allows us to compute $\overline{Z}(P)$ solely in terms of the initial polynomial f and the lattice geometry of P . In particular, no triangulation is needed.

Similarly, we obtain a triangulation-free description for the valuations in Val_i^{d+i} for $i > 1$ as the $(d+i)$ -summand of the series in (25). For $i = 1$, we have this description only for valuations of the form Z_f (as defined in (11)), where the polynomial $f \in \mathbb{R}[x, y]_{d+1}^{\mathbf{G}}$ is such that $(x+y)f + \mathbb{R}[x, y]_{r+1}^{\mathbf{G}}$ intersects $\mathbb{R}[x, y]_{r+1}^{\mathbf{D}}$.

8. THE REMAINING CASES

In this section, we consider the cases in Theorem 2 and Theorem 4 where the respective vector spaces are one-dimensional. The arguments used here essentially appear in [19] already; we recollect them to keep the paper self-contained.

Proposition 34. *Let $r \in \mathbb{Z}_{\geq 0}$ and $i \in \{1, \dots, r+2\}$. If $r \leq i \leq r+2$ or $r-i$ is odd, then $\text{Val}_i^r = \text{span}\{L_i^r\}$. Moreover, L_i^r is not identically zero in these cases and $L_i^r([0, e_1]) \neq 0$ unless $i = r+2$.*

Proof. We start by showing that L_i^r for $r-i$ odd is not identically zero on $\mathcal{P}(\mathbb{Z}^1)$ and thus not identically zero on $\mathcal{P}(\mathbb{Z}^2)$. Using Faulhaber's formula (see [19] for

more information), we obtain for $m \geq 1$ that

$$L^r(m[0, e_1])(x, y) = \frac{1}{r!} \sum_{k=0}^m k^r x^r = \frac{x^r}{r!(r+1)} \sum_{\ell=0}^r (-1)^\ell \binom{r+1}{\ell} B_\ell m^{r+1-\ell},$$

where B_ℓ denotes the ℓ th Bernoulli number. Since $B_\ell \neq 0$ for ℓ even, this shows that $L_i^r([0, e_1]) \neq 0$ for $r - i$ odd. Since also $B_1 \neq 0$, we obtain the claim that $L_i^r([0, e_1]) \neq 0$ holds for all i and r as in the proposition with $i \neq r + 2$. Moreover, L_{r+2}^r is not identically zero by [19, Lemma 26].

Next, for $r = 0$, it is easy to see that L_i^0 is not identically zero on $\mathcal{P}(\mathbb{Z}^2)$ for $i \in \{0, 1, 2\}$. Thus, for $r \leq i \leq r + 2$, using Lemma 7, we see that L_i^r is not translation invariant since L_{i-r}^0 is among its associated functions. So, in particular, $L_i^r \neq 0$. So, we have established $L_i^r \neq 0$ for the indices occurring in the proposition.

We note that $\text{Val}_i^0 = \text{span}\{L_i^0\}$ by the Betke–Kneser theorem. For $r \geq 2$ even, consider the one-homogeneous valuation Z_1^r , which is translation invariant by Lemma 9. Define the valuation $Z': \mathcal{P}(\mathbb{Z}^1) \rightarrow \mathbb{R}[x, y]_r$ by $Z'(S) := Z_1^r(S \times \{0\})$. By [19, Lemma 5], we have $Z'(S) \in \mathbb{R}[x]_r$. It is elementary to check that there is only one even, one-homogeneous, translationally polynomial valuation from $\mathcal{P}(\mathbb{Z}^1)$ to $\mathbb{R}[x]_r$. Thus, $Z' = cL_1^r$ for some $c \in \mathbb{R}$. It follows that the valuation $Y := Z_1^r - cL_1^r$ is simple. Lemma 24 from [19] yields that $Y([0, 1]^2) = 0$. Since we have $[0, 1]^2 = \mathbf{T} \sqcup (-\mathbf{T} + e_1 + e_2)$, it follows from the assumption that r is even, and the fact that Y is simple, translation invariant, and $\text{GL}_2(\mathbb{Z})$ equivariant that $Y(\mathbf{T}) = 0$. But this also implies $Y = 0$. So, we have established the statement for one-homogeneous valuations of even rank.

Next, suppose that $i = r = 1$. Then, L_1^1 is called the *discrete Steiner point* and it follows from Theorem 5 of [8] that $\text{Val}_1^1 = \text{span}\{L_1^1\}$. Hence, we proved the statement for $i = 1$.

Finally, let $r > 0$ and $i > 1$ satisfy the proposition's assumptions. Let $Z_i^r \in \text{Val}_i^r$. Then, $r - 1$ and $i - 1$ also satisfy the assumptions. By induction on i , we have $Z_{i-1}^{r-1} = cL_{i-1}^{r-1}$. Since the associated functions are hereditary by Lemma 5, this means that

$$Z_{i-j}^{r-j} = cL_{i-j}^{r-j}$$

for all $0 < j \leq \min\{r, i\}$. Consequently, $Z_i^r - cL_i^r$ is translation invariant. Using the homogeneous decomposition theorem for translation invariant valuations [20], we obtain that the polynomial $m \mapsto (Z_i^r - cL_i^r)(mP)$ has degree 2. This yields $Z_i^r - cL_i^r = 0$ as desired, except for $i = 2$. But then the claim follows from [19, Proposition 23]. $\square$

For $d \geq -2$, we set $\bar{L}_d := \sum_{r \geq 0} L_{r-d}^r$. By Lemma 24 and Lemma 25, we have $\bar{L}_d \in \overline{\text{Val}}_d$. Proposition 34 allows us to determine the space Val_d for d odd or non-positive.

Corollary 35. *If $d > 0$ is odd or $d \in \{-2, -1, 0\}$, then $\overline{\text{Val}}_d = \text{span}\{\bar{L}_d\}$. Moreover, $\bar{L}_d([0, e_1])$ is not identically zero in these cases and $\bar{L}_d([0, e_1]) \neq 0$ unless $d = -2$.*

Proof. The fact that $\bar{L}_d([0, e_1]) \neq 0$ for $d = 0$ or d odd follows from $L_1^{d+1}([0, e_1]) \neq 0$. To see that $\bar{L}_{-2}$ is not identically zero, it suffices to note that its constant part is L_2^0 , the two-dimensional volume.

Let $\bar{Z} \in \overline{\text{Val}}_d$ and consider for $r > d$ its r -summand Z^r . By Lemma 25, we have $Z^r \in \text{Val}_{r-d}^r$. By the assumption on d , Proposition 34 applies, so there exists a constant $c_r \in \mathbb{R}$ with $Z^r = c_r L_{r-d}^r$. By Lemma 24, the valuations Z^r for $r \geq 0$ are associated functions of one another. Since also L_{r-d}^r for $r \geq 0$ are associated functions of one another, we see that $c_r = c$ for all $r \geq 0$ and some $c \in \mathbb{R}$ that does not depend on r . Thus, $\bar{Z} = c \bar{L}_d$, as claimed. $\square$

9. PROOFS OF THE MAIN RESULTS

Here, we summarize how the results of the previous sections imply the main results Theorem 2 and Theorem 4.

Proof of Theorem 2. The first case follows from Proposition 34 for $i \geq 1$, and from Theorem 1 for $i = r = 0$. The second case is Corollary 17. The third case is Corollary 33. The fourth case consists of the two subcases, $i = 0 < r$ and $i > r + 2$. In the former case, the statement follows from Lemma 8, and in the latter case, it follows from Theorem 6. $\square$

Proof of Theorem 3. The polynomials $\mathbf{f}_{k,\ell} := \mathbf{p}_2^k \mathbf{p}_3^\ell$ with $2k + 3\ell = r$ form a basis of $\mathbb{R}[x, y]_r^{\mathbf{G}}$ by Lemma 16. By Theorem 15, the evaluation map Θ^r is an isomorphism from Val_1^r to $\mathbb{R}[x, y]_r^{\mathbf{G}}$, if $r > 1$ is odd. Hence,

$$\begin{aligned} \text{Val}_1^r &= \text{span}\{(\Theta^r)^{-1}(\mathbf{f}_{k,\ell}) : 2k + 3\ell = r\} \\ &= \text{span}\{Z_{\mathbf{f}_{k,\ell}} : 2k + 3\ell = r\}, \end{aligned}$$

where $Z_{\mathbf{f}_{k,\ell}}$ is given by (11). Note that $L_1^r(\mathbf{T}) \in \mathbb{R}[x, y]_r^{\mathbf{G}}$ also if r is even. From Lemma 16, we obtain $\mathbb{R}[x, y]_r^{\mathbf{G}} = \text{span}\{\mathbf{p}_r\}$ for $r \in \{2, 3\}$. Hence, $\mathbf{f}_{k,\ell}$ is a multiple of $L_1^2(\mathbf{T})^k L_1^3(\mathbf{T})^\ell$. By [19, Lemma 28], this multiple is non-zero, which concludes the proof. $\square$

Proof of Theorem 4. The statement on the dilative decomposition is proven in Proposition 26. For the determination of dimensions, the first case is proved in Corollary 35, the second case in Corollary 33, and the third case in Lemma 25. $\square$

10. CONSEQUENCES

We collect some consequences of our results.

Minkowski additive functions. Theorem 3 can be rewritten as a classification of Minkowski additive functions on $\mathcal{P}(\mathbb{Z}^2)$. A function Z on $\mathcal{P}(\mathbb{Z}^2)$ with values in a vector space is *Minkowski additive* if

$$Z(P + Q) = Z(P) + Z(Q)$$

for all $P, Q \in \mathcal{P}(\mathbb{Z}^2)$, where $P + Q = \{v + w : v \in P, w \in Q\}$ is the Minkowski sum of P and Q . Since

$$P + Q = (P \cup Q) + (P \cap Q)$$

for $P, Q \in \mathcal{P}(\mathbb{Z}^2)$ such that also $P \cap Q, P \cup Q \in \mathcal{P}(\mathbb{Z}^2)$, every Minkowski additive function is a valuation, which is easily seen to be one-homogeneous. We remark that every translation invariant, one-homogeneous valuation is Minkowski additive (cf. [20, Theorem 6]).

Corollary 36. *A function $Z: \mathcal{P}(\mathbb{Z}^2) \rightarrow \mathbb{R}[x, y]$ is translation invariant, $\mathrm{GL}_2(\mathbb{Z})$ equivariant, and Minkowski additive if and only if there are $m \in \mathbb{Z}_{\geq 0}$ and constants $c_r, c_{k,\ell} \in \mathbb{R}$ with $r, k, \ell \in \mathbb{Z}_{\geq 0}$ such that*

$$Z(P) = \sum_{r=0}^m c_{2r} L_1^{2r}(P) + \sum_{r=0}^3 c_{2r+1} L_1^{2r+1}(P) + \sum_{r=4}^m \sum_{2k+3\ell=2r+1} c_{k,\ell} L_1^{2k,3\ell}(P)$$

for every $P \in \mathcal{P}(\mathbb{Z}^2)$.

Here, we have used again that $p_{2,3}(r) = 1$ for $r \in \{3, 5, 7\}$.

Simple, translatively exponential valuations. By Lemma 13, each valuation in Val_i^{d+i} is simple, if $d > 0$ is even. Hence, Lemma 24 implies that each valuation in $\overline{\mathrm{Val}}_d$ is also simple. In fact, for each i , the i -summand Z_i of a valuation $\bar{Z} \in \overline{\mathrm{Val}}_d$ is in Val_i^{d+i} and thus simple by (17).

If $d \geq -2$ is *not* a positive and even number, then, by Corollary 35, we have $\overline{\mathrm{Val}}_d = \mathrm{span}\{\bar{L}_d\}$. If $d = -2$, the valuation $\bar{L}_{(-2)}$ is, by definition, the sum of the top-degree components L_{r+2}^r in (1). Since these correspond to the (continuous) moments by [19, Lemma 26], we have

$$(26) \quad \bar{L}_{(-2)}(P) = \sum_{r \geq 0} L_{r+2}^r(P) = \sum_{r \geq 0} \frac{1}{r!} \int_P v^r dv = \int_P e^v dv.$$

This valuation is called *the* exponential valuation in [2]. We introduced it as $\bar{E}(P)$ in the introduction. It is the Laplace transform of the indicator function of $-P$ (see [18]). For our purposes, it is interesting to observe that (26) implies that $\bar{L}_{(-2)}$ is simple.

For $d > -2$ odd and for $d = 0$, the valuations $\bar{L}_d$ are not simple by Corollary 35. Since also their first non-zero r -summand is not simple by Proposition 34, it follows that no sum of the valuations $\bar{L}_d$, where d is odd or zero, is simple. Hence, we obtain the following characterization of simple, translatively exponential, $\mathrm{GL}_2(\mathbb{Z})$ equivariant valuations from Theorem 4.

Corollary 37. *We have*

$$\{\bar{Z} \in \overline{\mathrm{Val}} : \bar{Z} \text{ is simple}\} = \prod_{\substack{d \geq -2, \\ d \text{ even}, d \neq 0}} \overline{\mathrm{Val}}_d.$$

Thus, the space of simple, translatively exponential, $\mathrm{GL}_2(\mathbb{Z})$ equivariant valuations consists of the valuations we constructed in Section 7 together with $\bar{L}_{(-2)}$. However, one should not regard $\bar{L}_{(-2)}$ as an “exceptional” valuation. In fact, it can be obtained in the same way as the valuations in $\overline{\mathrm{Val}}_d$ with $d > 0$ even by choosing $R(x, y) = \frac{1}{xy}$ in (18) and using the previous construction from there on. For this particular case, it is also described in [2, Chapter 8] how to evaluate $\bar{L}_{(-2)}(P)$ based on an extension to rational polyhedra.

Acknowledgments. A. Freyer and M. Ludwig were supported, in part, by the Austrian Science Fund (FWF): 10.55776/P34446. We used **SageMath** [25] to produce examples that motivated our results.

REFERENCES

- [1] BAJO, E., DAVIS, R., DE LOERA, J., GARBER, A., GARZÓN MORA, S., JOCHEMKO, K., AND YU, J. Weighted Ehrhart theory: Extending Stanley’s nonnegativity theorem. *Adv. Math.* 444 (2024), Paper No. 109627.
- [2] BARVINOK, A. *Integer Points in Polyhedra*. Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2008.
- [3] BECK, M., AND ROBINS, S. *Computing the Continuous Discretely*. Undergraduate Texts in Mathematics. Springer, New York, 2007.
- [4] BERG, S., JOCHEMKO, K., AND SILVERSTEIN, L. Ehrhart tensor polynomials. *Linear Algebra Appl.* 539 (2018), 72–93.
- [5] BETKE, U., AND KNESER, M. Zerlegungen und Bewertungen von Gitterpolytopen. *J. Reine Angew. Math.* 358 (1985), 202–208.
- [6] BÖRÖCZKY, K. J., DOMOKOS, M., HABERL, C., HARCOS, G., AND LI, J. Exponential valuations on lattice polygons. *Preprint* (2024).
- [7] BÖRÖCZKY, K. J., AND LUDWIG, M. Valuations on lattice polytopes. In *Tensor valuations and their applications in stochastic geometry and imaging*, vol. 2177 of *Lecture Notes in Math.* Springer, Cham, 2017, pp. 213–234.
- [8] BÖRÖCZKY, K. J., AND LUDWIG, M. Minkowski valuations on lattice polytopes. *J. Eur. Math. Soc. (JEMS)* 21 (2019), 163–197.
- [9] BRUNS, W., AND GUBELADZE, J. *Polytopes, Rings, and K-theory*. Springer Monographs in Mathematics. Springer, Dordrecht, 2009.
- [10] EHRHART, E. Sur les polyèdres rationnels homothétiques à n dimensions. *C. R. Acad. Sci. Paris* 254 (1962), 616–618.
- [11] GRUBER, P. M. *Convex and Discrete Geometry*, vol. 336 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Berlin, 2007.
- [12] JOCHEMKO, K. A brief introduction to valuations on lattice polytopes. In *Algebraic and geometric combinatorics on lattice polytopes*. World Sci. Publ., Hackensack, NJ, 2019, pp. 38–55.
- [13] JOCHEMKO, K., AND SANYAL, R. Combinatorial mixed valuations. *Adv. Math.* 319 (2017), 630–652.
- [14] JOCHEMKO, K., AND SANYAL, R. Combinatorial positivity of translation-invariant valuations and a discrete Hadwiger theorem. *J. Eur. Math. Soc. (JEMS)* 20 (2018), 2181–2208.
- [15] KHOVANSKIĬ, A. G., AND PUKHLIKOV, A. V. Finitely additive measures of virtual polyhedra. *Algebra i Analiz* 4 (1992), 161–185.
- [16] LAWRENCE, J. Rational-function-valued valuations on polyhedra. In *Discrete and computational geometry (New Brunswick, NJ, 1989/1990)*, vol. 6 of *DIMACS Ser. Discrete Math. Theoret. Comput. Sci.* Amer. Math. Soc., Providence, RI, 1991, pp. 199–208.
- [17] LAWSON, C. Transforming triangulations. *Discrete Math.* 3 (1972), 365–372.
- [18] LI, J., AND MA, D. Laplace transforms and valuations. *J. Funct. Anal.* 272 (2017), 738–758.
- [19] LUDWIG, M., AND SILVERSTEIN, L. Tensor valuations on lattice polytopes. *Adv. Math.* 319 (2017), 76–110.
- [20] MCMULLEN, P. Valuations and Euler-type relations on certain classes of convex polytopes. *Proc. London Math. Soc. (3)* 35 (1977), 113–135.
- [21] MCMULLEN, P. Valuations on lattice polytopes. *Adv. Math.* 220 (2009), 303–323.
- [22] MCMULLEN, P. *Convex Polytopes and Polyhedra*. Preprint, 2024.
- [23] NEUSEL, M. D., AND SMITH, L. *Invariant Theory of Finite Groups*, vol. 94 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2002.
- [24] ODA, T. *Convex Bodies and Algebraic Geometry*, vol. 15 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1988.
- [25] THE SAGE DEVELOPERS. *SageMath, the Sage Mathematics Software System (Version 10.4)*, 2024. <https://www.sagemath.org>.

INSTITUT FÜR DISKRETE MATHEMATIK UND GEOMETRIE, TECHNISCHE UNIVERSITÄT WIEN,
WIEDNER HAUPTSTRASSE 8-10/1046, 1040 WIEN, AUSTRIA

Email address: `ansgar.freyer@tuwien.ac.at`

INSTITUT FÜR DISKRETE MATHEMATIK UND GEOMETRIE, TECHNISCHE UNIVERSITÄT WIEN,
WIEDNER HAUPTSTRASSE 8-10/1046, 1040 WIEN, AUSTRIA

Email address: `monika.ludwig@tuwien.ac.at`

INSTITUT FÜR DISKRETE MATHEMATIK UND GEOMETRIE, TECHNISCHE UNIVERSITÄT WIEN,
WIEDNER HAUPTSTRASSE 8-10/1046, 1040 WIEN, AUSTRIA

Email address: `martin.rubey@tuwien.ac.at`