

ON THE SEMICONTINUITY OF FUNCTIONALS ON FUNCTION SPACES

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ABSTRACT. Results on the upper and lower semicontinuity of functionals defined on spaces of convex and more general functions are established. In particular, the following result is obtained. Let $\phi(v; \cdot)$ be the density of the absolutely continuous part of a Radon measure $\Phi(v; \cdot)$ associated to a function $v: X \rightarrow \mathbb{R}$ defined on the topological measure space (X, λ) . For concave $\zeta: [0, \infty) \rightarrow [0, \infty)$ with $\lim_{t \rightarrow 0} \zeta(t) = 0$ and $\lim_{t \rightarrow \infty} \zeta(t)/t = 0$, it is shown that the functional $v \mapsto \int_X \zeta(\phi(v; x)) d\lambda(x)$ depends upper semicontinuously on v . Examples include functional affine surface areas for convex functions.

1. INTRODUCTION

For a non-empty, compact, convex $K \subset \mathbb{R}^{n+1}$, define the affine surface area as

$$\int_{\text{bd } K} \kappa(K, x)^{\frac{1}{n+2}} dx,$$

where $\kappa(K, x)$ is the generalized Gaussian curvature of K at $x \in \text{bd } K$, the boundary of K , and dx stands for integration with respect to the n -dimensional Hausdorff measure on the boundary of K . Let \mathcal{K}^{n+1} denote the space of convex bodies (non-empty, compact, convex sets) in \mathbb{R}^{n+1} equipped with the topology induced by the Hausdorff metric. Since affine surface area vanishes on polytopes, it is not continuous on \mathcal{K}^{n+1} . However, as shown by Lutwak [18], affine surface area is an upper semicontinuous functional on \mathcal{K}^{n+1} . The upper semicontinuity of a wider class of curvature functionals was established in [14, 19].

Let D be a bounded convex domain of \mathbb{R}^n and $v \in C^2(D)$ convex. The affine surface area of the epi-graph of v is

$$(1) \quad \int_D \det(D^2 v(x))^{\frac{1}{n+2}} dx,$$

where $D^2 v(x)$ is the Hessian matrix of v at x and dx stands for integration with respect to n -dimensional Lebesgue measure. By a classical theorem of Aleksandrov (see [10, Theorem 6.4.1]), a convex function v is twice differentiable almost everywhere, and the integral in (1) is well-defined (but possibly infinite).

In [25, Lemma 6.4], Trudinger and Wang show that the functional defined in (1) is upper semicontinuous for a given bounded convex domain D with respect to locally uniform convergence, using that $\det(D^2 v)$ is the density of the absolutely continuous part of the Monge–Ampère measure of v and the weak*-continuity of Monge–Ampère measures (see Section 2). For further definitions of functional affine surface area, see, for example, [1, 4, 11, 24].

This paper aims to generalize this result to a specific class of extended real-valued convex functions without restricting to a bounded set D . We also extend the result to more general functionals. The main motivation comes from the quest for classification results of valuations on convex functions. Such classification results have been recently established for continuous valuations on convex functions (see, for example, [5, 7]). On convex bodies, they are classical for continuous valuations (see [23, Chapter 6]) and have also been established for upper semicontinuous valuations in certain cases (see [12, 13, 16, 17]).

Let $\text{Conv}(\mathbb{R}^n)$ be the space of extended real-valued functions $u: \mathbb{R}^n \rightarrow (-\infty, \infty]$ that are convex and lower semicontinuous and such that $u \not\equiv \infty$. The (effective) domain of u is $\text{dom}(u) = \{x \in \mathbb{R}^n : u(x) < \infty\}$ and we write int for interior. We consider the subspace

$$\text{Conv}_{\text{ld}}(\mathbb{R}^n) = \{u \in \text{Conv}(\mathbb{R}^n) : \text{dom}(u) \text{ is compact, } u \text{ is Lipschitz on } \text{int } \text{dom}(u)\}.$$

We use epi-convergence on $\text{Conv}(\mathbb{R}^n)$ and its subspaces, and we also use so-called τ -convergence on $\text{Conv}_{\text{Id}}(\mathbb{R}^n)$ (see Section 2).

First, we state the following result, which proves in particular the τ -upper semicontinuity of the functional defined in (1) on $\text{Conv}_{\text{Id}}(\mathbb{R}^n)$. Let $\text{Conc}([0, \infty))$ be the set of concave functions $\zeta: [0, \infty) \rightarrow [0, \infty)$ such that $\lim_{t \rightarrow 0} \zeta(t) = 0$ and $\lim_{t \rightarrow \infty} \zeta(t)/t = 0$.

Theorem 1. *For $\zeta \in \text{Conc}([0, \infty))$ and $u \in \text{Conv}_{\text{Id}}(\mathbb{R}^n)$,*

$$Z(u) = \int_{\text{dom}(u)} \zeta(\det(D^2 u(x))) dx$$

is finite, and

$$Z(u) \geq \limsup_{k \rightarrow \infty} Z(u_k)$$

for every sequence of functions u_k in $\text{Conv}_{\text{Id}}(\mathbb{R}^n)$ with uniformly bounded Lipschitz constants on their domains that epi-converges to $u \in \text{Conv}_{\text{Id}}(\mathbb{R}^n)$.

We remark that the restriction to sequences with uniformly bounded Lipschitz constants is necessary (see Example 1).

We will prove Theorem 1 and its generalizations in the dual setting obtained by using the Legendre transform. In the dual setting, we look at convex functions $v: \mathbb{R}^n \rightarrow \mathbb{R}$ whose Monge–Ampère measures have compact supports and at sequences $v_k: \mathbb{R}^n \rightarrow \mathbb{R}$ with uniformly bounded supports of their Monge–Ampère measures. We establish the results in greater generality for densities of the absolutely continuous part of Radon measures associated to functions on topological spaces in Theorem 4 and also provide results on lower semicontinuity in Theorem 6.

2. PRELIMINARIES

We collect results on Radon measures and on convex functions.

2.1. Radon Measures on Metric Spaces. Let (X, λ) be a metric space with a (non-negative) σ -finite Radon measure λ . We will consider σ -finite Radon measures depending on certain functions on X and use the Radon–Nikodým and Lebesgue decomposition theorems (see, for example, [10, Theorems 1.6.2 and 1.6.3]).

Let $\mathcal{F}(X)$ be a space of functions $v: X \rightarrow \mathbb{R}$ equipped with a notion of convergence. For $v \in \mathcal{F}(X)$, let $\Phi(v; \cdot)$ be a Radon measure depending on v defined on Borel sets of X such that we have weak*-star convergence of $\Phi(v_k; \cdot)$ to $\Phi(v; \cdot)$ as the sequence v_k converges to $v \in \mathcal{F}(X)$. Here, we say that $\Phi(v_k; \cdot)$ is weak*-convergent to $\Phi(v; \cdot)$ if for any sequence v_k converging in $\mathcal{F}(X)$ to v , we have

$$\lim_{k \rightarrow \infty} \int_X \beta(x) d\Phi(v_k; x) = \int_X \beta(x) d\Phi(v; x)$$

for every function $\beta \in C_c(X)$, the set of continuous functions with compact support on X . We need the following simple result. For a sequence of functions $v_k \in \mathcal{F}(X)$ converging to v , the weak*-convergence of $\Phi(v_k; \cdot)$ to $\Phi(v; \cdot)$ implies that

$$(2) \quad \limsup_{k \rightarrow \infty} \Phi(v_k; C) \leq \Phi(v; C)$$

for every compact set $C \subset X$ (cf. [2, Theorem 4.5.1]).

By the Lebesgue decomposition theorem, the measure $\Phi(v; \cdot)$ can be decomposed into measures absolutely continuous and singular with respect to λ on X , say $\Phi(v; \cdot) = \Phi^a(v; \cdot) + \Phi^s(v; \cdot)$. By the Radon–Nikodým theorem, we have for the absolutely continuous part

$$\Phi^a(v; A) = \int_A \phi(v; x) d\lambda(x),$$

where $A \subseteq X$ is a λ -measurable set and $\phi(v; \cdot)$ is λ -measurable, while for the singular part, there is a set $C_0 \subset X$ such that $\lambda(C_0) = 0$ and $\Phi^s(v; A \setminus C_0) = 0$, for every λ -measurable set $A \subseteq X$.

2.2. Convex Functions. Let $\text{Conv}(\mathbb{R}^n; \mathbb{R})$ denote the space of finite-valued, convex functions on \mathbb{R}^n , and we also consider the space of Lipschitz functions

$$\text{Conv}_{\text{lip}}(\mathbb{R}^n; \mathbb{R}) = \{v \in \text{Conv}(\mathbb{R}^n; \mathbb{R}) : v \text{ is Lipschitz}\}$$

and the space

$$\text{Conv}_{\text{bd}}(\mathbb{R}^n) = \{u \in \text{Conv}(\mathbb{R}^n) : \text{dom}(u) \text{ is bounded}\}.$$

We say that a sequence $u_k \in \text{Conv}(\mathbb{R}^n)$ or any of its subspaces epi-converges to $u \in \text{Conv}(\mathbb{R}^n)$ if for every sequence x_k that converges to x , we have

$$u(x) \leq \liminf_{k \rightarrow \infty} u_k(x_k),$$

and there exists a sequence x_k that converges to x such that

$$u(x) = \lim_{k \rightarrow \infty} u_k(x_k).$$

Epi-convergence is directly related to uniform convergence since u_k epi-converges to u if and only if u_k converges uniformly to u on every compact set that does not contain a boundary point of $\text{dom}(u)$ (see [21, Theorem 7.17]). In this paper, we also use the following notion of convergence: For $u, u_k \in \text{Conv}_{\text{ld}}(\mathbb{R}^n)$ with $k \in \mathbb{N}$, we say that $u_k \xrightarrow{\tau} u$ if

- (i) u_k epi-converges to u ;
- (ii) the Lipschitz constants of u_k are uniformly bounded on $\text{int dom}(u)$ by some constant that does not depend on k .

Condition (ii) for the space of real-valued Lipschitz continuous maps defined on the unit sphere S^{n-1} has been considered in [8, 9] with the additional requirement of uniform convergence on the unit sphere.

Example 1. The assumption of uniformly bounded Lipschitz constants is necessary for Theorem 1 to hold. For $x \in \mathbb{R}^n$, consider

$$u_k(x) = k \sum_{i=1}^n \langle x, x \rangle + \mathbf{I}_{B^n}(x),$$

where B^n is the n -dimensional Euclidean unit ball, $\langle \cdot, \cdot \rangle$ is the inner product on \mathbb{R}^n , and $\mathbf{I}_{B^n}(x) = 0$ if $x \in B^n$ and ∞ otherwise. Then u_k epi-converges to the function $\mathbf{I}_{\{0\}}$, which is ∞ for every nonzero x . This shows that Theorem 1 fails if we replace τ -convergence by epi-convergence since Z is, in general, not upper semicontinuous with respect to epi-convergence.

For $u \in \text{Conv}(\mathbb{R}^n)$, let $\partial u(x)$ be the subdifferential of u at $x \in \text{dom}(u)$,

$$\partial u(x) = \{y \in \mathbb{R}^n : u(z) \geq u(x) + \langle y, z - x \rangle \text{ for all } z \in \mathbb{R}^n\},$$

and set $\partial u(x) = \emptyset$ for $x \notin \text{dom}(u)$. Given a subset $A \subset \mathbb{R}^n$, the image of A through the subdifferential of u is defined as

$$\partial u(A) = \bigcup_{x \in A} \partial u(x).$$

For $v \in \text{Conv}(\mathbb{R}^n; \mathbb{R})$, the Monge–Ampère measure is given by

$$(3) \quad \text{MA}(v; B) = V_n(\partial v(B))$$

for every Borel set $B \subseteq \mathbb{R}^n$, where V_n is the n -dimensional volume. It is a Radon measure on \mathbb{R}^n . Let

$$\text{Conv}_{\text{MA}}(\mathbb{R}^n; \mathbb{R}) = \{v \in \text{Conv}(\mathbb{R}^n; \mathbb{R}) : \text{supp}(\text{MA}(v; \cdot)) \text{ is compact}\},$$

where supp stands for support.

Let $K \subset \mathbb{R}^{n+1}$ be a non-empty, closed, convex set with support function $h_K : \mathbb{R}^{n+1} \rightarrow (-\infty, \infty]$, that is,

$$h_K(x, x_{n+1}) = \sup \{ \langle (x, x_{n+1}), (y, y_{n+1}) \rangle : (y, y_{n+1}) \in K \}$$

for $x \in \mathbb{R}^n$ and $x_{n+1} \in \mathbb{R}$, where the inner product $\langle \cdot, \cdot \rangle$ is taken in \mathbb{R}^{n+1} . Note that if $K \in \mathcal{K}^{n+1}$, we have $h_K(\cdot, -1) \in \text{Conv}(\mathbb{R}^n; \mathbb{R})$. Let

$$\text{Conv}_{\text{MA,h}}(\mathbb{R}^n; \mathbb{R}) = \{h_K(\cdot, -1) \in \text{Conv}_{\text{MA}}(\mathbb{R}^n; \mathbb{R}) : K \in \mathcal{K}^{n+1}\}.$$

The inclusion $\text{Conv}_{\text{MA,h}}(\mathbb{R}^n; \mathbb{R}) \subset \text{Conv}_{\text{MA}}(\mathbb{R}^n; \mathbb{R})$ is strict. Functions $v \in \text{Conv}_{\text{MA,h}}(\mathbb{R}^n; \mathbb{R})$ may have $\text{dom } v^*$ not full-dimensional but always bounded, whereas in $\text{Conv}_{\text{MA}}(\mathbb{R}^n; \mathbb{R})$ the domain of v^* can be unbounded, which only occurs when it is not full-dimensional.

We equip $\text{Conv}_{\text{MA}}(\mathbb{R}^n; \mathbb{R})$ with the following notion of convergence: we say that a sequence v_k in $\text{Conv}_{\text{MA}}(\mathbb{R}^n; \mathbb{R})$ is τ^* -convergent to $v \in \text{Conv}_{\text{MA}}(\mathbb{R}^n; \mathbb{R})$ if

- (i) v_k epi-converges to v ;
- (ii) there exists a compact set in \mathbb{R}^n containing the supports of $\text{MA}(u; \cdot)$ and $\text{MA}(u_k; \cdot)$ for all $k \in \mathbb{N}$.

In this case we write $v_k \xrightarrow{\tau^*} v$. The notation will be justified in Lemma 3.

3. THE DUAL SPACE

For $u \in \text{Conv}(\mathbb{R}^n)$, the Legendre transform or convex conjugate of u is the function $u^* : \mathbb{R}^n \rightarrow (-\infty, \infty]$ given as

$$u^*(y) = \sup\{\langle x, y \rangle - u(x) : x \in \mathbb{R}^n\}$$

for $y \in \mathbb{R}^n$. If $u \in \text{Conv}_{\text{bd}}(\mathbb{R}^n)$, then $u^* \in \text{Conv}(\mathbb{R}^n; \mathbb{R})$ (see [21, Theorem 11.8], and $u^{**} = u$ (see [23, Theorem 1.6.13])). Note that convex conjugation is continuous w.r.t. epi-convergence,

$$(4) \quad u_k \text{ is epi-convergent to } u \Leftrightarrow u_k^* \text{ is epi-convergent to } u^*$$

for $u_k, u \in \text{Conv}(\mathbb{R}^n)$ (see [21, Theorem 11.43]).

The epigraph of $u \in \text{Conv}(\mathbb{R}^n)$ is the non-empty, closed, convex set

$$\text{epi}(u) = \{(x, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} \geq u(x)\}.$$

Hence,

$$u^*(y) = \sup\{\langle (y, -1), (x, u(x)) \rangle : x \in \mathbb{R}^n\} = h_{\text{epi}(u)}(y, -1).$$

If $\text{dom } u$ is compact, then there is $K \in \mathcal{K}^{n+1}$ such that $h_K(\cdot, -1) = h_{\text{epi}(u)}(\cdot, -1)$, which can be chosen as the intersection of $\text{epi}(u)$ and $\{x_{n+1} \leq \max_{x \in \text{dom}(u)} u(x)\}$. Hence, the Legendre transform of a function in $\text{Conv}(\mathbb{R}^n)$ with compact domain is of the form

$$(5) \quad h_K(\cdot, -1) \text{ with } K \in \mathcal{K}^{n+1}.$$

For the space of Legendre transforms of functions from $\text{Conv}_{\text{ld}}(\mathbb{R}^n)$, we write

$$(6) \quad \text{Conv}_{\text{ld}}^*(\mathbb{R}^n) = \{u^* : u \in \text{Conv}_{\text{ld}}(\mathbb{R}^n)\}$$

and use corresponding notation for further spaces. By [20, Corollary 13.3.3],

$$(7) \quad \text{Conv}_{\text{lip}}^*(\mathbb{R}^n; \mathbb{R}) = \text{Conv}_{\text{bd}}(\mathbb{R}^n).$$

Using this in its dual form and (5), we obtain

$$(8) \quad \text{Conv}_{\text{MA,h}}(\mathbb{R}^n; \mathbb{R}) \subset \text{Conv}_{\text{lip}}(\mathbb{R}^n; \mathbb{R}).$$

By [20, Corollary 23.5.1], we have the following relationship

$$(9) \quad p \in \partial u(x) \Leftrightarrow x \in \partial u(p)$$

for $u \in \text{Conv}(\mathbb{R}^n)$. If both u and u^* are differentiable, then we can rewrite (9) as $\nabla u^*(\nabla u(p)) = p$. For u and u^* twice differentiable and $\det(D^2 u(p)) > 0$, differentiating gives

$$\det(D^2 u^*(\nabla u(p))) = \frac{1}{\det(D^2 u(p))}.$$

Thus, if $u \in \text{Conv}_{\text{Id}}(\mathbb{R}^n)$ is such that $\det(D^2 u(x)) > 0$ for all $x \in \text{dom}(u)$ and $\zeta \in \text{Conc}([0, \infty))$, we have

$$\begin{aligned}
\int_{\text{dom}(u)} \zeta(\det(D^2 u(x))) \, dx &= \int_{\text{dom}(u)} \zeta\left(\frac{1}{\det(D^2 u^*(\nabla u(x)))}\right) \, dx \\
(10) \qquad \qquad \qquad &= \int_{\mathbb{R}^n} \zeta\left(\frac{1}{\det(D^2 u^*(y))}\right) \det(D^2 u^*(y)) \, dy \\
&= \int_{\mathbb{R}^n} \tilde{\zeta}(\det(D^2 u^*(x))) \, dx,
\end{aligned}$$

where $\tilde{\zeta}(t) = \zeta(1/t)t$ for $t \geq 0$. Note that we also have $\tilde{\zeta} \in \text{Conc}([0, \infty))$. Using (9), we observe that the last equality holds for every $u \in \text{Conv}_{\text{Id}}(\mathbb{R}^n)$. Indeed, let $u \in \text{Conv}_{\text{Id}}(\mathbb{R}^n)$ and let U be the set of points $x \in \text{dom}(u)$ such that $\det(D^2 u(x)) = 0$. By assumption, U also contains all points where u is not twice differentiable, since we set $\det(D^2 u(x)) = 0$ at such points. If $U_1 \subset U$ is the set of points where u is not twice differentiable, then by Aleksandrov's theorem, $V_n(U_1) = 0$. If $D = \partial u(U_1)$, we get by (3) that $\text{MA}(u^*; D) = V_n(U_1) = 0$, which implies $\text{MA}^a(u^*; D) = 0$. Now, if there exists a convex set $U_2 \subset U$ such that $\det(D^2 u(x)) = 0$ for all $x \in U_2$ and $V_n(U_2) > 0$, then there exists a vector $p \in \mathbb{R}^n$ such that $p \in \partial u(U_2)$ and by (9) we have $U_2 \subset \partial u^*(p)$, that is, u^* is not twice differentiable at p . Using (10), we conclude that

$$\begin{aligned}
\int_{\text{dom}(u)} \zeta(\det(D^2 u(x))) \, dx &= \int_U \zeta(\det(D^2 u(x))) \, dx + \int_{\text{dom}(u) \setminus U} \zeta(\det(D^2 u(x))) \, dx \\
(11) \qquad \qquad \qquad &= \int_{\mathbb{R}^n} \tilde{\zeta}(\det(D^2 u^*(x))) \, dx.
\end{aligned}$$

Let $u_k, u \in \text{Conv}_{\text{Id}}(\mathbb{R}^n)$. We aim to show that if the functions $u_k, u \in \text{Conv}_{\text{Id}}(\mathbb{R}^n)$ and $u_k \xrightarrow{\tau} u$, then the functions $u_k^*, u^* \in \text{Conv}_{\text{MA},h}(\mathbb{R}^n; \mathbb{R})$ and $u_k^* \xrightarrow{\tau^*} u^*$. Additionally, we will prove the reverse implication. To do this, we will use the following result (see, for example, [3, Theorem 3.61]).

Lemma 2. A function $u \in \text{Conv}(\mathbb{R}^n)$ has a Lipschitz constant L in the interior of its domain if and only if $\|p\| \leq L$ for every $p \in \partial u(x)$, where $x \in \text{int dom}(u)$.

Here, $\|\cdot\|$ denotes the Euclidean norm on \mathbb{R}^n . We now prove our main lemma.

Lemma 3. A sequence u_k in $\text{Conv}_{\text{Id}}(\mathbb{R}^n)$ is τ -convergent to $u \in \text{Conv}_{\text{Id}}(\mathbb{R}^n)$ if and only if u_k^* and u^* belong to $\text{Conv}_{\text{MA},h}(\mathbb{R}^n; \mathbb{R})$ and u_k^* is τ^* -convergent to u^* .

Proof. First, we show that for $\text{Conv}_{\text{MA},h}^*(\mathbb{R}^n; \mathbb{R})$, defined similar to (6),

$$(12) \qquad \qquad \qquad \text{Conv}_{\text{MA},h}^*(\mathbb{R}^n; \mathbb{R}) \subset \text{Conv}_{\text{Id}}(\mathbb{R}^n).$$

Let $u \in \text{Conv}_{\text{MA},h}^*(\mathbb{R}^n; \mathbb{R})$. By (8) and (7), we see that $u \in \text{Conv}_{\text{bd}}(\mathbb{R}^n)$. Since u is the Legendre transform of $h_K(\cdot, -1)$ for some $K \in \mathcal{K}^{n+1}$, the domain of u is compact. If $\text{int dom}(u)$ is empty, then we do not need the Lipschitz condition on u and hence $u \in \text{Conv}_{\text{Id}}(\mathbb{R}^n)$. For $\text{int dom}(u)$ non-empty, assume, for the sake of contradiction, that u is not Lipschitz on $\text{int dom}(u)$. By Lemma 2, for every constant $M > 0$, there exists $z \in \partial u(\text{int dom}(u))$ such that $\|z\| > M$. Using the continuity and convexity of u , we deduce that for every sufficiently small neighborhood $D \subset \mathbb{R}^n$ of z , there exists a set $U \subset \text{int dom}(u)$ with positive Lebesgue measure such that $D \subset \partial u(U)$. By (9),

$$D \subset \partial u(U) \quad \Leftrightarrow \quad U \subset \partial u^*(D),$$

which implies that $\text{MA}(u^*; D) > 0$. Since this holds for every sufficiently small neighborhood of z , we obtain that $z \in \text{supp}(\text{MA}(u^*; \cdot))$. Therefore, we have $\text{supp}(\text{MA}(u^*; \cdot)) \not\subset MB^n$. Since $M > 0$ was arbitrary, this leads to a contradiction to $u^* \in \text{Conv}_{\text{MA},h}(\mathbb{R}^n; \mathbb{R})$. Hence, we conclude that u is Lipschitz on $\text{int dom}(u)$ and, thus, $u \in \text{Conv}_{\text{Id}}(\mathbb{R}^n)$.

Now, assume that $u_k^*, u^* \in \text{Conv}_{\text{MA},h}(\mathbb{R}^n; \mathbb{R})$ for every $k \in \mathbb{N}$ and that $u_k^* \xrightarrow{\tau^*} u^*$ as $k \rightarrow \infty$. From (12), it follows that $u_k, u \in \text{Conv}_{\text{Id}}(\mathbb{R}^n)$ for every $k \in \mathbb{N}$. Let MB^n be the ball containing the

support of the Monge–Ampère measures of u_k^* and u^* for every $k \in \mathbb{N}$. We have seen that

$$\partial u_k(\text{int dom}(u_k)) \not\subseteq MB^n \quad \Rightarrow \quad \text{supp}(\text{MA}(u_k^*; \cdot)) \not\subseteq MB^n.$$

Consequently, we have $\partial u_k(\text{int dom}(u_k)), \partial u(\text{int dom}(u)) \subseteq MB^n$. By Lemma 2, this is equivalent to saying that the Lipschitz constants of u_k and u are bounded by M . Using (4), we conclude that $u_k \xrightarrow{\tau} u$.

For the converse, assume that $u \in \text{Conv}_{\text{Id}}(\mathbb{R}^n)$ with Lipschitz constant L_u . By [20, p. 227], we have

$$(13) \quad \text{MA}(u^*; \mathbb{R}^n) = V_n(\partial u^*(\mathbb{R}^n)) = V_n(\text{dom}(u)).$$

If $V_n(\text{dom}(u)) = 0$, this implies that $\text{supp}(\text{MA}(u^*; \cdot))$ is compact. For the case $V_n(\text{dom}(u)) > 0$, set $D = \partial u(\text{int dom}(u))$, and note that, by [20, p. 227], we have $\text{MA}(u^*; D) = V_n(\text{dom}(u))$. Hence, it follows from (13) that $\text{MA}(u^*; \cdot)$ is supported on D . By Lemma 2, we have $D \subseteq L_u B^n$, which implies that

$$(14) \quad \text{supp}(\text{MA}(u^*; \cdot)) \subseteq L_u B^n.$$

Combined with the fact that u has compact domain and (5), this implies that $u^* \in \text{Conv}_{\text{MA}, \text{h}}(\mathbb{R}^n; \mathbb{R})$. Hence, $\text{Conv}_{\text{Id}}^*(\mathbb{R}^n) = \text{Conv}_{\text{MA}, \text{h}}(\mathbb{R}^n; \mathbb{R})$.

To conclude the proof, it remains to show that if $u_k, u \in \text{Conv}_{\text{Id}}(\mathbb{R}^n)$ for every $k \in \mathbb{N}$ and $u_k \xrightarrow{\tau} u$ as $k \rightarrow \infty$, then $u_k^* \xrightarrow{\tau^*} u^*$. Let $M > 0$ be such that the Lipschitz constants of u_k and u are bounded by M . By (14), we know that

$$\text{supp}(\text{MA}(u^*; \cdot)) \subset L_u B^n \quad \text{and} \quad \text{supp}(\text{MA}(u_k^*; \cdot)) \subset L_{u_k} B^n,$$

where L_u and L_{u_k} are the Lipschitz constants of u and u_k , respectively. Since $L_{u_k}, L_u \leq M$ for every $k \in \mathbb{N}$, we conclude that

$$\text{supp}(\text{MA}(u^*; \cdot)), \text{supp}(\text{MA}(u_k^*; \cdot)) \subset MB^n,$$

and therefore, by (4), we have $u_k^* \xrightarrow{\tau^*} u^*$. This completes the proof. \square

4. UPPER AND LOWER SEMICONTINUITY

Let X be a topological space with σ -finite Radon measure λ . Let $\mathcal{F}(X)$ be a family of real-valued functions on X equipped with a notion of convergence, and $\Phi(v; \cdot)$ a family of σ -finite Radon measures depending weak*-continuously on the functions $v \in \mathcal{F}(X)$. Let $\mathcal{F}_{\Phi}(X)$ denote the set of functions $v \in \mathcal{F}(X)$ such that the support of $\Phi(v; \cdot)$ is compact, and let $\phi(v; \cdot)$ denote the density of the absolutely continuous part of $\Phi(v; \cdot)$ for $v \in \mathcal{F}(X)$. We equip $X \times \mathbb{R}$ with the product topology. The following result is a functional version of Theorem 2 in [14].

Theorem 4. *For $\zeta \in \text{Conc}([0, \infty))$ and non-negative $\omega \in C(X \times \mathbb{R})$,*

$$Z(v) = \int_X \zeta(\phi(v; x)) \omega(x, v(x)) \, d\lambda(x),$$

is finite for every $v \in \mathcal{F}_{\Phi}(X) \cap C(X)$, and

$$Z(v) \geq \limsup_{k \rightarrow \infty} Z(v_k)$$

for every sequence of functions v_k in $\mathcal{F}_{\Phi}(X) \cap C(X)$ with uniformly bounded supports $\text{supp}(\Phi(v_k; \cdot))$ that converges to $v \in \mathcal{F}_{\Phi}(X) \cap C(X)$.

Hence we can say that Z is τ -upper semicontinuous on $\mathcal{F}_\Phi(X) \cap C(X)$. In the proof of Theorem 4 and Theorem 6, we will use Jensen's inequality, which guarantees that for a concave function $\zeta: [0, \infty) \rightarrow \mathbb{R}$ and an integrable function $v: C \rightarrow \mathbb{R}$, where C is a compact subset of X with positive λ -measure,

$$\frac{1}{\lambda(C)} \int_C \zeta(v(x)) \, d\lambda(x) \leq \zeta \left(\frac{1}{\lambda(C)} \int_C v(x) \, d\lambda(x) \right).$$

(cf. [22, Theorem 3.3]). If ζ is convex, the inequality is reversed.

Note that $\zeta \in \text{Conc}([0, \infty))$ is continuous on $[0, \infty)$ and that $\zeta(0) = 0$. Since ζ is concave and non-negative on $[0, \infty)$, the function ζ is non-decreasing. Using these properties, we get $\zeta(rt) \geq r\zeta(t)$ for every $t > 0$ and $0 < r < 1$. In particular, if we take $s = rt < t$ we obtain

$$\frac{\zeta(s)}{s} \geq \frac{\zeta(t)}{t},$$

and this means that $t \mapsto \zeta(t)/t$ is non-increasing.

Proof of Theorem 4. Let $\varepsilon > 0$ be given, $v \in \mathcal{F}_\Phi(X) \cap C(X)$, and $C \subset X$ compact. Let C_0 be the set where the singular part of $\Phi(v; \cdot)$ is concentrated. Since $\lambda(C_0) = 0$ and $\phi(v; \cdot)$ is measurable λ -a.e. on C , we can choose by Lusin's theorem (see, for example, [2, Corollaries 4.3.17]) a compact set $C_\varepsilon \subset C$ where $\phi(v; \cdot)$ is continuous as a function restricted to C_ε such that $C_\varepsilon \cap C_0 = \emptyset$ and

$$(15) \quad \lambda(C \setminus C_\varepsilon) \leq \varepsilon.$$

Let v_k be a sequence in $\mathcal{F}_\Phi(X) \cap C(X)$ converging to v . First, we show that

$$(16) \quad \limsup_{k \rightarrow \infty} \int_{C_\varepsilon} \zeta(\phi(v_k; x)) \omega(x, v_k(x)) \, d\lambda(x) \leq \int_{C_\varepsilon} \zeta(\phi(v; x)) \omega(x, v(x)) \, d\lambda(x)$$

holds. Set $a = \inf\{\phi(v; x) : x \in C_\varepsilon\}$ and $b = \sup\{\phi(v; x) : x \in C_\varepsilon\}$. Note that $b < \infty$ since $\phi(v; \cdot)$ is continuous on the compact set C_ε . Hence, ζ is uniformly continuous on $[a, b]$. For $\eta > 0$ given, there exists a number $\delta > 0$ such that

$$(17) \quad |\zeta(s) - \zeta(t)| \leq \eta$$

whenever $s, t \in [a, b]$ are such that $|s - t| \leq \delta$.

Since $\lambda(\{x \in C_\varepsilon : \phi(v; x) = t\}) > 0$ holds only for countably many t , we can choose a subdivision $a = t_1 \leq t_2 \leq \dots \leq t_{m+1} = b$ of $[a, b]$, such that

$$(18) \quad \max_{i=1, \dots, m} (t_{i+1} - t_i) \leq \delta$$

and such that

$$(19) \quad \lambda(\{x \in C_\varepsilon : \phi(v; x) = t_i\}) = 0$$

for $i = 2, \dots, m$. Setting $C_i = \{x \in C_\varepsilon : t_i \leq \phi(v; x) \leq t_{i+1}\}$, so that $C_\varepsilon = \cup_{i=1}^m C_i$, and using the linearity of the integral, that ζ is non-decreasing, and (19), we get

$$(20) \quad \int_{C_\varepsilon} \zeta(\phi(v; x)) \omega(x, v(x)) \, d\lambda(x) \geq \sum_{i=1}^m \zeta(t_i) \int_{C_i} \omega(x, v(x)) \, d\lambda(x).$$

Consider $J_k \subseteq \{1, \dots, m\}$, depending on k such that $\int_{C_i} \omega(x, v_k(x)) \, d\lambda(x) > 0$ precisely if $i \in J_k$. By Jensen's inequality and the concavity of ζ ,

$$\begin{aligned} & \int_{C_i} \zeta(\phi(v_k; x)) \omega(x, v_k(x)) \, d\lambda(x) \\ & \leq \zeta \left(\frac{\int_{C_i} \omega(x, v_k(x)) \phi(v_k; x) \, d\lambda(x)}{\int_{C_i} \omega(x, v_k(x)) \, d\lambda(x)} \right) \int_{C_i} \omega(x, v_k(x)) \, d\lambda(x) \end{aligned}$$

holds for each $i \in J_k$. Hence,

$$\int_{C_\varepsilon} \zeta(\phi(v_k; x)) \omega(x, v_k(x)) \, d\lambda(x) \leq \sum_{i \in J_k} \zeta \left(\frac{\int_{C_i} \omega(x, v_k(x)) \phi(v_k; x) \, d\lambda(x)}{\int_{C_i} \omega(x, v_k(x)) \, d\lambda(x)} \right) \int_{C_i} \omega(x, v_k(x)) \, d\lambda(x).$$

Since $\phi(v; \cdot)$ is continuous on the compact set C_ε , the sets C_i are compact for $i = 1, \dots, m$. Hence, the functions $\omega(\cdot, v_k(\cdot))$ and $\omega(\cdot, v(\cdot))$ are uniformly continuous on C_i . Then, for any $\tilde{\varepsilon} > 0$, there exists k_0 such that for all $k \geq k_0$,

$$\int_{C_i} \omega(x, v_k(x)) d\Phi(v_k; x) \leq \int_{C_i} \omega(x, v(x)) d\Phi(v_k; x) + \tilde{\varepsilon} \Phi(v_k; C_i).$$

Using that $\Phi(v_k; \cdot)$ weak*-converges to $\Phi(v; \cdot)$, and in particular (2), we obtain

$$\limsup_{k \rightarrow \infty} \int_{C_i} \omega(x, v_k(x)) d\Phi(v_k; x) \leq \int_{C_i} \omega(x, v(x)) d\Phi(v; x) + \tilde{\varepsilon} \Phi(v; C_i).$$

Since $\tilde{\varepsilon} > 0$ was arbitrary, it follows that

$$\limsup_{k \rightarrow \infty} \int_{C_i} \omega(x, v_k(x)) d\Phi(v_k; x) \leq \int_{C_i} \omega(x, v(x)) d\Phi(v; x).$$

Hence,

$$\begin{aligned} & \limsup_{k \rightarrow \infty} \int_{C_\varepsilon} \zeta(\phi(v_k; x)) \omega(x, v_k(x)) d\lambda(x) \\ & \leq \limsup_{k \rightarrow \infty} \sum_{i \in J_k} \zeta \left(\frac{\int_{C_i} \phi(v_k; x) \omega(x, v_k(x)) d\lambda(x)}{\int_{C_i} \omega(x, v_k(x)) d\lambda(x)} \right) \int_{C_i} \omega(x, v_k(x)) d\lambda(x) \\ (21) \quad & \leq \sum_{i=1}^m \zeta(t_{i+1}) \int_{C_i} \omega(x, v(x)) d\lambda(x) \\ & = \sum_{i=1}^m (\zeta(t_i) + (\zeta(t_{i+1}) - \zeta(t_i))) \int_{C_i} \omega(x, v(x)) d\lambda(x). \end{aligned}$$

Note that we used the definition of C_i and the monotonicity of ζ in the second inequality. Now using (21), (20), (18), and (17), we conclude that

$$\begin{aligned} \limsup_{k \rightarrow \infty} \int_{C_\varepsilon} \zeta(\phi(v_k; x)) \omega(x, v_k(x)) d\lambda(x) \\ \leq \int_{C_\varepsilon} \zeta(\phi(v; x)) \omega(x, v(x)) d\lambda(x) + \eta \int_{C_\varepsilon} \omega(x, v(x)) d\lambda(x). \end{aligned}$$

Since $\eta > 0$ is arbitrary, this proves (16).

The second step is to show that

$$(22) \quad \limsup_{k \rightarrow \infty} \int_C \zeta(\phi(v_k; x)) \omega(x, v_k(x)) d\lambda(x) \leq \int_C \zeta(\phi(v; x)) \omega(x, v(x)) d\lambda(x).$$

Since ζ is non-decreasing and $\zeta(t)/t$ is non-increasing, we see that for every $t > 0$,

$$\begin{aligned} & \int_{C \setminus C_\varepsilon} \zeta(\phi(v_k; x)) \omega(x, v_k(x)) d\lambda(x) \\ & = \int_{\{x \in C \setminus (C_0 \cup C_\varepsilon) : \phi(v_k; x) \leq t\}} \zeta(\phi(v_k; x)) \omega(x, v_k(x)) d\lambda(x) \\ & \quad + \int_{\{x \in C \setminus (C_0 \cup C_\varepsilon) : \phi(v_k; x) > t\}} \zeta(\phi(v_k; x)) \omega(x, v_k(x)) d\lambda(x) \\ & \leq \zeta(t) \int_{C \setminus C_\varepsilon} \omega(x, v_k(x)) d\lambda(x) + \frac{\zeta(t)}{t} \int_{C \setminus C_\varepsilon} \omega(x, v_k(x)) \phi(v, x) d\lambda(x), \end{aligned}$$

where we use that $\lambda(C_0) = 0$. This implies, combined with (16) that

$$\begin{aligned} \limsup_{k \rightarrow \infty} \int_C \zeta(\phi(v_k; x)) \omega(x, v_k(x)) \, d\lambda(x) &\leq \int_C \zeta(\phi(v; x)) \omega(x, v(x)) \, d\lambda(x) \\ &\quad + \zeta(t) \int_{C \setminus C_\varepsilon} \omega(x, v(x)) \, d\lambda(x) + \frac{\zeta(t)}{t} \int_C \omega(x, v(x)) \, d\Phi(v, x) \end{aligned}$$

for every $t > 0$. Since $\omega(\cdot, v(\cdot))$ is a non-negative continuous function and C is compact, we have $0 \leq \max_{x \in C} \omega(x, v(x)) < \infty$. By this and (15), we obtain

$$\begin{aligned} \limsup_{k \rightarrow \infty} \int_C \zeta(\phi(v_k; x)) \omega(x, v_k(x)) \, d\lambda(x) &\leq \\ &\int_C \zeta(\phi(v; x)) \omega(x, v(x)) \, d\lambda(x) + (\varepsilon \zeta(t) + \frac{\zeta(t)}{t} \Phi(v; C)) \max_{x \in C} \omega(x, v(x)). \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary and since t does not depend on ε , for every $t > 0$

$$(23) \quad \begin{aligned} \limsup_{k \rightarrow \infty} \int_C \zeta(\phi(v_k; x)) \omega(x, v_k(x)) \, d\lambda(x) \\ \leq \int_C \zeta(\phi(v; x)) \omega(x, v(x)) \, d\lambda(x) + \frac{\zeta(t)}{t} \Phi(v; C) \max_{x \in C} \omega(x, v(x)). \end{aligned}$$

Using that $\Phi(v; \cdot)$ is a Radon measure and that $\lim_{t \rightarrow \infty} \zeta(t)/t = 0$, we can make $\zeta(t)/t$ arbitrarily small by choosing t suitably large. Therefore (23) proves (22).

To finish, if $\phi(v; \cdot)$ is supported on the compact set C_v with $\lambda(C_v) > 0$ and $\int_{C_v} \omega(x, v(x)) \, d\lambda(x) > 0$, then using that $\zeta(0) = 0$, the monotonicity of ζ and that there exists a compact set $\tilde{C} \subset X$ such that $C_{v_k}, C_v \subseteq \tilde{C}$ for all $k \in \mathbb{N}$, where C_{v_k} is the support of $\phi(v_k; \cdot)$, we get

$$\int_X \zeta(\phi(v_k; x)) \omega(x, v_k(x)) \, d\lambda(x) = \int_{\tilde{C}} \zeta(\phi(v_k; x)) \omega(x, v_k(x)) \, d\lambda(x)$$

and, using Jensen's inequality,

$$\begin{aligned} \int_X \zeta(\phi(v; x)) \omega(x, v(x)) \, d\lambda(x) &= \int_{\tilde{C}} \zeta(\phi(v; x)) \omega(x, v(x)) \, d\lambda(x) \\ &\leq \zeta \left(\frac{\int_{\tilde{C}} \phi(v; x) \omega(x, v(x)) \, d\lambda(x)}{\int_{\tilde{C}} \omega(x, v(x)) \, d\lambda(x)} \right) \int_{\tilde{C}} \omega(x, v(x)) \, d\lambda(x) \\ &\leq \zeta \left(\frac{\max_{x \in \tilde{C}} \omega(x, v(x)) \Phi(v; \tilde{C})}{\int_{\tilde{C}} \omega(x, v(x)) \, d\lambda(x)} \right) \int_{\tilde{C}} \omega(x, v(x)) \, d\lambda(x). \end{aligned}$$

Using again that $\Phi(v; \cdot)$ is a Radon measure, that \tilde{C} is compact, and that $\omega(\cdot, v(\cdot))$ is a non-negative continuous function, we obtain that the integral is finite. \square

We used the continuity of v_k only to show that $\omega(\cdot, v_k(\cdot))$ is continuous. Hence, the proof of Theorem 4 gives also the following result.

Theorem 5. For $\zeta \in \text{Conc}([0, \infty))$,

$$Z(v) = \int_X \zeta(\phi(v; x)) \, d\lambda(x)$$

is finite for every $v \in \mathcal{F}_\Phi(X)$, and

$$Z(v) \geq \limsup_{k \rightarrow \infty} Z(v_k)$$

for every sequence of functions v_k in $\mathcal{F}_\Phi(X)$ with uniformly bounded supports $\text{supp}(\Phi(v_k; \cdot))$ that converges to $v \in \mathcal{F}_\Phi(X)$.

Let $\text{Cvx}([0, \infty))$ be the set of decreasing convex functions $\zeta : [0, \infty) \rightarrow (0, \infty]$ such that we have $\lim_{t \rightarrow 0} \zeta(t) = \zeta(0)$ and $\lim_{t \rightarrow \infty} \zeta(t) = 0$. The following result is a functional version of [15, Theorem 6].

Theorem 6. Let $\zeta \in \text{Cvx}([0, \infty))$ and $\omega \in C(X \times \mathbb{R})$ be non-negative while $C \subseteq X$ is compact. Define $Z(\cdot, C): \mathcal{F}(X) \cap C(X) \rightarrow [0, \infty]$ by

$$Z(v, C) = \int_C \zeta(\phi(v; x)) \omega(x, v(x)) \, d\lambda(x).$$

If $v \in \mathcal{F}(X) \cap C(X)$, then

$$(24) \quad \liminf_{k \rightarrow \infty} Z(v_k, C) \geq Z(v, C)$$

for every sequence $v_k \in \mathcal{F}(X) \cap C(X)$ converging to $v \in \mathcal{F}(X) \cap C(X)$.

We allow that $\zeta(0) = \infty$. In this case, $Z(v, C) < \infty$ only if C is contained in the support of $\Phi(v; \cdot)$, otherwise, the integrand attains the value ∞ .

In the proof, we will also use the following version of Fatou's Lemma (see, for example, [22, Lemma 1.28]), which says that if μ_c is the counting measure, then

$$(25) \quad \sum_{y \in Y} \liminf_{k \rightarrow \infty} f_k(y) = \int_Y \left(\liminf_{k \rightarrow \infty} f_k(y) \right) \, d\mu_c(y) \leq \liminf_{k \rightarrow \infty} \sum_{y \in Y} f_k(y)$$

where $Y \subseteq \mathbb{Z}$ and $f_k : Y \rightarrow [0, \infty]$ for each $k \in \mathbb{Z}$.

Proof. Similar to the proof of Theorem 4, for $C \subset X$ compact, let $C_0 \subset X$ be the λ -null set, where the singular part of $\Phi(v; \cdot)$ restricted to C is concentrated. Choose by Lusin's theorem pairwise disjoint compact sets $C_l \subseteq C$ for $l \in \mathbb{N}$ such that $\phi(v; \cdot)$ is continuous as a function restricted to C_l and such that for every $l \in \mathbb{N}$

$$(26) \quad C_l \cap C_0 = \emptyset$$

and such that

$$(27) \quad \lambda\left(C \setminus \bigcup_{l=1}^{\infty} C_l\right) = 0.$$

We may assume that $\liminf_{k \rightarrow \infty} Z(v_k, C) < \infty$ and that $Z(v_k, C) < \infty$ for k sufficiently large. Otherwise (24) clearly holds.

Let v_k be a sequence on $\mathcal{F}(X) \cap C(X)$ converging to $v \in \mathcal{F}(X) \cap C(X)$. We will start by proving that for each $l \in \mathbb{N}$ the inequality

$$(28) \quad \liminf_{k \rightarrow \infty} \int_{C_l} \zeta(\phi(v_k; x)) \omega(x, v_k(x)) \, d\lambda(x) \geq \int_{C_l} \zeta(\phi(v; x)) \omega(x, v(x)) \, d\lambda(x)$$

holds. Let $\eta > 0$. We choose an increasing monotone sequence $t_i \in (0, \infty)$ for $i \in \mathbb{Z}$ such that $\lim_{i \rightarrow \infty} t_i = \sup\{\zeta(t) : t \in [0, \infty)\}$ and $\lim_{i \rightarrow -\infty} t_i = 0$ while

$$(29) \quad \max_{i \in \mathbb{Z}} |\zeta(t_{i+1}) - \zeta(t_i)| \leq \eta$$

and such that for $i \in \mathbb{Z}$ and $k \geq 0$,

$$(30) \quad \lambda(\{x \in C : \phi(v_k; x) = t_i\}) = 0,$$

where $v_0 = v$. Set

$$C_{li} = \{x \in C_l : t_i \leq \phi(v; x) \leq t_{i+1}\}.$$

Since $\phi(v; \cdot)$ is continuous on C_l and C_l is compact, the sets C_{li} are compact for $i \in \mathbb{Z}$. Hence, $\omega(\cdot, v_k(\cdot))$ and $\omega(\cdot, v(\cdot))$ are uniformly continuous on C_{li} , and using (2) we get

$$(31) \quad \limsup_{k \rightarrow \infty} \int_{C_{li}} \phi(v_k; x) \omega(x, v_k(x)) \, d\lambda(x) \leq \int_{C_{li}} \phi(v; x) \omega(x, v(x)) \, d\lambda(x).$$

By (26) and the definition of C_{li} ,

$$(32) \quad \int_{C_{li}} \phi(v; x) \omega(x, v(x)) \, d\lambda(x) \leq t_{i+1} \int_{C_{li}} \omega(x, v(x)) \, d\lambda(x).$$

Since ζ is decreasing, we obtain

$$(33) \quad \int_{C_l} \zeta(\phi(v; x)) \omega(x, v(x)) \, d\lambda(x) \leq \sum_{i \in \mathbb{Z}} \zeta(t_i) \int_{C_{li}} \omega(x, v(x)) \, d\lambda(x).$$

Let $J_k \subset \mathbb{Z}$ be such that $\int_{C_{li}} \omega(x, v_k(x)) \, d\lambda(x) > 0$ precisely if $i \in J_k$, and set $v_0 = v$. If $i \in J_0$, then $i \in J_k$ for k sufficiently large. By (30), Jensen's inequality, and the convexity of ζ , we obtain

$$(34) \quad \begin{aligned} & \int_{C_l} \zeta(\phi(v_k; x)) \omega(x, v_k(x)) \, d\lambda(x) \\ & \geq \sum_{i \in J_k} \zeta \left(\frac{\int_{C_{li}} \phi(v_k; x) \omega(x, v_k(x)) \, d\lambda(x)}{\int_{C_{li}} \omega(x, v_k(x)) \, d\lambda(x)} \right) \int_{C_{li}} \omega(x, v_k(x)) \, d\lambda(x). \end{aligned}$$

Using (34), (25), the monotonicity of ζ , and the classical form of Fatou's lemma, we obtain

$$\begin{aligned} & \liminf_{k \rightarrow \infty} \int_{C_l} \zeta(\phi(v_k; x)) \omega(x, v_k(x)) \, d\lambda(x) \\ & \geq \liminf_{k \rightarrow \infty} \sum_{i \in J_k} \zeta \left(\frac{\int_{C_{li}} \phi(v_k; x) \omega(x, v_k(x)) \, d\lambda(x)}{\int_{C_{li}} \omega(x, v_k(x)) \, d\lambda(x)} \right) \int_{C_{li}} \omega(x, v_k(x)) \, d\lambda(x) \\ & \geq \sum_{i \in J_0} \liminf_{k \rightarrow \infty} \left(\zeta \left(\frac{\int_{C_{li}} \phi(v_k; x) \omega(x, v_k(x)) \, d\lambda(x)}{\int_{C_{li}} \omega(x, v_k(x)) \, d\lambda(x)} \right) \int_{C_{li}} \omega(x, v_k(x)) \, d\lambda(x) \right) \\ & \geq \sum_{i \in J_0} \zeta \left(\frac{\limsup_{k \rightarrow \infty} \int_{C_{li}} \phi(v_k; x) \omega(x, v_k(x)) \, d\lambda(x)}{\int_{C_{li}} \omega(x, v(x)) \, d\lambda(x)} \right) \int_{C_{li}} \omega(x, v(x)) \, d\lambda(x), \end{aligned}$$

and by (31), (32), (33), and (29), we conclude that

$$\begin{aligned} & \liminf_{k \rightarrow \infty} \int_{C_l} \zeta(\phi(v_k; x)) \omega(x, v_k(x)) \, d\lambda(x) \\ & \geq \sum_{i \in J_0} \zeta \left(\frac{\int_{C_{li}} \phi(v; x) \omega(x, v(x)) \, d\lambda(x)}{\int_{C_{li}} \omega(x, v(x)) \, d\lambda(x)} \right) \int_{C_{li}} \omega(x, v(x)) \, d\lambda(x) \\ & \geq \sum_{i \in \mathbb{Z}} \zeta(t_{i+1}) \int_{C_{li}} \omega(x, v(x)) \, d\lambda(x) \\ & = \sum_{i \in \mathbb{Z}} (\zeta(t_i) - (\zeta(t_i) - \zeta(t_{i+1}))) \int_{C_{li}} \omega(x, v(x)) \, d\lambda(x) \\ & \geq \int_{C_l} \zeta(\phi(v; x)) \omega(x, v(x)) \, d\lambda(x) - \eta \int_{C_l} \omega(x, v(x)) \, d\lambda(x). \end{aligned}$$

Since $\eta > 0$ is arbitrarily, this proves (28). Finally, (27), (30), (28), and Fatou's Lemma imply

$$\begin{aligned} \liminf_{k \rightarrow \infty} \int_C \zeta(\phi(v_k; x)) \omega(x, v_k(x)) \, d\lambda(x) &= \liminf_{k \rightarrow \infty} \sum_{l=1}^{\infty} \int_{C_l} \zeta(\phi(v_k; x)) \omega(x, v_k(x)) \, d\lambda(x) \\ &\geq \sum_{l=1}^{\infty} \liminf_{k \rightarrow \infty} \int_{C_l} \zeta(\phi(v_k; x)) \omega(x, v_k(x)) \, d\lambda(x) \\ &= \int_C \zeta(\phi(v; x)) \omega(x, v(x)) \, d\lambda(x), \end{aligned}$$

as we wanted to prove. \square

We used the continuity of v_k only to show that $\omega(\cdot, v_k(\cdot))$ is continuous. Hence, we also have the following version of Theorem 6.

Theorem 7. *Let $\zeta \in \text{Cvx}([0, \infty))$ and $C \subseteq X$ compact. For $v \in \mathcal{F}(X)$, define $Z(v, C) = \int_C \zeta(\phi(v; x)) d\lambda(x)$ (as an element of $[0, \infty]$). If $v \in \mathcal{F}(X)$, then*

$$\liminf_{k \rightarrow \infty} Z(v_k, C) \geq Z(v, C)$$

for every sequence $v_k \in \mathcal{F}(X)$ that converges to v .

5. PROOF OF THEOREM 1 AND FURTHER APPLICATIONS

We obtain the following result as special case of Theorem 5.

Theorem 8. *For $\zeta \in \text{Conc}([0, \infty))$ and $v \in \text{Conv}_{\text{MA}}(\mathbb{R}^n; \mathbb{R})$,*

$$Z(v) = \int_{\mathbb{R}^n} \zeta(\det(D^2 v(x))) dx$$

is finite, and

$$Z(v) \geq \limsup_{k \rightarrow \infty} Z(v_k)$$

for every sequence v_k in $\text{Conv}_{\text{MA}}(\mathbb{R}^n; \mathbb{R})$ that τ^* -converges to v .

A version of Theorem 8 also applies to $\text{Conv}_{\text{MA}, h}(\mathbb{R}^n; \mathbb{R})$, which implies by (11) and Lemma 3 that Theorem 1 holds. We apply Theorem 5 also to the weighted functional affine surface area from [24] and remark that more general weighted affine surface areas [11, 24] are also covered by Theorem 5 on $\text{Conv}_{\text{MA}}(\mathbb{R}^n; \mathbb{R})$.

Theorem 9. *The functional $Z: \text{Conv}_{\text{MA}}(\mathbb{R}^n; \mathbb{R}) \rightarrow [0, \infty)$, defined by*

$$Z(v) = \int_{\mathbb{R}^n} \det(D^2 v(x))^{\frac{1}{n+2}} e^{-\frac{n}{n+2}v(x)} dx,$$

is τ^* -upper semicontinuous.

The following result is a special case of Theorem 7.

Theorem 10. *Let $\zeta \in \text{Cvx}([0, \infty))$ and $C \subset \mathbb{R}^n$ compact. Define*

$$Z(v, C) = \int_C \zeta(\det(D^2 v(x))) dx$$

for $v \in \text{Conv}_{\text{MA}}(\mathbb{R}^n; \mathbb{R})$ (as an element of $[0, \infty]$). If $v \in \text{Conv}_{\text{MA}}(\mathbb{R}^n; \mathbb{R})$, then

$$Z(v, C) \leq \liminf_{k \rightarrow \infty} Z(v_k, C)$$

for every sequence v_k in $\text{Conv}_{\text{MA}}(\mathbb{R}^n; \mathbb{R})$ that τ^* -converges to v .

We remark that instead of Monge–Ampère measures mixed Monge–Ampère measures and, in particular, Hessian measures can be used here. Such measures also appear in the characterization of continuous valuations on convex functions [6].

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