AFFINE FRACTIONAL L^p SOBOLEV INEQUALITIES

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Abstract. Sharp affine fractional L^p Sobolev inequalities for functions on $\mathbb{R}^n$ are established. The new inequalities are stronger than (and directly imply) the sharp fractional L^p Sobolev inequalities. They are fractional versions of the affine L^p Sobolev inequalities of Lutwak, Yang, and Zhang. In addition, affine fractional asymmetric L^p Sobolev inequalities are established.

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1. Introduction

Sharp fractional L^2 Sobolev inequalities are receiving increasing attention in the last decades. They are central in the study of solutions of equations involving the fractional Laplace operator $(-\Delta)^{1/2}$ which arises naturally in many non-local problems such as the stationary form of reaction-diffusion equations [9], the Signorini problem (and its equivalent formulation as the thin obstacle problem) [3], and the Dirichlet-to-Neumann operator of harmonic functions in the half space [29]. Also, the general operators $(-\Delta)^s$ for $s \in (0,1)$ arise in stochastic theory, associated with symmetric Levy processes (see [29] and the references therein).

Let $0 < s < 1$ and $1 \le p < n/s$. The fractional L^p Sobolev inequalities state that

$$\|f\|^p_{\frac{np}{n-ps}} \le \sigma_{n,p,s} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x-y|^{n+ps}} \,\mathrm{d}x \,\mathrm{d}y \tag{1}$$

for $f \in W^{s,p}(\mathbb{R}^n)$, the fractional L^p Sobolev space of functions $f \in L^p(\mathbb{R}^n)$ with finite right side in (1) (see, for example, [27]). In general, the optimal constants $\sigma_{n,p,s}$ and extremal functions are not known (see [6] for a conjecture). Equality is always attained in (1). For $p = 1$, the extremal functions of (1) are multiples of indicator functions of balls and the constants are explicitly known. The only further known case is $p = 2$, where the constants $\sigma_{n,2,s}$ can be obtained by duality from Lieb's sharp Hardy–Littlewood–Sobolev inequalities [18] (see, for example, [10]). The asymptotic behavior of $\sigma_{n,p,s}$ as $s \to 1^-$ was studied in [5]. Almgren and Lieb [1] and Frank and Seiringer [12] showed that the extremal functions of (1) are radially symmetric and of constant sign.

By a result of Bourgain, Brezis, and Mironescu [4],

$$\lim_{s \to 1^-} p(1-s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x-y|^{n+ps}} \,\mathrm{d}x \,\mathrm{d}y = \alpha_{n,p} \int_{\mathbb{R}^n} |\nabla f(x)|^p \,\mathrm{d}x$$

for $f \in W^{1,p}(\mathbb{R}^n)$, the Sobolev space of L^p functions f with weak L^p gradient ∇f, where

$$\alpha_{n,p} = \int_{\mathbb{S}^{n-1}} |\langle \xi, \eta \rangle|^p \,\mathrm{d}\xi \tag{2}$$

for any $\eta \in \mathbb{S}^{n-1}$. Here, integration on the unit sphere $\mathbb{S}^{n-1}$ is with respect to the $(n-1)$-dimensional Hausdorff measure, ω_n is the volume of the n-dimensional unit ball and $\langle \cdot, \cdot \rangle$ is the inner product on $\mathbb{R}^n$. For $p=1$ and $p=2$, this allows to deduce the sharp L^p Sobolev inequalities from (1) by calculating the limit of $\sigma_{n,p,s}/(1-s)$ as $s \to 1^-$.

Zhang [32] and Lutwak, Yang, and Zhang [24] obtained the following sharp affine L^p Sobolev inequality that is significantly stronger than the classical L^p Sobolev inequality:

$$\|f\|^p_{\frac{np}{n-p}} \le \sigma_{n,p} \frac{n\omega_n^{\frac{n+p}{n}}}{\alpha_{n,p}} |\,\Pi_p^* f|^{-\frac{p}{n}} \le \sigma_{n,p} \int_{\mathbb{R}^n} |\nabla f(x)|^p \,\mathrm{d}x \tag{3}$$

for $f \in W^{1,p}(\mathbb{R}^n)$ and $1 < p < n$, where the inequality between the first and third terms is the classical L^p Sobolev inequality and the optimal constants $\sigma_{n,p}$ were determined by Aubin [2] and Talenti [30]. We have rewritten the explicit constant for the first inequality from [24] using (2). Here $\Pi_p^* f$ is the L^p polar projection body of f, a convex body associated to f that was introduced with different notation in [24] (see Section 2.5), and $|\cdot|$ is the n-dimensional Lebesgue measure.

The main aim of this paper is to establish affine fractional L^p Sobolev inequalities that are stronger than the Euclidean fractional L^p Sobolev inequalities from (1) and are fractional counterparts of (3). The case $p=1$ was studied in [16], so from now on let $p > 1$.

Theorem 1. *Let $0 < s < 1$ and $1 < p < n/s$. For $f \in W^{s,p}(\mathbb{R}^n)$,*

$$\begin{aligned} \|f\|^p_{\frac{np}{n-ps}} &\le \sigma_{n,p,s} n\omega_n^{\frac{n+ps}{n}} \Big(\frac{1}{n} \int_{\mathbb{S}^{n-1}} \Big(\int_0^\infty t^{ps-1} \int_{\mathbb{R}^n} |f(x+t\xi) - f(x)|^p \,\mathrm{d}x \,\mathrm{d}t \Big)^{-\frac{n}{ps}} \mathrm{d}\xi \Big)^{-\frac{ps}{n}} \\ &\le \sigma_{n,p,s} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+ps}} \,\mathrm{d}x \,\mathrm{d}y. \end{aligned}$$

There is equality in the first inequality if and only if $f = h_{s,p} \circ \phi$ for some $\phi \in \mathrm{GL}(n)$, where $h_{s,p}$ is an extremal function of (1)*. There is equality in the second inequality if f is radially symmetric.*

In order to prove Theorem 1, we introduce the s-fractional L^p polar projection body $\Pi_p^{*,s} f$ associated to f, defined as the star-shaped set whose gauge function for $\xi \in \mathbb{S}^{n-1}$ is

$$\|\xi\|^{ps}_{\Pi_p^{*,s} f} = \int_0^\infty t^{-ps-1} \int_{\mathbb{R}^n} |f(x+t\xi) - f(x)|^p \,\mathrm{d}x \,\mathrm{d}t$$

(see Section 3 for details). The affine fractional Sobolev inequality now can be written as

$$\|f\|^p_{\frac{np}{n-ps}} \le \sigma_{n,p,s} n\omega_n^{\frac{n+ps}{n}} |\,\Pi_p^{*,s} f|^{-\frac{ps}{n}}. \tag{4}$$

Since both sides of (4) are invariant under translations of f, and for volume-preserving linear transformations $\phi : \mathbb{R}^n \to \mathbb{R}^n$,

$$\Pi_p^{*,s}(f \circ \phi^{-1}) = \phi\, \Pi_p^{*,s} f,$$

it follows that (4) is an affine inequality. In Theorem 10, we will show that

$$\lim_{s\to 1^-} p(1-s)|\,\Pi_p^{*,s} f|^{-\frac{ps}{n}} = |\,\Pi_p^* f|^{-\frac{p}{n}},$$

which establishes the connection to the L^p polar projection bodies introduced by Lutwak, Yang and Zhang [24].

In Section 4 we introduce fractional asymmetric L^p polar projection bodies as fractional counterparts of the asymmetric L^p polar projection bodies of Haberl and Schuster [14], which in turn are functional versions of the asymmetric L^p polar projection bodies of convex bodies introduced in [19]. We obtain affine fractional asymmetric L^p Sobolev inequalities for non-negative functions that are stronger than the inequalities for the symmetric fractional L^p polar projection bodies.

In the proofs of the main results, we use anisotropic fractional Sobolev norms, which were introduced in [20, 21] and depend on a star-shaped set $K \subset \mathbb{R}^n$. In Section 10 we discuss which choice of K (with given volume) gives the minimal fractional Sobolev norm and connect it to the corresponding quest for an optimal L^p Sobolev norm solved by Lutwak, Yang, and Zhang [25].

2. Preliminaries

We collect results on function spaces, Schwarz symmetrization, star-shaped sets, anisotropic Sobolev norms and L^p polar projection bodies, that will be used in the following.

2.1. **Function spaces.** For $p \ge 1$ and measurable $f : \mathbb{R}^n \to \mathbb{R}$, let

$$\|f\|_p = \Big(\int_{\mathbb{R}^n} |f(x)|^p \,\mathrm{d}x \Big)^{1/p}.$$

We set $\{f \ge t\} = \{x \in \mathbb{R}^n : f(x) \ge t\}$ for $t \in \mathbb{R}$ and use similar notation for level sets, etc. We say that f is non-zero, if $\{f \neq 0\}$ has positive measure, and we identify functions that are equal up to a set of measure zero. For $p \ge 1$, let

$$L^p(\mathbb{R}^n) = \Big\{ f : \mathbb{R}^n \to \mathbb{R} : f \text{ is measurable}, \|f\|_p < \infty \Big\}.$$

Here and below, when we use measurability and related notions, we refer to the n-dimensional Lebesgue measure on $\mathbb{R}^n$.

For $0 < s < 1$ and $p \ge 1$, we define the fractional Sobolev space $W^{s,p}(\mathbb{R}^n)$ as

$$W^{s,p}(\mathbb{R}^n) = \Big\{ f \in L^p(\mathbb{R}^n) : \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x-y|^{n+ps}} \,\mathrm{d}x \,\mathrm{d}y < \infty \Big\}.$$

For $p \ge 1$, we set

$$W^{1,p}(\mathbb{R}^n) = \big\{ f \in L^p(\mathbb{R}^n) : |\nabla f| \in L^p(\mathbb{R}^n) \big\},$$

where ∇f is the weak gradient of f.

2.2. **Symmetrization.** For a set $E \subset \mathbb{R}^n$, the indicator function 1_E is defined by $1_E(x) = 1$ for $x \in E$ and $1_E(x) = 0$ otherwise. Let $E \subseteq \mathbb{R}^n$ be a Borel set of finite measure. The Schwarz symmetral of E, denoted by $E^\star$, is the closed centered Euclidean ball with same volume as E.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a non-negative measurable function with super-level sets $\{f \ge t\}$ of finite measure. The layer cake formula states that

$$f(x) = \int_0^\infty 1_{\{f \ge t\}}(x) \,\mathrm{d}t$$

for almost every $x \in \mathbb{R}^n$ and allows us to recover the function from its super-level sets. The Schwarz symmetral of f, denoted by $f^\star$, is defined by

$$f^\star(x) = \int_0^\infty 1_{\{f \geq t\}^\star}(x) \, \mathrm{d}t$$

for $x \in \mathbb{R}^n$. Hence, $f^\star$ is determined by the properties of being radially symmetric, decreasing and having super-level sets of the same measure as those of f. Note that $f^\star$ is also called the symmetric decreasing rearrangement of f.

The proofs of our results make use of the Riesz rearrangement inequality, which is stated in full generality, for example, in [7].

Theorem 2 (Riesz's rearrangement inequality)**.** *For $f, g, k : \mathbb{R}^n \to \mathbb{R}$ non-negative, measurable functions with super-level sets of finite measure,*

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x) k(x-y) g(y) \, \mathrm{d}x \, \mathrm{d}y \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f^\star(x) k^\star(x-y) g^\star(y) \, \mathrm{d}x \, \mathrm{d}y.$$

We will use the characterization of equality cases of the Riesz rearrangement inequality due to Burchard [8].

Theorem 3 (Burchard)**.** *Let A, B and C be sets of finite positive measure in $\mathbb{R}^n$ and denote by α, β and γ the radii of their Schwarz symmetrals $A^\star, B^\star$ and $C^\star$. For $|\alpha - \beta| < \gamma < \alpha + \beta$, there is equality in*

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} 1_A(y) \, 1_B(x-y) \, 1_C(x) \, \mathrm{d}x \, \mathrm{d}y \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} 1_{A^\star}(y) \, 1_{B^\star}(x-y) \, 1_{C^\star}(x) \, \mathrm{d}x \, \mathrm{d}y$$

if and only if, up to sets of measure zero,

$$A = a + \alpha D, \; B = b + \beta D, \; C = c + \gamma D,$$

where D is a centered ellipsoid, and a, b and $c = a + b$ are vectors in $\mathbb{R}^n$.

2.3. **Star-shaped sets and star bodies.** A set $K \subseteq \mathbb{R}^n$ is star-shaped (with respect to the origin), if the interval $[0, x] \subset K$ for every $x \in K$. The gauge function $\|\cdot\|_K : \mathbb{R}^n \to [0, \infty]$ of a star-shaped set is defined as

$$\|x\|_K = \inf\{\lambda > 0 : x \in \lambda K\},$$

and the radial function $\rho_K : \mathbb{R}^n \setminus \{0\} \to [0, \infty]$ as

$$\rho_K(x) = \|x\|_K^{-1} = \sup\{\lambda \geq 0 : \lambda x \in K\}.$$

The n-dimensional Lebesgue measure or volume of a star-shaped set K in $\mathbb{R}^n$ with measurable radial function is given by

$$|K| = \frac{1}{n} \int_{\mathbb{S}^{n-1}} \rho_K(\xi)^n \, \mathrm{d}\xi.$$

We call a star-shaped set $K \subset \mathbb{R}^n$ a star body if its radial function is strictly positive and continuous in $\mathbb{R}^n \setminus \{0\}$. On the set of star bodies, the q-radial sum for $q \neq 0$ of $K, L \subset \mathbb{R}^n$ is defined by

$$\rho^q(K \,\tilde{+}_q\, L, \xi) = \rho^q(K, \xi) + \rho^q(L, \xi)$$

for $\xi \in \mathbb{S}^{n-1}$ (cf. [28, Section 9.3]). The dual Brunn–Minkowski inequality (cf. [28, (9.41)]) states that for star bodies $K, L \subset \mathbb{R}^n$ and $q > 0$,

$$|K \,\tilde{+}_{-q}\, L|^{-q/n} \geq |K|^{-q/n} + |L|^{-q/n}, \tag{5}$$

with equality precisely if K and L are dilates, that is, there is $\lambda > 0$ such that $K = \lambda L$.

Let $\alpha \in \mathbb{R}\backslash\{0,n\}$. For star-shaped sets $K, L \subseteq \mathbb{R}^n$ with measurable radial functions, the dual mixed volume is defined as

$$\tilde{V}_\alpha(K,L) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} \rho_K(\xi)^{n-\alpha}\rho_L(\xi)^\alpha \,\mathrm{d}\xi.$$

Notice that

$$\tilde{V}_\alpha(K,K) = |K|$$

and that

$$\tilde{V}_\alpha(K, L_1 \,\tilde{+}_\alpha\, L_2) = \tilde{V}_\alpha(K,L_1) + \tilde{V}_\alpha(K,L_2)$$

for star-shaped sets $K, L_1, L_2 \subseteq \mathbb{R}^n$ with measurable radial functions.

For star-shaped sets $K, L \subseteq \mathbb{R}^n$ of finite volume and $0 < \alpha < n$, the dual mixed volume inequality states that

$$\tilde{V}_\alpha(K,L) \le |K|^{(n-\alpha)/n}|L|^{\alpha/n}. \tag{6}$$

Equality holds if and only if K and L are dilates, where we say that star-shaped sets K and L are dilates if $\rho_K = \lambda\,\rho_L$ almost everywhere on $\mathbb{S}^{n-1}$ for some $\lambda > 0$. The definition of dual mixed volume for star bodies is due to Lutwak [22], where also the dual mixed volume inequality is derived from Hölder's inequality (also see [28, Section 9.3] or [13, B.29]).

2.4. **Anisotropic fractional Sobolev norms.** Let $0 < s < 1$ and $p \ge 1$. For $K \subset \mathbb{R}^n$ a star body and $f \in W^{s,p}(\mathbb{R}^n)$, the anisotropic fractional L^p Sobolev norm of f with respect to K is

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{\|x-y\|_K^{n+ps}} \,\mathrm{d}x\,\mathrm{d}y. \tag{7}$$

It was introduced in [21] for K a convex body (also, see [20]). For $K = B^n$, the Euclidean unit ball, we obtain the classical s-fractional L^p Sobolev norm of f. The limit as $s \to 1^-$ was determined in [4] in the Euclidean case and in [21] in the anisotropic case. We will also consider the following asymmetric versions of (7),

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(f(x)-f(y))_+^p}{\|x-y\|_K^{n+ps}} \,\mathrm{d}x\,\mathrm{d}y, \quad \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(f(x)-f(y))_-^p}{\|x-y\|_K^{n+ps}} \,\mathrm{d}x\,\mathrm{d}y,$$

where $a_+ = \max\{a, 0\}$ and $a_- = \max\{-a, 0\}$ for $a \in \mathbb{R}$. The limits as $s \to 1^-$ were determined in [26].

2.5. **L^p polar projection bodies.** For $p \ge 1$ and $f \in W^{1,p}(\mathbb{R}^n)$, the L^p polar projection body is defined as the star body with gauge function given by

$$\|\xi\|_{\Pi_p^* f}^p = \int_{\mathbb{R}^n} |\langle \nabla f(x), \xi\rangle|^p \,\mathrm{d}x$$

for $\xi \in \mathbb{S}^{n-1}$, were $\langle\cdot,\cdot\rangle$ denotes the inner product. It is the polar body of a convex body. The definition is due to Lutwak, Yang, and Zhang [24]. For a convex body $K \subset \mathbb{R}^n$, they defined the L^p polar projection body (with a different normalization) in [23] by

$$\|\xi\|_{\Pi_p^* K}^p = \int_{\mathbb{S}^{n-1}} |\langle \xi, \eta\rangle|^p \,\mathrm{d}S_p(K,\eta), \tag{8}$$

where $S_p(K,\cdot)$ is the L^p surface area measure of K (for the definition of L^p surface area measures, see, for example, [28, Section 9.1]).

Asymmetric L^p polar projection bodies of convex bodies were introduced in [19]. For $f \in W^{1,p}(\mathbb{R}^n)$, the asymmetric L^p polar projection bodies of f are defined as the star bodies with gauge function given by

$$\|\xi\|^p_{\Pi^*_{p,\pm} f} = \int_{\mathbb{R}^n} \langle \nabla f(x), \xi \rangle_\pm^p \,\mathrm{d}x$$

for $\xi \in \mathbb{S}^{n-1}$.

3. Fractional L^p Polar Projection Bodies

Let $0 < s < 1$ and $1 < p < n/s$. For $f \in W^{s,p}(\mathbb{R}^n)$, define the s-fractional L^p polar projection body $\Pi_p^{*,s} f$ as the star-shaped set given by the gauge function

$$\|\xi\|^{ps}_{\Pi_p^{*,s} f} = \int_0^\infty t^{-ps-1} \int_{\mathbb{R}^n} |f(x+t\xi) - f(x)|^p \,\mathrm{d}x\,\mathrm{d}t \tag{9}$$

for $\xi \in \mathbb{R}^n$. Note that $\|\cdot\|_{\Pi_p^{*,s} f}$ is a one-homogeneous function on $\mathbb{R}^n$.

Let $K \subset \mathbb{R}^n$ be a star body. The following simple calculation turns out to be useful. For $f \in W^{s,p}(\mathbb{R}^n)$,

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{\|x-y\|_K^{n+ps}} \,\mathrm{d}x\,\mathrm{d}y \\
&\quad= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(y+z)-f(y)|^p}{\|z\|_K^{n+ps}} \,\mathrm{d}z\,\mathrm{d}y \\
&\quad= \int_{\mathbb{S}^{n-1}}\int_0^\infty \|t\xi\|_K^{-n-ps} \int_{\mathbb{R}^n} |f(y+t\xi)-f(y)|^p\, t^{n-1} \,\mathrm{d}y\,\mathrm{d}t\,\mathrm{d}\xi \\
&\quad= \int_{\mathbb{S}^{n-1}}\int_0^\infty \|\xi\|_K^{-n-ps} t^{-ps-n} \,\|f(\cdot+t\xi)-f\|_p^p\, t^{n-1} \,\mathrm{d}t\,\mathrm{d}\xi \\
&\quad= \int_{\mathbb{S}^{n-1}} \rho_K(\xi)^{n+ps} \int_0^\infty t^{-ps-1} \,\|f(\cdot+t\xi)-f\|_p^p \,\mathrm{d}t\,\mathrm{d}\xi \\
&\quad= \int_{\mathbb{S}^{n-1}} \rho_K(\xi)^{n+ps} \rho_{\Pi_p^{*,s} f}(\xi)^{-ps} d\xi.
\end{aligned}$$

Hence,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{\|x-y\|_K^{n+ps}} \,\mathrm{d}x\,\mathrm{d}y = n\, \tilde{V}_{-ps}(K, \Pi_p^{*,s} f) \tag{10}$$

in this case.

Next, we establish basic properties of fractional L^p polar projection bodies.

Proposition 4. *For non-zero $f \in W^{s,p}(\mathbb{R}^n)$, the set $\Pi_p^{*,s} f$ is an origin-symmetric star body with the origin in its interior. Moreover, there is $c > 0$ depending only on f and p such that $\Pi_p^{*,s} f \subseteq c\, B^n$ for every $s \in (0,1)$.*

Proof. First, note that since for $\xi \in \mathbb{R}^n$ and $t > 0$,

$$\int_{\mathbb{R}^n} |f(x-t\xi) - f(x)|^p \,\mathrm{d}x = \int_{\mathbb{R}^n} |f(x) - f(x+t\xi)|^p \,\mathrm{d}x,$$

the set $\Pi_p^{*,s} f$ is origin-symmetric.

Next, we show that $\Pi_p^{*,s} f$ is bounded. We take $r > 1$ large enough so that $\|f\|_{L^p(rB^n)} \geq \frac{2}{3}\|f\|_p$ and easily see that for $t > 2r$,

$$\begin{aligned}\|f(\cdot + t\xi) - f(\cdot)\|_p &\geq \|f(\cdot + t\xi) - f(\cdot)\|_{L^p(rB^n - t\xi)} \\ &= \|f(\cdot) - f(\cdot - t\xi)\|_{L^p(rB^n)} \\ &\geq \|f\|_{L^p(rB^n)} - \|f(\cdot - t\xi)\|_{L^p(rB^n)} \\ &\geq \frac{2}{3}\|f\|_p - \frac{1}{3}\|f\|_p.\end{aligned}$$

Hence,

$$\int_0^\infty t^{-ps-1} \int_{\mathbb{R}^n} \big|f(x + t\xi) - f(x)\big|^p \,\mathrm{d}x \,\mathrm{d}t \geq \frac{\|f\|_p^p}{3^p} \int_r^\infty t^{-ps-1} \,\mathrm{d}t \geq \frac{\|f\|_p^p}{3^p} \frac{r^{-ps}}{ps} \geq c,$$

which implies that $\Pi_p^{*,s} f \subseteq c\,B^n$ for $c > 0$ independent of s.

Now, we show that $\Pi_p^{*,s} f$ has the origin in its interior. First observe that for $\xi, \eta \in \mathbb{R}^n$, by the triangle inequality and a change of variables,

$$\begin{aligned}&\|\xi+\eta\|_{\Pi_p^{*,s} f}^{ps} \\ &\quad= \int_0^\infty t^{-ps-1}\|f(\cdot + t\xi + t\eta) - f(\cdot)\|_p^p \,\mathrm{d}t \\ &\quad\leq \int_0^\infty t^{-ps-1} \left(\|f(\cdot + t\xi + t\eta) - f(\cdot + t\xi)\|_p + \|f(\cdot + t\xi) - f(\cdot)\|_p\right)^p \,\mathrm{d}t \\ &\quad\leq \int_0^\infty t^{-ps-1} 2^{p-1}(\|f(\cdot + t\eta) - f(\cdot)\|_p^p + \|f(\cdot + t\xi) - f(\cdot)\|_p^p) \,\mathrm{d}t \\ &\quad= 2^{p-1}\|\xi\|_{\Pi_p^{*,s} f}^{ps} + 2^{p-1}\|\eta\|_{\Pi_p^{*,s} f}^{ps}.\end{aligned} \tag{11}$$

Using the relation (10) with $K = B^n$, we get

$$\int_{\mathbb{S}^{n-1}} \|\xi\|_{\Pi_p^{*,s} f}^{ps} \,\mathrm{d}\xi = \frac{1}{n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x - y|^{n+ps}} \,\mathrm{d}x \,\mathrm{d}y,$$

which is finite since $f \in W^{s,p}(\mathbb{R}^n)$. We choose $r > 0$ large enough so that the set $A = \{\xi \in \mathbb{S}^{n-1} : \|\xi\|_{\Pi_p^{*,s} f}^s < r\}$ has positive $(n-1)$-dimensional Hausdorff measure and contains a basis $\{\xi_1, \ldots, \xi_n\} \subseteq A$ of $\mathbb{R}^n$. Applying (if necessary) a linear transformation to $\Pi_p^{*,s} f$, we may assume without loss of generality that $\xi_i = e_i$ are the canonical basis vectors. For every $x \in \mathbb{R}^n$, writing $x = \sum x_i e_i$ and using (11), we get

$$\|x\|_{\Pi_p^{*,s} f} \leq \Big(2^{n(p-1)} \sum_{i=1}^n |x_i|^{ps} \|e_i\|_{\Pi_p^{*,s} f}^{ps}\Big)^{\frac{1}{ps}} \leq d\,|x|, \tag{12}$$

where $d > 0$ is independent of x. This shows that $\Pi_p^{*,s} f$ has the origin as interior point.

Finally, we show that $\|\cdot\|_{\Pi_p^{*,s} f}$ is continuous. For $\xi,\eta\in\mathbb{R}^n$, by the triangle inequality and (12), we have

$$\begin{aligned}
\|\xi+\eta\|_{\Pi_p^{*,s} f}^{ps} & \\
&= \int_0^\infty t^{-1-ps}\|f(\cdot+t\xi+t\eta)-f(\cdot)\|_p^p\,\mathrm{d}t\\
&\le \int_0^\infty t^{-1-ps}\big(\|f(\cdot+t\eta)-f(\cdot)\|_p+\|f(\cdot+t\xi)-f(\cdot)\|_p\big)^p\,\mathrm{d}t\\
&\le \big(1+|\eta|^{\frac{s}{2}\frac{p}{p-1}}\big)^{p-1}\int_0^\infty t^{-1-ps}\left(\frac{\|f(\cdot+t\eta)-f(\cdot)\|_p^p}{|\eta|^{\frac{ps}{2}}}+\|f(\cdot+t\xi)-f(\cdot)\|_p^p\right)\mathrm{d}t\\
&= \big(1+|\eta|^{\frac{s}{2}\frac{p}{p-1}}\big)^{p-1}\big(|\eta|^{-\frac{ps}{2}}\|\eta\|_{\Pi_p^{*,s} f}^{ps}+\|\xi\|_{\Pi_p^{*,s} f}^{ps}\big)\\
&\le \big(1+|\eta|^{\frac{s}{2}\frac{p}{p-1}}\big)^{p-1}\big(d\,|\eta|^{\frac{ps}{2}}+\|\xi\|_{\Pi_p^{*,s} f}^{ps}\big),
\end{aligned}$$

where we used the inequality $a+b\le(1+r^{p/(p-1)})^{(p-1)/p}((r^{-1}a)^p+b^p)^{1/p}$ for $a,b,r>0$, which is a consequence of Hölder's inequality.

We obtain

$$\|\xi+\eta\|_{\Pi_p^{*,s} f}^{ps}\le\big(1+|\eta|^{\frac{s}{2}\frac{p}{p-1}}\big)^{p-1}\big(d\,|\eta|^{\frac{ps}{2}}+\|\xi\|_{\Pi_p^{*,s} f}^{ps}\big). \tag{13}$$

Applying inequality (13) to the vectors $\xi+\eta$ and $-\eta$, we get

$$\|\xi\|_{\Pi_p^{*,s} f}^{ps}=\|\xi+\eta-\eta\|_{\Pi_p^{*,s} f}^{ps}\le\big(1+|-\eta|^{\frac{s}{2}\frac{p}{p-1}}\big)^{p-1}\big(d\,|-\eta|^{\frac{ps}{2}}+\|\xi+\eta\|_{\Pi_p^{*,s} f}^{ps}\big),$$

which implies

$$\|\xi+\eta\|_{\Pi_p^{*,s} f}^{ps}\ge\big(1+|\eta|^{\frac{s}{2}\frac{p}{p-1}}\big)^{p-1}\|\xi\|_{\Pi_p^{*,s} f}^{ps}-d\,|\eta|^{\frac{ps}{2}}. \tag{14}$$

The continuity of $\|\cdot\|_{\Pi_p^{*,s} f}$ now follows from (13) and (14). □

4. Fractional Asymmetric L^p Polar Projection Bodies

Let $0<s<1$ and $1<p<n/s$. For $f\in W^{s,p}(\mathbb{R}^n)$, define the asymmetric s-fractional L^p polar projection bodies $\Pi_{p,+}^{*,s} f$ and $\Pi_{p,-}^{*,s} f$ as the star-shaped sets given by the gauge functions

$$\|\xi\|_{\Pi_{p,\pm}^{*,s} f}^{ps}=\int_0^\infty t^{-ps-1}\int_{\mathbb{R}^n}(f(x+t\xi)-f(x))_\pm^p\,\mathrm{d}x\,\mathrm{d}t$$

for $\xi\in\mathbb{R}^n$. We have $\Pi_{p,-}^{*,s} f=\Pi_{p,+}^{*,s}(-f)=-\Pi_{p,+}^{*,s} f$ and state our results just for $\Pi_{p,+}^{*,s} f$. Note that, as in the symmetric case, $\|\cdot\|_{\Pi_{p,+}^{*,s} f}^{ps}$ is a one-homogeneous function on $\mathbb{R}^n$. Also note that

$$\|\xi\|_{\Pi_p^{*,s} f}^{ps}=\|\xi\|_{\Pi_{p,+}^{*,s} f}^{ps}+\|\xi\|_{\Pi_{p,-}^{*,s} f}^{ps} \tag{15}$$

for $\xi\in\mathbb{R}^n$.

Let $K\subset\mathbb{R}^n$ be a star body and $f\in W^{s,p}(\mathbb{R}^n)$. As in (10), we obtain that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(f(x)-f(y))_+^p}{\|x-y\|_K^{n+ps}}\,\mathrm{d}x\,\mathrm{d}y=n\,\tilde V_{-ps}(K,\Pi_{p,+}^{*,s} f). \tag{16}$$

In the following proposition, we derive the basic properties of fractional asymmetric L^p polar projection bodies.

Proposition 5. *For non-zero $f \in W^{s,p}(\mathbb{R}^n)$, the set $\Pi^{*,s}_{p,+} f$ is a star body with the origin in its interior. Moreover, there is $c > 0$ depending only on f and p such that $\Pi^{*,s}_{p,+} f \subseteq c\, B^n$ for every $s \in (0,1)$.*

Proof. Since the functions $(a)_+^p$ and $(a)_-^p$ are convex, the inequalities $(a+b)_+^p \geq (a)_+^p + p(a)_+^{p-1} b$ and $(a+b)_-^p \geq (a)_-^p + p(a)_-^{p-1} b$ hold for $a, b \in \mathbb{R}$.

If $\int_{\mathbb{R}^n} (f(x))_+^p \,\mathrm{d}x > 0$, take $\varepsilon > 0$ so small that $\varepsilon + p\varepsilon^{1/p} \|f\|_p^{p-1} \leq \frac{1}{2} \int_{\mathbb{R}^n} (f(x))_+^p \,\mathrm{d}x$, and take $r > 0$ so large that $\int_{\mathbb{R}^n \setminus rB^n} |f(x)|^p \,\mathrm{d}x < \varepsilon$. For $z \in \mathbb{R}^n \setminus 2rB^n$, we obtain by Hölder's inequality that

$$\begin{aligned}
&\int_{rB^n} (f(x) - f(x+z))_+^p \,\mathrm{d}x \\
&\qquad \geq \int_{rB^n} (f(x))_+^p - p\,(f(x))_+^{p-1} f(x+z) \,\mathrm{d}x \\
&\qquad \geq \int_{rB^n} (f(x))_+^p \,\mathrm{d}x - p\Big(\int_{rB^n} (f(x))_+^p \,\mathrm{d}x \Big)^{\frac{p-1}{p}} \Big(\int_{rB^n} |f(x+z)|^p \,\mathrm{d}x \Big)^{\frac{1}{p}} \\
&\qquad \geq \int_{rB^n} (f(x))_+^p \,\mathrm{d}x - p\Big(\int_{\mathbb{R}^n} |f(x)|^p \,\mathrm{d}x \Big)^{\frac{p-1}{p}} \Big(\int_{\mathbb{R}^n \setminus rB^n} |f(x)|^p \,\mathrm{d}x \Big)^{\frac{1}{p}} \\
&\qquad \geq \int_{\mathbb{R}^n} (f(x))_+^p \,\mathrm{d}x - \varepsilon - p\,\|f\|_p^{p-1} \varepsilon^{\frac{1}{p}} \\
&\qquad \geq \frac{1}{2} \int_{\mathbb{R}^n} (f(x))_+^p \,\mathrm{d}x.
\end{aligned}$$

In case $\int_{\mathbb{R}^n} (f(x))_+^p \,\mathrm{d}x = 0$ the previous inequality holds trivially for any $r > 0$.

By an analogous calculation, and eventually increasing the value of r, we obtain that

$$\begin{aligned}
\int_{rB^n - z} (f(x) - f(x+z))_+^p \,\mathrm{d}x &= \int_{rB^n} (f(x) - f(x-z))_-^p \,\mathrm{d}x \\
&\geq \frac{1}{2} \int_{\mathbb{R}^n} (f(x))_-^p \,\mathrm{d}x.
\end{aligned}$$

It follows that $\int_{\mathbb{R}^n} (f(x) - f(x+z))_+^p \,\mathrm{d}x \geq \frac{1}{2} \|f\|_p^p$ for every $z \in \mathbb{R}^n \setminus 2rB^n$ with $r > 0$ depending only on f. Finally,

$$\begin{aligned}
\|\xi\|^{ps}_{\Pi^{*,s}_{p,+} f} &\geq \int_{2r}^{\infty} t^{-1-ps} \int_{\mathbb{R}^n} (f(x) - f(x+z))_+^p \,\mathrm{d}x \,\mathrm{d}t \\
&\geq \int_{2r}^{\infty} t^{-1-ps} \,\mathrm{d}t \,\frac{1}{2} \int_{\mathbb{R}^n} |f(x)|^p \,\mathrm{d}x \\
&\geq \frac{(2r)^{-ps}}{ps} \frac{1}{2} \int_{\mathbb{R}^n} |f(x)|^p \,\mathrm{d}x \\
&\geq \frac{(2r)^{-p}}{2p} \|f\|_p^p.
\end{aligned}$$

Note that $\Pi^{*,s}_p f \subset \Pi^{*,s}_{p,+} f$. Hence, it follows from Proposition 4 that $\Pi^{*,s}_{p,+} f$ contains the origin in its interior, that is, there is $d > 0$ such that

$$\|x\|_{\Pi^{*,s}_{p,+} f} \leq d\,|x| \tag{17}$$

for every $x \in \mathbb{R}^n$.

Finally, we show that $\|\cdot\|_{\Pi^{*,s}_{p,+} f}$ is continuous. Observe that the inequality $(a+b)_+^p \le (a_+ + b_+)^p$ holds for any $a, b \in \mathbb{R}$. Hence, for $\xi, \eta \in \mathbb{R}^n$, we obtain that

$$\begin{aligned}
&\int_{\mathbb{R}^n} (f(x+t\xi+t\eta)-f(x))_+^p \,\mathrm{d}x \\
&= \int_{\mathbb{R}^n} (f(x+t\xi+t\eta)-f(x+t\xi)+f(x+t\xi)-f(x))_+^p \,\mathrm{d}x \\
&\le \int_{\mathbb{R}^n} \big((f(x+t\xi+t\eta)-f(x+t\xi))_+ + (f(x+t\xi)-f(x))_+\big)^p \,\mathrm{d}x \\
&\le \int_{\mathbb{R}^n} (1+|\eta|^{\frac{s}{2}\frac{p}{(p-1)}})^{p-1}\Big(\frac{(f(x+t\xi+t\eta)-f(x+t\xi))_+^p}{|\eta|^{\frac{ps}{2}}} + (f(x+t\xi)-f(x))_+^p\Big)\,\mathrm{d}x \\
&\le (1+|\eta|^{\frac{s}{2}\frac{p}{(p-1)}})^{p-1}\Big(\frac{\|(f(\cdot+t\eta)-f(\cdot))_+\|_p^p}{|\eta|^{\frac{ps}{2}}} + \|(f(\cdot+t\xi)-f(\cdot))_+\|_p^p\Big),
\end{aligned}$$

where we used the inequality $a+b \le (1+r^{p/(p-1)})^{(p-1)/p}((r^{-1}a)^p + b^p)^{1/p}$ for $a, b, r > 0$, which is a consequence of Hölder's inequality. Thus, integrating and using (17), we obtain

$$\|\xi+\eta\|^{ps}_{\Pi^{*,s}_{p,+} f} \le (1+|\eta|^{\frac{s}{2}\frac{p}{p-1}})^{p-1}(d\,|\eta|^{\frac{ps}{2}} + \|\xi\|^{ps}_{\Pi^{*,s}_{p,+} f}). \tag{18}$$

Applying inequality (18) to the vectors $\xi+\eta$ and $-\eta$, we get

$$\|\xi\|^{ps}_{\Pi^{*,s}_{p,+} f} = \|\xi+\eta-\eta\|^{ps}_{\Pi^{*,s}_{p,+} f} \le (1+|-\eta|^{\frac{s}{2}\frac{p}{p-1}})^{p-1}(d\,|-\eta|^{\frac{ps}{2}} + \|\xi+\eta\|^{ps}_{\Pi^{*,s}_{p,+} f}),$$

which implies

$$\|\xi+\eta\|^{ps}_{\Pi^{*,s}_{p,+} f} \ge (1+|\eta|^{\frac{s}{2}\frac{p}{p-1}})^{-(p-1)}\|\xi\|^{ps}_{\Pi^{*,s}_{p,+} f} - d\,|\eta|^{\frac{ps}{2}}. \tag{19}$$

The continuity of $\|\cdot\|_{\Pi^{*,s}_{p,+} f}$ now follows from (18) and (19). □

5. The Limit of Fractional L^p Polar Projection Bodies

We establish the limiting behavior of s-fractional L^p polar projection bodies for $1 < p < n/s$ as $s \to 1^-$ in the symmetric and asymmetric case. For $p = 1$, a corresponding result was proved in [16].

Let $0 < s < 1$ and $1 < p < n/s$. Set $p' = p/(p-1)$. We say that $f_k \to f$ weakly in $L^p(\mathbb{R}^n)$ if

$$\int_{\mathbb{R}^n} f_k(x)g(x)\,\mathrm{d}x \to \int_{\mathbb{R}^n} f(x)g(x)\,\mathrm{d}x$$

for every $g \in L^{p'}(\mathbb{R}^n)$ as $k \to \infty$. Set $B_{p',+} = \{g \in L^{p'}(\mathbb{R}^n) : g \ge 0, \|g\|_{p'} \le 1\}$.

We require the following lemmas.

Lemma 6. *The following statements hold.*

(1) For $f \in L^p(\mathbb{R}^n)$,

$$\|f_+\|_p = \sup_{g \in B_{p',+}} \int_{\mathbb{R}^n} f(x)g(x)\,\mathrm{d}x.$$

(2) Let $f_k, f \in L^p(\mathbb{R}^n)$. If $f_k \to f$ weakly in $L^p(\mathbb{R}^n)$ as $k \to \infty$, then

$$\liminf_{k\to\infty} \|(f_k)_+\|_p \geq \|f_+\|_p.$$

(3) Assume f_k is a bounded sequence in $L^p(\mathbb{R}^n)$. If

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} f_k(x)g(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} f(x)g(x)\,\mathrm{d}x$$

for every g in a dense subset $D \subseteq L^{p'}(\mathbb{R}^n)$, then $f_k \to f$ weakly in $L^p(\mathbb{R}^n)$ as $k \to \infty$.

Proof. First we prove (1). Let $g \in B_{p',+}$ and write $f = f_+ - f_-$. Since f_- and g are non-negative, it follows from Hölder's inequality that

$$\int_{\mathbb{R}^n} f(x)g(x)\,\mathrm{d}x \leq \int_{\mathbb{R}^n} f_+(x)g(x)\,\mathrm{d}x \leq \|f_+\|_p.$$

For the opposite inequality, take $g = \|f_+\|_p^{-p/p'} f_+^{p/p'}$ and notice that $g \in B_{p',+}$ and

$$\int_{\mathbb{R}^n} f(x)g(x)\,\mathrm{d}x = \|f_+\|_p^{-\frac{p}{p'}} \int_{\mathbb{R}^n} f(x)f_+(x)^{\frac{p}{p'}}\,\mathrm{d}x \leq \|f_+\|_p^{-\frac{p}{p'}} \int_{\mathbb{R}^n} f_+(x)^p\,\mathrm{d}x = \|f_+\|_p.$$

Next we prove (2). Fix k_0 and $g_0 \in B_{p',+}$. By (1), we have

$$\int_{\mathbb{R}^n} f_{k_0}(x)g_0(x)\,\mathrm{d}x \leq \sup_{g \in B_{p',+}} \int_{\mathbb{R}^n} f_{k_0}(x)g(x)\,\mathrm{d}x = \|(f_{k_0})_+\|_p.$$

Since this inequality holds for every k_0,

$$\int_{\mathbb{R}^n} f(x)g_0(x)\,\mathrm{d}x = \lim_{k\to\infty} \int_{\mathbb{R}^n} f_k(x)g_0(x)\,\mathrm{d}x \leq \liminf_{k\to\infty} \|(f_k)_+\|_p.$$

Thus, by (1),

$$\|f_+\|_p = \sup_{g \in B_{p',+}} \int_{\mathbb{R}^n} f(x)g(x)\,\mathrm{d}x \leq \liminf_{k\to\infty} \|(f_k)_+\|_p.$$

Finally, we prove (3). Take $c \geq \max\{\|f_k\|_p, \|f\|_p\}$. Let $\varepsilon > 0$ and $g \in L^{p'}(\mathbb{R}^n)$. Take $h \in D$ such that $\|g - h\|_{p'} < \varepsilon/(2c)$. Then

$$\begin{aligned}
&\Big| \int_{\mathbb{R}^n} f_k(x)g(x)\,\mathrm{d}x - \int_{\mathbb{R}^n} f(x)g(x)\,\mathrm{d}x \Big| \\
&\leq \Big| \int_{\mathbb{R}^n} f_k(x)(g(x) - h(x))\,\mathrm{d}x \Big| + \Big| \int_{\mathbb{R}^n} f_k(x)h(x)\,\mathrm{d}x - \int_{\mathbb{R}^n} f(x)h(x)\,\mathrm{d}x \Big| \\
&\quad + \Big| \int_{\mathbb{R}^n} f(x)(g(x) - h(x))\,\mathrm{d}x \Big| \\
&\leq c\varepsilon/(2c) + \Big| \int_{\mathbb{R}^n} f_k(x)h(x)\,\mathrm{d}x - \int_{\mathbb{R}^n} f(x)h(x)\,\mathrm{d}x \Big| + c\varepsilon/(2c)
\end{aligned}$$

and the statement follows. ∎

Lemma 7. *For $f \in W^{1,p}(\mathbb{R}^n)$ and fixed $\xi \in \mathbb{S}^{n-1}$,*

$$\lim_{t\to 0} \Big\| \Big(\frac{f(\cdot + t\xi) - f(\cdot)}{t} \Big)_+ \Big\|_p^p = \int_{\mathbb{R}^n} \langle \nabla f(x), \xi\rangle_+^p \,\mathrm{d}x.$$

Proof. Let $g : \mathbb{R}^n \to \mathbb{R}$ be a smooth function with compact support. Write div_x for the divergence taken with respect to the variable x. Using integration by parts, we obtain for $\xi \in \mathbb{S}^{n-1}$ and $t > 0$,

$$\begin{aligned}
\int_{\mathbb{R}^n} g(x) \frac{f(x + t\xi) - f(x)}{t} \,\mathrm{d}x &= \int_{\mathbb{R}^n} f(x) \frac{g(x - t\xi) - g(x)}{t} \,\mathrm{d}x \\
&= -\int_{\mathbb{R}^n} f(x) \int_0^1 \langle \nabla g(x - rt\xi), \xi\rangle \,\mathrm{d}r \,\mathrm{d}x \\
&= -\int_{\mathbb{R}^n} f(x) \operatorname{div}_x \Big(\int_0^1 g(x - rt\xi) \,\mathrm{d}r \,\xi \Big) \,\mathrm{d}x \\
&= \int_{\mathbb{R}^n} \Big(\int_0^1 g(x - rt\xi) dr \Big) \langle \nabla f(x), \xi\rangle \,\mathrm{d}x.
\end{aligned}$$

By Minkowski's integral inequality $\| \int_0^1 g(\cdot - rt\xi) \,\mathrm{d}r \|_{p'} \le \|g\|_{p'}$, and we deduce

$$\Big\| \frac{f(\cdot + t\xi) - f(\cdot)}{t} \Big\|_p \le \| \langle \nabla f(\cdot), \xi\rangle \|_p < \infty.$$

Hence, $\frac{f(\cdot + t\xi) - f(\cdot)}{t}$ is uniformly bounded in $L^p(\mathbb{R}^n)$ on $(0, \infty)$.

By Lemma 6 (3),

$$\lim_{t\to 0} \int_{\mathbb{R}^n} g(x) \frac{f(x + t\xi) - f(x)}{t} \,\mathrm{d}x = \int_{\mathbb{R}^n} g(x) \langle \nabla f(x), \xi\rangle \,\mathrm{d}x$$

for every $g \in L^{p'}(\mathbb{R}^n)$. Hence, $\frac{f(\cdot + t\xi) - f(\cdot)}{t}$ converges weakly to $\langle \nabla f(\cdot), \xi\rangle$ as $t \to 0$.

By Lemma 6 (2),

$$\liminf_{t\to 0} \Big\| \Big(\frac{f(\cdot + t\xi) - f(\cdot)}{t} \Big)_+ \Big\|_p \ge \| \langle \nabla f(\cdot), \xi\rangle_+ \|_p.$$

For the opposite inequality we recall that for any $g \in B_{p',+}$, the function $x \mapsto \int_0^1 g(x - rt\xi) \,\mathrm{d}r$ is in $B_{p',+}$ as well. Hence,

$$\begin{aligned}
\int_{\mathbb{R}^n} g(x) \frac{f(x + t\xi) - f(x)}{t} \,\mathrm{d}x &= \int_{\mathbb{R}^n} \Big(\int_0^1 g(x - rt\xi) \,\mathrm{d}r \Big) \langle \nabla f(x), \xi\rangle \,\mathrm{d}x \\
&\le \| \langle \nabla f(x), \xi\rangle_+ \|_p.
\end{aligned}$$

Again by Lemma 6 (1),

$$\Big\| \Big(\frac{f(\cdot + t\xi) - f(\cdot)}{t} \Big)_+ \Big\|_p \le \| \langle \nabla f(\cdot), \xi\rangle_+ \|_p$$

for each $t > 0$. □

The following result is Lemma 4 in [16].

Lemma 8. *If $\varphi : [0, \infty) \to [0, \infty)$ be a measurable function with $\lim_{t\to 0^+} \varphi(t) = \varphi(0)$ and such that $\int_0^\infty t^{-s_0} \varphi(t) \,\mathrm{d}t < \infty$ for some $s_0 \in (0, 1)$, then*

$$\lim_{s\to 1^-} (1 - s) \int_0^\infty t^{-s} \varphi(t) \,\mathrm{d}t = \varphi(0).$$

We are now able to prove the main result of this section.

Theorem 9. *Let $f \in W^{1,p}(\mathbb{R}^n)$. For $\xi \in \mathbb{S}^{n-1}$,*

$$\lim_{s\to 1^-} (p(1-s))^{\frac{1}{p}} \|\xi\|_{\Pi^{*,s}_{p,+} f} = \|\xi\|_{\Pi^{*}_{p,+} f}.$$

Moreover,

$$\lim_{s\to 1^-} p(1-s) |\,\Pi^{*,s}_{p,+} f|^{-\frac{ps}{n}} = |\,\Pi^{*}_{p,+} f|^{-\frac{p}{n}},$$

and

$$\lim_{s\to 1^-} p(1-s)\tilde{V}_{-ps}(K, \Pi^{*,s}_{p,+} f) = \tilde{V}_{-p}(K, \Pi^{*}_{p,+} f)$$

for every star body $K \subset \mathbb{R}^n$.

Proof. Define $\varphi : [0,\infty) \to [0,\infty)$ by

$$\varphi(t) = \Big\| \Big(\frac{f(\cdot + t\xi) - f(\cdot)}{t} \Big)_+ \Big\|_p^p,$$

and note that $\varphi(t) \le \Big(\frac{2\|f\|_p}{t}\Big)^p$ for $t > 0$. By Lemma 8 and Lemma 7,

$$\lim_{s\to 1^-} p(1-s) \int_0^\infty t^{p(1-s)-1} \Big\| \Big(\frac{f(\cdot + t\xi) - f(\cdot)}{t} \Big)_+ \Big\|_p^p \,\mathrm{d}t = \int_{\mathbb{R}^n} \langle \nabla f(x), \xi \rangle_+^p \,\mathrm{d}x.$$

By Proposition 4 we can use the dominated convergence theorem to obtain

$$\begin{aligned}
&\lim_{s\to 1^-} n\,|(p(1-s))^{-\frac{1}{ps}}\, \Pi^{*,s}_{p,+} f| \\
&\quad = \lim_{s\to 1^-} \int_{\mathbb{S}^{n-1}} \Big(p(1-s) \int_0^\infty t^{p(1-s)-1} \Big\| \Big(\frac{f(\cdot + t\xi) - f(\cdot)}{t} \Big)_+ \Big\|_p^p \mathrm{d}t \Big)^{-\frac{n}{ps}} \mathrm{d}\xi \\
&\quad = \int_{\mathbb{S}^{n-1}} \Big(\int_{\mathbb{R}^n} \langle \nabla f(x), \xi \rangle_+^p \,\mathrm{d}x \Big)^{-\frac{n}{p}} \mathrm{d}\xi \\
&\quad = n\,|\,\Pi^{*}_{p,+} f|,
\end{aligned}$$

and

$$\begin{aligned}
\lim_{s\to 1^-} np(1-s)\tilde{V}_{-ps}(K, \Pi^{*,s}_{p,+} f) &= \lim_{s\to 1^-} p(1-s) \int_{\mathbb{S}^{n-1}} \|\xi\|_K^{n+ps} \|\xi\|^{ps}_{\Pi^{*,s}_{p,+} f} \,\mathrm{d}\xi \\
&= \int_{\mathbb{S}^{n-1}} \|\xi\|_K^{n} \, \|\xi\|^{p}_{\Pi^{*}_{p,+} f} \,\mathrm{d}\xi \\
&= n\,\tilde{V}_{-p}(K, \Pi^{*}_{p,+} f),
\end{aligned}$$

which completes the proof of the theorem. □

The following result is an immediate consequence of Theorem 9 and (15).

Theorem 10. *Let $f \in W^{1,p}(\mathbb{R}^n)$. For $\xi \in \mathbb{S}^{n-1}$,*

$$\lim_{s\to 1^-} (p(1-s))^{\frac{1}{p}} \|\xi\|_{\Pi^{*,s}_{p} f} = \|\xi\|_{\Pi^{*}_{p} f}.$$

Moreover,

$$\lim_{s\to 1^-} p(1-s) |\,\Pi^{*,s}_{p} f|^{-\frac{ps}{n}} = |\,\Pi^{*}_{p} f|^{-\frac{p}{n}},$$

and

$$\lim_{s\to 1^-} p(1-s)\tilde{V}_{-ps}(K, \Pi^{*,s}_{p} f) = \tilde{V}_{-p}(K, \Pi^{*}_{p} f) \tag{20}$$

for every star body $K \subset \mathbb{R}^n$.

6. Anisotropic Fractional Pólya–Szegő Inequalities

We will establish anisotropic Pólya–Szegő inequalities for fractional L^p Sobolev norms and their asymmetric counterparts.

Theorem 11. *If $f \in L^p(\mathbb{R}^n)$ is non-negative and $K \subset \mathbb{R}^n$ a star body, then*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(f(x)-f(y))_+^p}{\|x-y\|_K^{n+ps}}\,\mathrm{d}x\,\mathrm{d}y \ge \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(f^\star(x)-f^\star(y))_+^p}{\|x-y\|_{K^\star}^{n+ps}}\,\mathrm{d}x\,\mathrm{d}y. \tag{21}$$

Equality holds for non-zero $f \in W^{s,p}(\mathbb{R}^n)$ if and only if K is a centered ellipsoid and f is a translate of $f^\star \circ \phi$ for some $\phi \in \mathrm{SL}(n)$.

Proof. Writing

$$\|z\|_K^{-n-ps} = \int_0^\infty k_t(z)\,\mathrm{d}t$$

where $k_t(z) = 1_{t^{-1/(n+ps)}K}(z)$, we obtain

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(f(x)-f(y))_+^p}{\|x-y\|_K^{n+ps}}\,\mathrm{d}x\,\mathrm{d}y = \int_0^\infty \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (f(x)-f(y))_+^p k_t(x-y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t.$$

Note that

$$(f(x)-f(y))_+^p = p\int_0^\infty (f(x)-r)_+^{p-1}\,1_{\{f<r\}}(y)\,\mathrm{d}r.$$

Hence, for $t>0$, it follows from Fubini's theorem that

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} &(f(x)-f(y))_+^p\,k_t(x-y)\,\mathrm{d}x\,\mathrm{d}y\\ &= p\int_0^\infty \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (f(x)-r)_+^{p-1} k_t(x-y)\,1_{\{f<r\}}(y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}r\\ &= p\int_0^\infty \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (f(x)-r)_+^{p-1} k_t(x-y)(1-1_{\{f\ge r\}}(y))\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}r.\end{aligned}$$

Let $r,t>0$. Note that $\int_{\mathbb{R}^n}(f(x)-r)_+^{p-1}\,\mathrm{d}x<\infty$ and that

$$\begin{aligned}&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (f(x)-r)_+^{p-1} k_t(x-y)(1-1_{\{f\ge r\}}(y))\,\mathrm{d}x\,\mathrm{d}y\\ &= p\,\|k_t\|_1 \int_{\mathbb{R}^n}(f(x)-r)_+^{p-1}\,\mathrm{d}x - p\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(f(x)-r)_+^{p-1}k_t(x-y)\,1_{\{f\ge r\}}(y)\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

The first term is finite since $\{f>r\}$ has finite measure, $f \in L^{\frac{np}{n-ps}}(\mathbb{R}^n)$ and $\frac{np}{n-ps} > p-1$. Clearly the first term is invariant under Schwarz symmetrization. For the second term, by the Riesz rearrangement inequality, Theorem 2, we have

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} &(f(x)-r)_+^{p-1}k_t(x-y)\,1_{\{f\ge r\}}(y)\,\mathrm{d}x\,\mathrm{d}y\\ &\le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(f^\star(x)-r)_+^{p-1}k_t^\star(x-y)\,1_{\{f^\star\ge r\}}(y)\,\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

for $r,t>0$. Note that

$$(f(x)-r)_+^{p-1} = (p-1)\int_0^\infty (\tilde r - r)_+^{p-2}\,1_{\{f\ge \tilde r\}}(x)\,\mathrm{d}\tilde r$$

and that the corresponding equation holds for $f^\star$. Hence, if there is equality in (21), then, for $(\tilde{r}, r, t) \in (0,\infty)^3 \backslash M$ with $|M| = 0$, we have

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} 1_{\{f\geq \tilde{r}\}}(x)\, 1_{t^{-1/(n+ps)}K}(x-y)\, 1_{\{f\geq r\}}(y)\,\mathrm{d}x\,\mathrm{d}y$$
$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} 1_{\{f^\star\geq \tilde{r}\}}(x)\, 1_{t^{-1/(n+ps)}K^\star}(x-y)\, 1_{\{f^\star\geq r\}}(y)\,\mathrm{d}x\,\mathrm{d}y.$$

For almost every $(\tilde{r}, r) \in (0,\infty)^2$, we have $(\tilde{r}, r, t) \in (0,\infty)^3\backslash M$ for almost every $t > 0$. For such $(\tilde{r}, r)$ with $\tilde{r} \leq r$ and $t > 0$ sufficiently large, the assumptions of Theorem 3 are fulfilled and therefore there are a centered ellipsoid D and $a, b \in \mathbb{R}^n$ (depending on $(\tilde{r}, r, t)$) such that

$$\{f \geq \tilde{r}\} = a + \alpha D, \quad t^{-1/(n+ps)}K = b + \beta D, \quad \{f \geq r\} = c + \gamma D$$

where $c = a + b$. Since $K = t^{1/(n+ps)}b + (|K|/|D|)^{1/n} D$, the centered ellipsoid D does not depend on $(\tilde{r}, r, t)$ and also a, c do not depend on t. It follows that $b = 0$ and that K is a multiple of D. Hence, $a = c$ is a constant vector which concludes the proof. ☐

The following result is a variation of [17, Theorem 3.1].

Theorem 12. *If $f \in L^p(\mathbb{R}^n)$ is non-negative and $K \subset \mathbb{R}^n$ a star body, then*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{\|x-y\|_K^{n+ps}}\,\mathrm{d}x\,\mathrm{d}y \geq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f^\star(x)-f^\star(y)|^p}{\|x-y\|_{K^\star}^{n+ps}}\,\mathrm{d}x\,\mathrm{d}y.$$

Equality holds for non-zero $f \in W^{s,p}(\mathbb{R}^n)$ if and only if K is a centered ellipsoid and f is a translate of $f^\star \circ \phi$ for some $\phi \in \mathrm{SL}(n)$.

Proof. Since

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(f(x)-f(y))_-^p}{\|x-y\|_K^{n+ps}}\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(f(x)-f(y))_+^p}{\|x-y\|_{-K}^{n+ps}}\,\mathrm{d}x\,\mathrm{d}y,$$

the result follows from Theorem 11 for K and $-K$. ☐

7. Affine Fractional Pólya–Szegő Inequalities

We establish affine Pólya–Szegő inequalities for fractional asymmetric and symmetric L^p polar projection bodies.

Theorem 13. *If $f \in W^{s,p}(\mathbb{R}^n)$ is non-negative, then*

$$|\,\Pi_{p,+}^{*,s} f|^{-ps/n} \geq |\,\Pi_{p,+}^{*,s} f^\star|^{-ps/n}.$$

Equality holds if and only if f is a translate of $f^\star \circ \phi$ for some $\phi \in \mathrm{SL}(n)$.

Proof. By Theorem 11, (16) and the dual mixed volume inequality, we obtain for $K \subset \mathbb{R}^n$ a star body that

$$\begin{aligned} \tilde{V}_{-ps}(K, \Pi_{p,+}^{*,s} f) &\geq \tilde{V}_{-ps}(K^\star, \Pi_{p,+}^{*,s} f^\star) \\ &\geq |K^\star|^{(n+ps)/n} |\,\Pi_{p,+}^{*,s} f^\star|^{-ps/n} \\ &= |K|^{(n+ps)/n} |\,\Pi_{p,+}^{*,s} f^\star|^{-ps/n}. \end{aligned} \tag{22}$$

Setting $K = \Pi_{p,+}^{*,s} f$, we see that

$$|\,\Pi_{p,+}^{*,s} f| = \tilde{V}_{-ps}(\Pi_{p,+}^{*,s} f, \Pi_{p,+}^{*,s} f) \geq |\,\Pi_{p,+}^{*,s} f|^{(n+ps)/n} |\,\Pi_{p,+}^{*,s} f^\star|^{-ps/n},$$

which completes the proof of the inequality. By Theorem 11, there is equality in (22) if and only if f is a translate of $f^\star \circ \phi$ for some $\phi \in \mathrm{SL}(n)$. $\square$

The following result is obtained in the same way as Theorem 13 by replacing Theorem 11 with Theorem 12.

Theorem 14. *If $f \in L^p(\mathbb{R}^n)$ is non-negative, then*

$$|\,\Pi_p^{*,s} f|^{-ps/n} \geq |\,\Pi_p^{*,s} f^\star|^{-ps/n}.$$

Equality holds for $f \in W^{s,p}(\mathbb{R}^n)$ if and only if f is a translate of $f^\star \circ \phi$ for some $\phi \in \mathrm{SL}(n)$.

We remark that by Theorem 10 we obtain from Theorem 14 in the limit as $s \to 1^-$ that

$$|\,\Pi_p^{*} f|^{-p/n} \geq |\,\Pi_p^{*} f^\star|^{-p/n},$$

which is equivalent to the Pólya–Szegő inequality for L^p projection bodies by Cianchi, Lutwak, Yang, and Zhang [11, Theorem 2.1]. Similarly, by Theorem 9 we obtain from Theorem 13 in the limit as $s \to 1^-$ that

$$|\,\Pi_{p,+}^{*} f|^{-p/n} \geq |\,\Pi_{p,+}^{*} f^\star|^{-p/n},$$

which is equivalent to the Pólya–Szegő inequality for asymmetric L^p projection bodies by Haberl, Schuster and Xiao [15, Theorem 1].

8. Affine Fractional Asymmetric L^p Sobolev Inequalities

We establish the following affine fractional asymmetric L^p Sobolev inequalities and show that they are stronger than Theorem 1.

Theorem 15. *Let $0 < s < 1$ and $1 < p < n/s$. For non-negative $f \in W^{s,p}(\mathbb{R}^n)$,*

$$\|f\|^p_{\frac{np}{n-ps}} \leq 2\,\sigma_{n,p,s} n\omega_n^{\frac{n+ps}{n}} |\,\Pi_{p,+}^{*,s} f|^{-\frac{ps}{n}} \leq 2\sigma_{n,p,s} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(f(x)-f(y))_+^p}{|x-y|^{n+ps}}\,\mathrm{d}x\,\mathrm{d}y.$$

There is equality in the first inequality if and only if $f = h_{s,p}\circ\phi$ for some $\phi \in \mathrm{GL}(n)$ where $h_{s,p}$ is an extremal function of (1). There is equality in the second inequality if f is radially symmetric.

Proof. By Theorem 13,

$$|\,\Pi_{p,+}^{*,s} f|^{-ps/n} \geq |\,\Pi_{p,+}^{*,s} f^\star|^{-ps/n},$$

with equality if f is a translate of $f^\star \circ \phi$ for some $\phi \in \mathrm{SL}(n)$. Since $f^\star$ is radially symmetric, $\Pi_{p,+}^{*,s} f^\star = \Pi_{p,-}^{*,s} f^\star$ is a ball. Hence, it follows from (16) that

$$\begin{aligned} 2n\omega_n^{\frac{n+ps}{n}} |\,\Pi_{p,+}^{*,s} f^\star|^{-\frac{ps}{n}} &= 2\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(f^\star(x)-f^\star(y))_+^p}{|x-y|^{n+ps}}\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f^\star(x)-f^\star(y)|^p}{|x-y|^{n+ps}}\,\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

The fractional Sobolev inequality (1) shows that

$$\sigma_{n,p,s}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f^\star(x)-f^\star(y)|^p}{|x-y|^{n+ps}}\,\mathrm{d}x\,\mathrm{d}y \geq \|f^\star\|^p_{\frac{np}{n-ps}}.$$

Combining these inequalities and their equality cases, we complete the proof of the first inequality of the theorem.

For the second inequality, we set $K = B^n$ in (16) and apply the dual mixed volume inequality (6) to obtain

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(f(x)-f(y))_+^p}{|x-y|^{n+ps}}\,\mathrm{d}x\,\mathrm{d}y = n\tilde{V}_{-ps}(B^n, \Pi_{p,+}^{*,s} f) \ge n\omega_n^{\frac{n+ps}{n}}\,|\,\Pi_{p,+}^{*,s} f|^{-\frac{ps}{n}}.$$

There is equality precisely if $\Pi_{p,+}^{*,s} f$ is a ball, which is the case for radially symmetric functions. □

Note that it follows from the definition of fractional symmetric and asymmetric L^p polar projection bodies that

$$\Pi_p^{*,s} f = \Pi_{p,+}^{*,s} f \,\tilde{+}_{-ps}\, \Pi_{p,-}^{*,s} f.$$

We use the dual Brunn–Minkowski inequality (5) and obtain that

$$|\,\Pi_p^{*,s} f|^{-\frac{ps}{n}} \ge |\,\Pi_{p,+}^{*,s} f|^{-\frac{ps}{n}} + |\,\Pi_{p,-}^{*,s} f|^{-\frac{ps}{n}},$$

with equality precisely if the star bodies $\Pi_{p,+}^{*,s} f$ and $\Pi_{p,-}^{*,s} f$ are dilates. Thus, it follows that for non-negative f, Theorem 15 implies Theorem 1 and it is, in general, substantially stronger than Theorem 1. Of course, they coincide for even functions.

9. Affine Fractional L^p Sobolev Inequalities: Proof of Theorem 1

For non-negative f, the first inequality in Theorem 1 follows from Theorem 15, as mentioned before. For general f and $x, y \in \mathbb{R}^n$, we use

$$|f(x) - f(y)| \ge \big||f(x)| - |f(y)|\big|,$$

where equality holds if and only if $f(x)$ and $f(y)$ are both non-negative or non-positive. We obtain

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y \ge \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\big||f(x)|-|f(y)|\big|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y,$$

with equality if and only if f has constant sign for almost every $x, y \in \mathbb{R}^n$. Using the result for $|f|$, we obtain the first inequality of the theorem and its equality case.

For the second inequality, we set $K = B^n$ in (10) and apply the dual mixed volume inequality (6) as in the proof of Theorem 15.

10. Optimal Fractional L^p Sobolev Bodies

The following important question was asked by Lutwak, Yang and Zhang [25] for a given $f \in W^{1,p}(\mathbb{R}^n)$ and $1 \le p < n$: For which origin-symmetric convex bodies $K \subset \mathbb{R}^n$ is

$$\inf\Big\{\int_{\mathbb{R}^n} \|\nabla f(x)\|_{K^*}^p\,\mathrm{d}x : K \text{ origin-symmetric convex body}, |K| = \omega_n\Big\} \tag{23}$$

attained? An optimal L^p Sobolev body of f is a convex body where the infimum is attained.

Lutwak, Yang ang Zhang [25] showed that the infimum in (23) is attained (up to normalization) at the unique origin-symmetric convex body $\langle f\rangle_p$ in $\mathbb{R}^n$ such that

$$\int_{\mathbb{S}^{n-1}} g(\xi)\,\mathrm{d}S_p(\langle f\rangle_p, \xi) = \int_{\mathbb{R}^n} g(\nabla f(x))\,\mathrm{d}x \tag{24}$$

for every even $g \in C(\mathbb{R}^n)$ that is positively homogeneous of degree p, where $S_p(K,\cdot)$ is the L_p surface area measure of K. Setting $g = \|\cdot\|_{K^*}$, they obtain from the L^p Minkowski inequality that

$$\frac{1}{n}\int_{\mathbb{R}^n} \|\nabla f(x)\|_{K^*}^p \,\mathrm{d}x = V_p(\langle f\rangle_p, K) \geq |\langle f\rangle_p|^{(n-p)/n}|K|^{p/n}, \tag{25}$$

with equality precisely if K and $\langle f\rangle_p$ are homothetic (see [28, Section 9.1] for the definition of the L_p mixed volume $V_p(\cdot,\cdot)$ and the L^p Minkowski inequality). Hence, they obtain from their solution to their functional version (24) of the L^p Minkowski problem that $\langle f\rangle_p$ is the optimal L^p Sobolev body associated to f. Tuo Wang [31] obtained corresponding results for $f \in BV(\mathbb{R}^n)$ and $p = 1$.

Let $0 < s < 1$ and $1 < p < n/s$. The results by Lutwak, Yang and Zhang [25] suggest the following question for a given $f \in W^{s,p}(\mathbb{R}^n)$: For which star bodies $L \subset \mathbb{R}^n$ is

$$\inf\Big\{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{\|x-y\|_L^{n+ps}}\,\mathrm{d}x\,\mathrm{d}y : L \text{ star body}, |L| = \omega_n\Big\} \tag{26}$$

attained? An optimal s-fractional L^p Sobolev body of f is a star body where the infimum is attained.

By (10) and the dual mixed volume inequality (6),

$$\frac{1}{n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{\|x-y\|_L^{n+ps}}\,\mathrm{d}x\,\mathrm{d}y = \tilde{V}_{-ps}(L, \Pi_p^{*,s} f) \geq |L|^{(n+ps)/n}|\,\Pi_p^{*,s} f|^{-(ps)/n},$$

and there is equality precisely if L is a dilate of $\Pi_p^{*,s} f$. Hence, $\Pi_p^{*,s} f$ is the unique optimal s-fractional L^p Sobolev body associated to f.

To understand how the solutions to (23) and (26) are related, we use the following result: For $f \in W^{1,p}(\mathbb{R}^n)$ and $L \subset \mathbb{R}^n$ a star body,

$$\lim_{s\to 1^-} p(1-s)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{\|x-y\|_L^{n+ps}}\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}^n} \|\nabla f(x)\|_{\mathrm{Z}_p^* L}\,\mathrm{d}x, \tag{27}$$

where the convex body $\mathrm{Z}_p K$, defined for $\xi \in \mathbb{S}^{n-1}$ by

$$h_{\mathrm{Z}_p L}(\xi)^p = \int_{\mathbb{S}^{n-1}} |\langle \xi, \eta\rangle|^p \rho_L(\eta)^{n+p}\,\mathrm{d}\eta,$$

is a multiple of the L^p centroid body of L. This can be proved as in [21], where the corresponding result was established for a convex body L (with a different normalization of $\mathrm{Z}_p L$). It also follows from Theorem 10. Indeed, by (10) and (20),

$$\lim_{s\to 1^-} p(1-s)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{\|x-y\|_L^{n+ps}}\,\mathrm{d}x\,\mathrm{d}y = \tilde{V}_{-p}(L, \Pi_p^* f).$$

Using that

$$\Pi_p^* f = \Pi_p^* \langle f\rangle_p \tag{28}$$

for $f \in W^{1,p}(\mathbb{R}^n)$, which follows from (24) by setting $g = |\langle \cdot, \eta\rangle|^p$ for $\eta \in \mathbb{S}^{n-1}$ and using (8) and (9) (cf. [25]), and that

$$V_p(K, \mathrm{Z}_p L) = \tilde{V}_{-p}(L, \Pi_p^* K) \tag{29}$$

for K a convex body and L a star body, a well-known relation that follows from Fubini's theorem, we now obtain (27) from the first equation in (25).

Using (27), we obtain from (26) in the limit as $s \to 1^-$ for a given $f \in W^{1,p}(\mathbb{R}^n)$, the following question: For which star bodies $L \subset \mathbb{R}^n$ is

$$\inf\Big\{ \int_{\mathbb{R}^n} \|\nabla f(x)\|_{\mathrm{Z}_p^* L}\,\mathrm{d}x : L \text{ star body}, |L| = \omega_n \Big\} \tag{30}$$

attained? By (25) and the dual mixed volume inequality (6), we have

$$\frac{1}{n}\int_{\mathbb{R}^n} \|\nabla f(x)\|^p_{\mathrm{Z}_p^* L}\,\mathrm{d}x = V_p(\langle f\rangle_p, \mathrm{Z}_p L) = \tilde V_{-p}(L, \Pi_p^* f) \geq |L|^{(n+p)/n} |\Pi_p^* f|^{-p/n},$$

with equality precisely if L and $\Pi_p^* f$ are dilates, where we have used (28) and (29). From Theorem 10, we obtain that a suitably scaled sequence of optimal s-fractional Sobolev bodies converges to a multiple of the optimal body for (30) as $s \to 1^-$.

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