SHARP CONVEX LORENTZ-SOBOLEV INEQUALITIES

MONIKA LUDWIG, JIE XIAO, AND GAOYONG ZHANG

ABSTRACT. New sharp Lorentz-Sobolev inequalities are obtained by convexifying level sets in Lorentz integrals via the L^p Minkowski problem. New L^p isocapacitary and isoperimetric inequalities are proved for Lipschitz star bodies. It is shown that the sharp convex Lorentz-Sobolev inequalities are analytic analogues of isocapacitary and isoperimetric inequalities.

1. Introduction

The optimal L^p Sobolev embedding states that for $1 \le p < n$ the inequality

$$\|\nabla f\|_p \ge \alpha_{n,p} \|f\|_{\frac{np}{n-p}} \tag{1}$$

holds for every $f \in C_0^\infty(\mathbb{R}^n)$, where $C_0^\infty(\mathbb{R}^n)$ denotes the set of functions on $\mathbb{R}^n$ that are smooth and have compact support. Here $\|\cdot\|_q$ denotes the usual L^q norm for functions on $\mathbb{R}^n$,

$$\alpha_{n,p} = n^{\frac{1}{p}} \omega_n^{\frac{1}{n}} \big(\frac{p-1}{n-p}\big)^{\frac{1-p}{p}} \Big(\frac{\Gamma\big(\frac{n}{p}\big)\Gamma\big(n+1-\frac{n}{p}\big)}{\Gamma(n)}\Big)^{\frac{1}{n}}$$

is the best constant, Γ the gamma function and ω_n the n-dimensional volume of the unit ball in $\mathbb{R}^n$. Inequality (1) was proved by Federer and Fleming [22] and Maz′ya [52] for $p = 1$ and by Aubin [5] and Talenti [72] for $1 < p < n$. For strengthened versions of (1), see, e.g., [7, 14, 17, 20, 30, 46, 75, 77] .

One of the most important lines of development with respect to (1) is the Lorentz-Sobolev embedding theorem [4] (with sharp constants): If $f \in C_0^\infty(\mathbb{R}^n)$ and $1 \le p < n$, then

$$\|\nabla f\|_p^p \ge p^{-p}(n-p)^{p-1} n\omega_n^{\frac{p}{n}} \int_0^\infty V([f]_t)^{\frac{n-p}{n}}\, dt^p, \tag{2}$$

where $[f]_t = \{x \in \mathbb{R}^n : |f(x)| \ge t\}$ and V is the n-dimensional volume (Lebesgue measure). See also [1, 6, 18, 23, 50, 56, 63, 64].

Maz′ya [56] deduced (2) from the Maz′ya L^p isocapacitary inequality (see [56]),

$$V(K)^{\frac{n-p}{n}} \le \big(\tfrac{p-1}{n-p}\big)^{p-1} (n\omega_n^{\frac{p}{n}})^{-1} C_p(K) \quad \text{for compact } K \subset \mathbb{R}^n, \tag{3}$$

and the Maz′ya capacity strong-type inequality [53],

$$\|\nabla f\|_p^p \ge p^{-p}(p-1)^{p-1} \int_0^\infty C_p([f]_t)\, dt^p, \tag{4}$$

where $C_p(K)$ is the L^p variational capacity of K defined by

$$C_p(K) = \inf\{\|\nabla f\|_p^p : \ f \in C_0^\infty(\mathbb{R}^n) \text{ with } f(x) \ge 1 \text{ for } x \in K\}.$$

For more connections of (3) to Sobolev-type embeddings, see [26, 55, 57–60, 62, 76].

The aim of this paper is to put forward *convex* Lorentz-Sobolev inequalities that strengthen (1) in a way modeled after (2) and (4). Instead of Lorentz integrals using level sets $[f]_t$, our approach is based on integrals using the L^p *convexification* of level sets $\langle f\rangle_t$ which arises via the solution of the L^p Minkowski problem. The L^p convexification of level sets was defined by Lutwak, Yang, and Zhang in [46, 77] and applied in [14, 30, 32, 49, 75]. The definition of the convex sets $\langle f\rangle_t$ and an interpretation using L^p mixed volumes is given in Sections 3.3 and 4.1. For an even function f and $p = 1$, the L^1 convexification $\langle f\rangle_t$ can be described in the following way. The push-forward of the $(n-1)$-dimensional Hausdorff measure on the boundary of $[f]_t$ using the Gauss map defines a measure on the unit sphere $\mathbb{S}^{n-1}$ and the solution to the classical Minkowski problems states that there is a unique origin-symmetric compact convex set that defines the same measure on $\mathbb{S}^{n-1}$. This set is the L^1 convexification of level sets $\langle f\rangle_t$.

A sharp convex Lorentz-Sobolev inequality corresponding to (4) is the principal result of this paper (see Theorem 2 below). To explain more generally the strength of convex Lorentz-Sobolev inequalities, we compare the Lorentz-Sobolev inequality (2) to the corresponding convex Lorentz-Sobolev inequality. Such a convex Lorentz-Sobolev inequality with sharp constants can be deduced from the proof of the sharp L^p affine Sobolev inequality [46, 77] and is strengthened and implied by Theorem 2: If $f \in C_0^\infty(\mathbb{R}^n)$ and $1 \le p < n$, then

$$\|\nabla f\|_p^p \ge n\omega_n^{\frac{p}{n}} \int_0^\infty V(\langle f\rangle_t)^{\frac{n-p}{n}}\, dt. \tag{5}$$

In [46], the following sharp inequality is proved,

$$\int_0^\infty V(\langle f\rangle_t)^{\frac{n-p}{n}}\, dt \ge \frac{\alpha_{n,p}^p}{n\omega_n^{p/n}}\, \|f\|_{\frac{np}{n-p}}^p \tag{6}$$

for all $f \in C_0^\infty(\mathbb{R}^n)$ and $1 \le p < n$. Since the inequalities (5) and (6) are both sharp, they imply (1) with best constants. On the other hand, an inequality of Hardy, Littlewood and Pólya [33] implies that

$$\int_0^\infty V([f]_t)^{\frac{n-p}{n}}\, dt^p \ge \|f\|_{\frac{np}{n-p}}^p.$$

However, this inequality is not sharp unless $p = 1$. In view of this, the Lorentz-Sobolev embedding (2) implies (1) but without sharp constant.

Further, whereas there is no known geometric inequality equivalent to (2), we will show in Remark 3 that (5) has as a geometric analogue the following L^p isoperimetric inequality [41],

$$S_p(K) \ge n\omega_n^{\frac{p}{n}} V(K)^{\frac{n-p}{n}}, \tag{7}$$

where $K \subset \mathbb{R}^n$ is compact, convex and origin-symmetric, $S_p(K)$ is the L^p surface area of K for $p > 1$ (see Section 5.3 for the definition) and $S_1(K) = S(K)$ is the surface area of K. Moreover, in Section 6.4, we will show that for $f \in C_0^\infty(\mathbb{R}^n)$ and $1 \le p < n$,

$$\frac{n\omega_n^{\frac{p}{n}}}{p^p(n-p)^{1-p}} \int_0^\infty V([f]_t)^{\frac{n-p}{n}}\, dt^p \le \big(n\omega_n^{\frac{p}{n}} \int_0^\infty V(\langle f\rangle_t)^{\frac{n-p}{n}}\, dt\big)^{\frac{1}{p}}\, \|\nabla f\|_p^{p-1}. \tag{8}$$

Thus the sharp convex Lorentz-Sobolev inequality (5) is stronger than the Lorentz-Sobolev inequality (2).

2. Statement of principal results

The geometric analogue of the L^1 Sobolev inequality is the Euclidean isoperimetric inequality (see [22,24,52]). For $E \subset \mathbb{R}^n$ whose boundary is rectifiable, it states that if E has surface area $S(E)$, then

$$S(E) \geq n\omega_n^{1/n} V(E)^{(n-1)/n} \tag{9}$$

with equality if E is a ball in $\mathbb{R}^n$.

Two invariants come to the picture for strengthening the isoperimetric inequality. The first is the L^1 variational capacity $C := C_1$, which is invariant under rotations and translations. For compact $E \subset \mathbb{R}^n$ with rectifiable boundary,

$$S(E) \geq C(E) \geq n\omega_n^{1/n} V(E)^{(n-1)/n} \tag{10}$$

with equality if E is a closed ball in $\mathbb{R}^n$ (see [54, p. 105, (7)]), where $S(E) = C(E)$ for E compact and convex (see Section 6.1).

The second is the integral affine surface area Φ, which is $\mathrm{SL}(n)$ and translation invariant (see Section 5.2 for the definition). For $E \subset \mathbb{R}^n$ with Lipschitz boundary,

$$S(E) \geq \Phi(E) \geq n\omega_n^{1/n} V(E)^{(n-1)/n} \tag{11}$$

with equality holding in the right side inequality if and only if E is an ellipsoid. For convex E, the right side inequality of (11) is known as the Petty projection inequality (see [25,43,65,68]). The general case of (11) was proved in [77].

Within the L^p Brunn-Minkowski theory (see [10,13–15,30–32,35,36,39–42,44–49,61,67,69–71,74]), the natural extension of integral affine surface area is the L^p integral affine surface area Φ_p (see Section 5.2 for the definition) and the natural extension of surface area is the L^p surface area S_p.

The following lemma is the key geometric result needed to establish our sharp convex Lorentz-Sobolev inequalities. It generalizes inequalities (10) and (11) and strengthens the L^p isoperimetric inequality (7). A set M is a Lipschitz star body if M is compact, star shaped and has Lipschitz boundary.

Lemma 1. *If M is a Lipschitz star body in $\mathbb{R}^n$ and $1 \leq p < n$, then*

$$S_p(M) \;\geq\; \left(\tfrac{p-1}{n-p}\right)^{p-1} C_p(M) \;\geq\; n\omega_n^{\frac{p}{n}} V(M)^{\frac{n-p}{n}} \tag{12}$$

and

$$S_p(M) \;\geq\; \Phi_p(M) \;\geq\; n\omega_n^{\frac{p}{n}} V(M)^{\frac{n-p}{n}}. \tag{13}$$

Equality in (12) holds if M is a ball centered at the origin. Equality in the right side of (13) holds for $p > 1$ if and only if M is an ellipsoid centered at the origin and for $p = 1$ if and only if M is an ellipsoid containing the origin.

The right-hand side inequality of (12) is due to Maz′ya [54] (the case $p = 2$ and $n = 3$ was proved earlier by Pólya and Szegö [66]). The case $p = 2$ and $n = 3$ of the left-hand side inequality of (12) goes back to Pólya and Szegö [66]. For compact convex sets, the right-hand side inequality of (13) was proved in [45] (see [10] for an alternate proof).

Sharp convex Lorentz-Sobolev inequalities are stated in the following theorem.

Theorem 2. *If $f \in C_0^\infty(\mathbb{R}^n)$ and $1 \le p < n$, then*

$$\|\nabla f\|_p^p \;\ge\; \left(\tfrac{p-1}{n-p}\right)^{p-1} \int_0^\infty C_p(\langle f\rangle_t)\, dt \;\ge\; n\omega_n^{\frac{p}{n}} \int_0^\infty V(\langle f\rangle_t)^{\frac{n-p}{n}}\, dt \tag{14}$$

and

$$\|\nabla f\|_p^p \;\ge\; \int_0^\infty \Phi_p(\langle f\rangle_t)\, dt \;\ge\; n\omega_n^{\frac{p}{n}} \int_0^\infty V(\langle f\rangle_t)^{\frac{n-p}{n}}\, dt. \tag{15}$$

Equality holds in all inequalities for $p = 1$, as f tends to the characteristic function of an origin-centered ball and for $p \in (1, n)$, as f tends to $(a + b|x|^{p/(p-1)})^{(p-n)/p}$ with positive constants a, b.

The left side inequality of (14) for $p = 1$ and the left side inequality of (15) were proved in [75].

The analytic inequalities (14) and (15) will be proved by using the geometric inequalities (12) and (13). The following remark, which is proved in Section 6.3, shows that (12) and (13) are geometric analogues of the analytic inequalities (14) and (15) respectively.

Remark 3. *Theorem 2 implies Lemma 1 for origin-symmetric convex bodies. In particular, the sharp convex Lorentz-Sobolev inequality (5) is the analytic analogue of the L^p isoperimetric inequality (7).*

The convex sets $\langle f\rangle_t$ can be used to define *convexified* Choquet spaces and thus the sharp convex Lorentz-Sobolev inequalities imply new embedding theorems. For a nonnegative set function, ψ, defined on a family $\mathcal{F}$ of subsets of $\mathbb{R}^n$, the Choquet integral, $\psi[f]$, of a function f on $\mathbb{R}^n$ with respect to ψ is defined as

$$\psi[f] = \int_0^\infty \psi([f]_t)\, dt.$$

For $p > 0$, the p-Choquet space associated with the set function ψ consists of functions with $\psi[|f|^p] < \infty$. The L^p Lorentz space is the p-Choquet space with the set function $\psi = V^{(n-p)/n}$. Choquet spaces are studied with respect to general capacities and find important applications in potential theory and partial differential equations (see Adams' survey paper [2]). Assume that $\mathcal{F}$ is the family of compact convex sets in $\mathbb{R}^n$ and ψ is a nonnegative set function on $\mathcal{F}$. Define the convexified Choquet integral, $\psi\langle f\rangle$, of a function f with respect to ψ as

$$\psi\langle f\rangle = \int_0^\infty \psi(\langle f\rangle_t)\, dt.$$

The set function ψ can be chosen as the integral affine surface area Φ_p or other important geometric functionals of compact convex sets, in particular, valuations (see [3, 28, 29, 37, 38, 40]). This may be useful for finding new connections between functional analysis and convex geometry. Note that the functional $f \mapsto \psi\langle f\rangle$ also depends on p. The convexified (p, q)-Choquet space consists of functions with $\psi\langle |f|^q\rangle < \infty$. The convexified L^p Lorentz space is the convexified $(p, 1)$-Choquet space with $\psi = V^{(n-p)/n}$. Hence, inequalities (14) and (15) induce sharp embeddings of Sobolev spaces into convexified Choquet spaces.

In the following sections, the notion of L^p convexification is discussed and tools about L^p geometric set functions are collected. The proofs of Lemma 1 and Theorem 2 are presented in Section 6. In Section 7, invariance properties of the L^p convexification and of the inequalities from Theorem 2 are discussed.

3. The notion of L^p mixed volume

3.1. **The L^p mixed volume of convex bodies.** A convex body is a compact convex set in $\mathbb{R}^n$ which is throughout assumed to contain the origin in its interior. We denote by $\mathcal{K}_0^n$ the space of convex bodies equipped with the Hausdorff metric. Each convex body K is uniquely determined by its support function $h_K : \mathbb{R}^n \to [0, \infty)$ defined by

$$h_K(x) = \sup\{x \cdot y \ : \ y \in K\}, \quad x \in \mathbb{R}^n,$$

and we also write $h(K, \cdot)$ for h_K.

For $p \geq 1$ and convex bodies K and L, the Minkowski-Firey L^p sum $K +_p L$ is the convex body whose support function is given by

$$h(K +_p L, \cdot)^p = h(K, \cdot)^p + h(L, \cdot)^p .$$

The L^p mixed volume $V_p(K, L)$ of $K, L \in \mathcal{K}_0^n$ is defined by

$$V_p(K, L) = \frac{p}{n} \lim_{\varepsilon \to 0^+} \frac{V(K +_p \varepsilon^{\frac{1}{p}} L) - V(K)}{\varepsilon} .$$

The existence of this limit was established in [41]. In particular,

$$V_p(K, K) = V(K) \tag{16}$$

for every convex body K.

The L^p Minkowski inequality [41] states that for convex bodies K and L

$$V_p(K, L)^n \geq V(K)^{n-p} V(L)^p \tag{17}$$

with equality if and only if K and L are dilates when $p > 1$ and if and only if K and L are homothetic when $p = 1$. A simple consequence of the equality conditions in (17) is that for $1 < p \neq n$, if K and K' are origin-symmetric convex bodies such that for all origin-symmetric convex bodies L,

$$V_p(K, L) = V_p(K', L), \text{ then } K = K'. \tag{18}$$

In [41], it was also shown that

$$V_p(K, L) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_L(u)^p \, dS_p(K, u) \tag{19}$$

for $K, L \in \mathcal{K}_0^n$, where $S_p(K, \cdot) = h_K^{1-p} S_K$ is the L^p surface area measure of K and S_K is the classical surface area measure of K (see [68] for information on the surface area measure). The Borel measure S_K on the $(n-1)$-dimensional unit sphere $\mathbb{S}^{n-1}$ can be defined as the measure such that

$$\int_{\mathbb{S}^{n-1}} g(u) \, dS_K(u) = \int_{\partial K} g(\nu_K(x)) \, dx \tag{20}$$

for all $g \in C(\mathbb{S}^{n-1})$, where $\nu_K(x)$ is the unit outer normal vector to K at x. For a convex body K, S_K is not supported on a great hypersphere. Hence, also $S_p(K, \cdot) = h_K^{1-p} S_K$ is not supported on a great hypersphere, that is,

$$\int_{\mathbb{S}^{n-1}} |u \cdot v|^p \, dS_p(K, u) > 0, \quad v \in \mathbb{S}^{n-1}. \tag{21}$$

Note that

$$S_p(t\,K,\cdot) = t^{n-p} S_p(K,\cdot) \tag{22}$$

for all $t > 0$ and convex bodies K.

3.2. **The L^p mixed volume of star bodies.** A star body M is a compact set in $\mathbb{R}^n$ which is star shaped with respect to the origin, i.e., if $x \in M$, then the line segment joining the origin to x is contained in M. Suppose that M is a Lipschitz star body, that is, M has Lipschitz boundary ∂M. Then the unit outer normal $\nu_M(x)$ to M exists for almost all x in ∂M. The radial function, $\rho_M : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}$, of M is defined for $x \neq 0$ by

$$\rho_M(x) = \max\{\lambda \geq 0 : \lambda x \in M\}.$$

Note that for a Lipschitz star body M, the radial function ρ_M is locally Lipschitz continuous when restricted to its core

$$D_M = \{t\,x : t > 0, x \in \partial M, |x \cdot \nu_M(x)| > 0\}. \tag{23}$$

For M a Lipschitz star body and L a convex body in $\mathbb{R}^n$, the L^p mixed volume $V_p(M, L)$ of M and L is defined by

$$V_p(M, L) = \frac{1}{n} \int_{\partial M} h_L(\nu_M(x))^p |x \cdot \nu_M(x)|^{1-p} dx. \tag{24}$$

While $V_p(M, L)$ is well defined, it is not necessarily finite.

For $p = 1$, the Minkowski inequality (17) was extended to more general sets (see [9, 73, 77]). We require the following special case: If M is a Lipschitz star body and L is a convex body in $\mathbb{R}^n$, then

$$V_1(M, L)^n \geq V(M)^{n-1} V(L) \tag{25}$$

with equality if and only if M and L are homothetic. The next lemma extends this to $p \geq 1$.

Lemma 4. *If M is a Lipschitz star body in $\mathbb{R}^n$ and L a convex body in $\mathbb{R}^n$, then for $1 \leq p < \infty$.*

$$V_p(M, L)^n \geq V(M)^{n-p} V(L)^p. \tag{26}$$

Equality holds if and only if M is a dilate of the convex body L when $p > 1$ and if and only if M is homothetic to the convex body L when $p = 1$.

Proof. The case $p = 1$ is just (25). Suppose $p > 1$. For a Lipschitz star body M,

$$nV(M) = \int_{\partial M} x \cdot \nu_M(x)\, dx. \tag{27}$$

The inequality (26) follows from (27), the Jensen inequality and (25),

$$\begin{aligned}\frac{V_p(M,L)}{V(M)} &= \frac{1}{nV(M)}\int_{\partial M}\left(\frac{h_L(\nu_M(x))}{x\cdot\nu_M(x)}\right)^p (x\cdot\nu_M(x))\,dx\\ &\geq \left(\frac{1}{nV(M)}\int_{\partial M} h_L(\nu_M(x))\,dx\right)^p\\ &= \left(\frac{V_1(M,L)}{V(M)}\right)^p\\ &\geq \left(\frac{V(L)}{V(M)}\right)^{\frac{p}{n}}.\end{aligned}$$

Suppose that there is equality in (26). The equality conditions in (25) show that up to translation by a vector $y\in\mathbb{R}^n$, M and L are dilates. Hence M is convex and $h_M(\nu_M(x)) = x\cdot\nu_M(x)$ almost everywhere on ∂M. Combined with the equality conditions in Jensen's inequality it follows that $h_L(\nu_M(x))/h_M(\nu_M(x))$ is constant almost everywhere on ∂M. Thus $y = 0$ and M and L are dilates. □

We also require the following lemmas.

Lemma 5. *If M is a Lipschitz star body in $\mathbb{R}^n$, then*

$$\nu_M(x) = -\frac{\nabla\rho_M(x)}{|\nabla\rho_M(x)|} \tag{28}$$

and

$$\nabla\rho_M(x) = -\frac{\nu_M(x)}{x\cdot\nu_M(x)} \tag{29}$$

for almost all $x\in\partial M\cap D_M$.

Proof. For $x\in\partial M\cap D_M$, the definition of radial function implies that $\rho_M(x) = 1$. Since the gradient $\nabla\rho_M(x)$ points in the direction of the normal vector $-\nu_M(x)$, we obtain (28). The function $\rho_M : \mathbb{R}^n\backslash\{0\}\to\mathbb{R}$ is homogeneous of degree -1, that is, for $x\neq 0$ and $t>0$,

$$\rho_M(tx) = \frac{1}{t}.$$

Differentiating above equation withrespect to t and taking $t = 1$, shows that for almost all $x\in\partial M\cap D_M$,

$$x\cdot\nabla\rho_M(x) = -1. \tag{30}$$

Combining (28) and (30), we obtain for almost all $x\in\partial M\cap D_M$,

$$x\cdot\nu_M(x) = \frac{1}{|\nabla\rho_M(x)|},$$

and thus (29).

□

Lemma 6. *If M is a Lipschitz star body in $\mathbb{R}^n$ and $g:(0,\infty)\to\mathbb{R}$ is strictly monotone, then for $f:\mathbb{R}^n\to\mathbb{R}$ defined by $f(x)=g(1/\rho_M(x))$,*

$$|\nabla f(x)| = \frac{|g'(\rho_M(x))|}{|z\cdot \nu_M(z)|}$$

holds for almost all $x\in\mathbb{R}^n$, where

$$z=\frac{x}{\rho_M(x)}\in\partial M\cap D_M \quad \textit{and} \quad \nu_M(z)=-\frac{\nabla f(x)}{|\nabla f(x)|}=\frac{\nabla(1/\rho_M)(z)}{|\nabla(1/\rho_M)(z)|}.$$

Proof. It is enough to handle the case that g is strictly decreasing. Assume this and consider the level sets

$$M_t=\{x\in\mathbb{R}^n:\ f(x)\geq t\},\quad t\in\mathbb{R}.$$

It is easy to see that

$$M_t=\{x\in\mathbb{R}^n:\ 1/\rho_M(x)\leq s\}=sM,\quad t=g(s).$$

For $x\in\partial M_t$, let $s=\rho_M(x)$ and $x=sz$. Then $z\in\partial M\cap D_M$ and hence $\rho_M(z)=1$. Note now that $1/\rho_M$ is homogeneous of degree 1. So we have

$$\nabla f(x)=g'(s/\rho_M(z))\nabla(1/\rho_M)(z)=g'(s)\nabla(1/\rho_M)(z).$$

Combined with Lemma 5, this completes the proof of the lemma. □

3.3. **The L^p mixed volume of functions.** A new approach to understand sharp L^p Sobolev inequalities was presented in [49]. For $1\leq p<n$, an origin-symmetric convex body $\langle f\rangle$ (depending on p) is associated to $f\in C_0^\infty(\mathbb{R}^n)$ such that

$$V_p(\langle f\rangle,L)=\frac{1}{n}\int_{\mathbb{R}^n} h(L,-\nabla f(x))^p\,dx \tag{31}$$

for all origin-symmetric convex bodies L. Note that (31) is the Sobolev norm of f with respect to the norm whose unit ball is the polar body of L. The approach uses the solution of the

Even L^p Minkowski Problem. *Suppose μ is an even Borel measure on $\mathbb{S}^{n-1}$ that is not supported on a great hypersphere of $\mathbb{S}^{n-1}$. Then for $1\leq p\neq n$ there exists a unique origin-symmetric convex body K such that*

$$\mu=S_p(K,\cdot),$$

that is, μ is the L^p surface area measure of K.

For $p=1$, the Minkowski problem and its solution are classical (see [68]). For $p>1$, the even L^p Minkowski problem was first posed and solved in [41] (see also [44, 46]). We remark that the question for not necessarily even data was solved by Chou and Wang [15] (The case of $p\geq n$ was solved by Guan and Lin [27]). Different approaches were given in [36]. For generalizations, see Hu, Ma and Shen [35].

For $f\in C_0^\infty(U)$, where $U\subset\mathbb{R}^n$ is open, and $1\leq p<n$, define the origin-symmetric convex body $\langle f\rangle$ (depending on p) as the unique origin-symmetric convex body such that

$$\int_{\mathbb{S}^{n-1}} g(u)\,dS_p(\langle f\rangle,u)=\int_U g(-\nabla f(x))\,dx \tag{32}$$

for all even $g \in C(\mathbb{S}^{n-1})$ extended to $\mathbb{R}^n$ to be homogeneous of degree p. To see that $\langle f \rangle$ exists and is unique, define the even Borel measure μ (depending on p) by

$$\int_{\mathbb{S}^{n-1}} g(u)\, d\mu(u) = \int_U g(-\nabla f(x))\, dx \tag{33}$$

for all even $g \in C(\mathbb{S}^{n-1})$ extended to $\mathbb{R}^n$ to be homogeneous of degree p. For f not vanishing identically, the measure μ in (33) is not supported on any great hypersphere $\{x \cdot v = 0\}$, since taking $g(u) = |u \cdot v|^p$ in (33) shows that

$$\int_{\mathbb{S}^{n-1}} |u \cdot v|^p\, d\mu(u) = \int_U \Big| \frac{\partial f}{\partial v}(x) \Big|^p\, dx > 0,$$

where the above integral is positive since f is compactly supported in U. Hence, by the solution to the even L^p Minkowski problem, $\langle f \rangle$ exists and is uniquely determined. Equation (31) follows from (32) and the definition of L^p mixed volume (19). More generally, for $C^\infty(U)$, where $U \subset \mathbb{R}^n$ is open and bounded, we use

$$\int_{\mathbb{S}^{n-1}} g(u)\, dS_p(\langle f \rangle, u) = \int_U g(-\nabla f(x))\, dx \tag{34}$$

to define $\langle f \rangle$, if

$$\int_U \big| \frac{\partial f}{\partial v}(x) \big|^p\, dx > 0 \tag{35}$$

for all $v \in \mathbb{S}^{n-1}$.

4. The notion of L^p convexification

4.1. The L^p convexification of level sets. By the co-area formula (see, e.g., [54, Theorem 1.2.4]) and (31), we obtain that for $1 \le p < n$, origin-symmetric $L \in \mathcal{K}_0^n$ and $f \in C_0^\infty(\mathbb{R}^n)$,

$$\begin{aligned} V_p(\langle f \rangle, L) &= \frac{1}{n} \int_{\mathbb{R}^n} h(L, -\nabla f(x))^p\, dx \\ &= \frac{1}{n} \int_{\mathbb{R}^n} h(L, \nu(x))^p\, |\nabla f(x)|^p\, dx \\ &= \frac{1}{n} \int_0^\infty \int_{\partial [f]_t} h(L, \nu(x))^p\, |\nabla f(x)|^{p-1}\, dx\, dt \end{aligned} \tag{36}$$

where $\nu(x) = -\nabla f(x)/|\nabla f(x)|$. By Sard's theorem, for almost all $t > 0$, we have $\nabla f(x) \neq 0$ on $\partial [f]_t$ and hence $\nu(x)$ is well defined. Equation (36) motivates the following definition of a local version of $\langle f \rangle$ (see [46] and [49] and for the case of $p = 1$, [8] and [77]).

Let $f : U \to \mathbb{R}$, where $U \subset \mathbb{R}^n$ is open, be locally Lipschitz, let $t > 0$, and suppose $\nabla f(x) \neq 0$ a.e. on $\partial [f]_t = \{x \in U : f(x) = t\}$. For $1 \le p < n$, define the L^p convexification $\langle f \rangle_t$ of the level set $[f]_t$ as the unique origin-symmetric convex body such that

$$\int_{\mathbb{S}^{n-1}} g(u)\, dS_p(\langle f \rangle_t, u) = \int_{\partial [f]_t} g(\nu(x))\, |\nabla f(x)|^{p-1} dx \tag{37}$$

for all even $g \in C(\mathbb{S}^{n-1})$. To see that $\langle f \rangle_t$ exists and is unique, define the even Borel measure μ_t (depending on p) by

$$\int_{\mathbb{S}^{n-1}} g(u)\, d\mu_t(u) = \int_{\partial [f]_t} g(\nu(x))\, |\nabla f(x)|^{p-1} dx \tag{38}$$

for all even $g \in C(\mathbb{S}^{n-1})$. The measure μ_t is not supported on any great hypersphere $\{x \cdot v = 0\}$, since taking $g(u) = |u \cdot v|$ in (38) shows that

$$\int_{\mathbb{S}^{n-1}} |u \cdot v|\, d\mu_t(u) = \int_{\partial [f]_t} |v \cdot \nu(x)|\, |\nabla f(x)|^{p-1}\, dx > 0.$$

Thus by the solution to the L^p Minkowski problem $\langle f \rangle_t$ exists and is uniquely determined. The L^p convexification $\langle f \rangle_t$ depends only on ∇f restricted to $\partial [f]_t$. Note that for $1 \le p < n$ and L an origin-symmetric convex body,

$$V_p(\langle f \rangle, L) = \int_0^\infty V_p(\langle f \rangle_t, L)\, dt. \tag{39}$$

A simple consequence of (39) is the following lemma, which shows that $\langle f \rangle_t$ is a local version of $\langle f \rangle$.

Lemma 7. *If $f \in C_0^\infty(\mathbb{R}^n)$ and $|\nabla f(x)| \neq 0$ on $\partial [f]_t$ for $t > 0$, then for $1 \le p < n$,*

$$\lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} S_p(\langle f_{t,\varepsilon} \rangle, \cdot) = S_p(\langle f \rangle_t, \cdot)$$

weakly, where $f_{t,\varepsilon}(x) = f(x)$ for $t - \varepsilon < |f(x)| < t + \varepsilon$ and $f_{t,\varepsilon}(x) = 0$ otherwise.

Proof. Since $|\nabla f(x)| \neq 0$ on $\partial [f]_t$, (35) holds for

$$U = \{x \in \mathbb{R}^n :\ t - \varepsilon < |f(x)| < t + \varepsilon\}.$$

Hence $\langle f_{t,\varepsilon} \rangle$ is well defined. By (34), the co-area formula and (37),

$$\begin{aligned} \int_{\mathbb{S}^{n-1}} g(u)\, dS_p(\langle f_{t,\varepsilon} \rangle, u) &= \int_U g(-\nabla f(x))\, dx \\ &= \int_{t-\varepsilon}^{t+\varepsilon} \int_{\partial [f]_\tau} g(\nu(x)) |\nabla f(x)|^{p-1}\, dx d\tau \\ &= \int_{t-\varepsilon}^{t+\varepsilon} \left(\int_{\mathbb{S}^{n-1}} g(u)\, dS_p(\langle f \rangle_\tau, u) \right) d\tau \end{aligned}$$

for all even $g \in C(\mathbb{S}^{n-1})$. Applying Lebesgue's differentiation theorem concludes the proof of the lemma. □

We also require the following result.

Lemma 8. *If K is an origin-symmetric convex body in $\mathbb{R}^n$ and $f(x) = g(1/\rho_K(x))$, where $g \in C^1(0, \infty)$ is strictly decreasing, then for $t > 0$ and $1 \le p < n$, the convex bodies of the L^p convexification of the level sets of f are dilates of K, that is,*

$$\langle f \rangle_t = c_p(t)\, K,$$

and $c_p(t)^{n-p} = |g'(s)|^{p-1} s^{n-1}$, where $t = g(s)$.

Proof. Clearly, $[f]_t = sK$ with $t = g(s)$. By Lemma 6, we get

$$|\nabla f(x)| = |g'(s)|/h_K(u),$$

where $u = -\nabla f/|\nabla f|$. Using this and (37), for any even function $h \in C(\mathbb{S}^{n-1})$,

$$\begin{aligned}\int_{\mathbb{S}^{n-1}} h(u)\, dS_p(\langle f\rangle_t, u) &= \int_{\partial [f]_t} h(\nu_K(x))|\nabla f(x)|^{p-1}\, dx \\ &= \int_{\partial K} h(\nu_K(x))\big(\frac{|g'(s)|}{h_K(\nu_K(x))}\big)^{p-1} s^{n-1}\, dx \\ &= |g'(s)|^{p-1} s^{n-1} \int_{\mathbb{S}^{n-1}} h(u)\, dS_p(K, u).\end{aligned}$$

Thus, the uniqueness of the solution of the even L^p Minkowski problem and (22) imply that $\langle f\rangle_t = c_p(t)K$ where $c_p(t)^{n-p} = |g'(s)|^{p-1}s^{n-1}$ and $t = g(s)$. □

4.2. **The L^p convexification of star bodies.** The following application of L^p convexification is of special interest. Let M be a Lipschitz star body in $\mathbb{R}^n$ and set

$$f(x) = \rho_M(x)$$

for $x \in D_M$, where D_M is defined in (23). Note that $\nabla f \neq 0$ a.e. on D_M. For $1 \le p < n$, define

$$\breve{M}_p = \langle f\rangle_1$$

We call $\breve{M}_p$ the L^p convexification of M. Thus, by Lemma 5 and (37), $\breve{M}_p$ is the unique convex body that satisfies for all even $g \in C(\mathbb{S}^{n-1})$ the equation

$$\int_{\mathbb{S}^{n-1}} g(u)\, dS_p(\breve{M}_p, u) = \int_{\partial M} g(\nu_M(x))|x\cdot \nu_M(x)|^{1-p} dx \tag{40}$$

for $1 \le p < n$. As an immediate consequence, we obtain the following two lemmas.

Lemma 9. *If M is a Lipschitz star body in $\mathbb{R}^n$ with L^p convexification $\breve{M}_p$, then for $1 \le p < n$,*

$$V_p(M, K) = V_p(\breve{M}_p, K) \tag{41}$$

for any convex body K.

Proof. Equation (41) follows from (24), (40) and (19). □

Lemma 10. *If M is a Lipschitz star body in $\mathbb{R}^n$ with L^p convexification $\breve{M}_p$, then for $1 \le p < n$,*

$$V(\breve{M}_p) \ge V(M).$$

Equality holds if and only if M and $\breve{M}_p$ are dilates when $p > 1$ and if and only if M and $\breve{M}_p$ are homothetic when $p = 1$.

Proof. By Lemmas 9 and 4, for $1 \le p < n$, we have

$$V(\breve{M}_p) = V_p(\breve{M}_p, \breve{M}_p) = V_p(M, \breve{M}_p) \ge V(M)^{\frac{n-p}{n}} V(\breve{M}_p)^{\frac{p}{n}}.$$

This completes the proof of the lemma. □

5. Geometric set functions in the L^p Brunn-Minkowski theory

5.1. **The L^p variational capacity.** Recall that for a compact subset E of $\mathbb{R}^n$ and $p \geq 1$, the L^p variational capacity $C_p(E)$ is defined by

$$C_p(E) = \inf\{\|\nabla f\|_p^p : \ f \in C_0^\infty(\mathbb{R}^n) \text{ with } f \geq 1_E\},$$

where 1_E is the indicator function of E. Note that for the unit ball B of $\mathbb{R}^n$, we have $C_p(B) = n\omega_n \big(\frac{n-p}{p-1}\big)^{p-1}$ (cf. [54, p. 106]). For more information on L^p variational capacity, see [16, 21, 54].

The L^p variational capacity has the following invariance properties:

$$\begin{array}{lcll} C_p(\lambda E) & = & \lambda^{n-p} C_p(E) & \text{for all } \lambda > 0 \\ C_p(\phi E) & = & C_p(E) & \text{for each rotation or translation } \phi. \end{array} \tag{42}$$

The sharp L^p Maz′ya isocapacitary inequality (see [54, p. 105]) states that

$$\big(\tfrac{p-1}{n-p}\big)^{p-1} C_p(E) \geq n\omega_n^{p/n}\, V(E)^{(n-p)/n} \tag{43}$$

for all compact $E \subset \mathbb{R}^n$ and $p \geq 1$. Equality in (43) holds for balls. At the end-point case $p = 1$,

$$S(M) \geq C(M) = C_1(M) \tag{44}$$

holds for all Lipschitz star bodies M (see [54, p. 107]).

5.2. **The L^p integral affine surface area.** For $u \in \mathbb{S}^{n-1}$, let $\bar{u}$ be the line segment whose support function is given by

$$h_{\bar{u}}(x) = \frac{1}{2}|u \cdot x|, \quad x \in \mathbb{R}^n, \tag{45}$$

that is, $\bar{u}$ is a line segment of unit length that is parallel to u and is centered at the origin.

For a Lipschitz star body M in $\mathbb{R}^n$ and $p \geq 1$, the L^p integral affine surface area $\Phi_p(M)$ is defined by

$$\Phi_p(M) = \frac{(n\omega_n)^{\frac{n+p}{n}}}{\beta_{n,p}} \left(\int_{\mathbb{S}^{n-1}} V_p(M, \bar{u})^{-\frac{n}{p}}\, du \right)^{-\frac{p}{n}}, \tag{46}$$

where

$$\beta_{n,p} = V_p(B, \bar{u}) = \frac{1}{2^p n} \int_{\mathbb{S}^{n-1}} |u \cdot v|^p\, dv. \tag{47}$$

The normalization of Φ_p is chosen so that $\Phi_p(B) = n\omega_n$.

When K is a convex body, the following L^p affine isoperimetric inequality was proved in [45] (see [10] for an alternate proof),

$$\Phi_p(K) \ \geq \ n\omega_n^{p/n}\, V(K)^{(n-p)/n}. \tag{48}$$

Equality holds for $p > 1$ if and only if K is an ellipsoid centered at the origin, and for $p = 1$ if and only if K is an ellipsoid that contains the origin in its interior.

In [45], it is proved that for a convex body K,

$$\Phi_p(\psi K) = |\det \psi|^{\frac{n-p}{n}}\, \Phi_p(K) \tag{49}$$

for $\psi \in \mathrm{GL}(n)$.

5.3. **The L^p surface area.** Let M be a Lipschitz star body in $\mathbb{R}^n$. For $p \geq 1$, the L^p surface area $S_p(M)$ of M is defined by

$$S_p(M) = n\, V_p(M, B) = \int_{\partial M} |x \cdot \nu_M(x)|^{1-p} dx, \tag{50}$$

where B is the closed unit ball in $\mathbb{R}^n$. Note that for $p > 1$, the L^p surface area $S_p(M)$ is not finite for all Lipschitz star bodies. It is normalized so that $S_p(B) = n\omega_n$.

By (50) and (45), the L^p surface area has the following integral formula

$$S_p(M) = \frac{1}{\beta_{n,p}} \int_{\mathbb{S}^{n-1}} V_p(M, \bar{u})\, du. \tag{51}$$

where $\beta_{n,p}$ is defined in (47). An important property of L^p surface areas is the following monotonicity result.

Proposition 11. *If M is a Lipschitz star body in $\mathbb{R}^n$, then we have for $1 \leq q < p < n$,*

$$\left(\frac{S_q(M)}{n\omega_n}\right)^{\frac{1}{n-q}} \leq \left(\frac{S_p(M)}{n\omega_n}\right)^{\frac{1}{n-p}} \tag{52}$$

with equality if and only if M is a ball centered at the origin.

Proof. To prove (52), it suffices to consider the case $S_p(M) < \infty$ (otherwise there is nothing to prove). Since $S_1(M) = S(M)$, (50) and Jensen's inequality for the probability measure $S_1(M)^{-1}dx$ on ∂M yield

$$\begin{aligned}\Big(\frac{S_q(M)}{S_1(M)}\Big)^{\frac{1}{q-1}} &= \left(\frac{1}{S_1(M)} \int_{\partial M} |x \cdot \nu_M(x)|^{1-q} dx\right)^{\frac{1}{q-1}} \\ &\leq \left(\frac{1}{S_1(M)} \int_{\partial M} |x \cdot \nu_M(x)|^{1-p} dx\right)^{\frac{1}{p-1}} \\ &= \Big(\frac{S_p(M)}{S_1(M)}\Big)^{\frac{1}{p-1}},\end{aligned}$$

which implies

$$S_q(M)^{\frac{p-1}{q-1}} \leq S_1(M)^{\frac{p-q}{q-1}} S_p(M) \tag{53}$$

with equality if and only if $|x \cdot \nu_M(x)|$ is constant on ∂M. Furthermore, an application of (27), the Jensen inequality and (50) imply

$$\begin{aligned}&\Big(\frac{S_q(M)}{nV(M)}\Big)^{-\frac{1}{q}} \\ =\ & \left(\big(nV(M)\big)^{-1} \int_{\partial M} |x \cdot \nu_M(x)|^{-q} |x \cdot \nu_M(x)| dx\right)^{-\frac{1}{q}} \\ \leq\ & \Big(\frac{S_1(M)}{nV(M)}\Big)^{-1}.\end{aligned}$$

Hence by the isoperimetric inequality (9),

$$S_q(M) \geq S_1(M)^q \big(nV(M)\big)^{1-q} \geq \Big((n\omega_n)^{\frac{q-1}{n-q}} S_1(M)\Big)^{\frac{n-q}{n-1}},$$

that is,

$$S_1(M) \le (n\omega_n)^{\frac{1-q}{n-q}} S_q(M)^{\frac{n-1}{n-q}}. \tag{54}$$

Combining (54) and (53) gives

$$S_q(M)^{\frac{n-p}{n-q}} \le (n\omega_n)^{\frac{q-p}{n-q}} S_p(M).$$

This yields (52). The equality condition follows from the equality condition of the isoperimetric inequality – since $|x \cdot \nu_M(x)|$ is constant on ∂M under this circumstance. □

6. Proofs of the principal results

6.1. **Proof of Lemma 1.** First, note that the right side inequality of (12) is a special case of (43). For $p = 1$, the left side inequality of (12) is just (44) since $S_1(M) = S(M)$. We prove the left side inequality for $1 < p < n$. For a Lipschitz star body M, let

$$f(x) = g(1/\rho_M(x)), \quad g(s) = \min\{1, s^{\frac{n-p}{1-p}}\}.$$

We have that $[f]_t = s\,M$ where $t = g(s)$. By the co-area formula, Lemma 6, Fubini's theorem and (50), we obtain

$$\begin{aligned}
\int_{\mathbb{R}^n} |\nabla f(x)|^p dx &= \int_0^\infty \int_{\partial [f]_t} |\nabla f(x)|^{p-1}\, dx\, dt \\
&= \int_0^1 \int_{\partial [f]_t} |g'(1/\rho_M(x))|^{p-1}\, |\nabla (1/\rho_M)(x)|^{p-1}\, dx\, dt \\
&= \int_1^\infty |g'(s)|^p\, s^{n-1} \int_{\partial M} |y \cdot \nu_M(y)|^{1-p}\, dy\, ds \\
&= \Big(\int_1^\infty |g'(s)|^p\, s^{n-1}\, ds\Big) S_p(M).
\end{aligned}$$

Since

$$\int_1^\infty |g'(s)|^p\, s^{n-1}\, ds = \Big(\frac{n-p}{p-1}\Big)^{p-1},$$

standard limiting arguments show that the left side inequality of (12) is valid.

Second, the left side inequality of (13) follows from the Jensen inequality, (51) and (46). To prove the right side inequality of (13), we consider the L^p convexification $\breve{M}_p$ of a Lipschitz star body M defined by (40). By Lemma 9, we have

$$V_p(M, \bar{u}) = V_p(\breve{M}_p, \bar{u}) \tag{55}$$

for $1 \le p < n$. By (46) and (55), it follows that

$$\Phi_p(M) = \Phi_p(\breve{M}_p). \tag{56}$$

By (56), (48), and Lemma 10, we have for $1 \le p < n$,

$$\Phi_p(M) = \Phi_p(\breve{M}_p) \ge n\omega_n^{\frac{p}{n}} V(\breve{M}_p)^{\frac{n-p}{n}} \ge n\omega_n^{\frac{p}{n}} V(M_p)^{\frac{n-p}{n}}.$$

This concludes the proof ot the right side inequality of (13).

Third, let us handle the equality conditions of inequalities (12) and (13). By the equality conditions in Lemma 10 and the equality conditions of the inequality (48), the equality in the right side of (13) holds if and only if M is an ellipsoid centered at the origin when $p > 1$ and M is an ellipsoid that contains the origin when $p = 1$. It is easily verified that equalities in (12) hold if M is a ball centered at the origin.

Complete equality conditions for (12) are not known. We always have $C_1(K) = S(K)$ whenever K is a convex body. The following is a proof of this fact (see also [54, p. 107]). Of course, due to (44) we are only required to show $C_1(K) \geq S(K)$. Let f be a function so that $f \geq 1_K$. Then, K is a subset of $[f]_t$ for $t \in (0, 1)$, and hence

$$V_1(K, \bar{u}) \leq V_1(\langle f \rangle_t, \bar{u}), \quad u \in \mathbb{S}^{n-1}.$$

By the co-area formula [54, Theorem 1.2.4], (37) and (19), it follows that

$$\begin{aligned}\int_{\mathbb{R}^n} |u \cdot \nabla f(x)|\, dx &= \int_0^\infty \int_{\partial [f]_t} |u \cdot \nabla f(x)| |\nabla f(x)|^{-1} dx\, dt \\ &= 2n \int_0^\infty V_1(\langle f \rangle_t, \bar{u})\, dt \\ &\geq 2n\, V_1(K, \bar{u}).\end{aligned}$$

Integration with respect to u, Fubini's theorem and (51) give

$$\|\nabla f\|_1 \geq S(K),$$

and thus $C_1(K) \geq S(K)$.

6.2. **Proof of Theorem 2.** By (31) and (50), we have

$$\int_{\mathbb{R}^n} |\nabla f(x)|^p dx = \int_0^\infty S_p(\langle f \rangle_t)\, dt. \tag{57}$$

By Lemma 1 we have

$$\int_0^\infty S_p(\langle f \rangle_t)\, dt \geq \left(\tfrac{p-1}{n-p}\right)^{p-1} \int_0^\infty C_p(\langle f \rangle_t)\, dt \geq n\omega_n^{\frac{p}{n}} \int_0^\infty V(\langle f \rangle_t)^{\frac{n-p}{n}}\, dt$$

and

$$\int_0^\infty S_p(\langle f \rangle_t)\, dt \geq \int_0^\infty \Phi_p(\langle f \rangle_t)\, dt \geq n\omega_n^{\frac{p}{n}} \int_0^\infty V(\langle f \rangle_t)^{\frac{n-p}{n}}\, dt.$$

The above inequalities imply (14) and (15). The equality conditions follow from those of (1).

6.3. **Explanation of Remark 3.** It is shown above that Lemma 1 implies Theorem 2. To see that Theorem 2 also implies Lemma 1 for origin-symmetric convex bodies, we take

$$f(x) = g(1/\rho_K(x)) \text{ where } g(s) = (1 + s^{\frac{p}{p-1}})^{1-\frac{n}{p}} \tag{58}$$

for origin-symmetric $K \in \mathcal{K}_0^n$. By (57), Lemma 8, and an elementary calculation, the analytic inequalities (14) and (15) imply the geometric inequalities (12) and (13) for origin-symmetric convex bodies.

Assume the L^p isoperimetric inequality (7). By (57), we have

$$\int_{\mathbb{R}^n} |\nabla f(x)|^p dx = \int_0^\infty S_p(\langle f \rangle_t)\, dt \geq n\omega_n^{\frac{p}{n}} \int_0^\infty V(\langle f \rangle_t)^{\frac{n-p}{n}}\, dt.$$

Thus, the sharp convex Lorentz-Sobolev inequality (5) holds. To show that the inequality (5) also implies the inequality (7), we again take the function f defined in (58).

6.4. **Proof of inequality (8).** Let $\bar{f}$ be the symmetric rearrangement of f, defined by

$$\bar{f}(x) = \inf\{t > 0 : V([f]_t) < \omega_n |x|^n\}, \quad x \in \mathbb{R}^n.$$

The Pólya-Szegö principle [72] says that

$$\|\nabla \bar{f}\|_p \le \|\nabla f\|_p. \tag{59}$$

Let $\hat{f}$ be the increasing function on $(0, \infty)$ defined by $\hat{f}(1/|x|) = \bar{f}(x)$. If $t = \hat{f}(s)$, then $V([f]_t) = \omega_n s^{-n}$. It was shown in [46, Lemma 5.1] that

$$\omega_n^{\frac{n-p}{n}} \int_0^\infty \hat{f}'(s)^p s^{2p-n-1}\, ds = n^{p-1} \int_0^\infty V([f]_t)^{\frac{p(n-1)}{n}} \big(-\frac{d}{dt} V([f]_t)\big)^{1-p}\, dt.$$

Hence the Hölder inequality gives that

$$\begin{aligned} & \int_0^\infty V([f]_t)^{\frac{n-p}{n}}\, dt^p \\ = \; & p\omega_n^{\frac{n-p}{n}} \int_0^\infty \hat{f}(s)^{p-1} \hat{f}'(s) s^{p-n}\, ds \\ \le \; & p\omega_n^{\frac{n-p}{n}} \Big(\int_0^\infty \hat{f}'(s)^p s^{2p-n-1}\, ds \Big)^{\frac{1}{p}} \Big(\int_0^\infty \hat{f}(s)^p s^{p-n-1}\, ds \Big)^{1-\frac{1}{p}} \\ = \; & p\omega_n^{\frac{(n-p)(p-1)}{np}} \Big(n^{p-1} \int_0^\infty V([f]_t)^{\frac{p(n-1)}{n}} \big(-\frac{d}{dt} V([f]_t)\big)^{1-p}\, dt \Big)^{\frac{1}{p}} \\ & \times \Big(\int_0^\infty \hat{f}(s)^p s^{p-n-1}\, ds \Big)^{1-\frac{1}{p}}. \end{aligned} \tag{60}$$

It was shown in [46, (6.3)] that the following differential inequality holds

$$n^{p-1} V([f]_t)^{\frac{(n-1)p}{n}} \Big(-\frac{d}{dt} V([f]_t)\Big)^{1-p} \le V(\langle f \rangle_t)^{\frac{n-p}{n}}. \tag{61}$$

Integrating both sides of the inequality (61) gives

$$n^{p-1} \int_0^\infty V([f]_t)^{\frac{p(n-1)}{n}} \big(-\frac{d}{dt} V([f]_t)\big)^{1-p}\, dt \le \int_0^\infty V(\langle f \rangle_t)^{\frac{n-p}{n}}\, dt. \tag{62}$$

The case $p = 1$ of (62) yields (8) right away. Note that the following Hardy inequality in $\mathbb{R}^n$ holds for $1 < p < n$ (see [19, 34, 51]),

$$\int_{\mathbb{R}^n} |f(x)|^p |x|^{-p} dx \le p^p (n-p)^{-p} \|\nabla f\|_p^p.$$

Using polar coordinates, the definition of $\bar f$, the Hardy inequality, and the Pólya-Szegö principle (59), we have that if $1 < p < n$ then

$$\begin{aligned}\int_0^\infty \hat f(s)^p s^{p-n-1}\,ds &= \int_0^\infty r^{n-1-p}\hat f(r^{-1})^p\,dr\\ &= (n\omega_n)^{-1}\int_0^\infty \Big(\int_{r\mathbb{S}^{n-1}} |\bar f|^p\,du\Big)\frac{dr}{r^p}\\ &= (n\omega_n)^{-1}\int_{\mathbb{R}^n} \big(\frac{|\bar f(x)|}{|x|}\big)^p\,dx\\ &\le (n\omega_n)^{-1}\Big(\frac{p}{n-p}\Big)^p\|\nabla \bar f\|_p^p\\ &\le (n\omega_n)^{-1}\Big(\frac{p}{n-p}\Big)^p\|\nabla f\|_p^p.\end{aligned}$$

By (60), (62) and (63), this concludes the proof of (8).

7. Invariance properties of the inequalities

Let $\mathrm{Aff}(n)$ denote the group of invertible affine transformations of $\mathbb{R}^n$, that is, every map $\Psi \in \mathrm{Aff}(n)$ is a general linear transformation followed by a translation. Let $\mathrm{Sim}(n)$ be the group of similarities of $\mathbb{R}^n$, that is, every map $\Psi \in \mathrm{Sim}(n)$ is a rotation followed by a dilation ($x \mapsto t\,x$ for $x \in \mathbb{R}^n$ and $t > 0$) followed by a translation. There is a natural left action of $\mathbb{R}\backslash\{0\} \times \mathrm{Aff}(n)$ and $\mathbb{R}\backslash\{0\} \times \mathrm{Sim}(n)$ on functions $f : \mathbb{R}^n \to \mathbb{R}$, given by

$$f \mapsto s\,f \circ \Psi^{-1} \tag{63}$$

for each (s, Ψ) in $\mathbb{R}\backslash\{0\}\times\mathrm{Aff}(n)$ and $\mathbb{R}\backslash\{0\}\times\mathrm{Sim}(n)$, respectively. An inequality $L[f] \le R[f]$ for a class of functions $f : \mathbb{R}^n \to \mathbb{R}$ is called affine if

$$\frac{L[s\,f\circ\Psi^{-1}]}{R[s\,f\circ\Psi^{-1}]} = \frac{L[f]}{R[f]} \tag{64}$$

for each $(s, \Psi) \in \mathbb{R}\backslash\{0\} \times \mathrm{Aff}(n)$. It is called similarity invariant if (64) holds for each $(s, \Psi) \in \mathbb{R}\backslash\{0\} \times \mathrm{Sim}(n)$. By (42) and (49), the inequalities (2) and the left side inequalities of (14) and (15) are similarity invariant. We show that the the right side inequality of (15) is affine and that the right side inequality of (14) is similarity invariant. We use the following transformation properties of the support function and the L^p mixed volume that follow immediately from their definition and (16), (19) and (22): for a convex body L and $u \in \mathbb{R}^n$,

$$h(\psi L, u) = h(L, \psi^t u) \tag{65}$$

for all $\psi \in \mathrm{GL}(n)$, where ψ^t denotes the transpose of ψ, and for convex bodies K and L,

$$\begin{aligned} V_p(\psi K, L) &= |\det\psi|\,V_p(K, \psi^{-1}L)\\ V_p(K, sL) &= s^p\,V_p(K, L)\\ V_p(sK, L) &= s^{n-p}\,V_p(K, L)\end{aligned} \tag{66}$$

for every $\psi \in \mathrm{GL}(n)$ and $s > 0$. We require the following two lemmas.

Lemma 12. *Let $p \in [1,n)$. If $f \in C_0^\infty(\mathbb{R}^n)$, then*

$$\langle s\, f \circ \Psi^{-1} \rangle = s^{\frac{p}{n-p}}\, \psi \langle f \rangle$$

for $s > 0$ and $\Psi \in \mathrm{Aff}(n)$ given by $\Psi(x) = \psi x + y$ where $\psi \in \mathrm{GL}(n)$ and $y \in \mathbb{R}^n$.

Proof. Note that

$$\nabla(s\, f \circ \Psi^{-1})(x) = s\, \psi^{-t}\, \nabla f(\Psi^{-1} x).$$

Hence, for L an origin-symmetric convex body, by (31), (65) and (66),

$$\begin{aligned} V_p(\langle s\, f \circ \Psi^{-1}\rangle, L) &= \frac{1}{n} \int_{\mathbb{R}^n} h(L, -s\, \psi^{-t} \nabla f(\Psi^{-1} x))^p \, dx \\ &= \frac{1}{n} |\det \psi| \int_{\mathbb{R}^n} h(s\, \psi^{-1} L, -\nabla f(y))^p \, dy \\ &= |\det \psi|\, V_p(\langle f \rangle, s\, \psi^{-1} L) \\ &= V_p(s^{\frac{p}{n-p}} \psi \langle f \rangle, L). \end{aligned}$$

Since these equations hold for all origin-symmetric convex bodies L, (18) implies the statements of the lemma. □

Lemma 13. *Let $p \in [1,n)$. If $f \in C_0^\infty(\mathbb{R}^n)$ and $\nabla f(x) \neq 0$ on $\partial [f]_t$ for $t > 0$, then*

$$\langle s\, f \circ \Psi^{-1} \rangle_t = s^{\frac{p}{n-p}}\, \psi \langle f \rangle_t$$

for $s > 0$ and $\Psi \in \mathrm{Aff}(n)$ given by $\Psi(x) = \psi x + y$ where $\psi \in \mathrm{GL}(n)$ and $y \in \mathbb{R}^n$.

Proof. Since $(s\, f \circ \Psi^{-1})_{t,\varepsilon} = s\, f_{t,\varepsilon} \circ \Psi^{-1}$, Lemma 12 implies

$$\langle (s\, f \circ \Psi^{-1})_{t,\varepsilon} \rangle = s^{\frac{p}{n-p}}\, \psi \langle f_{t,\varepsilon} \rangle.$$

Hence, by (22),

$$S_p(\langle (s\, f \circ \Psi^{-1})_{t,\varepsilon} \rangle, \cdot) = s^p S_p(\psi \langle f_{t,\varepsilon} \rangle, \cdot).$$

By (19), this implies

$$V_p(\langle (s\, f \circ \Psi^{-1})_{t,\varepsilon} \rangle, L) = s^p\, V_p(\psi \langle f_{t,\varepsilon} \rangle, L).$$

for all convex bodies L. Hence Lemma 7 and (66) give

$$V_p(\langle s\, f \circ \Psi^{-1} \rangle_t, L) = s^p\, V_p(\psi \langle f \rangle_t, L) = V_p(s^{\frac{p}{n-p}} \psi \langle f \rangle_t, L).$$

Since these equations hold for all origin-symmetric convex bodies L, (18) implies the statements of the lemma. □

Proposition 14. *For $p \in [1,n)$ and $f \in C_0^\infty(\mathbb{R}^n)$,*

$$\Big(\int_0^\infty \Phi_p(\langle f \rangle_t)\, dt \Big) \Big/ \Big(\int_0^\infty V(\langle f \rangle_t)^{\frac{n-p}{n}}\, dt \Big)$$

is invariant under the natural action of $\mathbb{R} \backslash \{0\} \times \mathrm{Aff}(n)$ and

$$\Big(\int_0^\infty C_p(\langle f \rangle_t)\, dt \Big) \Big/ \Big(\int_0^\infty V(\langle f \rangle_t)^{\frac{n-p}{n}}\, dt \Big)$$

is invariant under the natural action of $\mathbb{R} \backslash \{0\} \times \mathrm{Sim}(n)$.

Proof. By Lemma 13 and (42), we get

$$C_p(\langle s\, f \circ \Psi^{-1} \rangle_t) = C_p(|s|^{\frac{p}{n-p}}\, \psi \langle f \rangle_t) = |s|^p\, |\det \psi|^{\frac{n-p}{n}} C_p(\langle f \rangle_t)$$

for $(s, \Psi) \in \mathbb{R}\backslash\{0\} \times \mathrm{Sim}(n)$. By Lemma 13 and (49), we get

$$\Phi_p(\langle s\, f \circ \Psi^{-1} \rangle_t) = \Phi_p(|s|^{\frac{p}{n-p}}\, \psi \langle f \rangle_t) = |s|^p\, |\det \psi|^{\frac{n-p}{n}} \Phi_p(\langle f \rangle_t).$$

for $(s, \Psi) \in \mathbb{R}\backslash\{0\} \times \mathrm{Aff}(n)$. Note also that Lemma 13 yields

$$V(\langle s\, f \circ \Psi^{-1} \rangle_t)^{\frac{n-p}{n}} = |s|^p\, |\det \psi|^{\frac{n-p}{n}} V(\langle f \rangle_t)^{\frac{n-p}{n}}$$

for $(s, \Psi) \in \mathbb{R}\backslash\{0\} \times \mathrm{Aff}(n)$. Combining the last three equations gives the statement of the proposition. ☐

8. A question

Since

$$S_1(K) = S(K) = C(K) = C_1(K) \quad \text{for all} \quad K \in \mathcal{K}_0^n,$$

it follows from (11) that

$$\Phi(K) \le C(K) \quad \text{for all} \quad K \in \mathcal{K}_0^n,$$

and thus

$$\int_0^\infty \Phi(\langle f \rangle_t)\, dt \le \int_0^\infty C(\langle f \rangle_t)\, dt \quad \text{for all} \quad f \in C_0^\infty(\mathbb{R}^n).$$

This naturally leads to the following

Question. *Are the geometric inequality*

$$\Phi_p(K) \le \left(\tfrac{p-1}{n-p}\right)^{p-1} C_p(K) \quad \textit{for all} \quad K \in \mathcal{K}_0^n,$$

and the analytic inequality

$$\int_0^\infty \Phi_p(\langle f \rangle_t)\, dt \le \left(\tfrac{p-1}{n-p}\right)^{p-1} \int_0^\infty C_p(\langle f \rangle_t)\, dt \quad \textit{for all} \quad f \in C_0^\infty(\mathbb{R}^n)$$

true for $1 < p < n$*?*

Acknowledgements

M. Ludwig and G. Zhang were supported in part by the National Science Foundation and J. Xiao by the Natural Sciences and Engineering Research Council of Canada.

References

[1] Adams, D.R.: A sharp inequality of J. Moser for higher order derivatives. Ann. Math 128, 385–398 (1988)
[2] Adams, D.R.: Choquet integrals in potential theory. Publ. Mat. 42, 3–66 (1998)
[3] Alesker, S.: Continuous rotation invariant valuations on convex sets. Ann. of Math. (2) 149, 977–1005 (1999)
[4] Alvino, A.: Sulla diseguaglianza di Sobolev in spazi di Lorentz. Boll. Un. Mat. Ital. A (5) 14, 148–156 (1977)
[5] Aubin, T.: Problèmes isopérimétriques et espaces de Sobolev. J. Differential Geom. 11, 573–598 (1976)
[6] Bastero, J., Milman, M., Ruiz Blasco, F.J.: A note on $L^{\infty,q}$ spaces and Sobolev embeddings. Indiana Univ. Math. J. 52, 1215–1230 (2003)

[7] Bobkov, S., Ledoux, M.: From Brunn-Minkowski to sharp Sobolev inequalities. Ann. Mat. Pura Appl. 187, 369–384 (2008)

[8] Böröczky, K., Bárány, I., Makai Jr., I., Pach, J.: Maximal volume enclosed by plates and proof of the chessboard conjecture. Discrete Math. 60, 101–120 (1986)

[9] Brothers, J., Morgan, F.: The isoperimetric theorem for general integrands. Michigan Math. J. 41, 419–431 (1994)

[10] Campi, S., Gronchi, P.: The L^p-Busemann-Petty centroid inequality. Adv. Math. 167, 128–141 (2002)

[11] Cassani, D.: Lorentz-Sobolev spaces and systems of Schrödinger equations in $\mathbb{R}^n$. Nonlinear Anal. 70, 2846–2854 (2009)

[12] Chavel, I.: Isoperimetric inequalities. Cambridge Tracts in Mathematics, 145. Cambridge University Press, Cambridge (2001)

[13] Chen, W. : L^p Minkowski problem with not necessarily positive data. Adv. Math. 201, 77–89 (2006)

[14] Cianchi, A., Lutwak, E., Yang, D., Zhang, G.: Affine Moser-Trudinger and Morrey-Sobolev inequalities. Calc. Var. Partial Differential Equations, 36, 419–436 (2009)

[15] Chou, K., Wang, X.-J.: The L_p-Minkowski problem and the Minkowski problem in centroaffine geometry. Adv. Math. 205 (2006), 33–83.

[16] Colesanti, A., Salani, P. : The Brunn-Minkowski inequality for p-capacity of convex bodies. Math. Ann. 327, 459–479 (2003)

[17] Cordero-Erausquin, D., Nazaret, D., Villani, C.: A mass-transportation approach to sharp Sobolev and Gagliardo-Nirenberg inequalities. Adv. Math. 182, 307–332 (2004)

[18] Cwickel, M., Pustylnik, E.: Sobolev type embeddings in the limiting case. J. Fourier Anal. Appl. 4, 433–446 (1998)

[19] Danielli, D., Garofalo, N., Phuc, N.C.: Inequalities of Hardy-Sobolev type in Carnot-Caratheodory Spaces. Sobolev Spaces in Mathematics I, Sobolev Type Inequalities, Vladimir Maz′ya Ed., International Mathematical Series 8, 117–151 (2009)

[20] Del Pino, M. , Dolbeault, J.: Best constants for Gagliardo-Nirenberg inequalities and applications to nonlinear diffusions. J. Math. Pures Appl. 81, 847–875 (2002)

[21] Evans, L.C., Gariepy, R.F.: Measure theory and fine properties of functions. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL (1992)

[22] Federer, H., Fleming, W.: Normal and integral currents. Ann. of Math. 72, 458–520 (1960)

[23] Frank, R.L., Seiringer, R.: Non-linear ground state representations and sharp Hardy inequalities. J. Funct. Anal. 255, 3407–3430 (2008)

[24] Gardner, R.: The Brunn-Minkowski inequality. Bull. Amer. Math. Soc. (N.S.) 39, 355–405 (2002)

[25] Gardner, R.: Geometric tomography. Second ed., Encyclopedia of Mathematics and its Applications, vol. 58, Cambridge University Press, Cambridge (2006)

[26] A.Grigor'yan, A.: Isoperimetric inequalities and capacities on Riemannian manifolds. The Maz'ya anniversary collection, Vol. 1 (Rostock, 1998), 139–153, Oper. Theory Adv. Appl., 109, Birkhäuser, Basel (1999)

[27] Guan, P., Li, C.S.: On the equation $\det(u_{ij} + \delta_{ij}u) = u^p f$ on S^n, No. 2000-7, NCTS in Tsing-Hua University.

[28] Haberl, C.: Blaschke valuations. Amer. J. Math., in press.

[29] Haberl, C., Ludwig, M.: A characterization of L_p intersection bodies. Int. Math. Res. Not. 10548, 1–29 (2006)

[30] Haberl, C., Schuster, F.: Asymmetric affine L_p Sobolev inequalities. J. Funct. Anal. 257, 641–658 (2009)

[31] Haberl, C., Schuster, F.: General L_p affine isoperimetric inequalities. J. Differential Geom. 83, 1–26 (2009)

[32] Haberl, C., Schuster, F., Xiao, J.: An asymmetric affine Pólya-Szegö principle. Preprint.

[33] Hardy, G. H. , Littlewood, J.E., Polya, G.: Some simple inequalities satisfied by convex functions. Messenger Math. 58, 145–152 (1929)

[34] Hardy, G. H. , Littlewood, J.E., Polya, G.: Inequalities. Cambridge Univ. Press, Cambridge (1952)

[35] Hu, C., Ma, X.-N., Shen, C.: On the Christoffel-Minkowski problem of Firey's p-sum. Calc. Var. Partial Differential Equations 21, 137–155 (2004)

[36] Hug, D., Lutwak, E., Yang, D., Zhang, G.: On the L_p Minkowski problem for polytopes. Discrete Comput. Geom. 33, 699–715 (2005)
[37] Ludwig, M.: Intersection bodies and valuations. Amer. J. Math. 128, 1409–1428 (2006)
[38] Ludwig, M.: Projection bodies and valuations. Adv. Math. 172, 158–168 (2002)
[39] Ludwig, M.: Ellipsoids and matrix-valued valuations. Duke Math. J. 119, 159–188 (2003)
[40] Ludwig, M., Reitzner, M.: A classification of $SL(n)$ invariant valuations. Ann. of Math., in press.
[41] Lutwak, E.: The Brunn-Minkowski-Firey theory. I. Mixed volumes and the Minkowski problem. J. Differential Geom. 38, 131–150 (1993)
[42] Lutwak, E.: The Brunn-Minkowski-Firey theory. II: Affine and geominimal surface areas. Adv. Math. 118, 244–294 (1996)
[43] Lutwak, E.: Selected affine isoperimetric inequalities. Handbook of Convex Geometry, Vol. A, P.M. Gruber and J.M. Wills, Eds., Elsevier, Horth-Holland, 151–176 (1993)
[44] Lutwak, E., Oliker, V.: On the regularity of solutions to a generalization of the Minkowski problem. J. Differential Geom. 41, 227–246 (1995)
[45] Lutwak, E., Yang, D., Zhang, G.: L_p affine isoperimetric inequalities. J. Differential Geom. 56, 111–132 (2000)
[46] Lutwak, E., Yang, D., Zhang, G.: Sharp affine L_p Sobolev inequalities. J. Differential Geom. 62, 17–38 (2002)
[47] Lutwak, E., Yang, D., Zhang, G.: The Cramer-Rao inequality for star bodies. Duke Math. J. 112, 59–81 (2002)
[48] Lutwak, E., Yang, D., Zhang, G.: On the L_p-Minkowski problem. Trans. Amer. Math. Soc. 356, 4359–4370 (2004)
[49] Lutwak, E., Yang, D., Zhang, G.: Optimal Sobolev norms and the L^p Minkowski problem. Int. Math. Res. Not. 21, Art. ID 62987 (2006)
[50] Malý, J., Pick, L.: An elementary proof of sharp Sobolev embeddings. Proc. Amer. Math. Soc. 130, 555–563 (2002)
[51] Matskewich, T., Sobolevskii, P.E.: The sharp constant in Hardy's inequality for complement of bounded domain. Nonlinear Anal. 33, 105–120 (1998)
[52] Maz′ya, V.G.: Classes of domains and imbedding theorems for function spaces. Soviet Math. Dokl. 1, 882–885 (1960)
[53] Maz′ya, V.G.: Certain integral inequalities for functions of many variables. Problems in Mathematical Analysis. Leningrad: LGU, No. 3, 33–68 (1972) (Russian). English translation: J. Soviet Math. 1, 205–234 (1973)
[54] Maz′ya, V.G.: Sobolev spaces. Springer-Verlag, Berlin (1985)
[55] Maz′ya, V.G.: Lectures on isoperimetric and isocapacitary inequalities in the theory of Sobolev spaces. Contemp. Math. 338, 307–340 (2003)
[56] Maz′ya, V.G.: Conductor and capacitary inequalities for functions on topological spaces and their applications to Sobolev-type imbeddings. J. Funct. Anal. 224, 408–430 (2005)
[57] Maz′ya, V.G.: Integral and isocapacitary inequalities. Linear and complex analysis, 85–107, Amer. Math. Soc. Transl. Ser. 2, 226, Amer. Math. Soc., Providence, RI (2009)
[58] Maz′ya, V.G., Verbitsky, I.E.: The Schrödinger operator on the energy space: boundedness and compactness criteria. Acta Math. 188, 263–302 (2002)
[59] Maz′ya, V.G., Verbitsky, I.E.: Infinitesimal form boundedness and Trudinger's subordination for the Schrödinger operator. Invent. Math. 162, 81–136 (2005)
[60] Maz′ya, V.G., Verbitsky, I.E.: Form boundedness of the general second-order differential operator. Comm. Pure Appl. Math. 59, 1286–1329 (2006)
[61] Meyer, M., Werner, E.: On the p-affine surface area. Adv. Math. 152, 288–313 (2000)
[62] Milman, E.: On the role of convexity in isoperimetry, spectral gap and concentration. Invent. Math. 177, 1–43 (2009)
[63] O'Neil, R.: Convolution operators and $L_{(p,q)}$ spaces. Duke Math. J. 30, 129–142 (1963)
[64] Peetre, J.: Espaces d'interpolation et theorème de Soboleff. Ann. Inst. Fourier 16, 279–317 (1966)

[65] Petty, C.M.: Isoperimetric problems. Proceedings of the Conference on Convexity and Combinatorial Geometry (Univ. Oklahoma, Norman, Okla., 1971), Dept. Math., Univ. Oklahoma, Norman, Okla., 26–41 (1971)
[66] Pólya, G., Szegö, G.: Isoperimetric inequalities in mathematical physics. Annals of Mathematics Studies, no. 27, Princeton University Press, Princeton, N. J. (1951)
[67] Ryabogin, D., Zvavitch, A.: The Fourier transform and Firey projections of convex bodies. Indiana Univ. Math. J. 53, 667–682 (2004)
[68] Schneider, R.: Convex bodies: the Brunn-Minkowski theory. Encyclopedia of Mathematics and its Applications, vol. 44, Cambridge University Press, Cambridge (1993)
[69] Schütt, C., Werner, E.: Surface bodies and p-affine surface area, phAdv. Math. 187, 98–145 (2004)
[70] Stancu, A. : The discrete planar L_0-Minkowski problem. Adv. Math. 167, 160–174 (2002)
[71] Stancu, A. : On the number of solutions to the discrete two-dimensional L_0-Minkowski problem. Adv. Math. 180, 290–323 (2003)
[72] Talenti, G.: Best constant in Sobolev inequality. Ann. Mat. Pura Appl. 110, 353–372 (1976)
[73] Taylor, J.: Crystalline variational problems. Bull. Amer. Math. Soc. 84, 568–588 (1978)
[74] Umanskiy, V.: On solvability of two dimensional L_p-Minkowski problem. Adv. Math. 180, 176–186 (2003)
[75] Xiao, J.: The sharp Sobolev and isoperimetric inequalities split twice. Adv. Math. 211, 417–435 (2007)
[76] Xiao, J.: The p-Faber-Krahn inequality noted. Topics Around Research of Vladimir Maz′ya, I. Function Spaces, pp. 373-390, Springer (2010)
[77] Zhang, G.: The affine Sobolev inequality. J. Differential Geom. 53, 183–202 (1999)

M. Ludwig, G. Zhang
Department of Mathematics
Polytechnic Institute of NYU
6 MetroTech Center
Brooklyn, NY 11201, USA
E-mail: {mludwig, gzhang}@poly.edu

J. Xiao
Department of Math & Stats
Memorial University
St. John's, NL, A1C 5S7
Canada
E-mail: jxiao@mun.ca