AFFINE HARDY–LITTLEWOOD–SOBOLEV INEQUALITIES

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Abstract. Sharp affine Hardy–Littlewood–Sobolev inequalities for functions on $\mathbb{R}^n$ are established, which are significantly stronger than (and directly imply) the sharp Hardy–Littlewood–Sobolev inequalities by Lieb and by Beckner, Dou, and Zhu. In addition, sharp reverse inequalities for the new inequalities and the affine fractional L^2 Sobolev inequalities are obtained for log-concave functions on $\mathbb{R}^n$.

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1. Introduction

Lieb [17] established the following sharp Hardy–Littlewood–Sobolev inequalities (HLS inequalities):

$$\gamma_{n,\alpha}\|f\|^2_{\frac{2n}{n+\alpha}} \geq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f(x)f(y)}{|x-y|^{n-\alpha}}\,\mathrm{d}x\,\mathrm{d}y \tag{1}$$

for $0<\alpha<n$ and non-negative $f\in L^p(\mathbb{R}^n)$ with $p=2n/(n+\alpha)$. There is equality if and only if $f(x)=a(1+\lambda\,|x-x_0|^2)^{-(n+\alpha)/2}$ for $x\in\mathbb{R}^n$ with $a\geq 0$, $\lambda>0$ and $x_0\in\mathbb{R}^n$. Here $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$ and $\|f\|_p^p=\int_{\mathbb{R}^n}|f(x)|^p\,\mathrm{d}x$ while $L^p(\mathbb{R}^n)$ is the space of measurable functions $f:\mathbb{R}^n\to\mathbb{R}$ with $\|f\|_p<\infty$. The constant is given by

$$\gamma_{n,\alpha}=\pi^{\frac{n-\alpha}{2}}\frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{n+\alpha}{2})}\Big(\frac{\Gamma(n)}{\Gamma(\frac{n}{2})}\Big)^{\frac{\alpha}{n}}, \tag{2}$$

where Γ is the gamma function. The HLS inequalities (1) can be considered as weak Young inequalities and are equivalent by duality to sharp fractional L^2 Sobolev inequalities (for more information, see [4, 5, 8, 18]).

The HLS inequalities are invariant under translations, rotations, and inversions, but not under volume-preserving linear transformations. For geometric questions, affine inequalities, that is, inequalities that are unchanged under translations and volume-preserving linear transformations turned out to be very powerful. The best-known example is the Petty projection inequality for convex bodies (that is, compact convex sets) in $\mathbb{R}^n$, which is stronger than the Euclidean isoperimetric inequality and directly implies it (see, for example, [9,29], and see [13,23,30,32] for related results in the L^p Brunn–Minkowski theory and for more general sets). Very recently, Milman and Yehudayoff [26] established isoperimetric inequalities for affine quermassintegrals, which are significantly stronger than the isoperimetric inequality for the classical quermassintegrals (thereby confirming a conjecture by Lutwak [22]). Gaoyong Zhang's affine Sobolev inequality [32] is an affine version of the classical L^1 Sobolev inequality. It was extended to functions of bounded variation by Tuo Wang [30], and corresponding results for L^p Sobolev inequalities were established

by Lutwak, Yang, and Zhang [24] and Haberl and Schuster [14]. Fractional Petty projection inequalities were recently obtained by the authors in [15] and affine fractional L^p Sobolev inequalities in [15, 16]. In all cases, the affine inequalities are significantly stronger than (and imply) their Euclidean counterparts.

The main aim of this paper is to establish sharp affine HLS inequalities that are stronger than Lieb's sharp HLS inequalities (1).

Theorem 1. *For $0 < \alpha < n$ and non-negative $f \in L^{2n/(n+\alpha)}(\mathbb{R}^n)$,*

$$\gamma_{n,\alpha}\|f\|^2_{\frac{2n}{n+\alpha}} \geq n\omega_n^{\frac{n-\alpha}{n}}\Big(\frac{1}{n}\int_{\mathbb{S}^{n-1}}\Big(\int_0^\infty t^{\alpha-1}\int_{\mathbb{R}^n} f(x)f(x+t\xi)\,\mathrm{d}x\,\mathrm{d}t\Big)^{\frac{n}{\alpha}}\,\mathrm{d}\xi\Big)^{\frac{\alpha}{n}}$$
$$\geq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{f(x)f(y)}{|x-y|^{n-\alpha}}\,\mathrm{d}x\,\mathrm{d}y.$$

There is equality in the first inequality precisely if $f(x) = a(1+|\phi(x-x_0)|^2)^{-(n+\alpha)/2}$ for $x \in \mathbb{R}^n$ with $a \geq 0$, $\phi \in \mathrm{GL}(n)$ and $x_0 \in \mathbb{R}^n$. There is equality in the second inequality if f is radially symmetric.

Here, $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$ and integration on $\mathbb{S}^{n-1}$ is with respect to the $(n-1)$-dimensional Hausdorff measure while ω_n is the n-dimensional volume of the n-dimensional unit ball.

To prove Theorem 1, for given $0 < \alpha < n$, we introduce the star-shaped set $\mathrm{S}_\alpha f$ associated to f, defined by its radial function for $\xi \in \mathbb{S}^{n-1}$ as

$$\rho_{\mathrm{S}_\alpha f}(\xi)^\alpha = \int_0^\infty t^{\alpha-1}\int_{\mathbb{R}^n} f(x)f(x+t\xi)\,\mathrm{d}x\,\mathrm{d}t \tag{3}$$

(see Section 3 for details). The first inequality from Theorem 1 now can be written as

$$\gamma_{n,\alpha}\|f\|^2_{\frac{2n}{n+\alpha}} \geq n\omega_n^{\frac{n-\alpha}{n}}|\,\mathrm{S}_\alpha f|^{\frac{\alpha}{n}}, \tag{4}$$

where $|\cdot|$ denotes n-dimensional Lebesgue measure. Since both sides of (4) are invariant under translations of f and

$$\mathrm{S}_\alpha(f\circ\phi^{-1}) = \phi\,\mathrm{S}_\alpha f$$

for volume-preserving linear transformations $\phi : \mathbb{R}^n \to \mathbb{R}^n$, it follows that inequality (4) is indeed an affine inequality.

As a critical tool in the proof of Theorem 1, we introduce an anisotropic version of the right side of (1). In addition, we use the Riesz rearrangement inequality, its equality case due to Burchard [3], and the original HLS inequalities for radially symmetric functions.

Recently, Dou and Zhu [7] and Beckner [2] obtained sharp HLS inequalities also for $\alpha > n$. In Section 5, we will establish sharp affine HLS inequalities for $\alpha > n$, which are stronger than (and directly imply) their results.

In Section 6, for $E \subset \mathbb{R}^n$ measurable, we consider $\mathrm{S}_\alpha 1_E$, where 1_E is the indicator function of E. For an n-dimensional convex body $E \subset \mathbb{R}^n$, we show that $\mathrm{S}_\alpha 1_E$ is proportional to the radial α-mean body of E, an important notion that was introduced by Gardner and Zhang [11]. Sharp isoperimetric inequalities for radial α-mean bodies were recently obtained in [15]. Sharp reverse inequalities were already established by Gardner and Zhang [11]. They generalize Zhang's reverse Petty projection inequality [31], and equality is attained precisely for n-dimensional simplices.

In Section 7, we establish reverse inequalities also in the functional setting.

Theorem 2. *For $0 < \alpha < n$ and log-concave $f \in L^2(\mathbb{R}^n)$,*

$$\frac{\Gamma(n+1)^{\frac{\alpha}{n}}}{\Gamma(\alpha)} |\,\mathrm{S}_\alpha f|^{\frac{\alpha}{n}} \geq \|f\|_2^{2-\frac{2\alpha}{n}} \|f\|_1^{\frac{2\alpha}{n}} \geq \|f\|^2_{\frac{2n}{n+\alpha}}.$$

There is equality in the first inequality if $f(x) = a\, e^{-\|x-x_0\|_\Delta}$ for $x \in \mathbb{R}^n$ with $a \geq 0$, $x_0 \in \mathbb{R}^n$, and Δ an n-dimensional simplex having a vertex at the origin.

Here, $\|\cdot\|_\Delta$ is the gauge function of $\Delta \subset \mathbb{R}^n$. The second inequality follows from Hölder's inequality; for the proof of the first inequality, see Section 7.

We remark that it is easy to see that for general, non-negative $f \in L^2(\mathbb{R}^n)$, no non-trivial reverse inequality can hold. In Section 7, we establish reverse inequalities also for $\alpha > n$ (see Theorem 15) and obtain results for s-concave functions for $s > 0$. Moreover, we will establish reverse affine fractional L^2 Sobolev inequalities.

2. Preliminaries

We collect results on symmetrization, star-shaped sets, log-concave and s-concave functions, and fractional polar projection bodies.

2.1. **Symmetrization.** Let $E \subseteq \mathbb{R}^n$ be a Borel set of finite measure. The Schwarz symmetral of E, denoted by $E^\star$, is the closed, centered Euclidean ball with the same volume as E.

Let f be a non-negative measurable function with superlevel sets of finite measure. Let $\{f \geq t\} = \{x \in \mathbb{R}^n : f(x) \geq t\}$ for $t \in \mathbb{R}$. We say that f is non-zero if $\{f \neq 0\}$ has positive measure, and we identify functions that are equal up to a set of measure zero. The layer cake formula states that

$$f(x) = \int_0^\infty 1_{\{f \geq t\}}(x)\,\mathrm{d}t \tag{5}$$

for almost every $x \in \mathbb{R}^n$ and allows us to recover the function from its superlevel sets. Here, for $E \subset \mathbb{R}^n$, the indicator function 1_E is defined by $1_E(x) = 1$ for $x \in E$ and $1_E(x) = 0$ otherwise.

The Schwarz symmetral of f, denoted by $f^\star$, is defined as

$$f^\star(x) = \int_0^\infty 1_{\{f \geq t\}^\star}(x)\,\mathrm{d}t$$

for $x \in \mathbb{R}^n$. Hence $f^\star$ is determined a.e. by the properties of being radially symmetric and having superlevel sets of the same measure as those of f. Note that $f^\star$ is also called the symmetric decreasing rearrangement of f. We say that $f^\star$ is strictly symmetric decreasing, if $f^\star(x) > f^\star(y)$ whenever $|x| < |y|$.

The proofs of our results make use of the Riesz rearrangement inequality (see, for example, [18, Theorem 3.7]).

Theorem 3 (Riesz's rearrangement inequality)**.** *For $f, g, k : \mathbb{R}^n \to \mathbb{R}$ non-negative, measurable functions with superlevel sets of finite measure,*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)k(x-y)g(y)\,\mathrm{d}x\,\mathrm{d}y \leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f^\star(x)k^\star(x-y)g^\star(y)\,\mathrm{d}x\,\mathrm{d}y.$$

We will use the characterization of equality cases of the Riesz rearrangement inequality due to Burchard [3].

Theorem 4 (Burchard)**.** *Let A, B and C be sets of finite positive measure in $\mathbb{R}^n$ and denote by α, β and γ the radii of their Schwarz symmetrals $A^\star, B^\star$ and $C^\star$. For $|\alpha - \beta| < \gamma < \alpha + \beta$, there is equality in*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} 1_A(y)\, 1_B(x-y)\, 1_C(x)\,\mathrm{d}x\,\mathrm{d}y \le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} 1_{A^\star}(y)\, 1_{B^\star}(x-y)\, 1_{C^\star}(x)\,\mathrm{d}x\,\mathrm{d}y$$

if and only if, up to sets of measure zero,

$$A = a + \alpha D,\ B = b + \beta D,\ C = c + \gamma D,$$

where D is a centered ellipsoid, and a, b and $c = a + b$ are vectors in $\mathbb{R}^n$.

Theorem 5 (Burchard)**.** *Let $f, g, k : \mathbb{R}^n \to \mathbb{R}$ non-negative, non-zero, measurable functions with superlevel sets of finite measure such that*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)k(x-y)g(y)\,\mathrm{d}x\,\mathrm{d}y < \infty.$$

If at least two of the Schwarz symmetrals $f^\star, g^\star, k^\star$ are strictly symmetric decreasing, then there is equality in

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)k(x-y)g(y)\,\mathrm{d}x\,\mathrm{d}y \le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f^\star(x)k^\star(x-y)g^\star(y)\,\mathrm{d}x\,\mathrm{d}y$$

if and only if there is a volume-preserving $\phi \in \mathrm{GL}(n)$ and $a, b, c \in \mathbb{R}^n$ with $c = a + b$ such that

$$f(x) = f^\star(\phi^{-1}x - a),\ k(x) = k^\star(\phi^{-1}x - b), g(x) = g^\star(\phi^{-1}x - c)$$

for $x \in \mathbb{R}^n$.

2.2. **Star-shaped sets and dual mixed volumes.** A closed set $K \subseteq \mathbb{R}^n$ is star-shaped (with respect to the origin) if the interval $[0, x] \subset K$ for every $x \in K$. The gauge function $\|\cdot\|_K : \mathbb{R}^n \to [0, \infty]$ of a star-shaped set is defined as

$$\|x\|_K = \inf\{\lambda > 0 : x \in \lambda K\}$$

and the radial function $\rho_K : \mathbb{R}^n \setminus \{0\} \to [0, \infty]$ as

$$\rho_K(x) = \|x\|_K^{-1} = \sup\{\lambda \ge 0 : \lambda x \in K\}.$$

For the n-dimensional unit ball B^n, we have $\|\cdot\|_{B^n} = |\cdot|$. The n-dimensional Lebesgue measure, or volume of a star-shaped set $K \subset \mathbb{R}^n$ with measurable radial function is given by

$$|K| = \frac{1}{n}\int_{\mathbb{S}^{n-1}} \rho_K(\xi)^n\,\mathrm{d}\xi.$$

We call K a star body if its radial function is strictly positive and continuous in $\mathbb{R}^n \setminus \{0\}$.

Let $\alpha \in \mathbb{R}\setminus\{0, n\}$. For star-shaped sets $K, L \subseteq \mathbb{R}^n$ with measurable radial functions, the dual mixed volume is defined as

$$\tilde{V}_\alpha(K, L) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} \rho_K(\xi)^{n-\alpha}\rho_L(\xi)^\alpha\,\mathrm{d}\xi.$$

Note that

$$\tilde{V}_\alpha(K, K) = |K|.$$

For $0 < \alpha < n$ and star-shaped sets $K, L \subseteq \mathbb{R}^n$ of finite volume, the dual mixed volume inequality states that

$$\tilde{V}_\alpha(K,L) \le |K|^{(n-\alpha)/n} |L|^{\alpha/n}. \tag{6}$$

Equality holds if and only if K and L are dilates, where we say that star-shaped sets K and L are dilates if $\rho_K = c\,\rho_L$ almost everywhere on $\mathbb{S}^{n-1}$ for some $c \ge 0$. The definition of dual mixed volume for star bodies is due to Lutwak [21], where also the dual mixed volume inequality (6) is derived from Hölder's inequality. For $0 < \alpha < n$ and star-shaped sets of finite volume, it follows from (6) that the dual mixed volume is finite. For star-shaped sets $K, L \subseteq \mathbb{R}^n$ and $\alpha > n$, the dual mixed volume inequality states that

$$\tilde{V}_\alpha(K,L) \ge |K|^{(n-\alpha)/n} |L|^{\alpha/n}. \tag{7}$$

It follows from the equality case of Hölder's inequality that equality holds for finite $\tilde{V}_\alpha(K,L)$ if and only if K and L are dilates. See [9, 29] for more information on dual mixed volumes.

2.3. **Log-concave and s-concave functions.** A function $f : \mathbb{R}^n \to [0,\infty)$ is log-concave, if $x \mapsto \log f(x)$ is a concave function on $\mathbb{R}^n$ with values in $[-\infty,\infty)$. For $s > 0$, a function $f : \mathbb{R}^n \to [0,\infty)$ is s-concave, if f^s is concave on its support.

The following result is a consequence of the Prékopa–Leindler inequality. It can be found as Theorem 11.3 in [10]. Log-concave functions are included as the case $s = 0$.

Lemma 6. *Let $s \ge 0$. If $f, g \in L^1(\mathbb{R}^n)$ are s-concave, then their convolution is $s/(ns+2)$-concave.*

Here, the convolution of $f, g \in L^1(\mathbb{R}^n)$ is defined by

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)\, g(y)\, \mathrm{d}y$$

for $x \in \mathbb{R}^n$.

We also consider the limiting case $s = \infty$. We say that a function $f : \mathbb{R}^n \to [0,\infty)$ is s-concave with $s = \infty$ if it is a multiple of the indicator function of a convex body.

2.4. **Fractional L^2 polar projection bodies.** For measurable $f : \mathbb{R}^n \to \mathbb{R}$ and $0 < \alpha < 1$, the α-fractional L^2 polar projection body of f, denoted by $\Pi_2^{*,\alpha} f$, was defined in [16] by its radial function for $\xi \in \mathbb{S}^{n-1}$ as

$$\rho_{\Pi_2^{*,\alpha} f}(\xi)^{-2\alpha} = \int_0^\infty t^{-2\alpha-1} \int_{\mathbb{R}^n} |f(x+t\xi) - f(x)|^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{8}$$

The fractional Sobolev space $W^{\alpha,2}(\mathbb{R}^n)$ is the set of all $f \in L^2(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(x)|^2}{|x-y|^{n+2\alpha}} \,\mathrm{d}x\,\mathrm{d}y < \infty.$$

The affine fractional L^2 Sobolev inequality [16, Theorem 1] states that

$$\|f\|^2_{\frac{2n}{n-2\alpha}} \le \sigma_{n,2,\alpha}\, n\omega_n^{\frac{n+2\alpha}{n}} |\Pi_2^{*,\alpha} f|^{-\frac{2\alpha}{n}} \tag{9}$$

for $f \in W^{\alpha,2}(\mathbb{R}^n)$ and $0 < \alpha < 1$, where $\sigma_{n,2,\alpha}$ is an explicitly known constant.

3. The Star-shaped Set $\mathrm{S}_\alpha f$

Let $f : \mathbb{R}^n \to [0,\infty)$ be measurable, $K \subset \mathbb{R}^n$ star-shaped with measurable radial function and $\alpha > 0$. Anisotropic fractional Sobolev norms were introduced in [19, 20] and used in [15]. Here, we introduce

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f(x)f(y)}{\|x-y\|_K^{n-\alpha}}\,\mathrm{d}x\,\mathrm{d}y,$$

an anisotropic version of the functional from (1). Using Fubini's theorem, polar coordinates, and (3), we obtain that

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f(x)f(y)}{\|x-y\|_K^{n-\alpha}}\,\mathrm{d}x\,\mathrm{d}y &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f(y)f(y+z)}{\|z\|_K^{n-\alpha}}\,\mathrm{d}y\,\mathrm{d}z\\ &= \int_{\mathbb{S}^{n-1}}\int_0^\infty \rho_K(t\xi)^{n-\alpha}\,t^{n-1}\int_{\mathbb{R}^n} f(y)f(y+t\xi)\,\mathrm{d}y\,\mathrm{d}t\,\mathrm{d}\xi\\ &= \int_{\mathbb{S}^{n-1}} \rho_K(\xi)^{n-\alpha}\int_0^\infty t^{\alpha-1}\int_{\mathbb{R}^n} f(y)f(y+t\xi)\,\mathrm{d}y\,\mathrm{d}t\,\mathrm{d}\xi\\ &= \int_{\mathbb{S}^{n-1}} \rho_K(\xi)^{n-\alpha}\rho_{\mathrm{S}_\alpha f}(\xi)^\alpha d\xi.\end{aligned}$$

Hence,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f(x)f(y)}{\|x-y\|_K^{n-\alpha}}\,\mathrm{d}x\,\mathrm{d}y = n\tilde{V}_\alpha(K,\mathrm{S}_\alpha f) \tag{10}$$

for measurable $f : \mathbb{R}^n \to [0,\infty)$ and star-shaped $K \subset \mathbb{R}^n$ with measurable radial function.

Note that, using polar coordinates and Fubini's theorem, we obtain that

$$\begin{aligned}|\,\mathrm{S}_n f| &= \frac{1}{n}\int_{\mathbb{S}^{n-1}} \rho_{\mathrm{S}_n f}(\xi)^n\,\mathrm{d}\xi\\ &= \frac{1}{n}\int_{\mathbb{S}^{n-1}}\int_0^\infty\int_{\mathbb{R}^n} t^{n-1}f(x)f(x+t\xi)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\xi\\ &= \frac{1}{n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)f(x+y)\,\mathrm{d}y\,\mathrm{d}x\\ &= \frac{1}{n}\|f\|_1^2\end{aligned} \tag{11}$$

for measurable $f : \mathbb{R}^n \to [0,\infty)$.

We remark that for $0 < \alpha < n$ and given measurable $f : \mathbb{R}^n \to [0,\infty)$, the dual mixed volume inequality (6) and (10) imply that

$$\sup\Big\{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f(x)f(y)}{\|x-y\|_K^{n-\alpha}}\,\mathrm{d}x\,\mathrm{d}y : K \subset \mathbb{R} \text{ star-shaped}, |K| = \omega_n\Big\} = n\omega_n^{\frac{n-\alpha}{n}}\,|\,\mathrm{S}_\alpha f|^{\frac{\alpha}{n}}$$

is attained precisely for a suitable dilate of $\mathrm{S}_\alpha f$ if $|\,\mathrm{S}_\alpha f|$ is finite. In this sense, $\mathrm{S}_\alpha f$ is the optimal choice of the star-shaped set K for given f. For $\alpha > n$ and given measurable $f : \mathbb{R}^n \to [0,\infty)$, the dual mixed volume inequality (7) and (10) imply that

$$\inf\Big\{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f(x)f(y)}{\|x-y\|_K^{n-\alpha}}\,\mathrm{d}x\,\mathrm{d}y : K \subset \mathbb{R} \text{ star-shaped}, |K| = \omega_n\Big\} = n\omega_n^{\frac{n-\alpha}{n}}\,|\,\mathrm{S}_\alpha f|^{\frac{\alpha}{n}}$$

is attained precisely for a suitable dilate of $\mathrm{S}_\alpha f$ if $|\mathrm{S}_\alpha f|$ is finite. Again $\mathrm{S}_\alpha f$ is the optimal choice in this sense.

We mention the following property of $\mathrm{S}_\alpha f$ for log-concave f.

Proposition 7. *If $f : \mathbb{R}^n \to [0,\infty)$ is log-concave and in $L^1(\mathbb{R}^n)$, then $\mathrm{S}_\alpha f$ is a convex body for every $\alpha > 0$.*

Proof. Since f is log-concave and in $L^1(\mathbb{R}^n)$, so is

$$y \mapsto \int_{\mathbb{R}^n} f(x) f(x+y)\,\mathrm{d}x.$$

This follows from the Young inequality (cf. [18, Theorem 4.2]) and Lemma 6. Hence, using a result of Ball [1, Theorem 5] (or see [11, Corollary 4.2]), we obtain the result. □

4. Affine HLS Inequalities for $0 < \alpha < n$

The following result is an immediate consequence of the Riesz rearrangement inequality and its equality case from Theorem 5.

Lemma 8. *Let $q > 0$ and $K \subset \mathbb{R}^n$ a star-shaped set with measurable radial function and $|K| > 0$. For a non-zero, measurable function $f : \mathbb{R}^n \to [0,\infty)$ such that*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f(x)f(y)}{\|x-y\|_K^q}\,\mathrm{d}x\,\mathrm{d}y < \infty$$

and strictly symmetric decreasing $f^\star$, there is equality in

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f(x)f(y)}{\|x-y\|_K^q}\,\mathrm{d}x\,\mathrm{d}y \le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f^\star(x)f^\star(y)}{\|x-y\|_{K^\star}^q}\,\mathrm{d}x\,\mathrm{d}y$$

if and only if K is a centered ellipsoid and f is a translate of $f^\star$.

We require the following lemmas for the proof of Theorem 1.

Lemma 9. *Let $0 < \alpha < n$ and $K \subset \mathbb{R}^n$ be star-shaped with measurable radial function. If $f : \mathbb{R}^n \to [0,\infty)$ is non-zero and measurable and*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f(x)f(y)}{\|x-y\|_K^{n-\alpha}}\,\mathrm{d}x\,\mathrm{d}y < \infty,$$

then

$$\tilde{V}_\alpha(K, \mathrm{S}_\alpha f) \le \tilde{V}_\alpha(K^\star, \mathrm{S}_\alpha f^\star).$$

For $|K| > 0$ and $f^\star$ strictly symmetric decreasing, there is equality if and only if K is a centered ellipsoid and f is a translate of $f^\star$.

Proof. By (10) and the Riesz rearrangement inequality, Theorem 3, we have

$$\tilde{V}_\alpha(K, \mathrm{S}_\alpha f) \le \tilde{V}_\alpha(K^\star, \mathrm{S}_\alpha f^\star).$$

By Lemma 8, there is equality if and only if K is a centered ellipsoid and f is a translate of $f^\star$. □

Lemma 10. *Let $0 < \alpha < n$ and $p = 2\alpha/(n+\alpha)$. For non-negative $f \in L^p(\mathbb{R}^n)$,*

$$|\,\mathrm{S}_\alpha f| \le |\,\mathrm{S}_\alpha f^\star|.$$

For $f^\star$ strictly symmetric decreasing with $|\,\mathrm{S}_\alpha f^\star| < \infty$, there is equality if and only if f is a translate of $f^\star$.

Proof. First, assume that $|\,\mathrm{S}_\alpha f| < \infty$. By Lemma 9 with $K = \mathrm{S}_\alpha f$ and the dual mixed volume inequality (6) for $0 < \alpha < n$, we have

$$\begin{aligned} |\,\mathrm{S}_\alpha f| &= \tilde{V}_\alpha(\mathrm{S}_\alpha f, \mathrm{S}_\alpha f) \\ &\le \tilde{V}_\alpha((\mathrm{S}_\alpha f)^\star, \mathrm{S}_\alpha f^\star) \\ &\le |(\mathrm{S}_\alpha f)^\star|^{\frac{n-\alpha}{n}} |\,\mathrm{S}_\alpha f^\star|^{\frac{\alpha}{n}} \\ &= |\,\mathrm{S}_\alpha f|^{\frac{n-\alpha}{n}} |\,\mathrm{S}_\alpha f^\star|^{\frac{\alpha}{n}}. \end{aligned}$$

The equality case follows from Lemma 9.

Second, assume that $|\,\mathrm{S}_\alpha f| = \infty$. For $k \ge 1$, define

$$f_{(k)}(x) = f(x)\, 1_{kB^n}(x).$$

Note that $f_{(k)}$ is non-decreasing with respect to k and converges to f pointwise. By the monotone convergence theorem, we obtain

$$\lim_{k\to\infty} \int_0^\infty t^{\alpha-1} \int_{\mathbb{R}^n} f_{(k)}(x) f_{(k)}(x+t\xi)\,\mathrm{d}x\,\mathrm{d}t = \int_0^\infty t^{\alpha-1} \int_{\mathbb{R}^n} f(x) f(x+t\xi)\,\mathrm{d}x\,\mathrm{d}t$$

and the convergence is monotone. A second application of the monotone convergence theorem shows that

$$\begin{aligned} \lim_{k\to\infty} \int_{\mathbb{S}^{n-1}} \left(\int_0^\infty t^{\alpha-1} \int_{\mathbb{R}^n} f_{(k)}(x) f_{(k)}(x+t\xi)\,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{n}{\alpha}} \mathrm{d}\xi \\ = \int_{\mathbb{S}^{n-1}} \left(\int_0^\infty t^{\alpha-1} \int_{\mathbb{R}^n} f(x) f(x+t\xi)\,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{n}{\alpha}} \mathrm{d}\xi. \end{aligned}$$

Hence,

$$\lim_{k\to\infty} |\,\mathrm{S}_\alpha f_{(k)}| = |\,\mathrm{S}_\alpha f| = \infty. \tag{12}$$

Since $f \in L^p(\mathbb{R}^n)$, the function $f_{(k)}$ has compact support and $|\,\mathrm{S}_\alpha f_{(k)}| < \infty$ for $k \ge 1$. It is easy to verify that $(f_{(k)})^\star \le f^\star$ a.e., so the first part of the lemma implies that

$$|\,\mathrm{S}_\alpha f_{(k)}| \le |\,\mathrm{S}_\alpha (f_{(k)})^\star| \le |\,\mathrm{S}_\alpha f^\star|$$

for $k \ge 1$. It follows from (12) that $|\,\mathrm{S}_\alpha f_{(k)}| \to \infty$, which shows that

$$|\,\mathrm{S}_\alpha f^\star| = \infty,$$

which is what we wanted to show. □

4.1. **Proof of Theorem 1.** Since $\|f\|_p = \|f^\star\|_p$, we obtain by the classical HLS inequality (1), by (10) and by Lemma 10 that

$$\begin{aligned}\gamma_{n,\alpha}\|f\|_p^2 &\geq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f^\star(x)f^\star(y)}{|x-y|^{n-\alpha}}\,\mathrm{d}x\,\mathrm{d}y\\ &= n\tilde{V}_\alpha(B^n, \mathrm{S}_\alpha f^\star)\\ &= n\omega_n^{\frac{n-\alpha}{n}}|\,\mathrm{S}_\alpha f^\star|^{\frac{\alpha}{n}}\\ &\geq n\omega_n^{\frac{n-\alpha}{n}}|\,\mathrm{S}_\alpha f|^{\frac{\alpha}{n}}.\end{aligned}$$

If there is equality throughout, then $f^\star$ realizes equality in the HLS inequality (1). Hence $f^\star(x) = a(1+\lambda|x|^2)^{-n/p}$ for some $a \geq 0$ and $\lambda > 0$. Consequently, $f^\star$ is strictly symmetric decreasing, and we may apply Lemma 10 to obtain the equality case in the first inequality in Theorem 1.

For the second inequality, we set $K = B^n$ in (10) and apply the dual mixed volume inequality (6) to obtain

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f(x)f(y)}{|x-y|^{n-\alpha}}\,\mathrm{d}x\,\mathrm{d}y = n\tilde{V}_\alpha(B^n, \mathrm{S}_\alpha f) \leq n\omega_n^{\frac{n-\alpha}{n}}|\,\mathrm{S}_\alpha f|^{\frac{\alpha}{n}}.$$

There is equality precisely if $\mathrm{S}_\alpha f$ is a ball, which is the case for radially symmetric functions.

5. Affine HLS Inequalities for $\alpha > n$

Jingbo Dou and Meijun Zhu [7] and William Beckner [2] established sharp HLS inequalities for $\alpha > n$ (also see [6, 28]):

$$\gamma_{n,\alpha}\|f\|_{\frac{2n}{n+\alpha}}^2 \leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f(x)f(y)}{|x-y|^{n-\alpha}}\,\mathrm{d}x\,\mathrm{d}y \tag{13}$$

for non-negative $f \in L^p(\mathbb{R}^n)$ with $p = 2n/(n+\alpha)$, where $\gamma_{n,\alpha}$ is defined in (2). There is equality if $f(x) = a(1+\lambda|x-x_0|^2)^{-(n-\alpha)/2}$ for $x \in \mathbb{R}^n$ with $a \geq 0$, $\lambda > 0$ and $x_0 \in \mathbb{R}^n$.

We will establish sharp HLS inequalities for $\alpha > n$ that strengthen and imply (13). We require the following lemmas.

The following result is a consequence of the Riesz rearrangement inequality and Theorem 4. Note that the middle function in (14) has superlevel sets of infinite measure.

Lemma 11. *Let $q > 0$ and $K \subset \mathbb{R}^n$ a star-shaped set with $0 < |K| < \infty$. For non-negative, non-zero $f \in L^p(\mathbb{R}^n)$ such that*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)\|x-y\|_K^q\, f(y)\,\mathrm{d}x\,\mathrm{d}y < \infty, \tag{14}$$

there is equality in

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)\|x-y\|_K^q\, f(y)\,\mathrm{d}x\,\mathrm{d}y \geq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f^\star(x)\|x-y\|_{K^\star}^q\, f^\star(y)\,\mathrm{d}x\,\mathrm{d}y \tag{15}$$

if and only if K is a centered ellipsoid and f is a translate of $f^\star$.

Proof. Writing

$$\|z\|_K^q = \int_0^\infty k_s(z)\,\mathrm{d}s$$

where $k_t(z) = 1_{s^{1/q}(\mathbb{R}^n\setminus K)}(z)$, and using the layer-cake formula (5) for f and g, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)\|x-y\|_K^q f(y)\,\mathrm{d}x\,\mathrm{d}y\\&= \int_0^\infty\int_0^\infty\int_0^\infty\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} 1_{\{f\ge r\}}(x)k_s(x-y)\,1_{\{f\ge t\}}(x)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}t.\end{aligned}$$

The Riesz rearrangement inequality, Theorem 3, implies that

$$\begin{aligned}&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} 1_{\{f\ge r\}}(x)k_s(x-y)\,1_{\{f\ge t\}}(y)\,\mathrm{d}x\,\mathrm{d}y\\&\qquad= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} 1_{\{f\ge r\}}(x)(1-1_{s^{1/q}K}(x-y))\,1_{\{f\ge t\}}(y)\,\mathrm{d}x\,\mathrm{d}y\\&\qquad= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} 1_{\{f\ge r\}}(x)\,1_{\{f\ge t\}}(y)\,\mathrm{d}x\,\mathrm{d}y\\&\qquad\quad- \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} 1_{\{f\ge r\}}(x)\,1_{s^{1/q}K}(x-y)\,1_{\{f\ge t\}}(y)\,\mathrm{d}x\,\mathrm{d}y\\&\qquad\ge \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} 1_{\{f\ge r\}^\star}(x)k_s^\star(x-y)\,1_{\{f\ge t\}^\star}(y)\,\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

for $r,s,t>0$. Note that $\int_{\mathbb{R}^n} 1_{\{f\ge r\}}(x)\,\mathrm{d}x<\infty$ for $r>0$, as $f\in L^p(\mathbb{R}^n)$. If there is equality in (15), then there is a null set $M\subset(0,\infty)^3$ such that

$$\begin{aligned}&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} 1_{\{f\ge r\}}(x)\,1_{s^{1/q}K}(x-y)\,1_{\{f\ge t\}}(y)\,\mathrm{d}x\,\mathrm{d}y\\&\qquad= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} 1_{\{f\ge r\}^\star}(x)\,1_{s^{1/q}K^\star}(x-y)\,1_{\{f\ge t\}^\star}(y)\,\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

for $(r,s,t)\in(0,\infty)^3\setminus M$.

For almost every $(r,t)\in(0,\infty)^2$, we have $(r,s,t)\in(0,\infty)^3\setminus M$ for almost every $s>0$. For such (r,t) with $r\ge t$ and $s>0$ sufficiently small, the assumptions of Theorem 4 are fulfilled and therefore there are a centered ellipsoid D and $a,b\in\mathbb{R}^n$ (depending on (r,s,t)) such that

$$\{f\ge r\}=a+\alpha D,\quad s^{1/q}K=b+\beta D,\quad \{f\ge t\}=c+\gamma D$$

where $c=a+b$. Since $K=s^{-1/q}b+(|K|/|D|)^{1/n}D$, the centered ellipsoid D does not depend on (r,s,t) and hence the vectors a and c do not depend on s. It follows that $b=0$ and that K is a multiple of D. Hence $a=c$ is a constant vector, which concludes the proof. □

Lemma 12. *Let $\alpha>n$ and $K\subset\mathbb{R}^n$ be star-shaped with measurable radial function. If $f:\mathbb{R}^n\to[0,\infty)$ is non-zero and measurable and*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{f(x)f(y)}{\|x-y\|_K^{n-\alpha}}\,\mathrm{d}x\,\mathrm{d}y<\infty,$$

then

$$\tilde{V}_\alpha(K,\mathrm{S}_\alpha f)\ge\tilde{V}_\alpha(K^\star,\mathrm{S}_\alpha f^\star).$$

For $|K| > 0$, there is equality if and only if K is a centered ellipsoid and f is a translate of $f^\star$.

Proof. By (10) and Lemma 11, we have

$$\tilde{V}_\alpha(K, \mathrm{S}_\alpha f) \geq \tilde{V}_\alpha(K^\star, \mathrm{S}_\alpha f^\star).$$

By Lemma 11, there is equality if and only if K is a centered ellipsoid and f is a translate of $f^\star$. □

Lemma 13. *Let $\alpha > n$ and $f : \mathbb{R}^n \to [0, \infty)$ measurable. If $|\mathrm{S}_\alpha f| < \infty$, then*

$$|\mathrm{S}_\alpha f| \geq |\mathrm{S}_\alpha f^\star|.$$

There is equality if and only if f is a translate of $f^\star$.

Proof. By Lemma 12 with $K = \mathrm{S}_\alpha f$ and the dual mixed volume inequality (7) for $\alpha > n$, we have

$$\begin{aligned} |\mathrm{S}_\alpha f| &= \tilde{V}_\alpha(\mathrm{S}_\alpha f, \mathrm{S}_\alpha f) \\ &\geq \tilde{V}_\alpha((\mathrm{S}_\alpha f)^\star, \mathrm{S}_\alpha f^\star) \\ &\geq |(\mathrm{S}_\alpha f)^\star|^{\frac{n-\alpha}{n}} |\mathrm{S}_\alpha f^\star|^{\frac{\alpha}{n}} \\ &= |\mathrm{S}_\alpha f|^{\frac{n-\alpha}{n}} |\mathrm{S}_\alpha f^\star|^{\frac{\alpha}{n}}. \end{aligned}$$

The equality case follows from Lemma 12. □

We are now in the position to prove affine HLS inequalities for $\alpha > n$.

Theorem 14. *For $\alpha > n$ and non-negative $f \in L^{2n/(n+\alpha)}(\mathbb{R}^n)$,*

$$\gamma_{n,\alpha} \|f\|^2_{\frac{2n}{n+\alpha}} \leq n\omega_n^{\frac{n-\alpha}{n}} |\mathrm{S}_\alpha f|^{\frac{\alpha}{n}} \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x) f(y)}{|x-y|^{n-\alpha}} \, \mathrm{d}x \, \mathrm{d}y.$$

There is equality in the first inequality precisely if $f(x) = a(1+|\phi(x-x_0)|^2)^{-(n-\alpha)/2}$ for $x \in \mathbb{R}^n$ with $a \geq 0$, $\phi \in \mathrm{GL}(n)$ and $x_0 \in \mathbb{R}^n$. There is equality in the second inequality if f is radially symmetric.

Proof. For the first inequality, we may assume that $|\mathrm{S}_\alpha f|$ is finite. Since $\|f\|_p = \|f^\star\|_p$, we obtain by the HLS inequality (13), by (10) and by Lemma 13 that

$$\begin{aligned} \gamma_{n,\alpha} \|f\|_p^2 &\leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f^\star(x) f^\star(y)}{|x-y|^{n-\alpha}} \, \mathrm{d}x \, \mathrm{d}y \\ &= n\tilde{V}_\alpha(B^n, \mathrm{S}_\alpha f^\star) \\ &= n\omega_n^{\frac{n-\alpha}{n}} |\mathrm{S}_\alpha f^\star|^{\frac{\alpha}{n}} \\ &\leq n\omega_n^{\frac{n-\alpha}{n}} |\mathrm{S}_\alpha f|^{\frac{\alpha}{n}}. \end{aligned}$$

If there is equality throughout, then $f^\star$ realizes equality in the HLS inequality (13). Hence $f^\star(x) = a(1 + \lambda|x|^2)^{-n/p}$ for some $a \geq 0$ and $\lambda > 0$. By Lemma 13, we obtain the equality case.

For the second inequality, assume that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{f(x)f(y)}{|x-y|^{n-\alpha}}\,\mathrm{d}x\,\mathrm{d}y<\infty.$$

We set $K=B^n$ in (10) and apply the dual mixed volume inequality (7) to obtain

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{f(x)f(y)}{|x-y|^{n-\alpha}}\,\mathrm{d}x\,\mathrm{d}y=n\tilde{V}_\alpha(B^n,\mathrm{S}_\alpha f)\geq n\omega_n^{\frac{n-\alpha}{n}}\,|\,\mathrm{S}_\alpha f|^{\frac{\alpha}{n}}.$$

There is equality precisely if $\mathrm{S}_\alpha f$ is a ball, which is the case for radially symmetric functions. □

Next, we state a sharp reverse of the first inequality from Theorem 14 for log-concave functions.

Theorem 15. *For $\alpha>n$ and log-concave $f\in L^2(\mathbb{R}^n)$,*

$$\frac{\Gamma(n+1)^{\frac{\alpha}{n}}}{\Gamma(\alpha)}\,|\,\mathrm{S}_\alpha f|^{\frac{\alpha}{n}}\leq\|f\|_2^{2-\frac{2\alpha}{n}}\|f\|_1^{\frac{2\alpha}{n}}\leq\|f\|^2_{\frac{2n}{n+\alpha}}.$$

There is equality in the first inequality if $f(x)=a\,e^{-\|x-x_0\|_\Delta}$ for $x\in\mathbb{R}^n$ with $a\geq 0, x_0\in\mathbb{R}^n$ and Δ an n-dimensional simplex having a vertex at the origin.

The proof of this result will be given in Section 7.

6. Radial Mean Bodies

Let $E\subset\mathbb{R}^n$ be a convex body. For $\alpha>-1$ and $\alpha\neq 0$, Gardner and Zhang [11] defined the radial α-th mean body of E, by its radial function for $\xi\in\mathbb{S}^{n-1}$, as

$$\rho_{\mathrm{R}_\alpha E}(\xi)^\alpha=\frac{1}{|E|}\int_E\rho_{E-x}(\xi)^\alpha\,\mathrm{d}x$$

for $\alpha\neq 0$ and as

$$\log(\rho_{\mathrm{R}_0E}(\xi))=\frac{1}{|E|}\int_E\log(\rho_{E-x}(\xi))\,\mathrm{d}x.$$

They showed that R_pE is a star body for $\alpha>-1$ and a convex body for $\alpha\geq 0$. This also follows from Proposition 7 and equation (16), which we will establish below. Gardner and Zhang [11] also showed that for $\alpha>-1$ and $\xi\in\mathbb{S}^{n-1}$,

$$\rho_{\mathrm{R}_\alpha E}(\xi)^\alpha=\frac{1}{(\alpha+1)|E|}\int_{E|\xi^\perp}|E\cap(\ell_\xi+y)|_1^{\alpha+1}\,\mathrm{d}y,$$

where $\ell_\xi=\{t\xi:t\in\mathbb{R}\}$ is the line in direction ξ and $|\cdot|_1$ denotes one-dimensional volume while $E|\xi^\perp$ is the image of the orthogonal projection of E to the hyperplane orthogonal to ξ.

Let $E \subset \mathbb{R}^n$ be convex. If $\alpha > 0$, then the definition of $\mathrm{S}_\alpha 1_E$ implies that

$$\begin{aligned}
\rho_{\mathrm{S}_\alpha 1_E}(\xi)^\alpha &= \int_0^\infty t^{\alpha-1} |E \cap (E + t\xi)|_1 \, \mathrm{d}t \\
&= \int_0^\infty t^{\alpha-1} \int_{E|\xi^\perp} (|E \cap (\ell_\xi + y)|_1 - t)_+ \, \mathrm{d}y \, \mathrm{d}t \\
&= \int_{E|\xi^\perp} \int_0^\infty t^{\alpha-1} (|E \cap (\ell_\xi + y)|_1 - t)_+ \, \mathrm{d}t \, \mathrm{d}y \\
&= \int_{E|\xi^\perp} \int_0^{|E\cap(\ell_\xi+y)|_1} t^{\alpha-1} (|E \cap (\ell_\xi + y)|_1 - t) \, \mathrm{d}t \, \mathrm{d}y \\
&= \frac{1}{\alpha(\alpha+1)} \int_{E|\xi^\perp} |E \cap (\ell_\xi + y)|_1^{\alpha+1} \, \mathrm{d}y.
\end{aligned}$$

Hence,

$$\mathrm{S}_\alpha 1_E = \Big(\frac{|E|}{\alpha}\Big)^{1/\alpha} \mathrm{R}_\alpha E \tag{16}$$

for $\alpha > 0$. If $-1 < \alpha < 0$, then, using (8), we obtain that

$$\begin{aligned}
\rho_{\Pi_2^{*,-\alpha/2} 1_E}(\xi)^\alpha &= \int_0^\infty t^{\alpha-1} |E \Delta (E + t\xi)| \, \mathrm{d}t \\
&= \int_0^\infty t^{\alpha-1} \int_{E|\xi^\perp} 2 \min\{|E \cap (\ell_\xi + y)|_1, t\} \, \mathrm{d}y \, \mathrm{d}t \\
&= 2 \int_{E|\xi^\perp} \int_0^\infty t^{\alpha-1} \min\{|E \cap (\ell_\xi + y)|_1, t\} \, \mathrm{d}t \, \mathrm{d}y \\
&= 2 \int_{E|\xi^\perp} \int_0^{|E\cap(\ell_\xi+y)|_1} t^\alpha \, \mathrm{d}t + \int_{|E\cap(\ell_\xi+y)|_1}^\infty |E \cap (\ell_\xi + y)|_1 \, t^{\alpha-1} \, \mathrm{d}t \, \mathrm{d}y \\
&= -\frac{2}{\alpha(\alpha+1)} \int_{E|\xi^\perp} |E \cap (\ell_\xi + y)|_1^{\alpha+1} \, \mathrm{d}y,
\end{aligned}$$

where $E \Delta F$ is the symmetric difference of $E, F \subset \mathbb{R}^n$. Hence, we obtain that

$$\Pi_2^{*,-\alpha/2} 1_E = \Big(\frac{2|E|}{-\alpha}\Big)^{1/\alpha} \mathrm{R}_\alpha E \tag{17}$$

for $-1 < \alpha < 0$.

See [11, 15] for information on sharp affine isoperimetric inequalities for radial mean bodies.

7. Reverse Affine HLS and Fractional L^2 Sobolev Inequalities

We prove Theorem 2 and Theorem 15 for log-concave functions and derive results for s-concave functions for $s > 0$. In addition, we establish reverse affine fractional L^2 Sobolev inequalities.

7.1. **An auxiliary result.**

We require the following result (see, for example, [27, Lemma 2.6], where it is said to be a consequence of [25]). A simple computation gives the equality case.

Lemma 16. *Let $\omega : [0,\infty) \to [0,\infty)$ be decreasing with*

$$0 < \int_0^\infty t^{\alpha-1}\omega(t)\,\mathrm{d}t < \infty \tag{18}$$

for $\alpha > 0$. If $\varphi : [0,\infty) \to [0,\infty)$ is non-zero, with $\varphi(0) = 0$, and such that $t \mapsto \varphi(t)$ and $t \mapsto \varphi(t)/t$ are increasing on $(0,\infty)$, then

$$\zeta(\alpha) = \left(\frac{\int_0^\infty t^{\alpha-1}\omega(\varphi(t))\,\mathrm{d}t}{\int_0^\infty t^{\alpha-1}\omega(t)\,\mathrm{d}t}\right)^{\frac{1}{\alpha}} \tag{19}$$

is a decreasing function of α on $(0,\infty)$. Moreover, ζ is constant on $(0,\infty)$ if $\varphi(t) = \lambda t$ on $[0,\infty)$ for some $\lambda > 0$.

Note that (18) and the assumptions on φ and ω imply that both integrals in (19) are finite and that the conditions of [27, Lemma 2.6] are fulfilled.

We will extend Lemma 16 to $\alpha \in (-1,\infty)$. Let ω be as in Lemma 16 and $\alpha > 0$. For any $t_0 > 0$, we have

$$\int_0^\infty t^{\alpha-1}\omega(t)\,\mathrm{d}t = \int_{t_0}^\infty t^{\alpha-1}\omega(t)\,\mathrm{d}t - \int_0^{t_0} t^{\alpha-1}(\omega(0)-\omega(t))\,\mathrm{d}t + \omega(0)\frac{t_0^\alpha}{\alpha}. \tag{20}$$

If, in addition,

$$\int_0^\infty t^{\alpha-1}(\omega(0)-\omega(t))\,\mathrm{d}t < \infty$$

for $\alpha \in (-1,0)$, then the right side of (20) is finite for $\alpha \in (-1,0)$. Moreover, it is equal to

$$\int_0^\infty t^{\alpha-1}(\omega(t)-\omega(0))\,\mathrm{d}t,$$

which is the analytic continuation of $\alpha \mapsto \int_0^\infty t^{\alpha-1}\omega(t)\,\mathrm{d}t$ to $(-1,0)$ (see, for example, [12, Section 1.3]).

For $\omega(t) = e^{-t}$ and $\omega(t) = (1-st)_+^{1/s}$ with $s > 0$, we obtain the well-known analytic continuation formulas for the Gamma and Beta functions,

$$\Gamma(\alpha) = \begin{cases} \int_0^\infty t^{\alpha-1}e^{-t}\,\mathrm{d}t & \text{for } 0 < \alpha, \\ \int_0^\infty t^{\alpha-1}(e^{-t}-1)\,\mathrm{d}t & \text{for } -1 < \alpha < 0, \end{cases} \tag{21}$$

and

$$s^{-\alpha}\,\mathrm{B}(\alpha, 1+\frac{1}{s}) = \begin{cases} \int_0^\infty t^{\alpha-1}(1-st)_+^{1/s}\,\mathrm{d}t & \text{for } 0 < \alpha, \\ \int_0^\infty t^{\alpha-1}((1-st)_+^{1/s}-1)\,\mathrm{d}t & \text{for } -1 < \alpha < 0. \end{cases} \tag{22}$$

We extend Lemma 16 to $\alpha \in (-1,\infty)$ using the analytic continuation formula from (20).

Lemma 17. *Let $\omega : [0,\infty) \to [0,\infty)$ be decreasing with*

$$0 < \int_0^\infty t^{\alpha-1}\omega(t)\,\mathrm{d}t < \infty$$

for $\alpha > 0$ and

$$0 < \int_0^\infty t^{\alpha-1}(\omega(0)-\omega(t))\,\mathrm{d}t < \infty$$

for $-1 < \alpha < 0$. If $\varphi : [0,\infty) \to [0,\infty)$ is non-zero, with $\varphi(0) = 0$, and such that $t \mapsto \varphi(t)$ and $t \mapsto \varphi(t)/t$ are increasing on $(0,\infty)$, then

$$\zeta(\alpha) = \begin{cases} \left(\dfrac{\int_0^\infty t^{\alpha-1}\omega(\varphi(t))\,\mathrm{d}t}{\int_0^\infty t^{\alpha-1}\omega(t)\,\mathrm{d}t} \right)^{\frac{1}{\alpha}} & \text{for } \alpha > 0 \\ \exp\Big(\displaystyle\int_0^\infty \frac{\omega(\varphi(t)) - \omega(t)}{t\,\omega(0)}\,\mathrm{d}t \Big) & \text{for } \alpha = 0 \\ \left(\dfrac{\int_0^\infty t^{\alpha-1}(\omega(\varphi(t)) - \omega(0))\,\mathrm{d}t}{\int_0^\infty t^{\alpha-1}(\omega(t) - \omega(0))\,\mathrm{d}t} \right)^{\frac{1}{\alpha}} & \text{for } -1 < \alpha < 0 \end{cases}$$

is a continuous, decreasing function of α on $(-1,+\infty)$. Moreover, ζ is constant on $(-1,\infty)$ if $\varphi(t) = \lambda t$ on $[0,\infty)$ for some $\lambda > 0$.

Proof. First, we show that ζ is well-defined. Take $t_0 > 0$ such that $\varphi(t_0) > 0$, and set $v_0 = \varphi(t_0)/t_0$. Since $t \mapsto \varphi(t)/t$ is increasing, we have $\varphi(t) \le v_0 t$ for $0 \le t \le t_0$ and $\varphi(t) \ge v_0 t$ for $t \ge t_0$.

For $\alpha \in (-1,\infty)$, we have

$$\begin{aligned} v_0 \int_{t_0}^\infty t^{\alpha-1}\omega(\varphi(t))\,\mathrm{d}t &\le v_0 \int_{t_0}^\infty t^{\alpha-1}\omega(v_0 t)\,\mathrm{d}t \\ &= \int_{v_0}^\infty \left(\frac{t}{v_0}\right)^{\alpha-1} \omega(t)\,\mathrm{d}t, \end{aligned} \tag{23}$$

where the last integral is finite and increasing with respect to α, and

$$\begin{aligned} v_0 \int_0^{t_0} t^{\alpha-1}(\omega(0) - \omega(\varphi(t)))\,\mathrm{d}t &\le v_0 \int_0^{t_0} t^{\alpha-1}(\omega(0) - \omega(v_0 t))\,\mathrm{d}t \\ &= \int_0^{v_0} \left(\frac{t}{v_0}\right)^{\alpha-1} (\omega(0) - \omega(t))\,\mathrm{d}t, \end{aligned} \tag{24}$$

where the last integral is finite and decreasing with respect to α. Now (23), (24), and (20) together with the dominated convergence theorem show that ζ is well-defined and continuous on $(-1,0)$ and $(0,\infty)$.

For $\alpha = 0$, we have

$$\begin{aligned} &\int_0^{t_0} \Big| \frac{\omega(\varphi(t)) - \omega(t)}{t} \Big|\,\mathrm{d}t \\ &\quad\le \int_0^{t_0} t^{-1}(\omega(0) - \omega(\varphi(t)))\,\mathrm{d}t + \int_0^{t_0} t^{-1}(\omega(0) - \omega(t))\,\mathrm{d}t \\ &\quad\le \int_0^{v_0} t^{-1}(\omega(0) - \omega(t))\,\mathrm{d}t + \int_0^{t_0} t^{-1}(\omega(0) - \omega(t))\,\mathrm{d}t \end{aligned} \tag{25}$$

and

$$\int_{t_0}^\infty \left| \frac{\omega(\varphi(t)) - \omega(t)}{t} \right| \mathrm{d}t \le \int_{v_0}^\infty t^{-1}\omega(t)\,\mathrm{d}t + \int_{t_0}^\infty t^{-1}\omega(t)\,\mathrm{d}t. \tag{26}$$

The monotonicity of the last integrals of (23) and (24) shows that the integrals in (25) and (26) are finite. Hence $\zeta(0)$ is well-defined.

It follows from Lemma 16 that ζ is decreasing on $(0,\infty)$. We show that it is decreasing on $(-1,0)$. By the change of variables $r=t^{-1}$, we get

$$\zeta(\alpha)^{-1}=\left(\frac{\int_0^\infty r^{-\alpha-1}(\omega(0)-\omega(\psi(r)^{-1}))\,\mathrm{d}r}{\int_0^\infty r^{-\alpha-1}(\omega(0)-\omega(r^{-1}))\,\mathrm{d}r}\right)^{-\frac{1}{\alpha}}$$

where $\psi(r)=\varphi(r^{-1})^{-1}$. Observe that $\omega(0)-\omega(r^{-1})$ is decreasing and non-negative, and that $\psi(r)/r=\left(\varphi(r^{-1})/r^{-1}\right)^{-1}$ is increasing. Since $-\alpha\in(0,1)$, Lemma 16 implies that $\zeta(\alpha)^{-1}$ is an increasing function of α.

Using (20) and the elementary relation,

$$\lim_{\alpha\to 0^\pm}\left(\frac{a(\alpha)+\frac{c}{\alpha}}{b(\alpha)+\frac{c}{\alpha}}\right)^{\frac{1}{\alpha}}=\exp\left(\frac{a(0)-b(0)}{c}\right),$$

which is valid for a constant $c\neq 0$ and continuous functions $\alpha\mapsto a(\alpha)$ and $\alpha\mapsto b(\alpha)$, we obtain that

$$\lim_{\alpha\to 0^\pm}\zeta(\alpha)=\exp\left(\int_0^\infty\frac{\omega(\varphi(t))-\omega(t)}{t\,\omega(0)}\,\mathrm{d}t\right).$$

So ζ is continuous at $\alpha=0$. □

7.2. **Radial mean bodies of functions.** Let $f\in L^2(\mathbb{R}^n)$ be non-zero and non-negative. We define

$$\mathrm{R}_\alpha f=\left(\frac{\alpha}{\|f\|_2^2}\right)^{\frac{1}{\alpha}}\mathrm{S}_\alpha f \tag{27}$$

for $\alpha>0$ and

$$\mathrm{R}_\alpha f=\left(\frac{|\alpha|}{2\|f\|_2^2}\right)^{\frac{1}{\alpha}}\Pi_2^{*,-\alpha/2}f \tag{28}$$

for $-1<\alpha<0$. In addition, we define $\mathrm{R}_0 f$ by its radial function for $\xi\in\mathbb{S}^{n-1}$ as

$$\log(\rho_{\mathrm{R}_0 f}(\xi))=-\gamma+\int_0^\infty\frac{1}{t}\left(\frac{1}{\|f\|_2^2}\int_{\mathbb{R}^n}f(x)f(x+t\xi)\,\mathrm{d}x-e^{-t}\right)\mathrm{d}t,$$

where γ is Euler's constant. The definitions (27) and (28) are compatible with (16) and (17) for $f=1_E$ and a convex body $E\subset\mathbb{R}^n$. Note that (11) implies that

$$|\,\mathrm{R}_n f|=\frac{\|f\|_1^2}{\|f\|_2^2} \tag{29}$$

for non-zero $f\in L^2(\mathbb{R}^n)$.

7.3. **Results for log-concave functions.** We start with a simple calculation, for which we need the following notation. For $f\in L^2(\mathbb{R}^n)$ and $y\in\mathbb{R}^n$, we set

$$\mathscr{G}f(y)=\int_{\mathbb{R}^n}f(x)\,f(x+y)\,\mathrm{d}x.$$

We define the simplex $\Delta_n\subset\mathbb{R}^n$ as the convex hull of the origin and the standard basis vectors $e_1,\dots,e_n$. In addition, let $B_1^n=\{(x_1,\dots,x_n):|x_1|+\cdots+|x_n|\le 1\}$ and $\mathbb{R}^n_+=\{(x_1,\dots,x_n)\in\mathbb{R}^n:x_1,\dots,x_n\ge 0\}$.

Lemma 18. *If $f(x) = e^{-\|x\|_{\Delta_n}}$ for $x \in \mathbb{R}^n$, then*

$$\mathscr{G}f(y) = \frac{1}{2^n} e^{-\|y\|_{B_1^n}}$$

for $y \in \mathbb{R}^n$.

Proof. We have

$$f(x) = \begin{cases} e^{-(x_1+\cdots+x_n)} & \text{for } x \in \mathbb{R}^n_+, \\ 0 & \text{for } x \notin \mathbb{R}^n_+. \end{cases}$$

Setting $g = \mathscr{G}f$, we obtain that

$$g(y) = e^{-(y_1+\cdots+y_n)} \prod_{i=1}^n \int_{\mathbb{R}_+ \cap (\mathbb{R}_+ - y_i)} e^{-2x_i} \, \mathrm{d}x_i.$$

Using

$$2 \int_{\mathbb{R}_+ \cap (\mathbb{R}_+ - y_i)} e^{-2x_i} \, \mathrm{d}x_i = \begin{cases} 1 & \text{for } y_i \geq 0 \\ e^{2y_i} & \text{for } y_i < 0 \end{cases}$$

and $I_-(y) = \{1 \leq i \leq n : y_i < 0\}$, we obtain that

$$g(y) = \frac{1}{2^n} \, e^{-(y_1+\cdots+y_n)+2\sum_{i\in I_-(y)}^n y_i} = \frac{1}{2^n} \, e^{-\|y\|_{B_1^n}}$$

for $y \in \mathbb{R}^n$. □

We establish the following inclusion relation.

Theorem 19. *If $f \in L^2(\mathbb{R}^n)$ is non-zero and log-concave, then*

$$\frac{1}{\Gamma(\beta+1)^{1/\beta}} \, \mathrm{R}_\beta f \subseteq \frac{1}{\Gamma(\alpha+1)^{1/\alpha}} \, \mathrm{R}_\alpha f$$

for $-1 < \alpha < \beta < \infty$. There is equality if $f(x) = a \, e^{-\|x-x_0\|_\Delta}$ for $x \in \mathbb{R}^n$ with $a > 0, x_0 \in \mathbb{R}^n$ and Δ an n-dimensional simplex having a vertex at the origin.

Proof. Note that $g = \mathscr{G}f$ is even, attains a maximum at $y = 0$ and that $g(0) = \|f\|_2^2$. Moreover, g is log-concave by Lemma 6.

We fix $\xi \in \mathbb{S}^{n-1}$ and see that $t \mapsto g(t\xi)$ is a positive, decreasing, log-concave function. We apply Lemma 17 to $\omega(t) = g(0)e^{-t}, \varphi(t) = -\log(g(t\xi)/g(0))$ and observe that

$$\zeta(\alpha) = \left(\frac{\int_0^\infty t^{\alpha-1} g(t\xi) \, \mathrm{d}t}{\int_0^\infty t^{\alpha-1} g(0) e^{-t} \, \mathrm{d}t} \right)^{1/\alpha} = \frac{\rho_{\mathrm{R}_\alpha f}(\xi)}{\Gamma(\alpha+1)^{1/\alpha}}$$

for $\alpha > 0$ and

$$\zeta(\alpha) = \left(\frac{\int_0^\infty t^{\alpha-1} (g(0) - g(t\xi)) \, \mathrm{d}t}{\int_0^\infty t^{\alpha-1} (g(0) - g(0)e^{-t}) \, \mathrm{d}t} \right)^{1/\alpha} = \frac{\rho_{\mathrm{R}_\alpha f}(\xi)}{\Gamma(\alpha+1)^{1/\alpha}}$$

for $-1 < \alpha < 0$, where we used formula (21). This proves the inclusion. There is equality in Lemma 17 if $g(t\xi) = e^{-h(\xi)t}$ for some function $h : \mathbb{S}^{n-1} \to (0, \infty)$. Hence, the equality case follows from Lemma 18 and the $\mathrm{SL}(n)$ and translation invariance and homogeneity of $|\, \mathrm{R}_\alpha f|$. □

Using (27) and (29), we obtain both Theorem 2 and Theorem 15 immediately from the above result. Using (28), (29), and Hölder's inequality, we also obtain the following inequality from Theorem 19.

Theorem 20. *For $0 < \alpha < 1/2$ and log-concave $f \in L^2(\mathbb{R}^n)$,*

$$\frac{\alpha}{\Gamma(n+1)^{\frac{2\alpha}{n}}\Gamma(1-2\alpha)}\,|\Pi_2^{*,\alpha} f|^{-\frac{2\alpha}{n}} \le \|f\|_2^{2+\frac{4\alpha}{n}}\|f\|_1^{-\frac{4\alpha}{n}} \le \|f\|^2_{\frac{2n}{n-2\alpha}}.$$

There is equality in the first inequality if $f(x) = a\,e^{-\|x-x_0\|_\Delta}$ for $x \in \mathbb{R}^n$ with $a \ge 0, x_0 \in \mathbb{R}^n$ and Δ an n-dimensional simplex having a vertex at the origin.

The inequality in the previous theorem is a reverse inequality to the affine fractional L^2 Sobolev inequality (9).

7.4. **Results for s-concave functions.** We obtain the following inclusion relation.

Theorem 21. *Let $s > 0$. If $f \in L^2(\mathbb{R}^n)$ is non-zero and s-concave, then*

$$\frac{1}{\big((n+\frac{2}{s})\,\mathrm{B}(\beta+1,n+\frac{2}{s})\big)^{1/\beta}}\,\mathrm{R}_\beta f \subseteq \frac{1}{\big((n+\frac{2}{s})\,\mathrm{B}(\alpha+1,n+\frac{2}{s})\big)^{1/\alpha}}\,\mathrm{R}_\alpha f$$

for $-1 < \alpha < \beta < \infty$.

Proof. As in the proof of Theorem 19, note that $g = \mathscr{G}f$ is even, continuous, and attains a maximum at $y = 0$. For $\xi \in S^{n-1}$, it follows that $t \mapsto g(t\xi)$ is positive and decreasing. By Lemma 6, the function g is r-concave with $r = s/(ns+2)$.

We apply Lemma 17 with $\omega(t) = g(0)(1-rt)_+^{1/r}$ and $\varphi(t) = (1-(g(t\xi)/g(0))^r)/r$. We obtain that

$$\zeta(\alpha) = \left(\frac{\int_0^\infty t^{\alpha-1} g(t\xi)\,\mathrm{d}t}{\int_0^\infty g(0)t^{\alpha-1}(1-rt)_+^{1/r}\,\mathrm{d}t}\right)^{1/\alpha} = \frac{r\,\rho_{\mathrm{R}_\alpha f}(\xi)}{(\alpha\,\mathrm{B}(\alpha,1+\frac{1}{r}))^{1/\alpha}}$$

for $\alpha > 0$ and

$$\zeta(\alpha) = \left(\frac{\int_0^\infty t^{\alpha-1}(g(0)-g(t\xi))\,\mathrm{d}t}{\int_0^\infty t^{\alpha-1}(g(0)(1-rt)_+^{1/r}-g(0))\,\mathrm{d}t}\right)^{1/\alpha} = \frac{r\,\rho_{\mathrm{R}_\alpha f}(\xi)}{(\alpha\,\mathrm{B}(\alpha,1+\frac{1}{r}))^{1/\alpha}}$$

for $-1 < \alpha < 0$, where we used formula (22). The result now follows from Lemma 17. □

For $s \to 0$, we recover Theorem 19 from Theorem 21. For $s \to \infty$ and $f = 1_E$ for a convex body $E \subset \mathbb{R}^n$, Theorem 21 implies that

$$\frac{1}{\big(n\,\mathrm{B}(\beta+1,n)\big)^{1/\beta}}\,\mathrm{R}_\beta E \subseteq \frac{1}{\big(n\,\mathrm{B}(\alpha+1,n)\big)^{1/\alpha}}\,\mathrm{R}_\alpha E \tag{30}$$

for $-1 < \alpha < \beta$. This recovers Theorem 5.5 by Gardner and Zhang [11], who showed that there is equality in (30) precisely for n-dimensional simplices. The problem to determine the precise equality conditions in Theorem 19 and Theorem 21 is open.

It follows from the proof of Theorem 21 under the assumptions given there that

$$\frac{\rho_{\mathrm{R}_\beta f}(\xi)}{\big((n+\frac{2}{s})\,\mathrm{B}(\beta+1,n+\frac{2}{s})\big)^{1/\beta}} \le \frac{\rho_{\mathrm{R}_\alpha f}(\xi)}{\big((n+\frac{2}{s})\mathrm{B}(\alpha+1,n+\frac{2}{s})\big)^{1/\alpha}} \tag{31}$$

with equality for $\xi \in \mathbb{S}^{n-1}$ if $\mathscr{G}f(t\xi) = a(1-\lambda t)_+^{1/s}$ for $t > 0$ with $a \geq 0$ and $\lambda > 0$. The following lemma shows that the inequality in (31) is sharp in some directions and that the constants in Theorem 21 are optimal.

Lemma 22. *Let $s > 0$. If $f(x) = (1 - \|x\|_{\Delta_n})_+^{1/s}$ for $x \in \mathbb{R}^n$, then*

$$\mathscr{G}f(y) = a\big(1 - \frac{1}{2}\|y\|_{B_1^n}\big)_+^{n+2/s}$$

for every $y \in \mathbb{R}^n$ with $y_1 + \cdots + y_n = 0$, where $a = \mathrm{B}(n, 1+2/s)/(n-1)!$.

Proof. Let $t_- = \max\{0, -t\}$ for $t \in \mathbb{R}$ and $y_- = ((y_1)_-, \dots, (y_n)_-)$ for $y \in \mathbb{R}^n$. Setting $\delta(t) = (1-t)_+^{1/s}$, we have

$$\begin{aligned}
\mathscr{G}f(y) &= \int_{\mathbb{R}^n_+ \cap (\mathbb{R}^n_+ - y)} \delta\big(\sum_{i=1}^n x_i\big)\, \delta\big(\sum_{i=1}^n (x_i + y_i)\big)\, \mathrm{d}x \\
&= \int_{\mathbb{R}^n_+ + y_-} \delta\big(\sum_{i=1}^n x_i\big)\, \delta\big(\sum_{i=1}^n (x_i + y_i)\big)\, \mathrm{d}x \\
&= \int_{\mathbb{R}^n_+} \delta\big(\sum_{i=1}^n (x_i + (y_i)_-)\big)\, \delta\big(\sum_{i=1}^n (x_i + y_i + (y_i)_-\big)\, \mathrm{d}x \\
&= \int_{\mathbb{R}^n_+} \delta\big(\sum_{i=1}^n x_i + \sum_{i=1}^n (y_i)_-\big)\, \delta\big(\sum_{i=1}^n x_i + \sum_{i=1}^n (y_i)_+\big)\, \mathrm{d}x \\
&= \frac{1}{(n-1)!} \int_0^\infty r^{n-1} \delta(r + \|y_-\|_{B_1^n})\, \delta(r + \|y_+\|_{B_1^n})\, \mathrm{d}r.
\end{aligned}$$

Now note that if $\|y_-\|_{B_1^n} = \|y_+\|_{B_1^n} = \frac{1}{2}\|y\|_{B_1^n} = t$, then

$$\begin{aligned}
\mathscr{G}f(y) &= \frac{1}{(n-1)!} \int_0^\infty r^{n-1} \delta(r+t)^2\, \mathrm{d}r \\
&= \frac{1}{(n-1)!} \int_0^\infty r^{n-1} (1-r-t)_+^{2/s}\, \mathrm{d}r.
\end{aligned}$$

For $t > 1$, this quantity is 0. Otherwise, we get

$$\begin{aligned}
\mathscr{G}f(y) &= \frac{1}{(n-1)!} \int_0^{1-t} r^{n-1} (1-r-t)^{2/s}\, \mathrm{d}r \\
&= \frac{1}{(n-1)!} (1-t) \int_0^1 ((1-t)s)^{n-1} ((1-t) - (1-t)r)^{2/s}\, \mathrm{d}r \\
&= \frac{1}{(n-1)!} (1-t)^{n+2/s} \int_0^1 r^{n-1} (1-r)^{2/s}\, \mathrm{d}r,
\end{aligned}$$

which completes the proof. □

Using (27), (29) and Hölder's inequality, we obtain the following inequalities from Theorem 21.

Corollary 23. *Let $s > 0$. If $f \in L^2(\mathbb{R}^n)$ is s-concave on its support, then*

$$\frac{(n\,\mathrm{B}(n, n+1+\frac{2}{s}))^{\frac{\alpha}{n}}}{\mathrm{B}(\alpha, n+1+\frac{2}{s})}\, |\,\mathrm{S}_\alpha f|^{\frac{\alpha}{n}} \geq \|f\|_2^{2-\frac{2\alpha}{n}}\, \|f\|_1^{\frac{2\alpha}{n}} \geq \|f\|^2_{\frac{2n}{n+\alpha}}$$

for $0 < \alpha < n$ and the inequalities are reversed for $\alpha > n$.

Using (28), (29), and Hölder's inequalities, we obtain the following inequality from Theorem 21.

Corollary 24. *Let $s > 0$. If $f \in L^2(\mathbb{R}^n)$ is s-concave on its support, then*

$$\frac{(n\,\mathrm{B}(n, n+1+\frac{2}{s}))^{-\frac{2\alpha}{n}}}{2|\,\mathrm{B}(-2\alpha, n+1+\frac{2}{s})|}\,|\Pi_2^{*,\alpha} f|^{-\frac{2\alpha}{n}} \leq \|f\|_2^{2+\frac{4\alpha}{n}}\,\|f\|_1^{-\frac{4\alpha}{n}} \leq \|f\|_{\frac{2n}{n-2\alpha}}^2$$

for $0 < \alpha < 1/2$.

The following problems remain open: Are the first inequalities in Corollary 23 and Corollary 24 sharp for $f \not\equiv 0$?

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