AFFINE FRACTIONAL SOBOLEV AND ISOPERIMETRIC INEQUALITIES

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ABSTRACT. Sharp affine fractional Sobolev inequalities for functions on $\mathbb{R}^n$ are established. For each $0 < s < 1$, the new inequalities are significantly stronger than (and directly imply) the sharp fractional Sobolev inequalities of Almgren and Lieb. In the limit as $s \to 1^-$, the new inequalities imply the sharp affine Sobolev inequality of Gaoyong Zhang. As a consequence, fractional Petty projection inequalities are obtained that are stronger than the fractional Euclidean isoperimetric inequalities, and a natural conjecture for radial mean bodies is proved.

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1. INTRODUCTION

Almgren and Lieb [1] established Pólya–Szegő inequalities for fractional Sobolev norms. The equality case was settled by Frank and Seiringer [11]. A consequence is the fractional Sobolev inequality: for $0 < s < 1$, there is $\alpha_{n,s} > 0$ such that for $f \in W^{s,1}(\mathbb{R}^n)$,

$$\|f\|_{\frac{n}{n-s}} \le \alpha_{n,s} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|}{|x-y|^{n+s}} \,\mathrm{d}x \,\mathrm{d}y, \tag{1}$$

and there is equality if and only if f is a constant multiple of the indicator function of a ball. Here $\|f\|_p$ denotes the L^p norm of f and $|\cdot|$ the Euclidean norm in $\mathbb{R}^n$ while $W^{s,1}(\mathbb{R}^n)$ denotes the fractional Sobolev space of L^1 functions with finite right side in (1). By a result of Bourgain, Brezis, and Mironescu [3], for $f \in W^{1,1}(\mathbb{R}^n)$,

$$\lim_{s\to 1^-} \alpha_{n,s} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|}{|x-y|^{n+s}} \,\mathrm{d}x \,\mathrm{d}y = \frac{1}{n\omega_n^{1/n}} \int_{\mathbb{R}^n} |\nabla f(x)| \,\mathrm{d}x, \tag{2}$$

where ω_n is the n-dimensional volume of the unit ball in $\mathbb{R}^n$ and $W^{1,1}(\mathbb{R}^n)$ denotes the Sobolev space of L^1 functions f with weak L^1 gradient ∇f. Dávila [7] proved a result corresponding to (2) for $f \in BV(\mathbb{R}^n)$, the space of L^1 functions with bounded variation. Hence, the sharp L^1 Sobolev inequalities on $W^{1,1}(\mathbb{R}^n)$ and $BV(\mathbb{R}^n)$ follow from the fractional Sobolev inequalities (1).

Gaoyong Zhang [39] established a sharp affine Sobolev inequality that is significantly stronger than the classical L^1 Sobolev inequality. For $n \ge 2$ and C^1 functions $f : \mathbb{R}^n \to \mathbb{R}$ with compact support, Zhang [39] proved that

$$\|f\|_{\frac{n}{n-1}} \le \frac{\omega_n}{2\omega_{n-1}} \Big(\frac{1}{n} \int_{\mathbb{S}^{n-1}} \|\langle \nabla f(\cdot), \xi \rangle\|_1^{-n} \,\mathrm{d}\xi \Big)^{-1/n} \le \frac{1}{n\omega_n^{1/n}} \int_{\mathbb{R}^n} |\nabla f(x)| \,\mathrm{d}x. \tag{3}$$

Here $\langle \cdot, \cdot \rangle$ denotes the inner product and integration on the $(n-1)$-dimensional unit sphere $\mathbb{S}^{n-1}$ is with respect to the $(n-1)$-dimensional Hausdorff measure. The first inequality in (3) is now called the affine Sobolev inequality or the Sobolev–Zhang inequality. It is affine since both terms remain invariant under volume-preserving affine transformations. Tuo Wang [35] extended the Sobolev–Zhang inequality to L^1 functions of bounded variation and showed that there is equality in this generalized Sobolev–Zhang inequality precisely for constant multiples of indicator functions of ellipsoids.

The main aim of this paper is to establish affine fractional Sobolev inequalities for $0 < s < 1$ that are stronger than the fractional Sobolev inequalities in (1) and imply, in the limit as $s \to 1^-$, the Sobolev–Zhang inequality.

Theorem 1. *For $0 < s < 1$ and $f \in W^{s,1}(\mathbb{R}^n)$,*

$$\|f\|_{\frac{n}{n-s}} \le \alpha_{n,s} n \omega_n^{(n+s)/n} \Big(\frac{1}{n}\int_{\mathbb{S}^{n-1}} \Big(\int_0^\infty t^{-s} \Big\|\frac{f(\cdot + t\xi) - f(\cdot)}{t}\Big\|_1 \,\mathrm{d}t\Big)^{-n/s} \mathrm{d}\xi\Big)^{-s/n}$$
$$\le \alpha_{n,s} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|}{|x-y|^{n+s}} \,\mathrm{d}x\,\mathrm{d}y.$$

There is equality in the first inequality if and only if f is a constant multiple of the indicator function of an ellipsoid. There is equality in the second inequality if f is radially symmetric.

In order to prove Theorem 1 we introduce the s-fractional polar projection body $\Pi^{*,s} f$ associated to f, defined as the star body whose gauge function for $\xi \in \mathbb{S}^{n-1}$ is

$$\|\xi\|^s_{\Pi^{*,s} f} = \int_0^\infty t^{-s} \int_{\mathbb{R}^n} \Big|\frac{f(x+t\xi) - f(x)}{t}\Big| \,\mathrm{d}x\,\mathrm{d}t \tag{4}$$

(see Section 3 for details). The affine fractional Sobolev inequality can now be written as

$$\|f\|_{\frac{n}{n-s}} \le \alpha_{n,s} n \omega_n^{(n+s)/n} |\,\Pi^{*,s} f|^{-s/n}, \tag{5}$$

where $|\cdot|$ denotes n-dimensional Lebesgue measure. Since both sides of (5) are invariant under translations of f, and for volume-preserving linear transformations $\phi : \mathbb{R}^n \to \mathbb{R}^n$,

$$\Pi^{*,s}(f \circ \phi^{-1}) = \phi\, \Pi^{*,s} f,$$

it follows that (5) is an affine inequality.

To show that Theorem 1 implies the Sobolev–Zhang inequality, we establish a result corresponding to (2). For $f \in W^{1,1}(\mathbb{R}^n)$, we will prove that

$$\lim_{s \to 1^-} (1-s) |\,\Pi^{*,s} f|^{-s/n} = 2\,|\,\Pi^* f|^{-1/n}, \tag{6}$$

where $\Pi^* f$ is the polar projection body of f which is defined (with different notation) in [39] (also see [24]) for $\xi \in \mathbb{S}^{n-1}$ by

$$\|\xi\|_{\Pi^* f} = \frac{1}{2}\int_{\mathbb{R}^n} |\langle \nabla f(x), \xi\rangle| \,\mathrm{d}x.$$

Moreover, we establish the corresponding statement for $f \in BV(\mathbb{R}^n)$ and also recover Wang's generalized Sobolev–Zhang inequality (without characterization of the equality case) from Theorem 1 (see Section 4).

We prove Theorem 1 by first establishing affine fractional Pólya–Szegő inequalities as a consequence of the Riesz rearrangement inequality. For the equality case, we use anisotropic fractional Sobolev norms (introduced in [21]) and the equality case of the Riesz rearrangement inequality due to Burchard [5]. For $0 < s < 1$ and $E \subset \mathbb{R}^n$ of finite s-perimeter (see Section 2), we define the fractional polar projection body as $\Pi^{*,s} E = \Pi^{*,s} 1_E$, where 1_E is the indicator function of E, and obtain from the affine fractional Pólya–Szegő inequalities that

$$\Big(\frac{|\,\Pi^{*,s} E|}{|\,\Pi^{*,s} B^n|}\Big)^{-s/n} \ge \Big(\frac{|E|}{|B^n|}\Big)^{(n-s)/n} \tag{7}$$

for $0 < s < 1$ and $E \subset \mathbb{R}^n$ of finite s-perimeter and finite measure, where B^n is the Euclidean unit ball in $\mathbb{R}^n$. From these fractional Petty projection inequalities, Theorem 1 follows by a co-area formula for the fractional Sobolev norm. In the limit as $s \to 1^-$, we obtain from (7) the generalized Petty projection inequality by Zhang [39] and Wang [35]. In addition, we prove versions of our functional and geometric inequalities for Steiner symmetrization.

In Section 9, we connect fractional polar projection bodies to radial mean bodies (introduced by Gardner and Zhang [13]) and obtain fractional Zhang inequalities. Moreover, we establish new affine isoperimetric inequalities for radial mean bodies, thereby proving a natural conjecture in the field.

2. Preliminaries

2.1. **Function spaces.**

For $p \ge 1$ and measurable $f : \mathbb{R}^n \to \mathbb{R}$, let

$$\|f\|_p = \Big(\int_{\mathbb{R}^n} |f(x)|^p \,\mathrm{d}x\Big)^{1/p}.$$

We set $\{f \ge t\} = \{x \in \mathbb{R}^n : f(x) \ge t\}$ for $t \in \mathbb{R}$ and use corresponding notation for level sets, etc. We say that f is non-zero if $\{f \ne 0\}$ has positive measure, and we identify functions that are equal up to a set of measure zero. For $p \ge 1$, let

$$L^p(\mathbb{R}^n) = \Big\{f : \mathbb{R}^n \to \mathbb{R} : f \text{ is measurable}, \|f\|_p < \infty\Big\}.$$

Here and below, when we use measurability and related notions, we refer to the n-dimensional Lebesgue measure on $\mathbb{R}^n$.

For $E \subset \mathbb{R}^n$, the indicator function 1_E is defined by $1_E(x) = 1$ for $x \in E$ and $1_E(x) = 0$ otherwise. We say that two measurable sets are equivalent if their indicator functions are equal up to a set of measure zero.

Let $0 < s < 1$. We define the s-fractional Sobolev space as

$$W^{s,1}(\mathbb{R}^n) = \Big\{f \in L^1(\mathbb{R}^n) : \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x) - f(y)|}{|x-y|^{n+s}} \,\mathrm{d}x\,\mathrm{d}y < \infty\Big\}.$$

For $f \in L^1(\mathbb{R}^n)$, we say that f admits a weak gradient $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ if

$$\int_{\mathbb{R}^n} \langle \nu(x), \nabla f(x)\rangle \,\mathrm{d}x = -\int_{\mathbb{R}^n} f(x) \operatorname{div} \nu(x)\,\mathrm{d}x$$

for every $\nu \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$, where $C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$ is the set of smooth vector fields $\nu : \mathbb{R}^n \to \mathbb{R}^n$ with compact support and $\operatorname{div}\nu$ denotes the divergence of ν. We set

$$W^{1,1}(\mathbb{R}^n) = \big\{f \in L^1(\mathbb{R}^n) : |\nabla f| \in L^1(\mathbb{R}^n)\big\}.$$

Let $\mathcal{B}(\mathbb{R}^n)$ denote the class of Borel sets in $\mathbb{R}^n$. For $f \in L^1(\mathbb{R}^n)$, we say that f is a function of bounded variation on $\mathbb{R}^n$ if there is a finite vector-valued Radon measure $\mathrm{D}f : \mathcal{B}(\mathbb{R}^n) \to \mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} \langle \nu(x), \sigma_f(x) \rangle \,\mathrm{d}|\mathrm{D}f|(x) = -\int_{\mathbb{R}^n} f(x) \operatorname{div} \nu(x) \,\mathrm{d}x \tag{8}$$

for every $\nu \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$, where $|\mathrm{D}f| : \mathcal{B}(\mathbb{R}^n) \to [0, \infty)$ denotes the variation measure of $\mathrm{D}f$ and $\sigma_f : \mathbb{R}^n \to \mathbb{R}^n$ the Radon–Nikodym derivative of $\mathrm{D}f$ with respect to $|\mathrm{D}f|$. We write $BV(\mathbb{R}^n)$ for the space of L^1 functions of bounded variation. For more information on functions of bounded variation, we refer to [8, Chapter 5].

2.2. **Symmetrization.** Let $E \subseteq \mathbb{R}^n$ be a Borel set of finite measure. The Schwarz symmetral of E, denoted by $E^\star$, is the centered Euclidean ball with the same volume as E.

For a non-negative integrable function f and $t > 0$, the superlevel set $\{f \geq t\}$ is a measurable set of finite measure. The layer cake formula states that

$$f(x) = \int_0^\infty 1_{\{f \geq t\}}(x) \,\mathrm{d}t$$

for almost every $x \in \mathbb{R}^n$ and allows us to recover the function from its superlevel sets. The Schwarz symmetral of f, denoted by $f^\star$, is defined by

$$f^\star(x) = \int_0^\infty 1_{\{f \geq t\}^\star}(x) \,\mathrm{d}t$$

for $x \in \mathbb{R}^n$. Hence $f^\star$ is determined by the properties of being radially symmetric and having superlevel sets that are balls of the same measure as the superlevel sets of f. Note that $f^\star$ is often called symmetric decreasing rearrangement of f.

Let $\xi \in S^{n-1}$. The Steiner symmetral of E in the direction ξ, denoted by E^ξ, is defined by the property that for every line L parallel to ξ, the set $L \cap E^\xi$ is an interval of the same length as $L \cap E$ and symmetric with respect to the subspace orthogonal to ξ.

For a non-negative integrable function f, the Steiner symmetral f^ξ is defined by

$$f^\xi(x) = \int_0^\infty 1_{\{f \geq t\}^\xi}(x) \,\mathrm{d}t$$

for $x \in \mathbb{R}^n$.

2.3. **Star bodies.** For a set $K \subseteq \mathbb{R}^n$ that is star-shaped (with respect to the origin), the gauge function $\|\cdot\|_K : \mathbb{R}^n \to [0, \infty]$ is defined as

$$\|x\|_K = \inf\{\lambda > 0 : x \in \lambda K\},$$

and the radial function $\rho_K : \mathbb{R}^n \setminus \{0\} \to [0, \infty]$ as

$$\rho_K(x) = \|x\|_K^{-1} = \sup\{\lambda \geq 0 : \lambda x \in K\}.$$

We call K a star body if its radial function is strictly positive and continuous in $\mathbb{R}^n \setminus \{0\}$.

We say that a star-shaped set is p-convex with $0 < p \leq 1$ if

$$\|x + y\|_K^p \leq \|x\|_K^p + \|y\|_K^p$$

for all $x, y \in \mathbb{R}^n$. A p-convex body is a star body that is also p-convex. For more information on p-convex sets, see, for example, [16].

For star bodies $K, L \subseteq \mathbb{R}^n$ and $p \in \mathbb{R}$, define the dual mixed volume as

$$\tilde{V}_p(K, L) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} \rho_K(\xi)^{n-p} \rho_L(\xi)^p \,\mathrm{d}\xi.$$

Notice that

$$\tilde{V}_p(K, K) = |K|.$$

The dual mixed volume inequality (see [31, Section 9.3] or [12, B.29]) states that for $p < 0$ or $p > n$,

$$\tilde{V}_p(K, L) \geq |K|^{(n-p)/n} |L|^{p/n}, \tag{9}$$

and the reverse inequality holds for $0 < p < n$. Equality holds for $p \neq 0, n$ if and only if K and L are dilates, where we say that star bodies K and L are dilates if $\rho_K = c\,\rho_L$ on $\mathbb{S}^{n-1}$ for some $c > 0$.

2.4. **Convex bodies and the Petty projection inequality.** A set $K \subseteq \mathbb{R}^n$ is called a (proper) convex body if it is compact and convex and has non-empty interior. For a convex body K, the support function $h_K : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$h_K(y) = \max\{\langle x, y \rangle : x \in K\}.$$

A convex body K that contains the origin in its interior is also a star body. The polar body of such K is defined as

$$K^* = \{y \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } x \in K\}$$

and is a convex body containing the origin in its interior while

$$\|x\|_K = h_{K^*}(x)$$

for $x \in \mathbb{R}^n$.

For $K \subset \mathbb{R}^n$ a convex body, the projection body is defined as the convex body whose support function for $\xi \in \mathbb{S}^{n-1}$ is

$$h_{\Pi K}(\xi) = |\operatorname{proj}_{\xi^\perp}(K)|_{n-1},$$

where $\operatorname{proj}_{\xi^\perp}$ denotes the orthogonal projection to the hyperplane, $\xi^\perp$, orthogonal to ξ and $|\cdot|_{n-1}$ stands for $(n-1)$-dimensional volume in $\xi^\perp$. Using the surface area measure $S(K, \cdot)$ of K, we can also write

$$h_{\Pi K}(\xi) = \frac{1}{2} \int_{\mathbb{S}^{n-1}} |\langle \xi, \eta \rangle| \,\mathrm{d}S(K, \eta)$$

for $\xi \in \mathbb{S}^{n-1}$. Projection bodies were introduced by Minkowski, and $h_{\Pi K}(\xi)$ is called the brightness of K in the direction ξ (see [12, 31] for more information).

The important Petty projection inequality [28] states that for a convex body $K \subset \mathbb{R}^n$,

$$|K|^{(n-1)/n} \leq \frac{\omega_n}{\omega_{n-1}} |\,\Pi^* K|^{-1/n} \tag{10}$$

with equality precisely for ellipsoids, where $\Pi^* K$ is the polar body of ΠK. A new proof using Theorem 1 (and hence the Riesz rearrangement inequality) is given in Section 7. A previous proof of (10) that also uses the Riesz rearrangement inequality and the so-called convolution square of a convex body was given by Schmuckenschläger [32]. For generalization of projection bodies of convex bodies, see, for example, [2,6,14,18,20,23,26], and for further results related to the Sobolev–Zhang inequality, see, for example, [15, 24, 25, 27, 36].

2.5. **Anisotropic fractional Sobolev norms and perimeters.** Let $0 < s < 1$ and let $K \subset \mathbb{R}^n$ be a star body. For $f \in W^{s,1}(\mathbb{R}^n)$, the anisotropic fractional Sobolev norm of f with respect to K is

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|}{\|x - y\|_K^{n+s}} \,\mathrm{d}x \,\mathrm{d}y.$$

It was introduced in [21] for K a convex body (also, see [22]). The s-fractional perimeter with respect to K of a measurable set $E \subseteq \mathbb{R}^n$ was introduced in [21] as

$$P_s(E, K) = \int_E \int_{E^c} \frac{1}{\|x - y\|_K^{n+s}} \,\mathrm{d}x \,\mathrm{d}y,$$

where $E^c = \mathbb{R}^n \backslash E$ is the complement of E in $\mathbb{R}^n$. The same definition will be used for $K \subset \mathbb{R}^n$ a star body. The anisotropic s-fractional perimeter of E with respect to K is half of the s-fractional anisotropic Sobolev norm of 1_E with respect to K. Since K is a star body, this implies for measurable $E \subset \mathbb{R}^n$ that $P_s(E, K) < \infty$ if and only if $1_E \in W^{s,1}(\mathbb{R}^n)$. If K is the Euclidean unit ball, we write $P_s(E)$ and obtain the well-known Euclidean s-fractional perimeter of E. We will use that

$$P_s(B^n) = \frac{2^{1-s}\pi^{(n-1)/2} n\omega_n}{s(n-s)} \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{n-s}{2}\right)} = \frac{2^{1-s} n\omega_n \omega_{n-s}}{s(1-s)\omega_{1-s}} \tag{11}$$

(cf. [9]), where $\omega_q = \pi^{q/2}/\Gamma(q/2 + 1)$ for $q > 0$ and Γ is the gamma function. A simple way to obtain this result is by using the Blaschke–Petkantschin formula [33, Theorem 7.2.7]:

$$\begin{aligned}
P_s(B^n) &= \int_{B^n \cap L \neq \emptyset} \int_{B^n \cap L} \int_{(\mathbb{R}^n \backslash B^n) \cap L} \frac{\mathrm{d}H^1(x) \,\mathrm{d}H^1(y)}{|x - y|^{s+1}} \,\mathrm{d}L \\
&= \frac{1}{2} \int_{\mathbb{S}^{n-1}} \int_{\xi^\perp} \int_{B^n \cap (z + \mathbb{R}\xi)} \int_{(\mathbb{R}^n \backslash B^n) \cap (z + \mathbb{R}\xi)} \frac{\mathrm{d}H^1(x) \,\mathrm{d}H^1(y)}{|x - y|^{1+s}} \,\mathrm{d}z \,\mathrm{d}\xi \\
&= \frac{n\omega_n (n-1)\omega_{n-1}}{2} \int_0^1 \int_{|x| \leq \sqrt{1-t^2}} \int_{|y| \geq \sqrt{1-t^2}} \frac{\mathrm{d}H^1(x) \,\mathrm{d}H^1(y)}{|x - y|^{1+s}} \, t^{n-2} \,\mathrm{d}t \\
&= \frac{2^{1-s}(n-1) n\omega_{n-1}\omega_n}{s(1-s)} \int_0^1 (1 - t^2)^{(1-s)/2} t^{n-2} \,\mathrm{d}t \\
&= \frac{(n-1) n\omega_{n-1}\omega_n}{2^s s(1-s)} \,\mathrm{B}\Big(\frac{1-s}{2} + 1, \frac{n-1}{2}\Big)
\end{aligned}$$

where the outer integration in the first integral is on the affine Grassmannian of lines in $\mathbb{R}^n$, and we use the $(n-1)$-dimensional Hausdorff measure for the direction of lines and the $(n-1)$-dimensional Lebesgue measure in the orthogonal complement of a line. Here H^1 denotes 1-dimensional Hausdorff measure, and $\mathbb{R}\xi$ is the line with direction ξ while B is the beta function. For the optimal constant in (1), we get

$$\alpha_{n,s} = \frac{\omega_n^{(n-s)/n}}{2P_s(B^n)}$$

since there is equality in (1) for indicator functions of balls.

The following fractional co-area formula is a consequence of Fubini's theorem (cf. [34, Example (2.9)]):

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f(x)-f(y)|}{\|x-y\|_K^{n+s}}\,\mathrm{d}x\,\mathrm{d}y &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_0^\infty \frac{|\,1_{\{f\geq t\}}(x)-1_{\{f\geq t\}}(y)|}{\|x-y\|_K^{n+s}}\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\\ &= 2\int_0^\infty P_s(\{f\geq t\},K)\,\mathrm{d}t\end{aligned}\tag{12}$$

for non-negative $f\in W^{s,1}(\mathbb{R}^n)$ and a star body $K\subset\mathbb{R}^n$.

3. Fractional Polar Projection Bodies

Let $0<s<1$ and $f\in BV(\mathbb{R}^n)$. We discuss properties of the s-fractional polar projection body $\Pi^{*,s}f$, that is, the star-shaped set defined through its gauge function for $\xi\in\mathbb{R}^n$ as

$$\|\xi\|_{\Pi^{*,s}f}^s=\int_0^\infty t^{-s}\int_{\mathbb{R}^n}\Big|\frac{f(x+t\xi)-f(x)}{t}\Big|\,\mathrm{d}x\,\mathrm{d}t.\tag{13}$$

Notice that $\|\cdot\|_{\Pi^{*,s}f}$ is a 1-homogeneous function.

We start with a simple observation. Let $K\subset\mathbb{R}^n$ be a star body. Using (13), we obtain

$$\begin{aligned}&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f(x)-f(y)|}{\|x-y\|_K^{n+s}}\,\mathrm{d}x\,\mathrm{d}y\\ &\quad=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f(y+z)-f(y)|}{\|z\|_K^{n+s}}\,\mathrm{d}z\,\mathrm{d}y\\ &\quad=\int_{\mathbb{S}^{n-1}}\int_0^\infty\|t\xi\|_K^{-n-s}\int_{\mathbb{R}^n}|f(y+t\xi)-f(y)|\,t^{n-1}\,\mathrm{d}y\,\mathrm{d}t\,\mathrm{d}\xi\\ &\quad=\int_{\mathbb{S}^{n-1}}\|\xi\|_K^{-n-s}\int_0^\infty t^{-s}\Big\|\frac{f(\cdot+t\xi)-f(\cdot)}{t}\Big\|_1\,\mathrm{d}t\,\mathrm{d}\xi\\ &\quad=\int_{\mathbb{S}^{n-1}}\rho_K(\xi)^{n+s}\rho_{\Pi^{*,s}f}(\xi)^{-s}\,\mathrm{d}\xi.\end{aligned}$$

Hence

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f(x)-f(y)|}{\|x-y\|_K^{n+s}}\,\mathrm{d}x\,\mathrm{d}y=n\,\tilde V_{-s}(K,\Pi^{*,s}f),\tag{14}$$

where we have already used the following result, which shows that $\Pi^{*,s}f$ is an origin-symmetric star body.

Proposition 2. *For non-zero $f\in W^{s,1}(\mathbb{R}^n)$, the set $\Pi^{*,s}f$ is an origin-symmetric s-convex body with the origin in its interior. Moreover, there is $c>0$ depending only on f such that $\Pi^{*,s}f\subseteq c\,B^n$ for every $s\in(0,1)$.*

Proof. First, note that since for $\xi\in\mathbb{R}^n$ and $t>0$,

$$\int_{\mathbb{R}^n}|f(x-t\xi)-f(x)|\,\mathrm{d}x=\int_{\mathbb{R}^n}|f(x)-f(x+t\xi)|\,\mathrm{d}x,$$

the set $\Pi^{*,s}f$ is origin-symmetric.

Next, we show that $\Pi^{*,s} f$ is bounded. We take $r > 1$ large enough so that $\int_{rB^n} |f(x)|\,\mathrm{d}x \ge \frac{2}{3}\|f\|_1$ and easily see that for $t > r$,

$$\begin{aligned}\int_{\mathbb{R}^n} |f(x+t\xi)-&f(x)|\,\mathrm{d}x\\ &\ge \int_{rB^n} (|f(x)| - |f(x+t\xi)|)\,\mathrm{d}x + \int_{rB^n-t\xi} (|f(x+t\xi)| - |f(x)|)\,\mathrm{d}x\\ &\ge 2(\frac{2}{3}\|f\|_1 - \frac{1}{3}\|f\|_1).\end{aligned}$$

Hence,

$$\int_0^\infty t^{-s} \left\| \frac{f(x+t\xi)-f(x)}{t} \right\|_1 \mathrm{d}t \ge \frac{2}{3}\|f\|_1 \int_r^\infty t^{-s-1}\,\mathrm{d}t \ge \frac{2}{3r}\|f\|_1,$$

which implies that $\Pi^{*,s} f \subseteq c\,B^n$ for $c > 0$ independent of s.

Next, we show that $\|\cdot\|^s_{\Pi^{*,s} f}$ is sublinear on $\mathbb{R}^n$. Indeed, for $\xi, \eta \in \mathbb{R}^n$, the triangle inequality and a change of variables show that

$$\begin{aligned}\|\xi+\eta\|^s_{\Pi^{*,s} f} &= \int_0^\infty t^{-s-1} \|f(\cdot + t\xi + t\eta) - f(\cdot)\|_1\,\mathrm{d}t\\ &\le \int_0^\infty t^{-s-1} (\|f(\cdot + t\xi + t\eta) - f(\cdot + t\xi)\|_1 + \|f(\cdot + t\xi) - f(\cdot)\|_1)\,\mathrm{d}t \qquad (15)\\ &= \|\xi\|^s_{\Pi^{*,s} f} + \|\eta\|^s_{\Pi^{*,s} f},\end{aligned}$$

which shows that $\Pi^{*,s} f$ is an s-convex set.

Now, we show that $\Pi^{*,s} f$ has the origin in its interior. Using the relation (14), we get

$$\int_{\mathbb{S}^{n-1}} \|\xi\|^s_{\Pi^{*,s} f}\,\mathrm{d}\xi = \frac{1}{n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x)-f(y)|}{|x-y|^{n+s}}\,\mathrm{d}x\,\mathrm{d}y,$$

which is finite since $f \in W^{s,1}(\mathbb{R}^n)$. We choose $r > 0$ large enough so that the set $A = \{\xi \in S^{n-1} : \|\xi\|^s_{\Pi^{*,s} f} < r\}$ has positive $(n-1)$-dimensional Hausdorff measure and a basis $\{\xi_1, \dots, \xi_n\} \subseteq A$ of $\mathbb{R}^n$. Applying (if necessary) a linear transformation to $\Pi^{*,s} f$, we may assume without loss of generality that $\xi_i = e_i$ are the canonical basis vectors. For every $x \in \mathbb{R}^n$ writing $x = \sum x_i e_i$ and using (15) we get

$$\|x\|_{\Pi^{*,s} f} \le \Big(\sum |x_i|^s \|e_i\|^s_{\Pi^{*,s} f} \Big)^{1/s} \le d\,|x|,$$

where $d > 0$ is independent of x. This shows that $\Pi^{*,s} f$ has the origin as interior point.

Finally, it is easy to see that a star-shaped set that is s-convex and has the origin in its interior also has a continuous gauge function, thus being an s-convex body. □

4. The Limit Case $s \to 1^-$ of Fractional Polar Projection Bodies

We prove (6) and the corresponding statement for functions of bounded variation. For $f \in BV(\mathbb{R}^n)$, the polar projection body is defined for $\xi \in \mathbb{S}^{n-1}$ by

$$\|\xi\|_{\Pi^* f} = \frac{1}{2} \int_{\mathbb{R}^n} |\langle \sigma_f(x), \xi\rangle|\,\mathrm{d}|\mathrm{D}f|(x)$$

(see [35]). Note that for $f \in W^{1,1}(\mathbb{R}^n)$, this definition coincides with (4).

Theorem 3. *Let $f \in BV(\mathbb{R}^n)$. For $\xi \in \mathbb{S}^{n-1}$,*

$$\lim_{s\to 1^-}(1-s)\|\xi\|^s_{\Pi^{*,s}f} = 2\,\|\xi\|_{\Pi^* f}.$$

Moreover,

$$\lim_{s\to 1^-}(1-s)|\,\Pi^{*,s}f|^{-s/n} = 2\,|\,\Pi^* f|^{-1/n}$$

and

$$\lim_{s\to 1^-}(1-s)\tilde{V}_{-s}(K,\Pi^{*,s}f) = 2\,\tilde{V}_{-1}(K,\Pi^* f)$$

for every star body $K \subset \mathbb{R}^n$.

We require the following two lemmas.

Lemma 4. *If $\varphi : [0,\infty) \to [0,\infty)$ is a measurable function with $\lim_{t\to 0^+}\varphi(t) = \varphi(0)$ and such that $\int_0^\infty t^{-s_0}\varphi(t)\,\mathrm{d}t < \infty$ for some $s_0 \in (0,1)$, then*

$$\lim_{s\to 1^-}(1-s)\int_0^\infty t^{-s}\varphi(t)\,\mathrm{d}t = \varphi(0).$$

Proof. Given $\varepsilon > 0$ take $\delta \in (0,1)$ so that $0 \le t \le \delta$ implies $|\varphi(t)-\varphi(0)| \le \varepsilon$. We have

$$\begin{aligned}\int_0^\infty t^{-s}\varphi(t)\,\mathrm{d}t &= \int_0^\delta t^{-s}\varphi(0)\,\mathrm{d}t + \int_0^\delta t^{-s}(\varphi(t)-\varphi(0))\,\mathrm{d}t + \int_\delta^\infty t^{-s}\varphi(t)\,\mathrm{d}t\\ &= \frac{\delta^{1-s}}{1-s}\varphi(0) + A + B\end{aligned}$$

where $(1-s)|A| \le \varepsilon\delta^{1-s} \le \varepsilon$ and

$$|B| \le \delta^{s_0-s}\int_\delta^\infty t^{-s_0}\varphi(t)\,\mathrm{d}t \le \delta^{s_0-s}\int_0^\infty t^{-s_0}\varphi(t)\,\mathrm{d}t$$

for $s \in (s_0,1)$. Since $(1-s)B \to 0$ and $\delta^{1-s} \to 1$ as $s \to 1^-$, we can take s sufficiently close to 1 so that $(1-s)|B| \le \varepsilon$ and $|\delta^{1-s}\varphi(0)-\varphi(0)| \le \varepsilon$, obtaining $\big|(1-s)\int_0^\infty t^{-s}\varphi(t)\,\mathrm{d}t - \varphi(0)\big| \le 3\varepsilon$. ☐

Lemma 5. *For $f \in BV(\mathbb{R}^n)$ and $\xi \in \mathbb{S}^{n-1}$,*

$$\lim_{t\to 0}\Big\|\frac{f(\cdot+t\xi)-f(\cdot)}{t}\Big\|_1 = \int_{\mathbb{R}^n}|\langle\sigma_f(x),\xi\rangle|\,\mathrm{d}|\mathrm{D}f|(x).$$

Proof. Let $g : \mathbb{R}^n \to \mathbb{R}$ be a smooth function with compact support, and write div_x for the divergence taken with respect to the variable x. Using (8), we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} g(x)\,\frac{f(x+t\xi)-f(x)}{t}\,\mathrm{d}x &= \int_{\mathbb{R}^n} f(x)\,\frac{g(x-t\xi)-g(x)}{t}\,\mathrm{d}x\\ &= -\int_{\mathbb{R}^n} f(x)\int_0^1\langle\nabla g(x-rt\xi),\xi\rangle\,\mathrm{d}r\,\mathrm{d}x\\ &= -\int_{\mathbb{R}^n} f(x)\,\mathrm{div}_x\Big(\int_0^1 g(x-rt\xi)\,\mathrm{d}r\,\xi\Big)\,\mathrm{d}x\\ &= \int_{\mathbb{R}^n}\Big(\int_0^1 g(x-rt\xi)dr\Big)\langle\sigma_f(x),\xi\rangle\,\mathrm{d}|Df|(x).\end{aligned}$$

Therefore

$$\lim_{t\to 0}\int_{\mathbb{R}^n} g(x)\,\frac{f(x+t\xi)-f(x)}{t}\,\mathrm{d}x = \int_{\mathbb{R}^n} g(x)\langle \sigma_f(x),\xi\rangle\,\mathrm{d}|Df|(x). \tag{16}$$

Let $\varepsilon > 0$. Since $|\mathrm{D}f|$ is a finite Radon measure, the set of smooth functions with compact support is dense in the space of L^1 functions w.r.t. $|\mathrm{D}f|$ (cf. [10, Proposition 7.9]). Hence, there is a smooth function $g:\mathbb{R}^n\to\mathbb{R}$ with compact support such that

$$\int_{\mathbb{R}^n} |g(x)-\operatorname{sgn}(\langle\sigma_f(x),\xi\rangle)|\,\mathrm{d}|\mathrm{D}f|(x) < \varepsilon$$

and $\|g\|_\infty \le 1+\varepsilon$. By (16),

$$\begin{aligned}
(1+\varepsilon)\liminf_{t\to 0}\int_{\mathbb{R}^n}&\Big|\frac{f(x+t\xi)-f(x)}{t}\Big|\,\mathrm{d}x\\
&\ge \lim_{t\to 0}\int_{\mathbb{R}^n} g(x)\frac{f(x+t\xi)-f(x)}{t}\,\mathrm{d}x\\
&= \int_{\mathbb{R}^n} g(x)\langle\sigma_f(x),\xi\rangle\,\mathrm{d}|\mathrm{D}f|(x)\\
&= \int_{\mathbb{R}^n} |\langle\sigma_f(x),\xi\rangle|\,\mathrm{d}|\mathrm{D}f|(x)\\
&\quad + \int_{\mathbb{R}^n} (g(x)-\operatorname{sgn}(\langle\sigma_f(x),\xi\rangle))\langle\sigma_f(x),\xi\rangle\,\mathrm{d}|\mathrm{D}f|(x)\\
&\ge \int_{\mathbb{R}^n} |\langle\sigma_f(x),\xi\rangle|\,\mathrm{d}|\mathrm{D}f|(x) - \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we obtain

$$\liminf_{t\to 0}\Big\|\frac{f(\cdot+t\xi)-f(\cdot)}{t}\Big\|_1 \ge \int_{\mathbb{R}^n} |\langle\sigma_f(x),\xi\rangle|\,\mathrm{d}|\mathrm{D}f|(x).$$

For the opposite inequality, notice that

$$\Big|\int_{\mathbb{R}^n}\Big(\int_0^1 h(x-rt\xi)\,\mathrm{d}r\Big)\langle\sigma_f(x),\xi\rangle\,\mathrm{d}|\mathrm{D}f|(x)\Big| \le \|h\|_\infty \int_{\mathbb{R}^n} |\langle\sigma_f(x),\xi\rangle|\,\mathrm{d}|\mathrm{D}f|(x)$$

for every $h\in L^\infty(\mathbb{R}^n)$. □

Proof of Theorem 3. Define $\varphi:[0,\infty)\to[0,\infty)$ by

$$\varphi(t) = \Big\|\frac{f(\cdot+t\xi)-f(\cdot)}{t}\Big\|_1$$

and note that $\varphi(t)\le \frac{2\|f\|_1}{t}$ for $t>0$. By Lemma 4 and Lemma 5,

$$\lim_{s\to 1^-}(1-s)\int_0^\infty t^{-s}\Big\|\frac{f(\cdot+t\xi)-f(\cdot)}{t}\Big\|_1\,\mathrm{d}t = \int_{\mathbb{R}^n} |\langle\sigma_f(x),\xi\rangle|\,\mathrm{d}|\mathrm{D}f|(x).$$

By Proposition 2, we can use the dominated convergence theorem to obtain,

$$\begin{aligned}\lim_{s\to 1^-} n\,|(1-s)^{-1/s}\,\Pi^{*,s} f| &= \lim_{s\to 1^-}\int_{\mathbb{S}^{n-1}}\Big((1-s)\int_0^\infty t^{-s}\Big\|\frac{f(\cdot+t\xi)-f(\cdot)}{t}\Big\|_1\,\mathrm{d}t\Big)^{-n/s}\mathrm{d}\xi\\ &= \int_{\mathbb{S}^{n-1}}\Big(\int_{\mathbb{R}^n}|\langle\sigma_f(x),\xi\rangle|\,\mathrm{d}|\mathrm{D}f|(x)\Big)^{-n}\mathrm{d}\xi\\ &= n\,|\tfrac12\,\Pi^* f|\end{aligned}$$

and

$$\begin{aligned}\lim_{s\to 1^-} n(1-s)\tilde V_{-s}(K,\Pi^{*,s} f) &= \lim_{s\to 1^-}(1-s)\int_{\mathbb{S}^{n-1}}\|\xi\|_K^{n+s}\|\xi\|^s_{\Pi^{*,s}f}\,\mathrm{d}\xi\\ &= 2\int_{\mathbb{S}^{n-1}}\|\xi\|_K^n\,\|\xi\|_{\Pi^* f}\,\mathrm{d}\xi\\ &= 2n\,\tilde V_{-1}(K,\Pi^* f),\end{aligned}$$

which completes the proof of the theorem. □

5. Anisotropic Fractional Pólya–Szegő Inequalities

Almgren and Lieb [1] established the following Pólya–Szegő inequality for fractional Sobolev norms. The equality case was settled by Frank and Seiringer [11]. Let $0 < s < 1$ and $n \ge 1$. For non-negative $f \in W^{s,1}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f(x)-f(y)|}{|x-y|^{n+s}}\,\mathrm{d}x\,\mathrm{d}y \ge \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f^\star(x)-f^\star(y)|}{|x-y|^{n+s}}\,\mathrm{d}x\,\mathrm{d}y, \tag{17}$$

with equality precisely if $\{f \ge t\}$ is equivalent to a ball for almost every $t > 0$.

The following improvement of (17) is key to proving the main theorems in Section 8. It is a variation of [17, Theorem 3.1].

Theorem 6. *Let $0 < s < 1$ and $K \subset \mathbb{R}^n$ a star body. For $f : \mathbb{R}^n \to \mathbb{R}$ a non-negative integrable function,*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f(x)-f(y)|}{\|x-y\|_K^{n+s}}\,\mathrm{d}x\,\mathrm{d}y \ge \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f^\star(x)-f^\star(y)|}{\|x-y\|_{K^\star}^{n+s}}\,\mathrm{d}x\,\mathrm{d}y.$$

If equality holds for non-zero $f \in W^{s,1}(\mathbb{R}^n)$, then K is a centered ellipsoid, and for almost every $t > 0$, the level set $\{f \ge t\}$ is equivalent to an ellipsoid.

The inequality in Theorem 6 follows rather directly from the Riesz rearrangement inequality, which is stated in full generality, for example, in [4].

Theorem 7 (Riesz's rearrangement inequality)**.** *For $f, g, k : \mathbb{R}^n \to \mathbb{R}$ non-negative measurable functions,*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)k(x-y)g(y)\,\mathrm{d}x\,\mathrm{d}y \le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f^\star(x)k^\star(x-y)g^\star(y)\,\mathrm{d}x\,\mathrm{d}y.$$

To establish the equality case in Theorem 6, we use the characterization of equality cases in a version of the Riesz rearrangement inequality due to Burchard [5].

Theorem 8 (Burchard)**.** *Let A, B and C be sets of finite positive measure in $\mathbb{R}^n$ and denote by α, β and γ the radii of their Schwarz symmetrals $A^\star, B^\star$ and $C^\star$. For $|\alpha-\beta| < \gamma < \alpha+\beta$, there is equality in*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} 1_A(y)\, 1_B(x-y)\, 1_C(x)\,\mathrm{d}x\,\mathrm{d}y \le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} 1_{A^\star}(y)\, 1_{B^\star}(x-y)\, 1_{C^\star}(x)\,\mathrm{d}x\,\mathrm{d}y$$

if and only if, up to sets of measure zero,

$$A = a+\alpha D, B = b+\beta D, C = c+\gamma D,$$

where D is a centered ellipsoid, and a, b and $c = a+b$ are vectors in $\mathbb{R}^n$.

By the co-area formula (12), Theorem 6 is an immediate consequence of the following result for anisotropic fractional perimeters.

Theorem 9. *Let $0<s<1$ and $K \subset \mathbb{R}^n$ a star body. For $E \subset \mathbb{R}^n$ measurable,*

$$\int_E\int_{E^c} \frac{1}{\|x-y\|_K^{n+s}}\,\mathrm{d}x\,\mathrm{d}y \ge \int_{E^\star}\int_{(E^\star)^c} \frac{1}{\|x-y\|_{K^\star}^{n+s}}\,\mathrm{d}x\,\mathrm{d}y. \tag{18}$$

If equality holds for E with finite positive s-perimeter, then K is a centered ellipsoid, and E is equivalent to an ellipsoid.

Proof. Write

$$\|z\|_K^{-n-s} = \int_0^\infty k_t(z)\,\mathrm{d}t, \text{ where } k_t(z) = 1_{t^{-1/(n+s)}K}(z)$$

and use Fubini's theorem to obtain

$$\int_E\int_{E^c} \frac{1}{\|x-y\|_K^{n+s}}\,\mathrm{d}x\,\mathrm{d}y = \int_0^\infty\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} 1_E(x)\, 1_{E^c}(y) k_t(x-y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t.$$

For fixed $t \in (0,\infty)$, we have

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} & 1_E(x)\, 1_{E^c}(y) k_t(x-y)\,\mathrm{d}x\,\mathrm{d}y\\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \big(1_E(x) - 1_E(x)\, 1_E(y)\big) k_t(x-y)\,\mathrm{d}x\,\mathrm{d}y\\ &= t^{-\frac{n}{n+s}}|K||E| - \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} 1_E(x)\, k_t(x-y)\, 1_E(y)\,\mathrm{d}x\,\mathrm{d}y.\end{aligned} \tag{19}$$

Clearly, the first term is invariant under Schwarz symmetrization. For the second term, we can apply Theorem 7 to show that the left side does not increase under Schwarz symmetrization. Thus

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} 1_E(x) k_t(x-y)\, 1_{E^c}(y)\,\mathrm{d}x\,\mathrm{d}y \ge \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} 1_{E^\star}(x)\, k_t^\star(x-y)\, 1_{(E^\star)^c}(y)\,\mathrm{d}x\,\mathrm{d}y$$

for $t>0$ and integrating this inequality for $t \in (0,\infty)$, we obtain (18).

If there is equality in (18), it follows from (19) that for almost every $t > 0$,

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} 1_E(x)\, 1_{t^{-1/(n+s)}K}(x-y)\, 1_E(y) \,\mathrm{d}x \,\mathrm{d}y$$

is invariant under Schwarz symmetrization. Now we may apply Theorem 8 with $A = C = E$ and $B = t^{-1/(n+s)}K$ for suitable $t > 0$. Since $A = C$, we have $a = c$, and this implies $b = 0$. Hence K is a centered ellipsoid, and E is equivalent to an ellipsoid. □

To establish corresponding inequalities for Steiner symmetrization, we require the following result.

Theorem 10 (Rogers [29])**.** *Let $\xi \in \mathbb{S}^{n-1}$. For $f, g, k : \mathbb{R}^n \to \mathbb{R}$ non-negative and measurable,*

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x)k(x-y)g(y) \,\mathrm{d}x \,\mathrm{d}y \le \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f^\xi(x)k^\xi(x-y)g^\xi(y) \,\mathrm{d}x \,\mathrm{d}y.$$

We remark that the above theorem is a special case of the inequalities of Rogers [30] and Brascamp–Lieb–Luttinger [4, Lemma 3.2].

As in the case of Schwarz symmetrization, we obtain the following consequences.

Theorem 11. *Let $0 < s < 1$ and $\xi \in \mathbb{S}^{n-1}$. If $K \subset \mathbb{R}^n$ is a star body, then*

$$\int_E \int_{E^c} \frac{1}{\|x-y\|_K^{n+s}} \,\mathrm{d}x \,\mathrm{d}y \ge \int_{E^\xi} \int_{(E^\xi)^c} \frac{1}{\|x-y\|_{K^\xi}^{n+s}} \,\mathrm{d}x \,\mathrm{d}y$$

for $E \subset \mathbb{R}^n$ measurable.

Theorem 12. *Let $0 < s < 1$ and $\xi \in \mathbb{S}^{n-1}$. If $K \subset \mathbb{R}^n$ is a star body, then*

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x)-f(y)|}{\|x-y\|_K^{n+s}} \,\mathrm{d}x \,\mathrm{d}y \ge \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f^\xi(x)-f^\xi(y)|}{\|x-y\|_{K^\xi}^{n+s}} \,\mathrm{d}x \,\mathrm{d}y$$

for $f : \mathbb{R}^n \to \mathbb{R}$ non-negative and integrable.

6. Affine Fractional Pólya–Szegő Inequalities

We establish the following affine version of the fractional Pólya–Szegő inequality (17) by Almgren and Lieb.

Theorem 13. *For $0 < s < 1$ and non-negative $f \in W^{s,1}(\mathbb{R}^n)$,*

$$|\,\Pi^{*,s} f|^{-s/n} \ge |\,\Pi^{*,s} f^\star|^{-s/n}, \tag{20}$$

with equality precisely if $\{f \ge t\}$ is equivalent to an ellipsoid for almost every $t > 0$.

Proof. By Theorem 6, (14), and (9), we have

$$\begin{aligned} \tilde V_{-s}(K, \Pi^{*,s} f) &\ge \tilde V_{-s}(K^\star, \Pi^{*,s} f^\star) \\ &\ge |K^\star|^{(n+s)/n} |\,\Pi^{*,s} f^\star|^{-s/n} \\ &= |K|^{(n+s)/n} |\,\Pi^{*,s} f^\star|^{-s/n}. \end{aligned}$$

Setting $K = \Pi^{*,s} f$, we obtain that

$$|\,\Pi^{*,s} f| = \tilde V_{-s}(\Pi^{*,s} f, \Pi^{*,s} f) \ge |\,\Pi^{*,s} f|^{(n+s)/n} |\,\Pi^{*,s} f^\star|^{-\frac{s}{n}},$$

which completes the proof of the inequality. The equality case follows from the equality case of the dual mixed volumes inequality and of Theorem 6. □

If we multiply (20) by $(1-s)$ and let $s \to 1^-$, we obtain from Theorem 13 and Theorem 3 the following affine Pólya–Szegő inequality by Cianchi, Lutwak, Yang, and Zhang [6] (see also [27]):

$$|\,\Pi^* f|^{-1/n} \geq |\,\Pi^* f^\star|^{-1/n}$$

for $f \in W^{1,1}(\mathbb{R}^n)$.

Using the same proof as for Theorem 13 for Steiner symmetrization and replacing Theorem 6 with Theorem 12, we obtain the following result.

Theorem 14. *Let $0 < s < 1$ and $\xi \in \mathbb{S}^{n-1}$. Then*

$$|\,\Pi^{*,s} f|^{-s/n} \geq |\,\Pi^{*,s} f^\xi|^{-s/n}$$

for non-negative $f \in W^{s,1}(\mathbb{R}^n)$.

As before, we obtain from Theorem 14 and Theorem 3 the following affine inequality:

$$|\,\Pi^* f|^{-1/n} \geq |\,\Pi^* f^\xi|^{-1/n}$$

for $f \in W^{1,1}(\mathbb{R}^n)$ and $\xi \in \mathbb{S}^{n-1}$.

7. Fractional Petty Projection Inequalities

A set $E \subset \mathbb{R}^n$ of finite measure is a set of finite perimeter if its indicator function 1_E is in $BV(\mathbb{R}^n)$. This allows us to translate results for functions in $BV(\mathbb{R}^n)$ to sets of finite perimeter and finite measure. The following result is an immediate consequence of the affine fractional Pólya–Szegő inequality from Theorem 13.

Theorem 15. *For $0 < s < 1$ and $E \subset \mathbb{R}^n$ of finite s-perimeter and finite measure,*

$$|\,\Pi^{*,s} E|^{-s/n} \geq |\,\Pi^{*,s} E^\star|^{-s/n}$$

with equality precisely if E is equivalent to an ellipsoid.

Here we write $\Pi^{*,s} E$ for the fractional polar projection body of 1_E. Note that

$$\|\xi\|^s_{\Pi^{*,s} E} = \int_0^\infty t^{-s} \frac{|E \triangle (E - t\xi)|}{t} \,\mathrm{d}t$$

for $E \subset \mathbb{R}^n$ of finite s-perimeter and finite measure. Also note that (14) implies that

$$2 \int_E \int_{E^c} \frac{1}{\|x-y\|_K^{n+s}} \,\mathrm{d}x \,\mathrm{d}y = n\, \tilde V_{-s}(K, \Pi^{*,s} E) \tag{21}$$

for $K \subset \mathbb{R}^n$ a star body.

From Theorem 15, we easily obtain the first inequality of the following result, which we call the fractional Petty projection inequality.

Theorem 16. *For $0 < s < 1$ and $E \subset \mathbb{R}^n$ of finite s-perimeter and finite measure,*

$$\Big(\frac{|E|}{|B^n|}\Big)^{(n-s)/n} \leq \Big(\frac{|\,\Pi^{*,s} E|}{|\,\Pi^{*,s} B^n|}\Big)^{-s/n} \leq \frac{P_s(E)}{P_s(B^n)}. \tag{22}$$

There is equality in the first inequality if and only if E is equivalent to an ellipsoid. There is equality in the second inequality if and only if E is equivalent to a set of constant s-fractional brightness.

Here, we say that $E \subset \mathbb{R}^n$ is of constant s-fractional brightness if $\Pi^{*,s} E$ is a dilate of B^n. The second inequality in (22) and its equality case follow directly from (14) and the dual mixed volume inequality (9).

Let $E \subset \mathbb{R}^n$ be a set of finite perimeter and finite measure. The perimeter of E is defined as $P(E) = |\mathrm{D}\, 1_E\,|$. By Dávila's version of (2),

$$\lim_{s\to 1^-} \frac{P_s(E)}{P_s(B^n)} = \frac{P(E)}{P(B^n)}.$$

Taking the limit $s \to 1^-$ in (22), we obtain from this and Theorem 3 that

$$\Big(\frac{|E|}{|B^n|}\Big)^{(n-1)/n} \le \Big(\frac{|\,\Pi^* E|}{|\,\Pi^* B^n|}\Big)^{-1/n} \le \frac{P(E)}{P(B^n)} \tag{23}$$

for $E \subset \mathbb{R}^n$ of finite perimeter and finite measure. The first inequality is the generalized Petty projection inequality, which was proved by Gaoyong Zhang [39] for compact sets with piecewise C^1 boundary and in full generality by Tuo Wang [35]. For E a convex body, it is the classical Petty projection inequality (10). There is equality in the second inequality of (23) if and only if E is of constant brightness.

The Steiner inequality for fractional polar projection bodies is contained in the following result, which is an immediate consequence of Theorem 14.

Theorem 17. *Let $0 < s < 1$ and $\xi \in \mathbb{S}^{n-1}$. Then*

$$|\,\Pi^{*,s} E|^{-s/n} \ge |\,\Pi^{*,s} E^{\xi}|^{-s/n}$$

for $E \subset \mathbb{R}^n$ of finite s-perimeter and finite Lebesgue measure.

Taking the limit as $s \to 1^-$ and using Theorem 3, we obtain from Theorem 17 the following Steiner inequality for polar projection bodies, which was recently established by Youjiang Lin [19],

$$|\,\Pi^* E|^{-1/n} \ge |\,\Pi^* E^{\xi}|^{-1/n}$$

for $E \subset \mathbb{R}^n$ of finite perimeter and finite Lebesgue measure and $\xi \in \mathbb{S}^{n-1}$. For convex bodies, this inequality was established by Lutwak, Yang, and Zhang [23] and for compact sets, by Wang and Xiao [37].

8. Proof of Theorem 1

A version of the following lemma for $s = 1$ is contained in [39].

Lemma 18. *Let $0 < s < 1$. If $g : \mathbb{R}^n \to [0, \infty)$ is measurable, then*

$$\Big(\int_{\mathbb{R}^n} g(x)^{n/(n-s)}\,\mathrm{d}x\Big)^{(n-s)/n} \le \int_0^\infty |\{g \ge t\}|^{(n-s)/n}\,\mathrm{d}t. \tag{24}$$

If the right side is finite, then there is equality precisely if $g = c\, 1_E$ for some $E \subset \mathbb{R}^n$ of finite measure and $c \ge 0$.

Proof. Let the right side of (24) be finite. By Fubini's theorem, we have

$$\int_{\mathbb{R}^n} g(x)^{n/(n-s)}\,\mathrm{d}x = \frac{n}{n-s}\int_0^\infty t^{s/(n-s)} |\{g \ge t\}|\,\mathrm{d}t.$$

Since $r \mapsto |\{g \ge r\}|$ is monotone decreasing, we obtain for $t > 0$,

$$\begin{aligned} t^{s/(n-s)}|\{g \ge t\}| &= \big(t\,|\{g \ge t\}|^{(n-s)/n}\big)^{s/(n-s)}|\{g \ge t\}|^{(n-s)/n} \\ &\le \Big(\int_0^t |\{g \ge r\}|^{(n-s)/n}\,\mathrm{d}r\Big)^{s/(n-s)}|\{g \ge t\}|^{(n-s)/n} \\ &= \frac{n-s}{n}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_0^t |\{g \ge r\}|^{(n-s)/n}\,\mathrm{d}r\Big)^{n/(n-s)}. \end{aligned}$$

Hence

$$\Big(\int_{\mathbb{R}^n} g(x)^{n/(n-s)}\,\mathrm{d}x\Big)^{(n-s)/n} \le \int_0^\infty |\{g \ge t\}|^{(n-s)/n}\,\mathrm{d}t$$

and there is equality precisely if $g = c\,1_E$ for some $E \subset \mathbb{R}^n$ of finite measure and $c \ge 0$. $\square$

Proof of Theorem 1. First, assume that f is non-negative. By the co-area formula (12) and (14),

$$\begin{aligned} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|}{\|x-y\|_K}\,\mathrm{d}x\,\mathrm{d}y &= 2\int_0^\infty P_s(\{f \ge t\}, K)\,\mathrm{d}t \\ &= n\int_0^\infty \tilde{V}_{-s}(K, \Pi^{*,s}\{f \ge t\})\,\mathrm{d}t. \end{aligned}$$

Hence, by the dual mixed volume inequality (9) and (14),

$$\begin{aligned} |\,\Pi^{*,s} f| &= \int_0^\infty \tilde{V}_{-s}(\Pi^{*,s} f, \Pi^{*,s}\{f \ge t\})\,\mathrm{d}t \\ &\ge |\,\Pi^{*,s} f|^{(n+s)/n}\int_0^\infty |\,\Pi^{*,s}\{f \ge t\}|^{-s/n}\,\mathrm{d}t \end{aligned}$$

and

$$|\,\Pi^{*,s} f|^{-s/n} \ge \int_0^\infty |\,\Pi^{*,s}\{f \ge t\}|^{-s/n}\,\mathrm{d}t.$$

The equality cases of inequality (1) and of the dual mixed volume inequality, applied to (21) for $K = B^n$, show that

$$\alpha_{n,s} n\omega_n^{(n+s)/n} = |\,\Pi^{*,s} B^n|^{s/n}|B^n|^{(n-s)/n}.$$

By the fractional Petty projection inequality from Theorem 16, it now follows that

$$\begin{aligned} |\,\Pi^{*,s} f|^{-s/n} &\ge \frac{|\,\Pi^{*,s} B^n|^{-s/n}}{|B^n|^{(n-s)/n}}\int_0^\infty |\{f \ge t\}|^{(n-s)/s}\,\mathrm{d}t \\ &= \frac{1}{\alpha_{n,s} n\omega_n^{(n+s)/n}}\int_0^\infty |\{f \ge t\}|^{(n-s)/s}\,\mathrm{d}t. \end{aligned}$$

In particular, the last term is finite. We apply Lemma 18 to obtain

$$\alpha_{n,s} n\omega_n^{(n+s)/n}|\,\Pi^{*,s} f|^{-s/n} \ge \|f\|_{\frac{n}{n-s}}. \tag{25}$$

Combining the equality cases of the fractional Petty projection inequality and Lemma 18, we obtain the equality case for non-negative f.

For general f and $x, y \in \mathbb{R}^n$, we use $|f(x) - f(y)| \geq ||f(x)| - |f(y)||$, where equality holds if and only if $f(x)$ and $f(y)$ are both non-negative or non-positive. Applying this inequality in the definition of $\Pi^{*,s} f$, we obtain

$$|\,\Pi^{*,s}|f||^{-s/n} \leq |\,\Pi^{*,s} f|^{-s/n}$$

with equality if and only if f has constant sign for almost every $x, y \in \mathbb{R}^n$. Using (25) for $|f|$, we obtain the first inequality of the theorem and its equality case.

For the second inequality, we set $K = B^n$ in (14) and apply the dual mixed volume inequality (9) to obtain

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x) - f(y)|}{|x-y|^{n+s}}\,\mathrm{d}x\,\mathrm{d}y = n\tilde{V}_{-s}(B^n, \Pi^{*,s} f) \geq n\omega_n^{(n+s)/n}|\,\Pi^{*,s} f|^{-s/n}.$$

There is equality precisely if $\Pi^{*,s} f$ is a ball, which is the case for radially symmetric functions. □

9. Fractional Zhang Projection Inequalities and Radial Mean Bodies

Let $E \subset \mathbb{R}^n$ be a convex body. For $p > -1$, Gardner and Zhang [13] defined the radial p-th mean body of E, by its radial function for $\xi \in \mathbb{S}^{n-1}$, as

$$\rho_{\mathrm{R}_p E}(\xi)^p = \frac{1}{|E|}\int_E \rho_{E-x}(\xi)^p\,\mathrm{d}x$$

for $p \neq 0$ and as

$$\log(\rho_{\mathrm{R}_0 E}(\xi)) = \frac{1}{|E|}\int_E \log(\rho_{E-x}(\xi))\,\mathrm{d}x.$$

They showed that $\mathrm{R}_p E$ is a star body for $p > -1$ and a convex body for $p \geq 0$.

Let $0 < s < 1$. For a star body $K \subset \mathbb{R}^n$ and a convex body $E \subset \mathbb{R}^n$, we obtain from Fubini's theorem that

$$\begin{aligned} \int_E\int_{E^c} \frac{1}{\|x-y\|_K^{n+s}}\,\mathrm{d}x\,\mathrm{d}y &= \int_E\int_{(E-x)^c} \frac{1}{\|z\|_K^{n+s}}\,\mathrm{d}z\,\mathrm{d}x \\ &= \frac{1}{s}\int_E\int_{\mathbb{S}^{n-1}} \rho_K(\xi)^{n+s}\rho_{E-x}(\xi)^{-s}\,\mathrm{d}\xi\,\mathrm{d}x \\ &= \frac{|E|}{s}\int_{\mathbb{S}^{n-1}} \rho_K(\xi)^{n+s}\rho_{\mathrm{R}_{-s}E}(\xi)^{-s}\,\mathrm{d}\xi. \end{aligned} \tag{26}$$

Hence, by (21),

$$\tilde{V}_{-s}(K, \Pi^{*,s} E) = \frac{2|E|}{s}\,\tilde{V}_{-s}(K, \mathrm{R}_{-s} E)$$

for every star body K. By the equality case of the dual mixed volume inequality (9), it follows that

$$\Pi^{*,s} E = \Big(\frac{s}{2|E|}\Big)^{\frac{1}{s}}\,\mathrm{R}_{-s} E$$

for every convex body $E \subset \mathbb{R}^n$.

Hence, we can reformulate inequalities obtained by Gardner and Zhang [13] to obtain inequalities for fractional polar projection bodies. In particular, Theorem 5.5 and Lemma 5.7 from [13] imply that

$$\Big(\frac{|\,\Pi^{*,s} E|}{|\,\Pi^{*,s}\Delta|}\Big)^{-s/n} \le \Big(\frac{|E|}{|\Delta|}\Big)^{(n-s)/n}$$

for $E \subset \mathbb{R}^n$ a convex body and $0 < s < 1$ with equality precisely if E is a simplex, where Δ is any n-dimensional simplex in $\mathbb{R}^n$. Letting $s \to 1^-$ and using Theorem 3, we obtain the Zhang projection inequality [38] (without the equality case), which was reproved in [13]:

$$\Big(\frac{|\,\Pi^{*} E|}{|\,\Pi^{*}\Delta|}\Big)^{-1/n} \le \Big(\frac{|E|}{|\Delta|}\Big)^{(n-1)/n}$$

for $E \subset \mathbb{R}^n$ a convex body.

Conversely, we can reformulate Theorem 16 for radial mean bodies and obtain sharp affine isoperimetric inequalities for radial p-th mean bodies for $-1 < p < 0$. Using a variation of our approach to Theorem 16, we obtain the following result, which proves a natural conjecture in the field. We have included the case $p = n$ from [13, Lemma 5.7] in the statement of the result.

Theorem 19. *If $E \subset \mathbb{R}^n$ is a convex body, then*

$$\begin{aligned} \frac{|\,\mathrm{R}_p E|}{|E|} &\le \frac{|\,\mathrm{R}_p B^n|}{|B^n|} \qquad \text{for } -1 < p < n,\\ \frac{|\,\mathrm{R}_p E|}{|E|} &\ge \frac{|\,\mathrm{R}_p B^n|}{|B^n|} \qquad \text{for } p > n, \end{aligned}$$

with equality if and only if E is an ellipsoid. Here,

$$\frac{|\,\mathrm{R}_p B^n|}{|B^n|} = \Big(\frac{2^{p+1}\omega_{n+p}}{(p+1)\omega_n\omega_{p+1}}\Big)^{n/p}$$

for $p > -1$ and $p \ne 0, n$, and

$$\frac{|\,\mathrm{R}_0 B^n|}{|B^n|} = 2^n e^{\frac{n}{2}(\psi(\frac12)-\psi(\frac{n}{2}+1))},$$

where ψ is the digamma function. Moreover,

$$\frac{|\,\mathrm{R}_n E|}{|E|} = 1$$

for every convex body $E \subset \mathbb{R}^n$.

Proof. Let $K \subset \mathbb{R}^n$ be a star body. It follows from Fubini's theorem that

$$\begin{aligned} \int_E\int_E \frac{1}{\|x-y\|_K^{n-p}}\,\mathrm{d}x\,\mathrm{d}y &= \int_E\int_{E-x}\frac{1}{\|z\|_K^{n-p}}\,\mathrm{d}z\,\mathrm{d}x\\ &= \frac{1}{p}\int_E\int_{\mathbb{S}^{n-1}}\rho_K(\xi)^{n-p}\rho_{E-x}(\xi)^p\,\mathrm{d}\xi\,\mathrm{d}x\\ &= \frac{|E|}{p}\int_{\mathbb{S}^{n-1}}\rho_K(\xi)^{n-p}\rho_{\mathrm{R}_pE}(\xi)^p\,\mathrm{d}\xi\\ &= \frac{n|E|}{p}\,\tilde V_p(K,\mathrm{R}_pE). \end{aligned} \tag{27}$$

For $0 < p < n$, the Riesz rearrangement inequality, Theorem 7, implies that

$$\begin{aligned}\int_E \int_E \frac{1}{\|x-y\|_K^{n-p}} \,\mathrm{d}x \,\mathrm{d}y \\ &\le \int_{E^\star} \int_{E^\star} \frac{1}{\|x-y\|_{K^\star}^{n-p}} \,\mathrm{d}x \,\mathrm{d}y \\ &= \Big(\frac{|E|}{|B^n|}\Big)^{(n+p)/n} \Big(\frac{|K|}{|B^n|}\Big)^{(n-p)/n} \int_{B^n} \int_{B^n} \frac{1}{|x-y|^{n-p}} \,\mathrm{d}x \,\mathrm{d}y.\end{aligned}$$

As in the proof of Theorem 9, we obtain from Theorem 8 that there is equality precisely if E is an ellipsoid. Hence, setting $K = \mathrm{R}_p E$, we obtain from (27) that

$$\begin{aligned}\frac{|\,\mathrm{R}_p E|}{|\,\mathrm{R}_p B^n|} &= \frac{p}{n|E|\,|\,\mathrm{R}_p B^n|} \int_E \int_E \frac{1}{\|x-y\|_{\mathrm{R}_p E}^{n-p}} \,\mathrm{d}x \,\mathrm{d}y \\ &\le \Big(\frac{|E|}{|B^n|}\Big)^{p/n} \Big(\frac{|\,\mathrm{R}_p E|}{|\,\mathrm{R}_p B^n|}\Big)^{(n-p)/n}\end{aligned}$$

and the result follows.

For $p > n$, we choose $c > 0$ so large that $c - \|x-y\|_K^{p-n}$ is non-negative on $E \times E$. This is possible since E is compact. We apply the Riesz rearrangement inequality and its equality case and obtain the result.

Next, we calculate the constants for $p > 0$ and $-1 < p < 0$. For $p > 0$, we obtain by (27) that

$$\Big(\frac{|\,\mathrm{R}_p B^n|}{|B^n|}\Big)^{1/n} = \Big(\frac{p}{n\omega_n^2} \int_{B^n} \int_{B^n} \frac{1}{|x-y|^{n-p}} \,\mathrm{d}x \,\mathrm{d}y\Big)^{1/p}$$

and by [33, Theorem 7.2.7 and Theorem 8.6.6] that

$$\int_{B^n} \int_{B^n} \frac{1}{|x-y|^{n-p}} \,\mathrm{d}x \,\mathrm{d}y = \frac{2^{p+1} n\omega_n \omega_{n+p}}{p(p+1)\omega_{p+1}}.$$

For $-1 < p < 0$, we obtain by (26) that

$$\Big(\frac{|\,\mathrm{R}_p B^n|}{|B^n|}\Big)^{1/n} = \Big(\frac{-p}{n\omega_n^2} \int_{B^n} \int_{(B^n)^c} \frac{1}{|x-y|^{n-p}} \,\mathrm{d}x \,\mathrm{d}y\Big)^{1/p}$$

and obtain the result from (11).

By (27), we have

$$\Big(\frac{|\,\mathrm{R}_p B^n|}{|B^n|}\Big)^{1/n} = \Big(\frac{p}{n\omega_n^2} \int_{B^n} \int_{B^n} \frac{1}{|x-y|^{n-p}} \,\mathrm{d}x \,\mathrm{d}y\Big)^{1/p}$$

for $p > 0$. By [33, Theorem 7.2.7 and Theorem 8.6.6],

$$\int_{B^n} \int_{B^n} \frac{1}{|x-y|^{n-p}} \,\mathrm{d}x \,\mathrm{d}y = \frac{2^{p+1} n\omega_n \omega_{n+p}}{p(p+1)\omega_{p+1}},$$

where $\omega_q = \pi^{q/2}/\Gamma(q/2+1)$ for $q > 0$. We obtain

$$\Big(\frac{|\,\mathrm{R}_0 B^n|}{|B^n|}\Big)^{1/n} = \lim_{p\to 0^+} \Big(\frac{2^{p+1}\omega_{n+p}}{(p+1)\omega_n \omega_{p+1}}\Big)^{1/p} = 2\, e^{\frac12 (\psi(\frac12) - \psi(\frac n2 + 1))}.$$

The case $p = n$ follows from the polar coordinate formula of volume combined with the definition of the radial n-th mean body. □

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