ADDITIONS/ALTERNATIVES TO KUNEN

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1. Review of logic in the setting of set theory

You should already know the contents of this section from an introductory course to logic.

1.A. **Coding the natural numbers.** The language of set theory is $\{\in\}$. Assume for simplicity that we add to our language: the constant symbol $\emptyset$ for the empty set, the successor function S, the union symbol $\cup$ and the pairing function symbol $\{\cdot,\cdot\}$ (and that we add to ZF$^-$ the according defining axioms for these functions; e.g., $\forall x\, x \notin \emptyset$ is now an axiom and not the "defining property" of $\emptyset$). As described in Kunen Ch. I, §13, this does not change anything, since these functions are definable anyway.

Then we can **code** each natural number n as a term $\ulcorner n \urcorner$, defined by induction: $\ulcorner 0 \urcorner := \emptyset$, $\ulcorner 1 \urcorner := S(\emptyset)$, $\ulcorner 1 \urcorner := S(S(\emptyset))$ etc. We do not claim that $\ulcorner n \urcorner$ is "really" the "true" form of n, or that the coding is in any way natural. The coding is just a reasonable way to talk about natural numbers in pure set theory without the need to add any objects that are not hereditery sets (and would violate the axiom of extensionality).

Recall that we define (in ZF$^-$) ω to be the first limit ordinal (bigger than 0).

Of course the "intention" is that ω should be the set $\{\ulcorner 0 \urcorner, \ulcorner 1 \urcorner, \dots\}$, but it is important to note that $\omega = \{\ulcorner 0 \urcorner, \ulcorner 1 \urcorner, \dots\}$ is **not** the definition of ω. This "definition" would be an infinite formula, i.e., not a valid first order formula. And we know that we cannot express $\{\ulcorner 0 \urcorner, \ulcorner 1 \urcorner, \dots\}$ in first order at all: Assume (as always) that ZF$^-$ is consistent. Then we get:

Lemma 1.1. *There is a model $\mathcal{M} = (M, E)$ of ZF$^-$ and a $c \in M$ such that $\mathcal{M} \models c \in \omega$ but $\mathcal{M} \models c \neq \ulcorner n \urcorner$ for all natural numbers n.*

Proof. This is just the usual construction of a nonstandard model: First extend the language by a new constant symbol c, and extend ZF$^-$ to a theory T by adding the sentences $c \in \omega$, $c > 0$, $c > 1$, etc. By the compactness theorem, T is consistent, i.e., has a model $\mathcal{M}$. $\square$

1.B. **Coding first order formulas.** Completely analogously to natural numbers, we can translate (code) formulas into terms in the language of set theory.

Fix (in the meta language) a first-order signature L. For simplicity, assume that L is finite,[1] as it is the case in set theory, where $L = \{\in\}$. (Also in many other interesting theories the signature is finite, e.g., in Peano arithmetic, where L is $\{0, 1, +, \cdot, <\}$).

[1] Of course it would not make any difference if we used a recursive infinite signature.

We start with coding the alphabet (i.e., the set of symbols), by "arbitrarily" assigning a natural number n (more exactly the corresponding term $\ulcorner n \urcorner$) to each symbol. For example, we could do the following:

- Assign terms of the form $\ulcorner 2n+1 \urcorner$ to the non-variable symbols: $\ulcorner \wedge \urcorner := \ulcorner 1 \urcorner$, $\ulcorner \neg \urcorner := \ulcorner 3 \urcorner$, $\ulcorner \exists \urcorner := \ulcorner 5 \urcorner$, and so on for $(,), \in, =$ which are mapped to $\ulcorner 7 \urcorner, \dots, \ulcorner 13 \urcorner$ (if the language L has other nonlogical symbols, we add them as well).
- Assign $\ulcorner 2n \urcorner$ to the variable symbol v_n, i.e., $\ulcorner v_3 \urcorner := \ulcorner 6 \urcorner$.

From here on, it is easy (and even "natural") to define all the other notions of mathematical logic inside ZF$^-$. (Remark: This is just an instance of the following general claim: All ("normal") mathematical concepts and proofs can be carried out naturally in ZF$^-$, somtimes additionally AC is needed.)

- We can define (in ZF) "x is a symbol", by the sentence
$$[(\exists n \in \omega) x = 2 \cdot n] \ \vee x = \ulcorner 1 \urcorner \vee x = \ulcorner 3 \urcorner \vee \cdots \vee x = \ulcorner 13 \urcorner$$
- We can define "x is a string", which just means $x \in \omega^{<\omega}$ and $x(l)$ is a symbol for each $l \in \mathrm{dom}(x)$.
- We can define "x is a formula", which is a rather long and tedious but entirely obvious definition, e.g., by induction on the length of the string x: $x \in \omega^{n+1}$ is a formula if either there is a formula $y \in \omega^n$ (this is already defined by induction) such that $x(0) = \ulcorner \neg \urcorner$ and $x(l+1) = y(l)$ for all $0 \le l < n$, or ... (add the other ways to build formulas here, including the atomic formulas).
- We can define all the other syntactical properties, such as "v is a variable symbol occuring freely in the formula x", "the formula x is the result of the conjunction of y and z" (or, informally, $x = y \wedge z$), etc etc.

Of course we can (in the object language) define all these notions for arbitary signatures (not neccessarily recursive or countable ones). However, if we start with a finite signature in the meta language and proceed as above, then we get a coding of all formulas ϕ (in the meta language) into terms $\ulcorner \phi \urcorner$ of the object language;[2] and (since all the simple syntactical properties are recursive) we get, e.g.: x is a free variable of ϕ iff ZF$^-$ proves "$\ulcorner x \urcorner$ is a free variable of $\ulcorner \phi \urcorner$", and the same holds for the other simple (recursive) syntactical properties of formulas.

But note that (just as in the case of natural numbers), the object language set of terms is not (and can not be) defined as $\{\ulcorner \phi_0 \urcorner, \ulcorner \phi_1 \urcorner, \dots\}$, where $\phi_0, \dots$ enumerates the (meta language) terms: If a ZF$^-$ model $\mathcal{M}$ has a nonstandard natural number c, then there will also be the nonstandard variable symbol x_c (more exactly: the code of this symbol, which we defined as $2 \cdot c$) and therefore the nonstandard formula $x_c = x_c$. This particular formula has "really finite" length, but there will of course also be a nonstandard term of nonstandard length c, e.g., the term corresponding to $x_0 = x_0 \wedge x_0 = x_0 \wedge \cdots \wedge x_0 = x_0$ (c many times).

1.C. **Representing recursive and recursively enumerable sets.** Actually, the previous claim that certain properties are expressible in ZF$^-$ are special cases of the following important fact:

[2] recall that we added the function symbols $\{\cdot, \cdot\}$ and $\cup$; so we can write elements of $\omega^{<}\omega$, i.e., sets of pairs of natural numbers, as terms.

Definition 1.2.
- *A is strongly definable in ZF, if there is a formula φ such that*
  - $n \in A$ *iff* ZF$\vdash \varphi(\ulcorner n \urcorner)$.
  - $n \in \mathbb{N} \setminus A$ *iff* ZF$\vdash \neg\varphi(\ulcorner n \urcorner)$.
- *A is weakly definable in ZF, if there is a formula φ such that*
  - $n \in A$ *iff* ZF$\vdash \varphi(\ulcorner n \urcorner)$.

Lemma 1.3. *$A \subseteq \mathbb{N}$ is recursive, iff A is strongly definable in ZF. $A \subseteq \mathbb{N}$ is recursively enumerable, iff A is weakly definable in ZF.*

Proof. The proof is rather straightforward, just describe in ZF what the computer is doing. □

(Of course we do not need ZF$^-$ here, we get exactly the same result for much weaker theories such as PA or even finite subsets of PA.)

So in particular, whenever T is (in the meta language) a recursive set of sentences, then we can strongly represent (i.e., define) T in ZF$^-$. (The usual disclaimer: As usual it cannot be guaranteed by ZF$^-$ that the represented T "is really the same" as the set T in the meta language, as the ZF$^-$-model might contain nonstandard natural numbers and thus nonstandard elemets of T.)

1.D. **Provability.** Once we have defined (in the object language) what formuals are, we can do the following (in ZF$^-$):

- We can define "x is a logical axiom" (again, a simple but tedious case destinction, or just use Lemma 1.3).
- We can define "s is a proof using the axioms T". (I.e., T is a set of formulas, s is a finite sequence of formulas, each one being a logical axiom or an element of T or follows from previous ones by modus ponens).
- So we can define the formula "$T \vdash x$" with two free variables T and x that expresses "the formula x is provable in T".

The set ZF$^-$ of formulas clearly is recursive (and so is, of course, ZF and ZFC). So according to Lemma 1.3, we can also define (more exactly: strongly represent) any of these theories T in ZF$^-$. (As usual: Generally a ZF$^-$-model will contain nonstandard T-axioms.)

In particualar, for a recursive T we can formulate $T \vdash \phi$ in ZF$^-$. However, note that this time we only have an r.e. property, not a recursive one, i.e., we only get weak representation: $T \vdash \phi$ holds ("really", i.e., in the meta language) iff ZF$^-$ proves $T \vdash \phi$. But $T \nvdash \phi$ does generally not imply that ZF$^-$ proves $T \nvdash \phi$. An important instance of this fact is the incompleteness theorem. (A direct argument for $T = \emptyset$ is the following: It is well known that $\{\phi :\ \emptyset \vdash \phi\}$ is r.e. but not recursive, i.e., the set $\{\phi :\ \not\emptyset \vdash \phi\}$ is not r.e. But the set of ϕ such that ZF$^-$ proves "$\emptyset \nvdash \ulcorner\phi\urcorner$" is easily seen to be r.e.)

1.E. **Defining semantics: satisfaction predicates.**

- We can define "$\mathcal{M}$ is an L-structure". In the case of $L = \{\in\}$ this just says: $\mathcal{M}$ is a pair $\langle M, E\rangle$, M is nonempty, and $E \subseteq M^2$. (If we have other non-logical function or relation symbols we have to modify this accordingly.)
- We can define "$\mathcal{M} \models \varphi(\bar{m})$" with free variables $\mathcal{M}$, φ and $\bar{m}$.
- So we can define the formula "$T \vDash \varphi$" with free variables T and φ that expresses that T semantically implies φ.

We will be most interested in the case that $L = \{\in\}$ and that the interpretation E of the relation symbol $\in$ is always as the "real" $\in$-relation restricted to the model M. So we can identify the $\{\in\}$-structure (called model) with its "universe" (ground set). So more formally the previous claims say:

Lemma 1.4. *There is a sentence $\varphi_{SAT}(A, \phi, \bar{p})$ with free variables $A, \phi, \bar{p}$ which expresses (when A is a nonempty set, ϕ a formula of the language $\{\in\}$ with free variables among $\{x_i : i < n\}$ for some $n \in \omega$ and $\bar{p}$ is in A^n) that $(A, \in) \models \phi$ under the assignment that maps x_i to $\bar{p}(i)$. We usually just write $A \models \phi(\bar{p})$ instead of $\varphi_{SAT}(A, \phi, \bar{p})$.*

It will be important later that the formula φ_{SAT} is quite simple and therefore absolute for transitive models (this is shown in Lemma 3.5).

1.F. **Completeness, compactness, Skolem Löwenheim.** We can now formulate (and prove) in ZF^- the completeness theorem:

Lemma 1.5. $(\forall T, \phi)$ $\quad T \vDash \varphi$ *iff* $T \vdash \varphi$.

This implies the compactness theorem in the usual way. We do not need AC, since we assume that the signature is finite (wellordered would be enough). We can of course also prove the other model theoretic sentences such as Skolem Löwenheim (but we might need AC for that).

1.G. **Undefinability of truth, incompleteness theorem.** The following is the "Fixed Point Lemma" (14.2 in Kunen):

Lemma 1.6. *Let $\phi(x)$ be any formula with one free variable x. Then there is a sentence ψ such that*

$$ZF \vdash \psi \leftrightarrow \phi(\ulcorner\psi\urcorner)$$

One important consequence of the fixed point lemma is Gödel's incompleteness theorem (Kunen 14.3):

Lemma 1.7. *Let T be any recursive, consistent set of formulas containing ZF. Then in T we (can formulate but cannot prove) Con(T).*

Another important consequence is that the truth of a formula cannot be descibed: We have seen that there is a ZF formula $M \models x$ (with two free variables M, x) that expresses that x is true in (the set) M; but there is no ZF formula $\mathbf{V} \models x$ (with one free variable x) that expresses that x is simply true, i.e., holds in the proper class $\mathbf{V}$ of all sets. (If $\mathrm{SAT}_{\mathbf{V}}$ were such a formula, then pick a ψ such that ZF proves $\psi \leftrightarrow \neg\mathrm{SAT}_{\mathbf{V}}(\ulcorner\psi\urcorner)$ holds, which obviously contradicts that $\mathrm{SAT}_{\mathbf{V}}$ is a truth predicate.) We summarize that as:

Lemma 1.8. *For a class $\mathbf{M}$ there is generally no truth predicate. In particular this is the case for $\mathbf{M} = \mathbf{V}$.*

So the important point to remember is: For *sets* M we *can* formulate $M \models x$ inside of ZF; but for proper classes $\mathbf{M}$ we *can not* formulate $\mathbf{M} \models x$ (with x a free parameter).

2. More logic in set theory

In this section, we give a few additions to Kunen I–IV.

2.A. **Relativisation.** So we have seen that we gernerally can not define a truth predicate for a class. What we will do instead is to relativise a formula φ to a class $\mathbf{M}$, by replacing all $\forall v$ by $\forall v \in \mathbf{M}$ (and the same for $\exists$), resulting in a new formula called $\varphi^{\mathbf{M}}$. (In the case of the universal class $\mathbf{V}$, $\varphi^{\mathbf{V}}$ is of course logically equivalent to φ.) Note that this is a process in the meta language: For a specific formula φ given in the meta language, we can construct the formula $\varphi^{\mathbf{M}}$.

If φ is a "standard" formula (from the meta language) and M is a set, then relativisation and the satisfaction predicate can both be used and actually are equivalent:

Lemma 2.1. *(Kunen IV 10.1.) For any $\varphi(x_1,\dots,x_n)$, ZF$^-$ proves the following*

$$(\forall M)(\forall p_1,\dots,p_n \in M)\ \varphi_{SAT}(M,\in,\ulcorner\varphi\urcorner,<p_1,\dots,p_n>)\ \leftrightarrow\ \varphi^M(p_1,\dots,p_n)$$

Note that we can apply relativisation only to one (or finitely many) formulas: we cannot create a single formula that expresses $\mathbf{M}\models T$ for an infinite (recursive) set T of formulas.

We will use the notation $\mathbf{M}\models\phi$ to denote $\phi^{\mathbf{M}}$; and despite the fact that we cannot formulate $\mathbf{M}\models T$ we will nevertheless use the notation $\mathbf{M}\models T$ (but should add "with respect to ZF"): By this, we mean: For all $\varphi\in T$ (in the meta language), ZF$\vdash\varphi^{\mathbf{M}}$. (We might use ZF$^-$ or ZFC instead of ZF.)

So in particular, note that it is a completely trivial statement that $\mathbf{V}\models$ZF (or ZFC etc), but we can not formulate in ZF "for all ZF-axioms x, $\mathbf{V}\models x$" (since there is no truth predicate). We *can* of course formulate in ZF "there is a (set) M which is a model of ZF", but we cannot prove it (by the incompleteness theorem).

The central observation for using relativisation in consistency proofs is Kunen IV 2.3, which we give in a special form:

Lemma 2.2. *Let T extend ZF and assume that $\mathbf{M}\models T$ (with respect to ZF, $\mathbf{M}$ nonempty). Then Con(ZF) implies Con(T).*

(This is proved "syntactically" in IV §8; but since we do believe in infinite methods in the meta theory we can give a more natural proof given in IV §9).

As an example, we will later show that $\mathbf{L}\models$ZFC+GCH; so this shows that Con(ZF) implies Con(ZFC+GCH).

Remark: it is possible to define satisfaction predicates for classes for a restricted family of formulas, see section 3.C.

2.B. **Existence of models vs. existance of wellfounded models.** Let T be (in the meta language) a recursive, consistent superset of ZF$^-$. The completeness theorem (which we can prove in ZF$^-$) says:

Con(T) iff $(\exists M,E)\ (M,E)\models T$.

Note however that Con(T) this is *NOT* equivalent to $\exists M\, M\models T$ (where we use the $\in$-relation). Rather, by Mostowski collapsing theorem (since T contains the axiom of extensionality) the following is proved in ZF to be equivalent:

- $(\exists M)\, M\models T$
- $(\exists M$ transitive$)\, M\models T$
- $(\exists M,E)\ (M,E)\models T$ and E is wellfounded

(Remark: Note that T can include the foundation axiom, i.e. the statement that $\in$ is wellfounded, without the T-model (M,E) actually having a wellfounded E. This can be seen by construcing a nonstandard model of T.)

And ZF proves:

Lemma 2.3. *If there is an M such that $M \models$ZF, then Con(ZF+Con(ZF)).*

Proof. If $M \models$ZF, then clearly Con(ZF) holds. But Con(ZF) is an absolute statement for transitive models (since it only quantifies over natural numbers, i.e., it is Δ_0 when ω is considered a constant). So $M \models$Con(ZF); and therefore ZF+Con(ZF) is consistent. □

Remark: In the same way $M \models$ZF implies also Con(ZF+Con(ZF+Con(ZF))), Con(ZF+Con(ZF+Con(ZF+Con(ZF)))), etc.

Lemma 2.4. *ZF does not prove $Con(ZF) \rightarrow (\exists M)\, M \models ZF$.*

Proof. Otherwise, ZF+Con(ZF) would prove Con(ZF+Con(ZF)), contradicting the incompleteness theorem. □

Remark: In the proof of Lemma 2.3 we used absoluteness: If ϕ is simple enough, then ZF proves: ϕ is absolute for transitive models. In other words: If $\mathbf{M}$ is a class (can be a set), and ϕ simple enough, then ZF$\vdash \phi \leftrightarrow \phi^{\mathbf{M}}$. In particular, if N is a model of ZF (we can also write $\mathbf{V}$ instead of N) and if $\mathbf{M}$ is a transitive class, then $\omega^{\mathbf{M}} = \omega^{\mathbf{V}}$, i.e., $\mathbf{M}$ and N contain exactly the same antural numbers. In particular, if N does not contain nonstandard natural numbers, then $\mathbf{M}$ cannot contain nonstandard numbers either (but if N contains nonstandard nonstandard natural numbers, then $\mathbf{M}$ has to contain the same nonstandard numbers as well). In any case, all sentences of elementary number theory are absolute between N and $\mathbf{M}$. In particular, Con(T) for any recursive T is absolute.

Remark: This remark also shows that the usual set theoretic methods to prove independence (inner models such as $\mathbf{L}$, and forcing) can never be used to prove the independence of elementary number theoretic statements (since they use transitive models). In particular, if one proves a number theoretic statement using the additional axiom $\mathbf{V} = \mathbf{L}$, then one can prove the same statement just in ZF. (In particular, the use of the axiom of choice can always be eliminated in all proofs for elementary number theoretic statements.)

2.C. **Reflection, elementary submodels.** Using the satisfaction relation, we can define "ϕ is absolutene between M, N" (for sets $M \subseteq N$) as a single formula with free variables ϕ, M, N. More formally, we call ϕ absolute between M,N if $(\forall \bar p \in M^n) M \vDash \varphi(\bar p)$ iff $N \vDash \varphi(\bar p)$, where n is minimal such that the free variables of φ are among $\{x_i : i < n\}$.

Definition 2.5. *$M \preceq N$ (M is an elementary submodel of N), if all formulas are absolute.*

Analogously to Kunen IV 7.3, one can proves (in ZF):

Lemma 2.6. *(Tarski Vaught criterion:) Assume that $M \subseteq N$. Then Then the following are equivalent:*
(a) $M \preceq N$
(b) If ϕ is of the form $\exists x \psi(x, y_1, \dots, y_m)$, then

$$(\forall \bar p \in M^m)\big[(\exists a \in N) N \models \psi(a, \bar p) \rightarrow (\exists a \in M) N \models \psi(a, \bar p)\big].$$

Kunen IV 7.8 can easily be modified to sets:

Lemma 2.7. *Let N be a (well-orderable) set and $X \subseteq N$. Then there is an elementary submodel M of N such that $|M| \leq \max(\aleph_0, |X|)$.*

Of course, ZF$^-$ proves the following: If M and M' are isomorphic, then M and M' satisfy exactly the same sentences (without free variables). This is a "set version" of 7.9.

So in particular, 7.10 can be formulated for sets the following way:

Lemma 2.8. *Let N be a (wellorderable) set and $X \subseteq N$ transitive. Then there is a transitive M satisfying the same setences as N and such that $X \subseteq M$ and $|M| \leq \max(\aleph_0, |X|)$.*

(We could also allow formulas with parameters in X, since these parameters are not moved by the Mostowsky collapse.)

2.D. **Further reading (Not required for this course.)** We already know (from IV 6.9) that we cannot prove in ZFC that there is a (strongly) inaccessible cardinal. Using the incompleteness theorem, even something much stronger can be shown: ZFC plus the existence of an inaccessible has higher consistency strenght than just ZFC. This is also proved in Kunen p.145.

Read the "curious example" on Kunen p.146.

3. Absoluteness of satisfaction, the constructible universe

In this section, we give some additions/alternatives to Kunen V–VI.

3.A. **The satisfaction predicate is absolute.** Recall the definition of "ϕ is absolute between $\mathbf{M}$ and $\mathbf{N}$" and of "ϕ is absolute for transitive models". In the following, if we just say "ϕ is absolute", we always mean absolute for transitive models. More formally:

Definition 3.1. *In the following, "$\phi(\bar{x})$ is absolute with respect to S" means: $S \subset ZF^-$ is a finite set of sentences, and if $\mathbf{M}$ is a class (possibly using parameters $\bar{p}$), then ZF proves:*
For all $\bar{p}$, if $\mathbf{M}$ is transitive, nonempty and satisfies S, then $(\forall \bar{x} \in \mathbf{M})\ \phi \leftrightarrow \phi^{\mathbf{M}}$.

Usually the theory S involved is "completely harmless", e.g., a finite subset of ZF$^-$\Powerset. And usually it is not neccessary to keep track of such harmless S, so we will just omit the "with respect to S".

We already know that Δ_0-formulas are absolute (Kunen IV 3.6).

Definition 3.2. *Let $\mathbf{M} \subseteq \mathbf{N}$ be nonempty classes (possibly sets). $\psi(\bar{x})$ is upwards absolute between $\mathbf{M}$ and $\mathbf{N}$, if $(\forall \bar{x} \in \mathbf{M})\ \psi^M \to \psi^N$. Similarly, ψ is downwards absolute, if $(\forall \bar{x} \in \mathbf{M})\ \psi^M \leftarrow \psi^N$.*

So by definition, ψ is absolute between $\mathbf{M}$ and $\mathbf{N}$ iff it is upwards and downwards absolute.

Lemma 3.3. *If $\mathbf{M} \subseteq \mathbf{N}$ and $\phi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ is absolute, then $\exists x_1, \ldots, x_n \phi$ is upwards absolute and $\forall x_1, \ldots, x_n \phi$ is downwards absolute.*

Proof. Straight. □

Lemma 3.4. *Let* $\mathbf{M} \subseteq \mathbf{N}$ *both satisfy some finite* $S \subset ZF$. *Let* $\zeta(\bar{y})$ *be any formula, and assume that there are absolute formulas* $\phi(\bar{x}, \bar{y})$ *and* $\psi(\bar{x}, \bar{y})$ *such that*

$$S \vdash \forall \bar{y} \ (\ \zeta \leftrightarrow \exists \bar{x} \phi \leftrightarrow \forall \bar{x} \psi \ ).$$

Then ζ *is absolute between* $\mathbf{M}$ *and* $\mathbf{N}$.

(This is in some way similar to Kunen IV 3.7. and 3.10)

Proof. $\zeta^M \underset{M \models S}{\leftrightarrow} (\exists \bar{x} \phi)^M \underset{\text{upw.abs.}}{\rightarrow} (\exists \bar{x} \phi)^N \underset{N \models S}{\leftrightarrow} \zeta^N$.
And $\zeta^N \underset{N \models S}{\leftrightarrow} (\forall \bar{x} \psi)^N \underset{\text{downw.abs.}}{\rightarrow} (\forall \bar{x} \psi)^M \underset{M \models S}{\leftrightarrow} \zeta^M$. □

Lemma 3.5. *The satisfaction relation is absolute.*

Proof. To see whether $A \models \varphi(\bar{p})$, we have to inductively calculate the truth value of $A \models \psi(\bar{a})$ for all formulas ψ and all possible parameters $\bar{a}$ (of course we do only need subformulas of φ, but that does not make much difference). Let us set Fml to be the set of forumals and $Z = \mathrm{Fml} \times A^{<\omega}$. We say that f is a truth function, if the following is satisfied:

- f is a function from Z to $\{\text{true}, \text{false}\}$.
- If ϕ is a formula of the form $x_n = x_m$ (for variables x_n, x_m) and if $p \in M^k$ for some $k > \max(n, m)$ then $f(\phi, p) = \text{true}$ iff $p(n) = p(m)$.
- Analogously for $\in$ instead of $=$.
- If ϕ is a formula of the form $\psi_1 \wedge \psi_2$, then $f(\phi, p) = \text{true}$ iff $f(\psi_1, p) = \text{true}$ and $f(\psi_2, p) = \text{true}$.
- Similarly for $\vee$, $\neg$, $\rightarrow$.
- If ϕ is a formula of the form $\exists x_n \psi$, then $f(\phi, \bar{p}) = \text{true}$ iff there is a k bigger than n and than all indices of free variables in ϕ and a $p' \in P^k$ such that for $\bar{p}(l) = \bar{p}'(l)$ for all $l \neq n$ in the domain of p such that $f(\psi, \bar{p}') = \text{true}$.
- Analogously for $\forall$.

So $A \models \varphi(\bar{p})$ iff there is some truth function f with $f(\varphi, \bar{p}) = \text{true}$, and equivalently iff for all truth functions f we have $f(\varphi, \bar{p}) = \text{true}$.

Note that the following are absolute for transitive models:

- $\varphi_1(\omega)$ saying "ω is the set of natural numbers". (Kunen IV 5.1)
- $\varphi_2(\mathrm{Fml})$ saying "Fml is the set of formulas".
- $\varphi_3(Z, \mathrm{Fml}, A)$ saying "$Z = \mathrm{Fml} \times A^{<\omega}$". (Kunen IV 3.10, 3.11, 5.3).
- $\zeta(f, A)$ saying "f is a truth function". Note that this formula is even Δ_0 when we consider Fml, ω and Z as constants, since in the definition of truth function we only quantify over formulas, parameters, and natural numbers.
- $\psi(f, \phi, \bar{p})$ which says "$f(\phi, \bar{p}) = \text{true}$. (This is obviously even Δ_0.)

So $A \models \varphi(\bar{p})$ is equivalent to

$$(\exists f) \ \zeta(f, A) \wedge \psi(f, \phi, \bar{p})$$

as well as to

$$(\forall f) \ \zeta(f, A) \rightarrow \psi(f, \phi, \bar{p})$$

Since both $\zeta(f, A) \wedge \psi(f, \phi, \bar{p})$ as $\zeta(f, A) \rightarrow \psi(f, \phi, \bar{p})$ are absolute for transitive models; $A \models \varphi(\bar{p})$ is absolute for transitive models as well, according to Lemma 3.4. □

3.B. Δ_1 **properties (Not needed for this course.)** We can define a more restricted class of absolute formulas, called Δ_1 formulas, and show:

- Δ_1 formulas are absolute for transitive models,
- all the formulas that we proved to be absolute for transitive models are in fact Δ_1.

Definition 3.6.
- *A formula ψ is Π_1 if it has the form $\forall x_1 \dots \forall x_n \phi$ where ϕ is Δ_0.*
- *ψ is Σ_1 if it has the form $\exists x_1 \dots \exists x_n \phi$ where ϕ is Δ_0.*
- *A formula ζ is called Δ_1 with respect ot a basic theory S, if there are a Π_1 formula ψ and a Σ_1 formula ϕ such that*

$$S \vdash \forall \bar{x}\ (\psi \leftrightarrow \phi \leftrightarrow \zeta).$$

The following is an immediate consequence of Lemmas 3.17 and 3.4:

Lemma 3.7. *Assume that $\mathbf{M} \subseteq \mathbf{N}$ are transitive (and nonempty). Then every Σ_1 formula is upwards absolute and every Π_1 formula is downwards absolute. If $\mathbf{M}$ and $\mathbf{N}$ both satisfy S, then every Δ_1 formula (with respect to S) is absolute.*

The following should be clear (using prenex normal form):

Lemma 3.8. *If φ_1, φ_2 are Σ_1 formulas, then $\varphi_1 \wedge \varphi_2$, $\varphi_1 \vee \varphi_2$ $\exists x \varphi_1$, $(\forall t \in x)\varphi_1$ are logically equivalent to a Σ_1 formula (and $\neg\varphi_1$ is logically equivalent to a Π_1 formula).*

Similarly, if φ_1, φ_2 are Π_1 formulas, then $\varphi_1 \wedge \varphi_2$, $\varphi_1 \vee \varphi_2$ $\forall x \varphi_1$, $(\exists t \in x)\varphi_1$ are logically equivalent to a Π_1 formula (and $\neg\varphi_1$ is logically equivalent to a Σ_1 formual).

The properties listet in Kunen IV 3.9, 3.11 and 5.1 are Δ_0. One can show that the other properties that are shown to be absolute for transitive models in Kunen IV 3.14, 5.3, 5.4, 5.5, 5.7 are in fact Δ_1. In particular:

Lemma 3.9. *The following are Δ_1:*

- *$z = A^{<\omega}$*
- *$A \models \phi(\bar{a})$*
- *R is a wellorder on A*
- *$\alpha + \beta$, α^β (ordinal)*

(Later, one can show: The function $\alpha \mapsto L(\alpha)$ is Δ_1.)

Note that "α is a cardinal", "α is regular" and "α is a limit cardinal" are Π_1 statements (but not Δ_1, since they are not absolute, which follows, e.g., from Kunen p. 141).

As with Δ_0, we can define Δ_1 for functions:

Definition 3.10. *A functions $\mathbf{F}$ is called Σ_1 if "$\mathbf{F}(\bar{x}) = y$" is Σ_1. Similarly for Π_1 and for Δ_1.*

It turns out that Σ_1, Π_1 and Δ_1 are the same for functions:

Lemma 3.11. *Assume that $\mathbf{F}$ is a Σ_1 or a Π_1 function. More exactly, assume that some S proves that the according Σ_1 (or Π_1) formula defines a function (on the ordinals, say). Then $\mathbf{F}$ is actually Δ_1 (with respect to S).*

Proof. By assumption, $z = \mathbf{F}(x)$ is expressed by a Σ_1 formula $\phi(x,z)$. We have to show that there is a Π_1 formula $\psi(x,z)$ which is equivalent to ϕ (modulo S). But $z = \mathbf{F}(x)$ iff

$$\forall t\ (\ t = z \lor\ t \neq \mathbf{F}(x))$$

This is Π_1 (since $t \neq \mathbf{F}(x)$ is Π_1). $\square$

Lemma 3.12. *The composition of Δ_1 functions is Δ_1.*

Proof. $\mathbf{F}(\mathbf{G}_1(\bar{x}), \dots, \mathbf{G}_n(\bar{x})) = z$ can be written as:

$$\forall y_1, \dots, y_n\ [\ (y_1 = \mathbf{G}_1(\bar{x}) \land \dots \land y_n = \mathbf{G}_n(\bar{x}))\ \rightarrow\ z = \mathbf{F}(y_1, \dots, y_y)\ ]$$

which can be written as Π_1 (since $\mathbf{G}_n$ can be written as Σ_1 and $\mathbf{F}$ as Π_1), but it is also equivalent to:

$$\exists y_1, \dots, y_n\ [\ (y_1 = \mathbf{G}_1(\bar{x}) \land \dots \land y_n = \mathbf{G}_n(\bar{x})) \land z = \mathbf{F}(y_1, \dots, y_y)\ ]$$

which can be written as Σ_1 (since $\mathbf{F}$ is also Σ_1). $\square$

Lemma 3.13. *If $\mathbf{F}$ is a Δ_1 function, and ϕ a Δ_1 property, then $\psi(\bar{x}) := (\exists t \in \mathbf{F}(\bar{x}))\phi(t,\bar{x})$ is Δ_1, and the same holds for $(\forall t \in \mathbf{F}(\bar{x}))\phi(t,\bar{x})$.*

Proof. $\psi(\bar{x})$ iff $\forall z\ (z = \mathbf{F}(\bar{x}) \rightarrow \exists t \in z\phi(t,\bar{x}))$ iff $\forall z\ (z = \mathbf{F}(\bar{x}) \land \exists t \in z\phi(t,\bar{x}))$. $\square$

So for example whenever ϕ is Δ_1, then so is $(\forall n \in \omega)\phi$ and $(\exists n \in \omega)\phi$, since ω can be interpreted as a (constant) Δ_1 (and in fact even Δ_0) function.

Lemma 3.14. *Assume that $\mathbf{F}$ is a function on the ordinals which is defined by recursion on a Δ_1 function $\mathbf{G}$, by*

$$\mathbf{F}(\alpha) = \mathbf{G}(\mathbf{F} \restriction \alpha).$$

Then $\mathbf{F}$ is Δ_1.

Proof. For an oridnal α (which is a Δ_0 property), $y = \mathbf{F}(\alpha)$ is equivalent to:

$$\exists f : \alpha \to \mathbf{V}\ (\forall \beta < \alpha\, f(\beta) = \mathbf{G}(f \restriction \beta)\ \land\ y = \mathbf{G}(f)).$$ $\square$

3.C. **Restricted satisfaction predicate for classes (Not needed for this course.)** For a class $\mathbf{M}$ it *is* possible to define satisfaction predicates for restricted set of formulas. Let us just give the example of Σ_1 formulas and $\mathbf{M} = \mathbf{V}$. In this case we can even define a Σ_1 formula $\varphi^{\Sigma_1}_{\mathrm{SAT}}\,(\phi, \bar{p})$ (with free two variables ϕ and $\bar{p}$) such that for all Σ_1 formulas ψ ZF proves the following

$$(\forall \bar{p})\ \psi\ \leftrightarrow\ \varphi^{\Sigma_1}_{\mathrm{SAT}}\,(\ulcorner \psi \urcorner, \bar{p})$$

(This does not contradict Kunen I 14.2, since the negation of a Σ_1 formula is not Σ_1 any more.)

How to define this formula? Recall that Δ_0 formulas ϕ are absolute for transitive models. If $\psi(\bar{p}) = \exists \bar{x}\phi(\bar{x},\bar{p})$ holds in $\mathbf{V}$, then pick a witness $\bar{x}$ ans let A be the transitive closure of $\{\bar{x},\bar{p}\}$. Then $A \models \phi(\bar{x},\bar{p})$ and therefore $A \models \psi(\bar{p})$. On the other hand, the Σ_1 formula ψ is upwards absolute; so we get: If ψ is a Σ_1 formula, then $\psi(\bar{p})$ holds (in $\mathbf{V}$) iff

$$(\exists A)\ A \text{ is transitive}\ \land\ \bar{p} \in A\ \land\ A \models \psi(\bar{p})$$

This is a Σ_1 formula in the variables $\psi, \bar{p}$ (since we can write $A \models \psi(\bar{p})$ as a Σ_1 formula). Similarly, there are Σ_n formulas that capture Σ_n-truth in $\mathbf{V}$ for any natural (meta language) number n.

There are several important applications of this fact; as an unimportant "toy application" we can slightly strengthen reflection: By reflection we can get the the absoluteness of the $\Sigma_{10^{10}}$ satisfaction formula between $R(\beta)$ and $\mathbf{V}$, which in turn implies absoluteness of all $\Sigma_{10^{10}}$ formulas.

3.D. **The definable subsets.** (This section replaces much of Kunen V. Note that we do not mention **HOD** in this course.)

Definition 3.15. *Given a set A, let $\mathcal{D}(A)$ be the family of definable subsets of A (by formulas with parameters in A).*

This is welldefined since we have the satisfaction relation for sets.

So $X \in \mathcal{D}(A)$ iff there is a formula $\phi(x, y_1, \dots, y_n)$ and there are $a_1, \dots, a_n \in A$ such that

$$X = \{t \in A : A \models \phi(t, a_1, \dots, a_n)\}$$

Lemma 3.16.
- *$\mathcal{D}(A) \subseteq \mathcal{P}(A)$.*
- *If A is transitive, then $\mathcal{D}(A)$ is a transitive superset of A .*
- *$A \in \mathcal{D}(A)$.*
- *Every finite subset of A is in $\mathcal{D}(A)$.*

Proof. If A is transitive, then $b \in A$ can be written as $\{t \in A : A \models t \in b\}$, so $A \subseteq \mathcal{D}(A)$ (and $\mathcal{D}(A)$ is trivially transitive as well).

$A = \{t \in A : t = t\}$.

If $\{a_1, \dots, a_n\} \subseteq A$, then use the formula $\phi(x)$ of the form $x = a_1 \vee \dots \vee x = a_n$. (Note: In Kunen VI 1.3(c) this fact has to be proved differently, since Kunen uses the satisfatcion relation only in the form of (meta-theoretic) relativised formulas.) $\square$

Lemma 3.17.
1. *$x = \mathcal{D}(A)$ (a formula with the two free variables x and A) is absolute for transitive models.*[3]
2. *From a wellorder $<_A$ on A we can construct/define a wellorder $<$ on $\mathcal{D}(A)$. This construction is also absolute.*[4]
3. *If A can be wellordered, then $|\mathcal{D}(A)| = \max(\aleph_0, |A|)$.*

Proof. $A \models \phi(a)$ is absolute for transitive models (see Lemma 3.5). So also the formula $X = \{t \in A : A \models \phi(t, a)\}$ (with free variables X, A, ϕ, a) is absolute (for transitive models). Also, the set of formulas and the set of parameters, $A^{<\aleph_0}$, is absolte, which implies that $\mathcal{D}(A)$ is absolte.

Let Fm be the set of formulas, i.e., $\mathrm{Fm} \subseteq \omega$. Given a wellorder of A, we can construct (in an absolute way) a wellorder $<_Z$ of $Z := \mathrm{Fm} \times A^{<\omega}$ (as in Kunen I 10.12 and 10.13, absoluteness follows from Kunen IV 5.6). And there is the obvious surjective map

$$f : Z \to \mathcal{D}(A)$$

which defines a wellorder on $\mathcal{D}$ (and since f is absolutene, the wellorder is defined absolutely as well).

In more detail:

[3] As always, assuming the model $\mathbf{M}$ satisfies some "harmless" finite S. Note that if $\mathbf{M}$ satisfies comprehension for the satisfaction formula, then $\mathcal{D}(A) \in \mathbf{M}$ for all $A \in \mathbf{M}$.

[4] I.e., there is a formula $\Psi(x, y, A, <_A)$ which expresses $x < y$ for the order constructed from $<_A$; this formula is absolute fore transitive models and defines a wellorder on $\mathcal{D}(A)$.

Given ϕ and $\bar{p}$, let $f(\phi,\bar{p})$ be $\{t\in A:\ A\models\phi(t,\bar{p})\}$ (if the length of p covers all free variables of ϕ, set $f(\phi,\bar{p})=0$ otherwise). As shown above, the function f is absolute.

Now fix $a_1\neq a_2$ in $\mathcal{D}(A)$. There is a $<_Z$-minimal b_1 such that $f(b_1)=a_1$, analogously define b_2. Obviously $b_1\neq b_2$. Set $a_1<a_2$ iff $b_1<b_2$. This shows that (given $<_Z$) we can define in an absolute way a wellorder on $\mathcal{D}(A)$.

Note that $|Z|=|\aleph_0\times A^{<\omega}|=|\aleph_0\times A|$ (Kunen I 10.13), so since $f: Z\to\mathcal{D}(A)$ is a surjection we get $|\mathcal{D}(A)|=\max(\aleph_0,|A|)$. $\square$

3.E. **L: Kunen VI.** As mentioned, our Section 3.D replaces Kunen V as well as Kunen VI 1.1–1.3:

- Instead of VI 1.1., we give our own Definition 3.15 of $\mathcal{D}(A)$.
- VI 1.2 follows from Lemma 2.1 (which is the same as Kunen IV 10.1.).
- VI 1.3 follows from Lemmas 3.16 and 3.17

From here on, we follow Kunen VI, with the following exceptions:

- Ignore all references to **HOD** (we did not define this class, if you are interested what it is read Kunen V §2).
- In the proof of VI 3.2, we of course use Lemma 3.17 instead of V 1.7.
- Instead of Definition V 4.1 and V 4.3, we use the following Definitions 3.18 and 3.20. We repeat Kunen VI 4.5(a) as Lemma 3.19.

Definition 3.18. *We define by induction the relation $<_\alpha$ satisfying:*

- *$<_\alpha$ is a wellorder on $L(\alpha)$.*
- *If $\alpha<\beta$, then $L(\alpha)$ is an initial segment of $L(\beta)$ under $<_\beta$; and $<_\alpha$ is the restriction of $<_\beta$ to $L(\alpha)$.*

. The induction proceeds as follows: If δ is a limit, then $<_\delta$ is $\bigcup_{\alpha<\delta}<_\alpha$. (I.e., for $x,y\in L(\delta)$, we set $x<_\delta y$ iff $x<_\alpha y$ for some (or: all) $\alpha<\delta$ with $x,y\in L(\alpha)$.) So assume that we deal with the successor case $\alpha+1$: Let $x,y\in L(\alpha+1)$.
We set $x<_{\alpha+1}y$, if one of the following holds:

- *$x,y\in L(\alpha)$ and $x<_\alpha y$.*
- *$x\in L(\alpha)$ and $y\notin L(\alpha)$.*
- *x and y both are not in $L(\alpha)$, and $x<y$ in the order that we define (in an absolute way) from the wellorder $<_\alpha$ on $L(\alpha)$ as in Lemma 3.17.*

.

Lemma 3.19. *The Defintion of $<_\alpha$ has the properties claimed above, and is absolute for transitive models. (I.e., the function that maps α to the set $<_\alpha$ is absolute; and also the relation $x<_\alpha y$ with three variables x,y,α is absolute.)*

Proof. We know that by Lemma 3.17) the wellorder on $\mathcal{D}(A)$ is defined absolutely, and the rest of the inductive construction uses absolute case distinctions and functions as well; so the inductively defined function is absolute. $\square$

Definition 3.20. *We define $x<_L y$ for $x,y\in L$ by $x<_\alpha y$ for some (or equivalently: all) α with $x,y\in L(\alpha)$.*

Note: Lemma 3.19 is Kunen VI 4.5 (a). Compare that to Kunen VI 4.5 (c): It is not true that $x<_L y$ (a formuly with only two variables x,y) is absolute: It is true for transitive proper classes, but generally not for transitive sets, simply because a set M can think that $x\notin\mathbf{L}$ while really $x\in L(\beta)$ for some $\beta\notin M$. That this can actually happen (and that in fact $\mathbf{V} \neq \mathbf{L}$ is consistent) can be proven only later on using forcing.