Note that $2^{\omega} = \omega$; this should not be confused with cardinal exponentiation (see §10).

A minor variant of Theorem 9.3 is transfinite recursion on an ordinal, δ. If $\mathbf{F}: \mathbf{V} \to \mathbf{V}$, there is a unique function g with domain δ such that $\forall \alpha < \delta\, [g(\alpha) = \mathbf{F}(g \restriction \alpha)]$; to see this, let $\mathbf{G}: \mathbf{ON} \to \mathbf{V}$ be the function satisfying (2), and let $g = \mathbf{G} \restriction \delta$. g is a set by the Axiom of Replacement.

An important special case, when $\delta = \omega$, is often used in arithmetic. For example, we define $n!$ by the clauses:

$$0! = 1.$$

$$(n+1)! = n! \cdot (n+1).$$

This may be cast more formally in the form of Theorem 9.3 as in the discussion of $\alpha + \beta$ above. Here there are only two clauses, as there are no limit ordinals $< \omega$.

§10. Cardinals

We use 1–1 functions to compare the size of sets.

10.1. Definition. (1) $A \preccurlyeq B$ iff there is a 1–1 function from A into B.

(2) $A \approx B$ iff there is a 1–1 function from A onto B.

(3) $A \prec B$ iff $A \preccurlyeq B$ and $B \not\preccurlyeq A$. □

It is easily seen that $\preccurlyeq$ is transitive and that $\approx$ is an equivalence relation. A much deeper result is given in the following theorem.

10.2. Theorem. *Schröder–Bernstein.*

$$A \preccurlyeq B, B \preccurlyeq A \to A \approx B.$$

Proof. See Exercise 8. □

One determines the size of a finite set by counting it. More generally, if A can be well-ordered, then $A \approx \alpha$ for some α (Theorem 7.6), and there is then a least such α, which we call the *cardinality* of A.

10.3. Definition. If A can be well-ordered, $|A|$ is the least α such that $\alpha \approx A$. □

If we write a statement involving $|A|$, such as $|A| < \alpha$, we take it to imply that A can be well-ordered.

Under AC, $|A|$ is defined for every A. Since $A \approx B \to |A| = |B|$ and $|A| \approx A$, the operation $|A|$ picks, under AC, a unique representative of each $\approx$-equivalence class.

Regardless of AC, $|\alpha|$ is defined and $\leq \alpha$ for all α.

10.4. DEFINITION. α is a *cardinal* iff $\alpha = |\alpha|$. $\square$

Equivalently, α is a cardinal iff $\forall \beta < \alpha (\beta \not\approx \alpha)$. We use κ and λ to range over cardinals.

10.5. LEMMA. *If* $|\alpha| \leq \beta \leq \alpha$, *then* $|\beta| = |\alpha|$.

PROOF. $\beta \subset \alpha$ so $\beta \preccurlyeq \alpha$, and $\alpha \approx |\alpha| \subset \beta$ so $\alpha \preccurlyeq \beta$. Thus, by Theorem 10.2, $\alpha \approx \beta$. $\square$

10.6. LEMMA. *If* $n \in \omega$, *then*

(1) $n \not\approx n + 1$.

(2) $\forall \alpha (\alpha \approx n \rightarrow \alpha = n)$.

PROOF. (1) is by induction on n. (2) follows using Lemma 10.5. $\square$

10.7. COROLLARY. *ω is a cardinal and each $n \in \omega$ is a cardinal.* $\square$

10.8. DEFINITION. A is *finite* iff $|A| < \omega$. A is *countable* iff $|A| \leq \omega$. *Infinite* means not finite. *Uncountable* means not countable. $\square$

One cannot prove on the basis of the axioms so far given that uncountable sets exist (see IV 6.7).

Cardinal multiplication and addition must be distinguished from ordinal multiplication.

10.9. DEFINITION. (1) $\kappa \oplus \lambda = |\kappa \times \{0\} \cup \lambda \times \{1\}|$.

(2) $\kappa \otimes \lambda = |\kappa \times \lambda|$. $\square$

Unlike the ordinal operations, $\oplus$ and $\otimes$ are commutative, as is easily checked from their definitions. Also, the definitions of $+$ and $\cdot$ (7.17 and 7.19) imply that $|\kappa + \lambda| = |\lambda + \kappa| = \kappa \oplus \lambda$ and $|\kappa \cdot \lambda| = |\lambda \cdot \kappa| = \kappa \otimes \lambda$. Thus, e.g., $\omega \oplus 1 = |1 + \omega| = \omega < \omega + 1$ and $\omega \otimes 2 = |2 \cdot \omega| = \omega < \omega \cdot 2$.

10.10. LEMMA. *For* $n, m \in \omega$, $n \oplus m = n + m < \omega$ *and* $n \otimes m = n \cdot m < \omega$.

PROOF. First show $n + m < \omega$ by induction on m. Then show $n \cdot m < \omega$ by induction on m. The rest follows by 10.6 (2). $\square$

We now consider $\oplus$ and $\otimes$ on infinite cardinals.

10.11. LEMMA. *Every infinite cardinal is a limit ordinal.*

PROOF. If $\kappa = \alpha + 1$, then since $1 + \alpha = \alpha$, $\kappa = |\kappa| = |1 + \alpha| = |\alpha|$, a contradiction. □

We remark that the principle of transfinite induction (Theorem 9.2) can be applied to prove results about cardinals, since every class of cardinals is a class of ordinals. This is illustrated by the following Theorem.

10.12. THEOREM. *If κ is an infinite cardinal, $\kappa \otimes \kappa = \kappa$.*

PROOF. By transfinite induction on κ. Assume this holds for smaller cardinals. Then for $\alpha < \kappa$, $|\alpha \times \alpha| = |\alpha| \otimes |\alpha| < \kappa$ (applying Lemma 10.10 when α is finite). Define a well-ordering $\lhd$ on $\kappa \times \kappa$ by $\langle \alpha, \beta \rangle \lhd \langle \gamma, \delta \rangle$ iff

$$\max(\alpha, \beta) < \max(\gamma, \delta) \vee [\max(\alpha, \beta) = \max(\gamma, \delta)$$
$$\wedge \langle \alpha, \beta \rangle \text{ precedes } \langle \gamma, \delta \rangle \text{ lexicographically}].$$

Each $\langle \alpha, \beta \rangle \in \kappa \times \kappa$ has no more than $|(\max(\alpha, \beta) + 1) \times (\max(\alpha, \beta) + 1)| < \kappa$ predecessors in $\lhd$, so type$(\kappa \times \kappa, \lhd) \leq \kappa$, whence $|\kappa \times \kappa| \leq \kappa$. Since clearly $|\kappa \times \kappa| \geq \kappa$, $|\kappa \times \kappa| = \kappa$. □

10.13. COROLLARY. *Let κ, λ be infinite cardinals, then*

(1) $\kappa \oplus \lambda = \kappa \otimes \lambda = \max(\kappa, \lambda)$.

(2) $|\kappa^{<\omega}| = \kappa$ (*see Definition* 7.21).

PROOF. For (2), use the proof of Theorem 10.12 to define, by induction on n, a 1–1 map $f_n : \kappa^n \to \kappa$. This yields a 1–1 map $f : \bigcup_n \kappa^n \to \omega \times \kappa$, whence $|\kappa^{<\omega}| \leq \omega \otimes \kappa = \kappa$. □

It is consistent with the axioms so far presented (ZFC$^-$ − P) that the only infinite cardinal is ω (see IV 6.7).

AXIOM 8. *Power Set.*

$$\forall x\, \exists y\, \forall z\, (z \subset x \to z \in y). \quad \square$$

10.14. DEFINITION. $\mathscr{P}(x) = \{z : z \subset x\}$. □

This definition is justified by the Power Set and Comprehension Axioms. The operation $\mathscr{P}$ gives us a way of constructing sets of larger and larger cardinalities.

10.15. THEOREM. *Cantor.* $x \prec \mathscr{P}(x)$. □

Under AC, it is immediate from 10.15 that there is a cardinal $>\omega$, namely $|\mathscr{P}(\omega)|$, but in fact AC is not needed here.

10.16. THEOREM. $\forall\alpha\,\exists\kappa\,(\kappa > \alpha$ *and* κ *is a cardinal*$)$.

PROOF. Assume $\alpha \geq \omega$. Let $W = \{R \in \mathscr{P}(\alpha \times \alpha)\colon R$ well-orders $\alpha\}$. Let $S = \{\text{type}(\langle \alpha, R\rangle)\colon R \in W\}$ (S exists by Replacement). Then $\sup(S)$ is a cardinal $>\alpha$. □

10.17. DEFINITION. α^+ is the least cardinal $>\alpha$. κ is a successor cardinal iff $\kappa = \alpha^+$ for some α. κ is a limit cardinal iff $\kappa > \omega$ and is not a successor cardinal. □

10.18. DEFINITION. $\aleph_\alpha = \omega_\alpha$ is defined by transfinite recursion on α by:

(1) $\omega_0 = \omega$.

(2) $\omega_{\alpha+1} = (\omega_\alpha)^+$.

(3) For γ a limit, $\omega_\gamma = \sup\{\omega_\alpha\colon \alpha < \gamma\}$. □

10.19. LEMMA. (1) *Each* ω_α *is a cardinal.*

(2) *Every infinite cardinal is equal to* ω_α *for some* α.

(3) $\alpha < \beta \to \omega_\alpha < \omega_\beta$.

(4) ω_α *is a limit cardinal iff* α *is a limit ordinal.* ω_α *is a successor cardinal iff* α *is a successor ordinal.* □

Many of the basic properties of cardinals need AC. See [Jech 1973] for a discussion of what can happen if AC is dropped.

10.20. LEMMA (AC). *If there is a function f from X onto Y, then* $|Y| \leq |X|$.

PROOF. Let R well-order X, and define $g\colon Y \to X$ so that $g(y)$ is the R-least element of $f^{-1}(\{y\})$. Then g is 1–1, so $Y \preccurlyeq X$. □

As in Theorem 10.16, one can prove without AC that there is a map from $\mathscr{P}(\omega)$ onto ω_1, but one cannot produce a 1–1 map from ω_1 into $\mathscr{P}(\omega)$.

10.21. LEMMA (AC). *If* $\kappa \geq \omega$ *and* $|X_\alpha| \leq \kappa$ *for all* $\alpha < \kappa$, *then* $|\bigcup_{\alpha<\kappa} X_\alpha| \leq \kappa$.

PROOF. For each α, pick a 1–1 map f_α from X_α into κ. Use these to define a 1–1 map from $\bigcup_{\alpha<\kappa} X_\alpha$ into $\kappa \times \kappa$. The f_α are picked using a well-ordering of $\mathscr{P}(\bigcup_\alpha X_\alpha \times \kappa)$. □

Lévy showed that is consistent with ZF that $\mathscr{P}(\omega)$ and ω_1 are countable unions of countable sets.

A very important modification of Lemma 10.21 is the downward Löwenheim–Skolem–Tarski theorem of model theory, which is frequently applied in set theory (see, e.g., IV 7.8). 10.23 is a purely combinatorial version of this theorem.

10.22. DEFINITION. An *n-ary function* on A is an $f: A^n \to A$ if $n > 0$, or an element of A if $n = 0$. If $B \subset A$, B is closed under f iff $f''B^n \subset B$ (or $f \in B$ when $n = 0$). A *finitary function* is an n-ary function for some n. If $\mathscr{S}$ is a set of finitary functions and $B \subset A$, the *closure* of B under $\mathscr{S}$ is the least $C \subset A$ such that $B \subset C$ and C is closed under all the functions in $\mathscr{S}$. □

Note that there is a least C, namely $\bigcap \{D: B \subset D \subset A \wedge D$ is closed under $\mathscr{S}\}$.

10.23. THEOREM (AC). *Let κ be an infinite cardinal. Suppose $B \subset A$, $|B| \leq \kappa$, and $\mathscr{S}$ is a set of $\leq \kappa$ finitary functions on A. Then the closure of B under $\mathscr{S}$ has cardinality $\leq \kappa$.*

PROOF. If $f \in \mathscr{S}$ and $D \subset A$, let $f * D$ be $f''(D^n)$ if f is n-place, or $\{f\}$ if f is 0-place. Note that $|D| \leq \kappa \to |f * D| \leq \kappa$. Let $C_0 = B$ and $C_{n+1} = C_n \cup \bigcup \{f * C_n : f \in \mathscr{S}\}$. By Lemma 10.21 and induction on n, $|C_n| \leq \kappa$ for all n. Let $C_\omega = \bigcup_n C_n$. Then C_ω is the closure of B under $\mathscr{S}$ and, by 10.21 again, $|C_\omega| \leq \kappa$. □

A simple illustration of Theorem 10.23 is the fact that every infinite group, G, has a countably infinite subgroup. To see this, let $B \subset G$ be arbitrary such that $|B| = \omega$, and apply 10.23 with $\mathscr{S}$ consisting of the 2-ary group multiplication and the 1-ary group inverse.

Our intended application of 10.23 is not with groups, but with models of set theory.

We turn now to cardinal exponentiation.

10.24. DEFINITION. $A^B = {}^BA = \{f : f$ is a function $\wedge$ $\mathrm{dom}(f) = B$ $\wedge$ $\mathrm{ran}(f) \subset A\}$. □

$A^B \subset \mathscr{P}(B \times A)$, so A^B exists by the Power Set Axiom.

10.25. DEFINITION (AC). $\kappa^\lambda = |{}^\lambda\kappa|$. □

The notations A^B and ${}^BA$ are both common in the literature. When discussing cardinal exponentiation, one can avoid confusion by using κ^λ for the cardinal and ${}^\lambda\kappa$ for the set of functions.

10.26. LEMMA. *If $\lambda \geq \omega$ and $2 \leq \kappa \leq \lambda$, then ${}^{\lambda}\kappa \approx {}^{\lambda}2 \approx \mathscr{P}(\lambda)$.*

PROOF. ${}^{\lambda}2 \approx \mathscr{P}(\lambda)$ follows by identifying sets with their characteristic functions, then

$$ {}^{\lambda}2 \preccurlyeq {}^{\lambda}\kappa \preccurlyeq {}^{\lambda}\lambda \preccurlyeq \mathscr{P}(\lambda \times \lambda) \approx \mathscr{P}(\lambda) \approx {}^{\lambda}2. \quad \square $$

Cardinal exponentiation is not the same as ordinal exponentiation (Definition 9.5). The ordinal 2^{ω} is ω, but the cardinal $2^{\omega} = |\mathscr{P}(\omega)| > \omega$. In this book, ordinal exponentiation is rarely used, and κ^{λ} denotes cardinal exponentiation unless otherwise stated.

If $n, m \in \omega$, the ordinal and cardinal exponentiations n^m are equal (Exercise 13).

The familiar laws for handling exponents for finite cardinals are true in general.

10.27. LEMMA (AC). *If κ, λ, σ are any cardinals,*

$$ \kappa^{\lambda \oplus \sigma} = \kappa^{\lambda} \otimes \kappa^{\sigma} \quad \textit{and} \quad (\kappa^{\lambda})^{\sigma} = \kappa^{\lambda \otimes \sigma}. $$

PROOF. One easily checks without AC that

$$ {}^{(B \cup C)}A \approx {}^{B}A \times {}^{C}A \quad (\text{if} \quad B \cap C = 0), $$

and

$$ {}^{C}({}^{B}A) \approx {}^{C \times B}A. \quad \square $$

Since Cantor could show that $2^{\omega_\alpha} \geqq \omega_{\alpha+1}$ (Theorem 10.15), and had no way of producing cardinals between ω_α and 2^{ω_α}, he conjectured that $2^{\omega_\alpha} = \omega_{\alpha+1}$.

10.28. DEFINITION (AC). CH (the Continuum Hypothesis) is the statement $2^{\omega} = \omega_1$. GCH (the Generalized Continuum Hypothesis) is the statement $\forall \alpha\, (2^{\omega_\alpha} = \omega_{\alpha+1})$. $\square$

Under GCH, κ^{λ} can be easily computed, but one must first introduce the notion of cofinality.

10.29. DEFINITION. If $f : \alpha \to \beta$, f maps α *cofinally* iff $\operatorname{ran}(f)$ is unbounded in β. $\square$

10.30. DEFINITION. The cofinality of β $(\operatorname{cf}(\beta))$ is the least α such that there is a map from α cofinally into β. $\square$

So $\operatorname{cf}(\beta) \leq \beta$. If β is a successor, $\operatorname{cf}(\beta) = 1$.

10.31. LEMMA. *There is a cofinal map $f : \mathrm{cf}(\beta) \to \beta$ which is strictly increasing* ($\xi < \eta \to f(\xi) < f(\eta)$).

PROOF. Let $g : \mathrm{cf}(\beta) \to \beta$ be any cofinal map, and define f recursively by

$$f(\eta) = \max(g(\eta), \sup\{f(\xi) + 1 : \xi < \eta\}). \quad \square$$

10.32. LEMMA. *If α is a limit ordinal and $f : \alpha \to \beta$ is a strictly increasing cofinal map, then* $\mathrm{cf}(\alpha) = \mathrm{cf}(\beta)$.

PROOF. $\mathrm{cf}(\beta) \leq \mathrm{cf}(\alpha)$ follows by composing a cofinal map from $\mathrm{cf}(\alpha)$ into α with f. To see $\mathrm{cf}(\alpha) \leq \mathrm{cf}(\beta)$, let $g : \mathrm{cf}(\beta) \to \beta$ be a cofinal map, and let $h(\xi)$ be the least η such that $f(\eta) > g(\xi)$; then $h : \mathrm{cf}(\beta) \to \alpha$ is a cofinal map. $\square$

10.33. COROLLARY. $\mathrm{cf}(\mathrm{cf}(\beta)) = \mathrm{cf}(\beta)$.

PROOF. Apply Lemma 10.32 to the strictly increasing cofinal map $f : \mathrm{cf}(\beta) \to \beta$ guaranteed by Lemma 10.31. $\square$

10.34. DEFINITION. β is *regular* iff β is a limit ordinal and $\mathrm{cf}(\beta) = \beta$. $\square$

So, by Corollary 10.33, $\mathrm{cf}(\beta)$ is regular for all limit ordinals β.

10.35. LEMMA. *If β is regular then β is a cardinal.* $\square$

10.36. LEMMA. *ω is regular.* $\square$

10.37. LEMMA (AC). *κ^+ is regular.*

PROOF. If f mapped α cofinally into κ^+ where $\alpha < \kappa^+$, then

$$\kappa^+ = \bigcup \{f(\xi) : \xi < \alpha\},$$

but a union of $\leq\kappa$ sets each of cardinality $\leq\kappa$ must have cardinality $\leq\kappa$ by Lemma 10.21. $\square$

Without AC, it is consistent that $\mathrm{cf}(\omega_1) = \omega$. It is unknown whether one can prove in ZF that there exists a cardinal of cofinality $> \omega$.

Limit cardinals often fail to be regular. For example, $\mathrm{cf}(\omega_\omega) = \omega$. More generally, the following holds.

10.38. LEMMA. *If α is a limit ordinal, then* $\mathrm{cf}(\omega_\alpha) = \mathrm{cf}(\alpha)$.

PROOF. By Lemma 10.32. $\square$

Thus, if ω_α is a regular limit cardinal, then $\omega_\alpha = \alpha$. But the condition $\omega_\alpha = \alpha$ is not sufficient. For example, let $\sigma_0 = \omega$, $\sigma_{n+1} = \omega_{\sigma_n}$, and $\alpha = \sup\{\sigma_n : n \in \omega\}$. Then α is the first ordinal to satisfy $\omega_\alpha = \alpha$ but $\mathrm{cf}(\alpha) = \omega$. Thus, the first regular limit cardinal is rather large.

10.39. DEFINITION. (1) κ is *weakly inaccessible* iff κ is a regular limit cardinal.

(2) (AC) κ is *strongly inaccessible* iff $\kappa > \omega$, κ is regular, and

$$\forall \lambda < \kappa (2^\lambda < \kappa). \quad \square$$

So, strong inaccessibles are weak inaccessibles, and under GCH the notions coincide. It is consistent that 2^ω is weakly inaccessible or that it is larger than the first weak inaccessible (see VII 5.16). One cannot prove in ZFC that weak inaccessibles exist (see VI 4.13).

A modification of Cantor's diagonal argument yields that $(\omega_\omega)^\omega > \omega_\omega$. More generally, the following holds.

10.40. LEMMA (AC). *König. If κ is infinite and* $\mathrm{cf}(\kappa) \le \lambda$, *then* $\kappa^\lambda > \kappa$.

PROOF. Fix any cofinal map $f : \lambda \to \kappa$. Let $G : \kappa \to {}^\lambda\kappa$. We show that G cannot be onto. Define $h : \lambda \to \kappa$ so that $h(\alpha)$ is the least element of

$$\kappa \smallsetminus \{(G(\mu))(\alpha) : \mu < f(\alpha)\}.$$

Then $h \notin \mathrm{ran}\, G$. $\square$

10.41. COROLLARY (AC). *If* $\lambda \ge \omega$, $\mathrm{cf}(2^\lambda) > \lambda$.

PROOF. $(2^\lambda)^\lambda = 2^{\lambda \otimes \lambda} = 2^\lambda$, so apply Lemma 10.40 with $\kappa = 2^\lambda$. $\square$

10.42. LEMMA (AC + GCH). *Assume that* $\kappa, \lambda \ge 2$ *and at least one of them is infinite, then*

(1) $\kappa \le \lambda \to \kappa^\lambda = \lambda^+$.

(2) $\kappa > \lambda \ge \mathrm{cf}(\kappa) \to \kappa^\lambda = \kappa^+$.

(3) $\lambda < \mathrm{cf}(\kappa) \to \kappa^\lambda = \kappa$.

PROOF. (1) is by Lemma 10.26. For (2), $\kappa^\lambda > \kappa$ by Lemma 10.40, but $\kappa^\lambda \le \kappa^\kappa = 2^\kappa = \kappa^+$. For (3), $\lambda < \mathrm{cf}(\kappa)$ implies that ${}^\lambda\kappa = \bigcup\{{}^\lambda\alpha : \alpha < \kappa\}$, and each $|{}^\lambda\alpha| \le \max(\alpha, \lambda)^+ \le \kappa$. $\square$

The following definitions are sometimes useful.

10.43. DEFINITION (AC). (a) ${}^{<\beta}A = A^{<\beta} = \bigcup\{{}^\alpha A : \alpha < \beta\}$.

(b) $\kappa^{<\lambda} = |{}^{<\lambda}\kappa|$. $\square$

When $\kappa \geq \omega$, $\kappa^{<\omega} = \kappa$ (10.13 (2)), and $\kappa^{<\lambda} = \sup\{\kappa^\theta : \theta < \lambda \wedge \theta$ is a cardinal$\}$ (Exercise 15), so 10.43 (b) is used mainly when λ is a limit cardinal.

10.44. DEFINITION (AC). $\beth_\alpha$ is defined by transfinite recursion on α by:
(1) $\beth_0 = \omega$.
(2) $\beth_{\alpha+1} = 2^{\beth_\alpha}$,
(3) For γ a limit, $\beth_\gamma = \sup\{\beth_\alpha : \alpha < \gamma\}$. □

Thus, GCH is equivalent to the statement $\forall\alpha(\beth_\alpha = \omega_\alpha)$.

§11. The real numbers

11.1. DEFINITION. $\mathbb{Z}$ is the ring of integers, $\mathbb{Q}$ is the field of rational numbers, $\mathbb{R}$ is the field of real numbers, and $\mathbb{C}$ is the field of complex numbers. □

Any reasonable way of defining these from the natural numbers will do, but for definiteness we take $\mathbb{Z} = \omega \times \omega/\sim$, where $\langle n, m\rangle$ is intended to represent $n - m$, the equivalence relation $\sim$ is defined appropriately, $\mathbb{Z}$ is the set of equivalence classes, and operations $+$ and $\cdot$ are defined appropriately. $\mathbb{Q} = (\mathbb{Z} \times (\mathbb{Z} \setminus \{0\}))/\simeq$ where $\langle x, y\rangle$ is intended to represent x/y.

$$\mathbb{R} = \{X \in \mathscr{P}(\mathbb{Q}) : X \neq 0 \wedge X \neq \mathbb{Q} \wedge \forall x \in X\, \forall y \in \mathbb{Q}(y < x \rightarrow y \in X)\}.$$

So $\mathbb{R}$ is the set of left sides of Dedekind cuts. $\mathbb{C} = \mathbb{R} \times \mathbb{R}$, with field operations defined in the usual way.

§12. Appendix 1: Other set theories

We discuss briefly two other systems of set theory which differ from ZF in that they give classes a formal existence. In both, all sets are classes, but not all classes are sets. Let us temporarily use capital letters to range over classes. We define X to be a set iff $\exists Y(X \in Y)$, and we use lower case letters to range sets. In both systems, the sets satisfy the usual ZF axioms, and the intersection of a class with a set is a set.

The system NBG (von Neumann–Bernays–Gödel, see [Gödel 1940]) has as a class comprehension axiom, the universal closure of

$$\exists X\, \forall y\, (y \in X \leftrightarrow \phi(y)),$$

where ϕ may have other free set and class variables, but the bound variables of ϕ may only range over sets. NBG is a conservative extension of ZF; that is, if ψ is a sentence with only set variables, NBG $\vdash \psi$ iff ZF $\vdash \psi$ (see [Wang 1949], [Shoenfield 1954]). Unlike ZF, NBG is finitely axiomatizable.