Solution to Exercise A3.

Let $f \in M$ be an injection from $2^{<\omega}$ to $\mathbb{P}$, satisfying the following condition:

$$\forall s, t \in 2^{<\omega} \; (s \perp t \iff f(s) \perp f(t)).$$

This is possible because $\mathbb{P}$ is non-atomic.

Next, let $\alpha > o(M)$ be a countable ordinal, let E_α be a relation on ω coding α, and let $z_\alpha \in 2^\omega$ canonically encode E_α (note that we can effectively identify $\mathcal{P}(\omega \times \omega)$, $2^{\omega \times \omega}$ and 2^ω). Now let H be the filter generated by $\{f(z_\alpha \restriction n) \mid n < \omega\}$. The assumption on f guarantees that all elements from this set are compatible, so H is indeed a filter.

But now, in $M[H]$, we can use the function f (which is in M and hence in $M[H]$) to decode the real z_α, by using $z_\alpha = \bigcup f^{-1}[H]$. Therefore $z_\alpha \in M[H]$ and hence also $E_\alpha \in M[H]$. Then (ω, E_α) is a well-ordered set of ordertype α, but on the other hand $o(M[H]) = o(M) < \alpha$, so $\alpha \notin M[H]$. But then $M[H]$ cannot be a model of ZFC, since it is a theorem of ZFC that every well-ordered set is isomorphic to an ordinal. $\square$