# ADDITIONS/ALTERNATIVES TO KUNEN, PART 2

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## 1. Satisfaction relation and relativization

1.1. **The satisfaction relation for sets.** Recall that we can formulate $A \models \phi$ in ZFC (cf. Kunen I§14 and IV§10).

For simplicity, we only use the first oder language with signature $\{\in\}$, and interpret the relation symbol $\in$ always as the "real" $\in$-relation restricted to the model $A$. So we can identify the $\{\in\}$-structure (called model) with its "universe" (ground set).

More formally we claim that there is a sentence $\varphi_{\mathrm{SAT}}(A, \phi, \bar{p})$ with free variables $A, \phi, \bar{p}$ which expresses (when $A$ is a nonempty set, $\phi$ a formula of the language $\{\in\}$ with free variables among $\{x_i : i < n\}$ for some $n \in \omega$ and $\bar{p}$ is in $A^n$) that $(A, \in) \models \phi$ under the assignment that maps $x_i$ to $\bar{p}(i)$. We usually just write $A \models \phi(\bar{p})$ instead of $\varphi_{\mathrm{SAT}}(A, \phi, \bar{p})$.

It is important, e.g., for the development of $\mathbf{L}$, that the formula $\varphi_{\mathrm{SAT}}$ is absolute for transitive models (this is shown in Lemma 5).

Recall that we can define $A \models T$ (for $T$ a set of sentences) to mean $(\forall \varphi \in T) A \models \varphi$. In particular, if $T$ is (in the metalanguage) a recursive set of sentences, then we can represent/define $T$ in ZF$^-$, and thus formulate $A \models T$. (Note that, as usual, it cannot be guaranteed that the represented $T$ "is really the same" as the set $T$ in the metalanguage, as the universe might contain nonstandard natural numbers and thus nonstandard elemets of $T$.)

As mentioned, we can prove the completeness and the incompleteness theorems inside of ZF (AC is not needed), so we get the following (in ZF):

From the incompleteness theorem (Kunen I 14.3) we get: If $T$ is a recursive, consistent superset of ZF$^-$, then

$$T \nvdash \mathrm{Con}(T).$$

The completeness theorem (which we can prove in ZF) says:

> Con($T$) iff there is a set $M$ and a relation $E \subseteq M^2$ such that $(M, E) \models T$.

So in particular

$$T \nvdash (\exists M, E)(M, E) \models T$$

(Note that this gives an alternative proof of Kunen IV 7.7 under the suitable assumptions.)

Note however that Con($T$) this is *NOT* equivalent to: there is a set $M$ such that $M \models T$ (where we use the $\in$-relation). Rather, the Mostowski collapsing theorem gives us (assuming that $T$ contains the axiom of extensionality):

$$(\exists M)\, M \models T \text{ iff } (\exists M \text{ transitive})\, M \models T \text{ iff } (\exists M, E)\, (M, E) \models T \text{ and } E \text{ is wellfounded}$$

And ZF proves:

**Lemma 1.** *If there is an $M$ such that $M \models$ZF, then Con(ZF+Con(ZF)).*

*Proof.* If $M \models$ZF, then clearly Con(ZF) holds. But Con(ZF) is an absolute statement for transitive models (since it only quantifies over natural numbers, i.e., it is $\Delta_0$ when $\omega$ is considered a constant). So $M \models$Con(ZF); and therefore ZF+Con(ZF) is consistent. □

Remark: In the same way $M \models$ZF implies also Con(ZF+Con(ZF+Con(ZF))), Con(ZF+Con(ZF+Con(ZF+Con(ZF)))), etc.

**Lemma 2.** *ZF does not prove that $(\exists M, E)\ (M, E) \models T$ implies $(\exists M)\, M \models T$.*

*Proof.* Otherwise, ZF+Con(ZF) would prove Con(ZF+Con(ZF)), contradicting the incompleteness theorem. □

1.2. **The satisfaction predicate is absolute.** In the following, let $\mathbf{M} \subseteq \mathbf{N}$ be nonempty classes (possibly sets).

- We already know that $\Delta_0$-formulas are absolute for transitive classes (Kunen IV 3.6).
- We call a formula $\psi$ is upwards absolute between $\mathbf{M}$ and $\mathbf{N}$, if $\psi^M(\bar{p}) \rightarrow \psi^N(\bar{p})$ for all $\bar{p} \in \mathbf{M}$.
- Similarly, $\psi$ is downwards absolute between $\mathbf{N}$ and $\mathbf{M}$, if $\psi^N(\bar{p}) \rightarrow \psi^M(\bar{p})$, again for all $\bar{p} \in \mathbf{M}$.
- So by definition, $\psi$ is absolute between $\mathbf{M}$ and $\mathbf{N}$ iff it is upwards and downwards absolute.

**Lemma 3.** *If $\mathbf{M} \subseteq \mathbf{N}$ and $\phi(x_1, \dots, x_n, y_1, \dots, y_m)$ is absolute, then $\exists x_1, \dots, x_n \phi$ is upwards absolute and $\forall x_1, \dots, x_n \phi$ is downwards absolute.*

(The proof is trival.)

**Lemma 4.** *Let $\mathbf{M} \subseteq \mathbf{N}$ both satisfy some basic theory $S \subset$ZFC. Let $\zeta(\bar{y})$ be any formula, and assume that there are absolute formulas $\phi(\bar{x}, \bar{y})$ and $\psi(\bar{x}, \bar{y})$ such that*

$$S \vdash \forall \bar{y}\ (\ \zeta \leftrightarrow \exists \bar{x} \phi \leftrightarrow \forall \bar{x} \psi\ ).$$

*Then $\zeta$ is absolute.*

(This is in some way similar to Kunen IV 3.7. and 3.10)

*Proof.* $\zeta^M \underset{M \models S}{\leftrightarrow} (\exists \bar{x} \phi)^M \underset{\text{u. abs.}}{\rightarrow} (\exists \bar{x} \phi)^N \underset{N \models S}{\leftrightarrow} \zeta^N \underset{N \models S}{\leftrightarrow} (\forall \bar{x} \psi)^N \underset{\text{d. abs.}}{\rightarrow} (\forall \bar{x} \psi)^M \underset{M \models S}{\leftrightarrow} \zeta^M$. □

Usually the theory $S$ involved is "completely harmless", e.g., a finite subset of ZF$^-\setminus$Powerset. And usually it is not neccessary to keep track of such harmless $S$.

**Lemma 5.** *The satisfaction relation is absolute between transitive models.*

*Proof.* To see whether $A \models \varphi(\bar{p})$, we have to inductively calculate the truth value of $A \models \psi(\bar{a})$ for all formulas $\psi$ and all possible parameters $\bar{a}$ (of course we do only need subformulas of $\varphi$, but that does not make much difference). Let us set Fml to be the set of forumals and $Z = \text{Fml} \times A^{<\omega}$. We say that $f$ is a truth function, if the following is satisfied:

- $f$ is a function from $Z$ to $\{\text{true}, \text{false}\}$.
- If $\phi$ is a formula of the form $x_n = x_m$ (for variables $x_n, x_m$) and if $p \in M^k$ for some $k > \max(n, m)$ then $f(\phi, p) = \text{true}$ iff $p(n) = p(m)$.

- Analogously for $\in$ instead of $=$.
- If $\phi$ is a formula of the form $\psi_1 \wedge \psi_2$, then $f(\phi, p) = \text{true}$ iff $f(\psi_1, p) = \text{true}$ and $f(\psi_2, p) = \text{true}$.
- Similarly for $\vee$, $\neg$, $\rightarrow$.
- If $\phi$ is a formula of the form $\exists x_n \psi$, then $f(\phi, \bar{p}) = \text{true}$ iff there is a $k$ bigger than $n$ and than all indices of free variables in $\phi$ and a $p' \in P^k$ such that for $\bar{p}(l) = \bar{p}'(l)$ for all $l \neq n$ in the domain of $p$ such that $f(\psi, \bar{p}') = \text{true}$.
- Analogously for $\forall$.

So $A \models \varphi(\bar{p})$ iff there is some truth function $f$ with $f(\varphi, \bar{p}) = \text{true}$, and equivalently iff for all truth functions $f$ we have $f(\varphi, \bar{p}) = \text{true}$.

Note that the following are absolute for transitive models:

- $\varphi_1(\omega)$ saying "$\omega$ is the set of natural numbers". )Kunen IV 5.1)
- $\varphi_2(\text{Fml})$ saying "Fml is the set of formulas".
- $\varphi_3(Z, \text{Fml}, A)$ saying "$Z = \text{Fml} \times A^{<\omega}$". (Kunen IV 3.10, 3.11, 5.3).
- $\zeta(f, A)$ saying "$f$ is a truth function". Note that this formula is even $\Delta_0$ when we consider Fml, $\omega$ and $Z$ as constants, since in the definition of truth function we only quantify over formulas, parameters, and natural numbers.
- $\psi(f, \phi, \bar{p})$ which says "$f(\phi, \bar{p}) = \text{true}$. (This is obviously even $\Delta_0$.)

So $A \models \varphi(\bar{p})$ is equivalent to

$$(\exists f)\ \zeta(f, A) \wedge \psi(f, \phi, \bar{p})$$

as well as to

$$(\forall f)\ \zeta(f, A) \rightarrow \psi(f, \phi, \bar{p})$$

Since both $\zeta(f, A) \wedge \psi(f, \phi, \bar{p})$ as $\zeta(f, A) \rightarrow \psi(f, \phi, \bar{p})$ are absolute for transitive models; $A \models \varphi(\bar{p})$ is absolute for transitive models as well. □

1.3. $\Delta_1$ **properties.** The following is not needed for the course. We can define a more restricted class of absolute formulas, called $\Delta_1$ formulas, and show:

- $\Delta_1$ formulas are absolute for transitive models,
- all the formulas that we proved to be absolute for transitive models are in fact $\Delta_1$.

**Definition 1.** • *A formula $\psi$ is $\Pi_1$ if it has the form $\forall x_1 \ldots \forall x_n \phi$ where $\phi$ is $\Delta_0$.*
*$\psi$ is $\Sigma_1$ if it has the form $\exists x_1 \ldots \exists x_n \phi$ where $\phi$ is $\Delta_0$. A formula $\zeta$ is called $\Delta_1$ with respect ot a basic theory $S$, if there are a $\Pi_1$ formula $\psi$ and a $\Sigma_1$ formula $\phi$ such that*

$$S \vdash \forall \bar{x}\ (\psi \leftrightarrow \phi \leftrightarrow \zeta).$$

The following is an immediate consequence of Lemma 4:

**Lemma 6.** *Assume that* $\mathbf{M} \subseteq \mathbf{N}$ *are transitive (and nonempty). Then every $\Sigma_1$ formula is upwards absolute and every $\Pi_1$ formula is downwards absolute. If* $\mathbf{M}$ *and* $\mathbf{N}$ *both satisfy $S$, then every $\Delta_1$ formula (with respect to $S$) is absolute.*

The following should be clear (using prenex normal form):

**Lemma 7.** *If $\varphi_1, \varphi_2$ are $\Sigma_1$ formulas, then $\varphi_1 \wedge \varphi_2$, $\varphi_1 \vee \varphi_2$ $\exists x \varphi_1$, $(\forall t \in x)\varphi_1$ are logically equivalent to a $\Sigma_1$ formula (and $\neg\varphi_1$ is logically equivalent to a $\Pi_1$ formual).*

*Similarly, if $\varphi_1, \varphi_2$ are $\Pi_1$ formulas, then $\varphi_1 \wedge \varphi_2$, $\varphi_1 \vee \varphi_2$ $\forall x \varphi_1$, $(\exists t \in x)\varphi_1$ are logically equivalent to a $\Pi_1$ formula (and $\neg\varphi_1$ is logically equivalent to a $\Sigma_1$ formual).*

The properties listet in Kunen IV 3.9, 3.11 and 5.1 are $\Delta_0$. One can show that the other properties that are shown to be absolute for transitive models in Kunen IV 3.14, 5.3, 5.4, 5.5, 5.7 are in fact $\Delta_1$. In particular:

**Lemma 8.** *The following are $\Delta_1$:*

- $z = A^{<\omega}$
- $A \models \phi(\bar{a})$
- *$R$ is a wellorder on $A$*
- *$\alpha + \beta$, $\alpha^\beta$ (ordinal)*

Later one can show

**Lemma 9.** *The function $\alpha \mapsto L(\alpha)$ is $\Delta_1$*

Note that "$\alpha$ is a cardinal", "$\alpha$ is regular" and "$\alpha$ is a limit cardinal" are $\Pi_1$ statements (but not $\Delta_1$, which follows, e.g., from Kunen p. 141).

Of course functions can be $\Delta_1$ as well:

**Definition 2.** *A functions $\mathbf{F}$ is called $\Sigma_1$ if "$\mathbf{F}(\bar{x}) = y$" is $\Sigma_1$. Similarly for $\Pi_1$ and for $\Delta_1$.*

**Lemma 10.** *The composition of $\Delta_1$ functions is $\Delta_1$.*

*Proof.* $\mathbf{F}(\mathbf{G}_1(\bar{x}), \dots, \mathbf{G}_n(\bar{x})) = z$ can be written as:

$$\forall y_1, \dots, y_n \ [\ (y_1 = \mathbf{G}_1(\bar{x}) \wedge \dots \wedge y_n = \mathbf{G}_n(\bar{x})) \ \rightarrow \ z = \mathbf{F}(y_1, \dots, y_y)\ ]$$

which can be written as $\Pi_1$ (since $\mathbf{G}_n$ can be written as $\Sigma_1$ and $\mathbf{F}$ as $\Pi_1$), but it is also equivalent to:

$$\exists y_1, \dots, y_n \ [\ (y_1 = \mathbf{G}_1(\bar{x}) \wedge \dots \wedge y_n = \mathbf{G}_n(\bar{x})) \wedge z = \mathbf{F}(y_1, \dots, y_y)\ ]$$

which can be written as $\Sigma_1$ (since $\mathbf{F}$ is also $\Sigma_1$). □

**Lemma 11.** *Assume that $\mathbf{F}$ is a $\Sigma_1$ function. More exactly, assume that some $S$ proves that the according $\Sigma_1$ formula defines a function (on the ordinals, say). Then $\mathbf{F}$ is actually $\Delta_1$ (with respect to $S$-models).*

*Proof.* By assumption, $z = \mathbf{F}(x)$ is expressed by a $\Sigma_1$ formula $\phi(x, z)$. We have to show that there is a $\Pi_1$ formula $\psi(x, z)$ which is equivalent to $\phi$ (modulo $S$). But $z = \mathbf{F}(x)$ iff

$$\forall t \ (\ t = z \vee \ t \neq \mathbf{F}(x))$$

This is $\Pi_1$ (since $t \neq \mathbf{F}(x)$ is $\Pi_1$). □

**Lemma 12.** *If $\mathbf{F}$ is a $\Delta_1$ function, and $\phi$ a $\Delta_1$ property, then $\psi(\bar{x}) := (\exists t \in \mathbf{F}(\bar{x}))\phi(t, \bar{x})$ is $\Delta_1$, and the same holds for $(\forall t \in \mathbf{F}(\bar{x}))\phi(t, \bar{x})$.*

*Proof.* $\psi(\bar{x})$ iff $\forall z \ (z = \mathbf{F}(\bar{x}) \rightarrow \exists t \in z \phi(t, \bar{x}))$ iff $\forall z \ (z = \mathbf{F}(\bar{x}) \wedge \exists t \in z \phi(t, \bar{x}))$. □

So for example whenever $\phi$ is $\Delta_1$, then so is $(\forall n \in \omega)\phi$ and $(\exists n \in \omega)\phi$, since $\omega$ can be interpreted as a (constant) $\Delta_1$ (and in fact even $\Delta_0$) function.

**Lemma 13.** *Assume that* $\mathbf{F}$ *is a function on the ordinals which is defined by recursion on a* $\Sigma_1$ *function* $\mathbf{G}$*, by*

$$\mathbf{F}(\alpha) = \mathbf{G}(\mathbf{F} \restriction \alpha).$$

*Then* $\mathbf{F}$ *is* $\Sigma_1$ *(and therefore* $\Delta_1$*).*

*Proof.* For an oridnal $\alpha$ (which is a $\Delta_0$ property), $y = \mathbf{F}(\alpha)$ is equivalent to:

$$\exists f : \alpha \to \mathbf{V}\ (\forall \beta < \alpha\, f(\beta) = \mathbf{G}(f \restriction \beta)\ \wedge\ y = \mathbf{G}(f)).$$

□

1.4. **Satisfaction for classes: relativization.** Note that we have a satisfaction predicate only for *sets* $M$, not for classes. In general, if $\mathbf{M}$ is a class, then there is no predicate $\varphi^{\mathbf{M}}_{\text{SAT}}(\phi)$ that expresses that $\phi$ holds in $\mathbf{M}$, see the remark after Kunen I 14.2 (which shows this fact for $\mathbf{M} = \mathbf{V}$). However, there is a different way to express that $\phi$ holds in $\mathbf{M}$ (where $\mathbf{M}$ can be a set or a proper class): Relativize $\phi$ to the formula $\phi^{\mathbf{M}}$. If $\varphi$ is a "standard" formula (from the meta theory) and $M$ is a set, then both notions can be used and actually are the same: For any $\varphi(x_1, \dots, x_n)$, ZF$^-$ proves the following (Kunen IV 10.1.):

$$(\forall M)(\forall p_1, \dots, p_n \in M)\ M \models \varphi(\bar{p})\ \leftrightarrow\ \varphi^M(p_1, \dots, p_n)$$

(More formally, here we should write $\varphi_{\text{SAT}}(M, \in, \ulcorner\varphi\urcorner, < p_1, \dots, p_n >)$ instead of $M \models \varphi(\bar{p})$.)

So for classes we can only use the form $\phi^{\mathbf{M}}$ (which is a new formula constructed from $\phi$ in the meta-theory). We will also write $\mathbf{M} \models \phi(\bar{p})$ instead of $\phi^{\mathbf{M}}(\bar{p})$, but you should be aware that this means something different than the satisfaction relation. In particular we can only talk about finitely many $\phi$, so we cannot really formulate $\mathbf{M} \models T$ for, e.g., infinite recursive $T$. We will still write $\mathbf{M} \models T$, by which we mean something like: For all $\phi \in T$ (in the metalanguage), ZF$^-$ (or another appropriate theory) proves $\phi^{\mathbf{M}}$. Note that in particular $\mathbf{V} \models$ZF$^-$ holds trivially (and the same for $\mathbf{V} \models$ZFC, if we assume ZFC as basic theory); and is not a contradiction to the incompleteness theorem.

1.5. **Restricted satisfaction predicate for classes.** The following is not required for the course:

For a class $\mathbf{M}$ it *is* possible to define satisfaction predicates for restricted set of formulas. Let us just give the example of $\Sigma_1$ formulas and $\mathbf{M} = \mathbf{V}$. In this case we can even define a $\Sigma_1$ formula $\varphi^{\mathbf{M}}_{\text{SAT } \Sigma_1}(\phi, \bar{p})$ (with free two variables $\phi$ and $\bar{p}$) such that for all $\Sigma_1$ formulas $\psi$ ZF proves the following

$$(\forall \bar{p})\ \psi^{\mathbf{M}}\ \leftrightarrow\ \varphi^{\mathbf{M}}_{\text{SAT } \Sigma_1}(\psi, \bar{p})$$

(This does not contradict Kunen I 14.2, since the negation of a $\Sigma_1$ formula is not $\Sigma_1$ any more.)

How to define this formula? Recall that any $\Sigma_1$ formula is upwards absolute for transitive models. So if $\psi$ is a $\Sigma_1$ formula, then $\psi(\bar{p})$ holds (in $V$) iff

$$(\exists A)\ A \text{ is transitive } \wedge\ \bar{p} \in A\ \wedge\ A \models \psi(\bar{p})$$

This is a $\Sigma_1$ formula (since we can write $A \models \psi(\bar{p})$ as a $\Sigma_1$ formula). Similarly, there are $\Sigma_n$ formulas that capture $\Sigma_n$-truth in $V$ for any natural (meta-language) number $n$.

There are several important applications of this fact; as a "toy application" (which is not important) we can slightly strengthen reflection: By reflection we can get the the absoluteness of the $\Sigma_{10^{10}}$ satisfaction formula between $R(\beta)$ and $\mathbf{V}$, which in turn implies absoluteness of all $\Sigma_{10^{10}}$ formulas.

## 2. Reflection, elementary submodels

Using the satisfaction relation, we can define (in the object language) for sets $M \subseteq N$: a formula $\phi$ (with free variables among $\{x_i : i < n\}$ with $n \in \omega$) is absolute between $M$,$N$ if $(\forall \bar{p} \in M^n) M \vDash \varphi(\bar{p})$ iff $N \vDash \varphi(\bar{p})$.

**Definition 3.**
*$M \preceq N$ ($M$ is an elementary submodel of $N$), if all formulas are absolute.*

Analogously to Kunen IV 7.3, one proves (all of the following are in the object language): Assume that $M \subseteq N$. Then Then the following are equivalent:
(a) $M \preceq N$
(b) If $\phi$ is of the form $\exists x \psi(x, y_1, \dots, y_m)$, then $(\forall \bar{p} \in M^m)\big[(\exists a \in N) N \models \psi(a, \bar{p}) \rightarrow (\exists a \in M) N \models \psi(a, \bar{p})\big]$. (This is called the Tarski Vaught criterion).
Kunen IV 7.8 can easily be modified to sets:

**Lemma 14.** *Let $N$ be a (well-orderable) set and $X \subseteq N$. Then there is an elementary submodel $M$ of $N$ such that $|M| \leq \max(\aleph_0, |X|)$.*

Of course, $\mathrm{ZF}^-$ proves the following: If $M$ and $M'$ are isomorphic, then $M$ and $M'$ satisfy exactly the same sentences (without free variables). This is a "set version" of 7.9.
So in particular, 7.10 can be formulated for sets the following way:

**Lemma 15.** *Let $N$ be a (wellorderable) set and $X \subseteq N$ transitive. Then there is a transitive $M$ satisfying the same setences as $N$ and such that $X \subseteq M$ and $|M| \leq \max(\aleph_0, |X|)$.*

(We could also allow formulas with parameters in $X$, since these parameters are not moved by the Mostowsky collapse.)

## 3. Further reading

The following is interesting but not important for the rest of the book (and will not be required in an exam):
We already know (from IV 6.9) that we cannot prove in ZFC that there is a (strongly) inaccessible cardinal. I also proved (using the incompleteness theorem) that even something stronger is true: ZFC plus the existence of an inaccessible even has higher consistency strenght than just ZFC. This is also proved in Kunen p.145.
I also recommend to read the "curious example" on p.146.

## 4. The definable subsets

This section replaces Kunen V.
Given a set $A$, let $\mathcal{D}(A)$ be the family of definable subsets of $A$ (by formulas with parameters in $A$). This is welldefined since we have the satisfaction relation for sets.

So $X \in \mathcal{D}(A)$ iff there is a formula $\phi(x, y_1, \dots, y_n)$ and there are $a_1, \dots, a_n \in A$ such that

$$X = \{t \in A : A \models \phi(t, a_1, \dots, a_n)\}$$

**Lemma 16.**
- *$\mathcal{D}(A) \subseteq \mathcal{P}(A)$.*
- *If $A$ is transitive, then $\mathcal{D}(A)$ is a transitive superset of $A$ .*
- *$A \in \mathcal{D}(A)$.*
- *Every finite subset of $A$ is in $\mathcal{D}(A)$.*

*Proof.* Any $b \in A$ can be written as $\{t \in A : A \models t \in b\}$ (since $A$ is transitive), so $A \subseteq \mathcal{D}(A)$.

If $X \in \mathcal{D}(A)$, then $X \subseteq A$, so if $t \in X$ then $t \in A$ and so $t \in \mathcal{D}(A)$, which shows that $\mathcal{D}(A)$ is again transitive.

$A = \{t \in A : t = t\}$.

If $\{a_1, \dots, a_n\} \subseteq A$, then use the formula $\phi(x)$ of the form $x = a_1 \vee \dots \vee x = a_n$. (Note: In Kunen VI 1.3(c) this fact has to be proved differently, since Kunen uses the satisfatcion relation only in the form of (meta-theoretic) relativized formulas.) □

**Lemma 17.**
(1) *$x = \mathcal{D}(A)$ (a formula with the two free variables $x$ and $A$) is absolute for transitive models (of some finite $S \subseteq$ZF).*
*If a transitive model* **M** *satisfies comprehension for the satisfaction formula, then $\mathcal{D}(A) \in \mathbf{M}$ for all $A \in \mathbf{M}$.*
(2) *From a wellorder $<_A$ on $A$ we can construct/define a wellorder on $\mathcal{D}(A)$. This construction is also absolute.*
(3) *If $A$ can be wellordered, then $|\mathcal{D}(A)| = \max(\aleph_0, |A|)$.*

*Proof.* $A \models \phi(a)$ is absolute for transitive models (see Lemma 5). So also the formula $X = \{t \in A : A \models \phi(t, a)\}$ (with free variables $X, A, \phi, a$) is absolute. Also, the set of formulas and the set of parameters, $A^{<\aleph_0}$, is absolte, which implies that $\mathcal{D}(A)$ is absolte.

Let Fm be the set of formulas, i.e., Fm $\subseteq \omega$. Given a wellorder of $A$, we can construct (in an absolute way) a wellorder $<_Z$ of $Z :=$ Fm $\times A^{<\omega}$ (as in Kunen I 10.12 and 10.13, absoluteness follows from Kunen IV 5.6). This in turn defines a wellorder on $\mathcal{D}$ in the obvious way.

In more detail: There is a n absolute, surjective map $f$ from $Z$ to $\mathcal{D}(A)$: Given $\phi$ and $\bar{p}$, let $f(\phi, \bar{p})$ be $\{t \in A : A \models \phi(t, \bar{p})\}$ (if the length of $p$ covers all free variables of $\phi$, set $f(\phi, \bar{p}) = 0$ otherwise). Now fix $a_1 \neq a_2$ in $\mathcal{D}(A)$. There is a $<_Z$-minimal $b_1$ such that $f(b_1) = a_1$, analogously define $b_2$. obviously $b_1 \neq b_2$. Then set $a_1 < a_2$ iff $b_1 < b_2$. This shows that (given $<_Z$) we can define in an absolute way a wellorder on $\mathcal{D}(A)$.

Note that $|Z| = |\aleph_0 \times A^{<\omega}| = |\aleph_0 \times A|$ (Kunen I 10.13), so since $f : Z \to \mathcal{D}(A)$ is a surjection we get $|\mathcal{D}(A)| = \max(\aleph_0, |A|)$. □

## 5. **L**

(The following corresponds to Kunen VI §1 and §2, details can be found there.)

### 5.1. **Definition and basic properties.**

- $L(0) = 0$,
- $L(\alpha + 1) = \mathcal{D}(L(\alpha))$,
- $L(\delta) = \bigcup_{\alpha<\delta} L(\alpha)$ for $\delta$ limit.

The class $\mathbf{L}$ is defined as $\bigcup_{\alpha\in\mathrm{ON}} L(\alpha)$.

**Lemma 18.**
- *$L(\alpha)$ is transitive.*
- *$L(\alpha) \subseteq L(\beta)$ whenever $\alpha < \beta$.*
- *$L(\alpha) \in L(\alpha+1)$.*
- *$L(\alpha) \subseteq R(\alpha)$.*
- *$ON \cap L(\alpha) = \alpha$.*
- *$L(n) = R(n)$ for $n \in \omega$, $L(\omega) = R(\omega)$.*
- *$|L(\alpha)| = |\alpha|$ for all $\alpha \geq \omega$.*

*Proof.* The proof is an easy induciton, using Lemmas 16 and 17 in successor steps. (For the last item, just use AC. However, AC is not neccessary, since $\mathcal{D}(A)$ can be *canonically* wellordered. We will come back to this later.) □

**Lemma 19.** $\mathbf{L} \models$*ZF. (More exactly: For each $\phi$ in ZF, we can prove (in ZF) that $\phi^{\mathbf{L}}$ holds.)*

*Proof.* All of this is straightforward, apart from Comprehension. For Comprehension, note that the $L$-hierarchy satisfies the requirements for reflection. This implies:

> Given $\phi_0, \dots, \phi_n$ (in the meta theory), we can prove in ZF: For each $\alpha$ there is a $\beta > \alpha$ such that $\phi_0, \dots, \phi_n$ are absolute between $\mathbf{L}$ and $L(\beta)$.

So we want to show that comprehension for $\phi$ holds in $\mathbf{L}$, i.e., that for all $z \in \mathbf{L}$ the set

$$X := \{t \in z : \mathbf{L} \models \phi(t)\}$$

is in $\mathbf{L}$. Chose $\alpha$ large enough so that $z$ (and any parameters) are in $L(\alpha)$. Reflection gives a $\beta > \alpha$ such that $\mathbf{L} \models \phi(t)$ iff $L(\beta) \models \phi(t)$ for all $t \in L(\beta)$ and therefore for all $t \in z$. This shows that $X$ is a definable subset of $L(\beta)$, i.e., $X \in L(\beta+1)$, and so $X \in \mathbf{L}$. □