1.1 SAT $\qquad M \models \varphi$

Mittwoch, 28. Oktober 2009
15:29

Recall def of first-order logic satisfaction, e.g., for signature $\{\in\}$: $(M, E) \models \varphi(m_1, \dots, m_n)$
In case $E = \in \upharpoonright M \times M$ we just write $M \models \varphi(m_1, \dots, m_n)$

Just as all other mathematical notions, we can formulate this Def in the language of ZFC:

To actually do this is a bit tedious:

First we can code formulas as, e.g., numbers (or strings, or trees), and find ZFC-Formulas $\mathrm{Fmla}(x)$ that formulate that x is Code for a Formula. In particular, for every (in the Metalanguage) Formula φ, we can calculate $\ulcorner\varphi\urcorner$ and we get: $ZFC \vdash \mathrm{Fmla}(\ulcorner\varphi\urcorner)$.

Then we can use the inductive Def of $\models$ to define a ZFC-Fmla $\mathrm{Sat}(M, E, \ulcorner\varphi\urcorner, \bar{p})$ expressing:
M is a set, $E \subseteq M \times M$, $\ulcorner\varphi\urcorner$ is Code of a fmla with free vars $x_1 \dots x_n$, and $p = (m_1, \dots, m_n)$ for $m_i \in M$, and $(M, E) \models \varphi(m_1, \dots, m_n)$

(Remark: We will later see that we can choose Sat to be either Σ_1 or Π_1, i.e., relatively simple.)

1.2 Relativization

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Given $\varphi(x_1, \dots, x_n)$ we can construct $\varphi'(M, x_1, \dots, x_n)$ [with new free var M] by ind, replacing $\exists x$ by $\exists x \in M$ and $\forall x$ by $\forall x \in M$.
We will write $\varphi^M(x_1 \dots x_n)$ instead of $\varphi'(M, x_1 \dots x_n)$

<u>Fact</u> $\overset{(*)}{\text{For all}}$ $\{\in\}$-sentences $\varphi(x_1 \dots x_n)$,

$$ZFC \vdash \forall m_1 \dots m_n \in M \left[(M \models \varphi(m_1 \dots m_n)) \longleftrightarrow \varphi^M(m_1 \dots m_n) \right]$$

Note: the quantifier $(*)$ is in the Metalanguage!
Why? We cannot formulate $\forall \varphi \dots \varphi^M(\dots)$.
Why? Of course we <u>can</u> map φ to φ^M, but there is no truth predicate $True(\ulcorner \varphi \urcorner)$ that expresses that φ is true, such as in:

$$\forall \ulcorner \varphi \urcorner \quad \underbrace{Sat(M, \ulcorner \varphi \urcorner)}_{OK} \longleftrightarrow True(\ulcorner \underbrace{\varphi^M}_{\varphi \mapsto \varphi^M \ OK} \urcorner)$$

Quant. over Fmlas OK

Does not exist!

So usually all statements in this lecture are meant to be (more correctly: translate to) sentences claimed to be provable in our underlying Theory, ZFC.
E.g., if I claim: $\forall \kappa: 2^\kappa > \kappa$, I mean
$ZFC \vdash \forall \kappa \; 2^\kappa > \kappa$.
But the fact above has to be interpreted partially in Metalanguage

1.3 Incompleteness, Truth

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Short Detour:

(a) Incompl. Thm

(b) 2nd incompl. Thm

(c) Why there is no truth predicate.

Ad (a): In the proof of the incompl. Thm, one shows:

(1) There is a formula $Pr(x)$ s.t.
$ZFC \vdash Pr(\ulcorner\varphi\urcorner)$ iff $ZFC \vdash \varphi$ (iff $\mathbb{N} \models Pr(\ulcorner\varphi\urcorner)$)

(2) For all Fmlas $\psi(x)$ there is Fmla φ s.t.
$ZFC \vdash \varphi \Leftrightarrow \neg\psi(\ulcorner\varphi\urcorner)$

(3) In part. there is φ s.t. $ZFC \vdash \varphi \Leftrightarrow \neg Pr(\ulcorner\varphi\urcorner)$
This φ states own unprovability and is true (but unprovable) sentence

Ad (b):

Let Con be the Fmla $\neg Pr(\ulcorner 0 \neq 0 \urcorner)$.

One can show:

$ZFC \nvdash Con$

We will write Pr_T for provable in T, and $Con(T)$ for $\neg Pr_T(\ulcorner 0 \neq 0\urcorner)$, then we generally get (for suitable T):

$T \nvdash Con(T)$

Ad (c):

There is no predicate $True(x)$ s.t.

$$\varphi \Leftrightarrow True(\ulcorner\varphi\urcorner).$$

[illegible] Let True be any predicate. Using (2), we get: $ZFC \vdash \varphi \Leftrightarrow \neg True(\ulcorner\varphi\urcorner)$

1.4 Rel. Classes

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Let C be a class def by $\psi(x,p)$
(there can be an additional parameter p).
I.e., C "=" $\{x : \varphi(x,p)\}$

Just as φ^M, we can def for each φ
a Fmla φ^C by ind: Repl. $\exists x$ by $\exists x \in C$ etc.

We will write $C \models \varphi$ as abbrev. for φ^C.

Caution: Generally there is __no__ formula
Sat_C s.t. $\mathrm{Sat}_C(\ulcorner\varphi\urcorner)$ expresses $C \models \varphi$!
(For $C = V$ this would be truth predicate!)

This means that we __can__ formulate
$M \models ZFC$
for sets M, as $\underbrace{\forall \ulcorner\varphi\urcorner}_{\text{ok}} \ \underbrace{\ulcorner\varphi\urcorner \in ZFC}_{\text{ok}} \rightarrow \underbrace{SAT(M, \ulcorner\varphi\urcorner)}_{\text{ok}}$

But we cannot formulate $C \models ZFC$
for a class C (e.g., $C = L$), since that
would require $\forall \ulcorner\varphi\urcorner \ \ulcorner\varphi\urcorner \in ZFC \rightarrow \cancel{C \models \varphi}$
e.g., via, $\forall \ulcorner\varphi\urcorner \ \ulcorner\varphi\urcorner \in ZFC \rightarrow \mathrm{True}(\underbrace{\ulcorner\varphi^C\urcorner}_{\text{ok}})$
$\uparrow$
X not ok X

We will of course still use $L \models ZFC$
and similar formulas. But formally
this has to be interpreted as:
For all φ in ZFC (Relativierung!)
$ZFC \vdash L \models \varphi$

Related: Trivially, for all $\varphi \in ZFC$ $ZFC \vdash \varphi$.
But $ZFC \vdash \forall \varphi \in ZFC \ \varphi$ does not make
sense: We would need a truth predicate
to formulate it (and even if such a
pred. existed we might get problems with
the 2nd Incompleteness Thm)

1.5 Models

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16:08

Trivial Model Theory:

$$[(\exists M) M \models T] \leftrightarrow \mathrm{Con}(T)$$

(this is, of course, provable in ZFC; the quantifier is of course over sets, not classes)

We are mainly interested in <u>wellfounded</u> models of the language $\{\in\}$:

(Mostowsky) If $(M, \in)$ is wellfounded, and if $(M, \in) \models \mathrm{Ext}$, then $\exists! M'$ transitive s.t. $\exists f : (M, \in) \to (M', \in)$ isom.

In part., $(M, \in) \models \varphi(m)$ iff $M' \models \varphi(f(m))$

Note that $(\exists M) M \models T \not\to (\exists M \text{ w.f.}) M \models T$

E.g., the following is provable from ZFC+Con(ZFC)

(1) Con(ZFC + ¬Con ZFC) (because of 2nd incompleteness)

(2) $\forall M$ trans, $M \models \mathrm{ZFC} \to M \models \mathrm{Con(ZFC)}$
(since "Con(ZFC)" is a simple statement and therefore absolute for transitive models, see Cohen)

and therefore

ZFC ⊢ Con(ZFC) → ∃ model of ZFC + ¬Con ZFC
but ∄ wellfounded model of ZFC + ¬Con ZFC

Note that $(M, \in) \models$ Foundation does <u>not</u> imply that $\in$ is "really" wellfounded,
(just look at model of ZFC + ¬Con ZFC above)

Note that $(M, \in)$ is always trivially wellfounded

1.6 Class models

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We will see: $L \models ZFC + CH$, therefore $\mathrm{Con}(ZFC + CH)$, i.e., $ZFC \nvdash \neg CH$.

This is, of course, a bit problematic. (It would be completely OK if L was a <u>set</u>).

What we mean is:

(1) For all $\varphi \in ZFC$, $\quad ZFC \vdash L \models \varphi$

(2) $ZFC \vdash L \models CH$

Therefore: Assume towards contradiction that $ZFC \vdash \neg CH$. Then there are $\varphi_1 .. \varphi_n \in ZFC$ st. $\{\varphi_1 .. \varphi_n\} \vdash \neg CH$. This proof can be relativized to L: $\{\varphi_1^L \dots \varphi_n^L\} \vdash \neg CH^L$ (just check that a proof remains valid if every $\exists x$ is replaced by $\exists x \in L$)

But by (1) and (2) we get

$$ZFC \vdash \underbrace{\underbrace{\varphi_1^L, \dots, \varphi_n^L}_{\vdash \neg CH^L}, CH^L}_{0 \neq 0},$$

So ZFC itself must be inconsistent.