Rend. Sem. Mat. Univ. Politec. Torino
Vol. 70, 3 (2012), 207 – 246

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NON ELEMENTARY PROPER FORCING

Abstract. We introduce a simplified framework for ord-transitive models and Shelah's non elementary proper (nep) theory. We also introduce a new construction for the countable support nep iteration.

Introduction

In this paper, we introduce a simplified, self contained framework for forcing with ord-transitive models and for non elementary proper (nep) forcing, and we provide a new construction for the countable support nep-iteration.

Judah and Shelah [3] introduced the notion "Suslin proper": A forcing notion $Q \subseteq \omega^\omega$ is Suslin proper if

(1) "$p \in Q$", "$q \leq p$" and "$q \perp p$" (i.e., p and q are incompatible) are all $\utilde{\mathbf{\Sigma}}^1_1$ statements (in some real parameter r), and if

(2) for all contable transitive models M (of some ZFC*, a sufficiently large fragment of ZFC) that contain the parameter r and for all $p \in Q^M := Q \cap M$ there is a $q \leq p$ which is M-generic, i.e., forces that the generic filter G meets every maximal antichain $A \in M$ of Q^M.

We always assume that $H(\chi)$ satisfies ZFC* (for sufficiently large regular cardinals χ). Then every Suslin proper forcing Q is proper. (Given an elementary submodel N of $H(\chi)$, apply the Suslin proper property to the transitive collapse of N.) So Suslin proper is a strengthening of properness for nicely definable forcings.

Shelah [9] introduced a generalization of Suslin proper which he called **non elementary proper (nep)**. Actually, it is a generalization in two "directions":

(a) In (1), we do not require "$p \in Q$" and "$p \leq q$" to be defined by $\utilde{\mathbf{\Sigma}}^1_1$ statements, but rather by some arbitrary formulas that happen to be sufficiently (upwards) absolute.[1]

(b) In (2), we do not require M to be a transitive model, but rather a so-called ord-transitive model (and we allow more general parameters r).

The motivation for (a) is straightforward: This way, we can include forcing notions that are not Suslin proper (such as Sacks forcing), while we can still prove many of the results that hold for Suslin forcing notions.

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*Supported by Austrian Science Fund (FWF): P21651-N13 and P23875-N13

[1] For incompatibility, we do not require absoluteness, although it will be satisfied in the "natural" examples (but not, e.g., in nep iterations). Of course, according to (2), if p_1 and p_2 are incompatible in M and q is M-generic, then there cannot be an $r \leq q, p_1, p_2$.

Why is (b) useful? To "approximate" a forcing notion Q by forcings $Q^M \in M$, it is necessary that Q is the union of Q^M for all possible models M. (This is of course the case if Q is Suslin proper: any $p \in Q$ is a real, and therefore element of some countable transitive M and thus of $Q^M = Q \cap M$.) So if we allow only countable transitive models M, we can only talk about forcings Q that are subsets of $H(\aleph_1)$. Of course there are many other interesting forcing notions, such as iterations of length $\geq \omega_2$, products of size $\geq \aleph_2$, or alternative creature forcing constructions of large size, etc. Switching from transitive models to ord-transitive models allows us to deal with some of these forcings as well.

Note that such ord-transitive models can be useful in a different (and simpler) setting as well: Instead of considering a forcing definition and the realizations Q^V and Q^M of this definition (in V and a countable model M, respectively), we can just use two arbitrary (and entirely different) forcings $Q^V \in V$ and $Q^M \in M$ and require that Q^M is an M-complete subforcing of Q^V. In the transitive case this concept is a central ingredient of Shelah's oracle-cc [8, IV], and it can be applied to ord-transitive models as well: An example is [2] (joint with Goldstern, Shelah and Wohofsky), which proves the consistency of the Borel Conjecture plus the dual Borel Conjecture. For this construction, nep forcing is not required, but ord-transitive models are. We very briefly comment on this in Section 1.3.

To summarize:

- Just as Suslin proper, nep has consequences that are not satisfied by all proper forcing notions. So when we know that a forcing is nep and not just proper, we know more about its behavior. And while nep implies all of the useful consequences of Suslin proper, nep is more general (i.e., weaker): Some popular forcings are nep, but not Suslin proper (e.g., Sacks forcing).
  For example, let us say that "Q preserves non-meager" if Q forces that the ground model reals are not meager (and analogously we define "Q preserves non-Lebesgue-null). Goldstern and Shelah [8, XVIII.3.11] proved that the proper countable support iteration (P_α, Q_α) of non-meager preserving forcing notions preserves non-meager, provided that all Q_α are Suslin proper.
  Shelah and the author [5, 9.4] proved that the same preservation theorem holds for Lebesgue-null instead of meager and that it is sufficient to assume (nicely definable) nep instead of Suslin proper. This has been applied by Roslanowski and Shelah in [7], which proves that consistently every real function is continuous on a set of positive outer Lebesgue measure.

- In particular, forcings that are not subsets of $H(\aleph_1)$ can be nep; for example big countable support products. In particular, we get the following preservation theorem: under suitable assumptions, the countable support iteration of nep forcings is nep.
  An example of how this can be used is Lemma 4.24 of this paper. (This fact was used by Shelah and Steprāns in [11, 4.5] to investigate Abelian groups).

Contents

Most of the notion and results in this paper are due to Shelah, and (most likely) can be found in [9], some of them explicitly (and sometimes in a more general setting), some at least "in spirit". However, the notation and many technical details are different: In many cases the notation here is radically simplified, in other cases the notions are just incomparable (for example the definition of nep-parameter). Most importantly, we work in standard set theory, not in a set theory with ordinals as urelements. The result of Subsection 3.5 is due to Zapletal.

1. Forcing with ord-transitive models

Whenever we use the notation $N \prec H(\chi)$, we imply that N is countable, and that χ is a sufficiently large regular cardinal. We write $H(\chi)$ for the sets that are hereditarily smaller than χ and R_α for the sets of rank less than α. (We will use the notation V_α for forcings extension of P_α, the α-th stage of some forcing iteration.)

1.1. Ord-transitive models

Let M be a countable set such that $(M,\in)$ satisfies ZFC*, a subset of ZFC.[2] We do not require M to be transitive or elementary. ON denotes the class of ordinals. We use ON^M to denote the set of $x \in M$ such that M thinks that $x \in$ ON; similarly for other definable classes. This notation can formally be inconsistent with the following notation (but as usual we assume that the reader knows which variant is used): [3] For a definable set such as ω_1, we use ω_1^M to denote the element x of M such that M thinks that x satisfies the according definition.

[2] We assume that ZFC* contains a sufficient part of ZFC, in particular extensionality, pairing, product, set-difference, emptyset, infinity and the existence of ω_1.

[3] If M is not transitive, then for example the set $x = \{\alpha \in M : M \vDash \alpha \in \omega_1\}$ will generally be different from the element $y \in M$ such that $M \vDash y = \omega_1$. In that case $x \notin M$.

Definition 1.1.

- M is ord-absolute, if $\omega^M = \omega$, $\omega \subseteq M$, and $\mathrm{ON}^M \subseteq \mathrm{ON}$ (and therefore $\mathrm{ON}^M = M \cap \mathrm{ON}$).
- M is ord-transitive, if it is ord-absolute and $x \in M \setminus \mathrm{ON}$ implies $x \subset M$.

An elementary submodel $N \prec H(\chi)$ is not ord-transitive. The simplest example of an ord-transitive model that is not transitive is the ord-collapse of an elementary submodel:

Definition 1.2. Define $\text{ord-col}^M : M \to V$ as the transitive collapse of M fixing the ordinals:

$$\text{ord-col}^M(x) = \begin{cases} x & \text{if } x \in \mathrm{ON} \\ \{\text{ord-col}^M(t) : t \in x \cap M\} & \text{otherwise.} \end{cases}$$

$\text{ord-col}(M) := \{\text{ord-col}^M(x) :\ x \in M\}$.

By induction one can easily show:

Fact 1.3. Assume that M is ord-absolute and set $i := \text{ord-col}^M$, $M' := \text{ord-col}(M)$. Then

- $i : M \to M'$ is an $\in$-isomorphism.
- $i(x) \in \mathrm{ON} \leftrightarrow x \in \mathrm{ON}$. In particular, $M \cap \mathrm{ON} = M' \cap \mathrm{ON}$.
- M' is ord-transitive.
- i is the identity iff M is ord-transitive.
- The ord-collapse "commutes" with the transitive collapse, i.e., the transitive collapse of the ord-collapse of M is the same as the transitive collapse of M.

So if $N \prec H(\chi)$ and $H(\chi) \vDash \mathrm{ZFC}^*$, then $M = \text{ord-col}(N)$ is an ord-transitive model. This example demonstrates that several simple formulas (that are absolute for transitive models), such as "$x \subset z$", "$x \cup y = z$" and "$x \cap y = z$", are *not* absolute for the ord-transitive models.[4] However, a few simple properties are absolute: In particular, if a formula $\varphi(r)$ about real numbers is absolute for all transitive models, then is absolute for all ord-transitive models as well (which can easily be seen using the transitive collaps, cf. the following Fact 1.5). We now mention some of these absolute properties for ord-transitive models M:

- $x \in \omega^\omega$ is absolute; every $\boldsymbol{\Sigma}^1_1$ formula is absolute;
- "Finite sets" are absolute: $z = \{x,y\}$ is absolute, if $x \in M$ and x is finite, then $x \subset M$ and $M \vDash$ "x is finite". $H^M(\aleph_0) = H(\aleph_0)$.

[4] "$\varphi(\bar{x})$ is absolute" means $M \vDash \varphi(\bar{m})$ iff $V \vDash \varphi(\bar{m})$ for all $\bar{m}$ from M. Let i be the ord-collapse from an elementary submodel N to M. Set $x = \omega_1$, $y = \{\{0\}\}$ and $z = x \cup y$. Then $x \in \mathrm{ON}$ and $z \notin \mathrm{ON}$, so $i(x) = x$ and $i(z)$ is countable. Therefore $i(x) \cup i(y) \neq i(z)$, and $i(x) \not\subseteq i(z)$. Also, $i(z) \cap i(x) \neq i(x)$.

- If $M \vDash f : A \to B$, then $f : A \cap M \to B \cap M$. If additionally M thinks that f is injective (or surjective), then f is injective (or surjective with respect to the new image).

- $x \in R_\alpha$ is upwards absolute. If additionally $x \notin \mathrm{ON}$, then $|x| \leq |\alpha|$ is upwards absolute.

- If either $x \in \mathrm{ON}$, or $x \cap \mathrm{ON} = \emptyset$, then $y \subset x$ is absolute.

Instead of ord-transitive models, we could equivalently use transitive models with an (ordinal) labeling on the ordinals:

Definition 1.4. A labeled model is a pair (M, f) consisting of a transitive, countable ZFC* model M and a strictly monotonic function $f : (M \cap \mathrm{ON}) \to \mathrm{ON}$ satisfying $f(\alpha) = \alpha$ for $\alpha \leq \omega$.
Given a labeled model (M, f), define a map $i : M \to V$ by

$$i(x) = \begin{cases} f(x) & \text{if } x \in \mathrm{ON} \\ \{i(y) : y \in x\} & \text{otherwise.} \end{cases}$$

Set $\mathrm{uncoll}(M, f) := i[M]$.
Given an ord-transitive model M, let $j : M \to M'$ be the transitive collapse (an $\in$-isomorphism) and let $f : M' \cap \mathrm{ON} \to \mathrm{ON}$ be the inverse of j. Define $\mathrm{labeledcoll}(M) := (M', f)$.

By induction, one can prove the following:

Fact 1.5. If M is an ord-transitive model, then $\mathrm{labeledcoll}(M)$ is a labeled model and $\mathrm{uncoll}(\mathrm{labeledcoll}(M)) = M$. If (M, f) is a labeled model, then $\mathrm{uncoll}(M, f)$ is an ord-transitive model and $\mathrm{labeledcoll}(\mathrm{uncoll}(M, f)) = (M, f)$.

We say that the ord-transitive model M and the labeled model (M', f') correspond to each other, if $M = \mathrm{uncoll}(M', f')$ or equivalently $(M', f') = \mathrm{labeledcoll}(M)$. So each ord-transitive model corresponds to exactly one labeled model and vice versa.

This also shows that is easy to create "weird" ord-transitive models; in particular "α is successor ordinal" and similarly simple formulas are generally not absolute for ord-transitive models. We will generally not be interested in such weird models:

Definition 1.6. Let M be ord-transitive.

- M is "successor-absolute", if "α is successor" and "$\alpha = \beta + 1$" both are absolute between M and V.

- M is $\mathrm{cf}\,\omega$-absolute, if M is successor absolute and "$\mathrm{cf}(\alpha) = \omega$" and "$A$ is a countable cofinal subset of α" both are absolute between M and V.

Fact 1.7. If M is $\mathrm{cf}\,\omega$-absolute and M thinks that x is countable, then $x \subset M$.

Proof. If $x \notin \mathrm{ON}$, then $x \subseteq M$. So assume towards a contradiction that $x \in \mathrm{ON}$ is minimal with $x \not\subset M$ (and $x < \omega_1^M$). M thinks that $y := x \setminus \{0\}$ (constructed in M) is countable and cofinal in x. Since $y \notin \mathrm{ON}$ we know $y \subset M$, so $x = \bigcup_{\alpha \in y} \alpha$ is a subset of M, since x was the minimal counterexample.) ◻

M is successor-absolute iff the corresponding labeled model (M', f') satisfies: $f(\alpha+1) = f(\alpha)+1$ and $f(\delta)$ is a limit ordinal for all limit ordinals δ.

Remark 1.8.

- We will see in the next section how to construct forcing extensions for ord-transitive models M, or equivalently labeled models (M', f'): If G is M-generic, and G' the image under the transitive collapse (which will be M'-generic), then the forcing extension $M[G]$ is just the ord-transitive model corresponding to $(M'[G'], f')$. Such forcing extensions are the most important "source" for ord-transitive models that are not just (the ord-transitive collapse of) an elementary model.
- In applications, we typically have to deal with ord-transitive models that are internal forcing extensions of elementary models (i.e., in the construction above G is in V and M is the ord-collapse of $N \prec H(\chi)$).
- All such models are successor-absolute (and satisfy many additional absoluteness properties). So for applications, it is enough to only consider successor-absolute models, and restrictions of this kind sometimes make notation easier.
- Ord-collapses M of elementary submodels are $\mathrm{cf}\,\omega$-absolute. The same holds for (internal) forcing extensions $M[G]$ by proper forcing notions. However, general (internal) forcing extensions $M[G]$ will not be $\mathrm{cf}\,\omega$-absolute: E.g. if G is generic for a Levy collapse, then $M[G]$ will think that ω_1^V is countable. In some applications (such as the the preservation theorem mentioned in the introduction) it is essential to use such collapses, therefore we generally cannot restrict ourselves to $\mathrm{cf}\,\omega$-absolute models. However, for other applications, $\mathrm{cf}\,\omega$-absolute models are sufficient (e.g., for the application mentioned in Section 1.3).

Every ord-transitive model is hereditarily countable modulo ordinals:

Definition 1.9.

- We define ord-clos by induction: $\text{ord-clos}(x) = x \cup \bigcup\{\text{ord-clos}(t) : t \in x \setminus \mathrm{ON}\}$.
- $\mathrm{hco}(\alpha) = \{x \in R_\alpha : |\text{ord-clos}(x)| \leq \aleph_0\}$
- $\mathrm{hco} = \bigcup_{\alpha \in \mathrm{ON}} \mathrm{hco}(\alpha)$.

For example, if $\alpha > \omega_1$, then ω_1 is element of $\mathrm{hco}(\alpha)$, but $\omega_1 \cup \{\{\emptyset\}\}$ or $\omega_1 \setminus \{\emptyset\}$ are not.

Facts 1.10.

- $\text{ord-clos}(M)$ is the smallest ord-transitive superset of M.
- An ord-absolute ZFC*-model M is ord-transitive iff $\text{ord-clos}(M) = M$.
- If M is ord-transitive and countable, then $M \in \mathrm{hco}$.

- If M is ord-transitive and $x \in M$, then $\text{ord-clos}(x) = \text{ord-clos}^M(x) \subseteq M$.
- "$x \in \text{hco}(\alpha)$" is upwards absolute for ord-transitive models.

As already mentioned, there is an ord-transitive model M such that ω_1^V is countable in M. So M thinks that ω_1^V is not just element of hco (which is true in V as well), but that it can also be constructed as countable set (which is false in V).

1.2. Forcing extensions

Forcing still works for ord-transitive models (but the evaluation of names has to be modified in the natural way). In the following, M always denotes an ord-transitive model.

Definition 1.11. Let M think that $\leq$ is a partial order on P. So in V, $\leq$ is a partial order on $P \cap M$. Then G is called P-generic over M (or just M-generic, or P-generic), if $G \cap P \cap M$ is a filter on $P \cap M$ and meets every dense subset $D \in M$ of P.[5]

To simplify notation, we will use the following assumption:

Assumption 1.12. $P \cap \text{ON}$ is empty. (Then in particular $P \subseteq M$, and we can write P instead of $P \cap M$. Also, if $D \subset P$ is in M, then $D \subset M$.)

In Definition 1.11 we do not assume $G \subseteq P$. This slightly simplifies notation later on. Obviously G is M-generic iff $G \cap P$ is M-generic. One could equivalently use maximal antichains, predense sets, or open dense sets instead of dense sets in the definition (and one can omit the "filter" part if one requires that a maximal antichain A in M meets the filter G in exactly one point).

Let $\text{labeledcoll}(M) = (M', f')$ be the labeled model corresponding to M, via the transitive collapse j. Let $G \subseteq P$ and set $P' := j(P)$ and $G' := j[G]$. Since the transitive collapse is an isomorphism, G' is P'-generic over M' iff G is P-generic over M. In that case we can form the forcing extension $M'[G']$ in the usual way, and define $M[G] = \text{uncoll}(M'[G'], f')$ as the ord-transitive model corresponding to $(M'[G'], f')$. Let $J : M[G] \to M'[G']$ be the transitive collapse, and I its inverse, then we can define $\underset{\sim}{\tau}[G]^M$ as $I(J(\underset{\sim}{\tau})[G'])$ for a P-name $\underset{\sim}{\tau}$ in M. Elementarity shows that this is a "reasonable" forcing extension.

We now describe this extension in more detail and using the ord-transitive model M more directly:

Basic forcing theory shows: If M is a transitive model, $P \in M$, and G a P-generic filter over M, then we can define the evaluation of names by

$$\underset{\sim}{\tau}[G] = \{\underset{\sim}{\sigma}[G] : (\underset{\sim}{\sigma}, p) \in \underset{\sim}{\tau},\ p \in G\}, \tag{1.1}$$

and $M[G]$ will be a (transitive) forcing extension of M.

[5] I.e.: If $p, q \in G \cap P \cap M$, then there is a $r \leq p, q$ in $G \cap P \cap M$; and if $D \in M$ and M thinks that D is a dense subset of P (or equivalently: $D \cap M$ is a dense subset of $P \cap M$) then $G \cap D \cap M$ is nonempty.

This evaluation of names works for elementary submodels as well, provided that G is not only N-, but also V-generic. More exactly: If $N \prec H(\chi)$ contains P, and if G is N- *and* V-generic, then $N[G]$ is a forcing extension of N (and in particular end-extension). Here it is essential that G is V-generic as well: If $N \prec H(\chi)$ and $G \in V$ is N-generic (for any nontrivial forcing P), then $N[G]$ is not an end-extension of N, since $G \in \mathcal{P}(P) \in N$, but $G \notin N$.

This can be summarized as follows:

Fact 1.13. Assume that either M is transitive and G is M-generic, or that $M \prec H(\chi)$ and G is M- and V-generic. Then

- $M[G] \supset M$ is an end-extension[6] (i.e., if $y \in M[G]$ and $y \in x \in M$, then $y \in M$), and $\mathrm{ON}^{M[G]} = \mathrm{ON}^M$.
- $M[G] \vDash \varphi(\underset{\sim}{\tau}[G])$ iff $M \vDash p \Vdash \varphi(\underset{\sim}{\tau})$ for some $p \in G$.

In the transitive case $M[G]$ is transitive; and in the elementary submodel case, we get:

- $(M[G], \epsilon, M) \prec (H^{V[G]}(\chi), \epsilon, H^V(\chi))$.
- Forcing extension commutes with transitive collapse: Let i be the transitive collapse of M, and I of $M[G]$. Then I extends i, $i[G]$ is $i[M]$ -generic and $i(\underset{\sim}{\tau})[i[G]] = I(\underset{\sim}{\tau}[G])$.

If one considers general ord-transitive candidates M (i.e., M is neither transitive nor an elementary submodel), then Definition (1.1) does not work any more. For example, if M is countable and thinks that $\underset{\sim}{\tau}$ is a standard name for the ordinal ω_1^V, then $\underset{\sim}{\tau} \subset M$ is countable, so $\underset{\sim}{\tau}[G]$ will always be countable and different from ω_1^V. This leads to the following natural modification of (1.1):

Definition 1.14. Let G be P-generic over M, and let M think that $\underset{\sim}{\tau}$ is a P-name.

$$\underset{\sim}{\tau}[G]^M := \begin{cases} x, \text{ if } x \in M \,\&\, (\exists p \in G \cap P)\, M \vDash \text{“} p \Vdash \underset{\sim}{\tau} = \check{x} \text{”} \\ \{\underset{\sim}{\sigma}[G]^M : (\exists p \in G \cap P)\, (\underset{\sim}{\sigma}, p) \in \underset{\sim}{\tau} \cap M\} \text{ otherwise.} \end{cases}$$

$M[G] := \{\underset{\sim}{\tau}[G]^M : \underset{\sim}{\tau} \in M,\ M \vDash \text{“}\underset{\sim}{\tau} \text{ is a } P\text{-name”}\}$.

(Note that being a P-name is absolute.)

We usually just write $\underset{\sim}{\tau}[G]$ instead of $\underset{\sim}{\tau}[G]^M$. There should be no confusion which notion of evaluation we mean, 1.14 or (1.1), which we can also write as $\underset{\sim}{\tau}[G]^V$:

- If M is transitive, then $\underset{\sim}{\tau}[G]^M = \underset{\sim}{\tau}[G]^V$.

[6] Any usual concept of forcing extension (with regard to pairs of $\in$-models) will require that $M[G]$ is an end-extension of M: If $\underset{\sim}{\tau}$ is forced to be in some x with $x \in V$, then the value of $\underset{\sim}{\tau}$ can be decided by a dense set. Similarly, we get: M is $M[G]$ intersected with the transitive closure of M.

- If M is elementary submodel (and G is M- and V-generic), then we use $\underset{\sim}{\tau}[G]^V$. ($\underset{\sim}{\tau}[G]^M$ does not lead to a meaningful forcing extensions.)[7]
- If M is ord-transitive, then we use $\underset{\sim}{\tau}[G]^M$.

Remark 1.15. The omission of M in $\underset{\sim}{\tau}[G]^M$ should not hide the fact that for ord-transitive models, $\underset{\sim}{\tau}[G]^M$ trivially *does* depend on M: If for example $M_1 \cap \beta = \alpha < \beta$ and M_1 thinks that $\underset{\sim}{\tau}$ is a standard name for β, and if M_2 contains P, $\underset{\sim}{\tau}$ and α, then then $\underset{\sim}{\tau}[G]^{M_1} = \beta \neq \underset{\sim}{\tau}[G]^{M_2}$.

$\underset{\sim}{\tau}[G]$ is well-defined only if G is M-generic, or at least a filter. (If G contains $p_0 \perp_P p_1$, then there is (in M) a name $\underset{\sim}{\tau}$ and $x_0 \neq x_1$ such that p_i forces $\underset{\sim}{\tau} = x_i$ for $i \in \{0, 1\}$.)

If M is ord-transitive then the basic forcing theorem works as usual (using the modified evaluation):

Theorem 1.16. Assume that M is ord-transitive and that G is M-generic. Then

- $M[G]$ is ord-transitive.
- $M[G] \supset M$ is an end-extension. $\mathrm{ON}^{M[G]} = \mathrm{ON}^M$.
- $M[G] \vDash \varphi(\underset{\sim}{\tau}[G])$ iff $M \vDash p \Vdash \varphi(\underset{\sim}{\tau})$ for some $p \in G \cap P$.

Moreover, the transitive collapse commutes with the forcing extension: Let (M', f') correspond to M, and G' the image of G under the transitive collapse. Then $(M'[G'], f')$ corresponds to $M[G]$.

(The proof is a straightforward induction.) So forcing extensions of ord-transitive models behave just like the usual extensions. For example, we immediately get:

Corollary 1.17. If M is countable and ord-transitive, then $M \vDash$ "$p \Vdash \varphi(\underset{\sim}{\tau})$" iff $M[G] \vDash \varphi(\underset{\sim}{\tau}[G])$ for every M-generic filter G (in V) containing p.

Fact 1.18. Assume that N is ord-transitive, $M \in N$, $P \in M$. Then the following are absolute between N and V (for $G \in N$ and $\underset{\sim}{\tau} \in M$):

- M is ord-transitive.
- G is M-generic, and
- (assuming M is ord-transitive and G is M-generic) $\underset{\sim}{\tau}[G]^M$.

The last item means that we get the same value for $\underset{\sim}{\tau}[G]^M$ whether we calculate it in N or V. It does *not* mean $\underset{\sim}{\tau}[G]^M = \underset{\sim}{\tau}[G]^N$. (If $\underset{\sim}{\tau}$ is in M, then $\underset{\sim}{\tau}[G]^N$ will generally not be an interesting or meaningful object.)

[7] If M is not ord-transitive, e.g., $M \prec H(\chi)$, then $\underset{\sim}{\tau}[G]^M$ does not lead to a meaningful forcing extension: Let P be the countable partial functions from ω_1 to ω_1, and let G be M-generic (G can additionally be V-generic as well). Let $\underset{\sim}{\Gamma} \in M$ be the canonical name for the generic filter G. So $\underset{\sim}{\Gamma}[G]^M$ is countable. Since P is σ-closed, $\underset{\sim}{\Gamma}[G]^M \in \mathcal{P}^V(P) \in M$, so $M[G]$ (using the modified evaluation) is not an end-extension of V.

Let us come back once more to the proper case. By induction on the rank of the names we get that the ord-collapse and forcing extension commute:

LEMMA 1.19. Assume that $N \prec H(\chi)$, and $P \in N$. Let $i : N \to M$ be the ord-collapse.

- $G \subseteq P$ is N-generic iff $i[G]$ is M-generic.
- Assume that G is N- and V-generic. Then the ord-collapse I of $N[G]$ extends i, and $I(\underset{\sim}{\tau}[G]) = (i(\underset{\sim}{\tau}))[i[G]]$.
- If $P \subseteq \mathrm{hco}$, then i is the identity on P.

1.3. M-complete subforcings

In the rest of the paper, we will use ord-transitive models in the context of definable proper forcings (similar to Suslin proper). But first let us briefly describe another, simpler, setting in which ord-transitive models can be used.

Let M be a countable transitive model and Q^M a forcing notion in M. We say that Q^M is an M-complete subforcing of $Q \in V$, if Q^M is a subforcing of Q and every maximal antichain $A \in M$ of Q^M is a maximal antichain in Q as well. So there are two differences to the "proper" setting: Q^M and Q do not have to be defined by the same formula,[8] and we do not just require that below every condition in Q^M we find a Q^M-generic condition in Q, but that already the empty condition is Q^M-generic.[9]

For transitive models, this concept has been used for a long time. It is, e.g., central to Shelah's oracle-cc [8, IV]. In oracle-cc forcing, one typically constructs a forcing notion Q of size $\aleph_1$ as follows: Construct (by induction on $\alpha \in \omega_1$) an increasing (non-continuous) sequence of countable transitive models M^α (we can assume that M^α knows that α is countable), and forcing notions $Q^\alpha \in M^\alpha$ such that $Q^\alpha \subseteq \alpha$ (so Q^α is forcing equivalent to Cohen forcing, but this is not the right way to think about Q^α). We require that $Q^\delta = \bigcup_{\beta<\delta} Q^\beta$ for limits δ and that $Q^{\beta+1}$ is an M^β-complete superforcing of Q^β. We set $Q = \bigcup_{\beta<\omega_1} Q^\beta$. So each Q^α will be M^α-complete subforcing of Q. So we use the pair (M^α, Q^α) as an approximation to the final forcing notion Q. Since we use transitive models, this Q has to be subset of $H(\aleph_1)$.

If we want to investigate larger forcing notions, we can try to use ord-transitive models instead. For example, in [2] we use a forcing iteration $\bar{P} = (P_\alpha, \underset{\sim}{Q}_\alpha)_{\alpha \le \omega_2}$ (where each $\underset{\sim}{Q}_\alpha$ consists of conditions in $H(\aleph_1)$), and we "approximate" $\bar{P}$ by pairs $(M^x, \bar{P}^x)$, where M^x is a countable ord-transitive model and M^x thinks that $\bar{P}^x$ is a forcing iteration of length ω_2^V. Instead of assuming that $P^x_{\omega_2}$ is a subforcing of P_{ω_2}, it is more natural to assume (inductively) that each P^x_α can be canonically (and in particular M^x-completely) embedded into P_α, and that P_α forces that $Q^x_\alpha[G^x_\alpha]$ (evaluated by the induced P^x_α-generic filter $[G^x_\alpha]$) is an $M^x[G^x_\alpha]$-complete subforcing of Q_α. We show that

[8] They do not have to be nicely definable at all, and furthermore Q^M and Q can be entirely different: E.g., Q^M could be Cohen forcing in M and Q could be (equivalent to) random forcing in V.

[9] In the proper case, this is equivalent to "Q is ccc".

given $\bar{P}^x$ in a countable ord-transitive model M^x we can find variants of the finite support and the countable support iterations $\bar{P}$ such that $\bar{P}^x$ canonically embeds into $\bar{P}$ (and we show that some preservation theorems that are known for proper countable support iterations also hold for this variant of countable support). For this application it is enough to consider cf ω-absolute models.

In the current paper, we do something very similar (in the nep setting, i.e., the definable/proper framework), in Sections 4.1 and 4.3. Let us again stress the obvious difference: In the nep case, we use definable forcings, and Q^x_α is the evaluation in $M^x[G^x_\alpha]$ of the same formula that defines Q_α in $V[G_\alpha]$, and we just get that below (the canonical image of) each $p \in P^x_\alpha$ there is some M^x-generic $q \in P_\alpha$.

In particular, the application of non-wellfounded models in [2] does not use any of the concepts that are introduced in the rest of this paper.

2. Nep forcing

2.1. Candidates

We now turn our attention to definable forcings. More particularly, we will require that for all suitable (ord-transitive) models M, "$x \in Q$" is upward absolute between M and V.[10] Also, we will require that for all $x \in Q$ there is a model M knowing that $x \in Q$. This is only possible if $Q \subset \mathrm{hco}$ (since every countable ord-transitive model is hereditarily countable modulo ordinals), but it is not required that $Q \subseteq H(\aleph_1)$ (as it is the case when using countable transitive models only).

It is natural to allow parameters other than just reals. The following is a simple example of a definable iteration using a function $\mathfrak{p} : \omega_1 \to 2$ as parameter: $(P_\beta, Q_\beta)_{\beta<\omega_1}$ is the countable support iteration such that Q_β is Miller forcing if $\mathfrak{p}(\beta) = 0$ and random forcing if $\mathfrak{p}(\beta) = 1$.

Once we use such a parameter $\mathfrak{p}$, we of course cannot assume that $\mathfrak{p}$ is in the model M (since M is countable and ord-transitive). Instead, we will assume that M contains its own version $\mathfrak{p}_M$ of the parameter; in our example we would require that $\delta := \omega_1^V \in M$ and that M thinks that $\mathfrak{p}_M$ is a function from δ to 2, (so really $\mathrm{dom}(\mathfrak{p}_M) = \delta \cap M$) and we require that $\mathfrak{p}_M(\beta) = \mathfrak{p}(\beta)$ for all $\beta \in M$.

More generally we define "$\mathfrak{p}$ is a nep parameter" by induction on the rank: $\emptyset$ is a nep-parameter, and

Definition 2.1. $\mathfrak{p}$ is a nep-parameter, if $\mathfrak{p}$ is a function with domain $\beta \in \mathrm{ON}$ and $\mathfrak{p}(\alpha)$ is a nep-parameter for all $\alpha \in \beta$.
Let M be an ord-transitive model. Then $\mathfrak{p}_M$ is the M-version of $\mathfrak{p}$, if $\mathrm{dom}(\mathfrak{p}_M) = \mathrm{dom}(\mathfrak{p}) \cap M$ and $\mathfrak{p}_M(\alpha)$ is the M-version of $\mathfrak{p}(\alpha)$ for all $\alpha \in \mathrm{dom}(\mathfrak{p}_M)$.

In other words: A nep-parameter is just an arbitrary set together with a hereditary wellorder.

[10] There are useful notions similar to nep without this property. Examples for such forcings appear naturally when iterating nep forcings, cf. Subsection 4.1.

If M contains $\mathfrak{p}_M$, then M thinks that $\mathfrak{p}_M$ is a nep-parameter (and if $\beta = \mathrm{dom}(\mathfrak{p})$, then $\beta \in M$ and M thinks $\beta = \mathrm{dom}(\mathfrak{p}_M)$).

We can canonically code a real r, an ordinal, or a subset of the ordinals as a nep-parameter.

Definition 2.2. Let $\mathfrak{p}$ be a nep-parameter. M is a $(\mathrm{ZFC}^*, \mathfrak{p})$-candidate, if M is a countable, ord-transitive, successor absolute model of ZFC* and contains $\mathfrak{p}_M$, the M-version of $\mathfrak{p}$.

We can require many additional absoluteness conditions for candidates, e.g., the absoluteness of the canonical coding of $\alpha \times \alpha$, or $\mathrm{cf}\,\omega$-absoluteness. The more conditions we require, the less candidates we will get, i.e., the weaker the properness notion "for all candidates, there is a generic condition" is going to be. In practice however, these distinctions do not seem to matter: All nep forcings will satisfy the (stronger) official definition, and for all applications weaker versions suffices.

To be more specific: Most applications will only use properness for candidates M that satisfy

(2.1) M is an internal forcing extension of an elementary submodel N.

More exactly: We start with $N \prec H(\chi)$, pick some $P \in N$, set $(N', P') = \text{ord-col}(N, P)$, and let $G \in V$ be P'-generic over N'. Some application might also use

(2.2) M is an elementary submodel in a P-extension, for a σ-complete P.

More exactly: Let P be σ-complete, pick in the P-extension $V[G]$ some $N \prec H^{V[G]}(\chi)$ and let N' be the ord-collapse. Then N' is in V (and an ord-transitive model).

Of course all these models satisfy a variety of absoluteness properties (such as the canonical coding of $\alpha \times \alpha$ etc). So for all applications, it would be enough to consider candidates that satisfy (2.1) (or some exotic application might need (2.2)), but we we do not make the properties (2.1) or (2.2) part of the official definition of "candidate", since both properties are much more complicated (and less absolute) than just "M is a countable, ord-transitive ZFC*-model".

Note however that generally we can *not* assume that the P used in (2.1) is proper or even just ω_1-preserving. For example in the application in [5], we need P to be a collapse of $\aleph_1$. So in particular we can not assume that all candidates are $\mathrm{cf}\,\omega$-absolutene.

We will only be interested in the normal case:

Definition 2.3. ZFC* is normal, if $H(\chi) \models \mathrm{ZFC}^*$ for sufficiently large regular χ.

Sometimes we will assume that ZFC* is element of a candidate M. This allows us to formulate, e.g., "M thinks that M' is a candidate". We can guarantee this by choosing ZFC* recursive, or by coding it into $\mathfrak{p}$.

Lemma 2.4. 1. (Assuming normality.) If $N \prec H(\chi)$ contains $\mathfrak{p}$, and $(M, \mathfrak{p}_M)$ is the ord-collapse of $(N, \mathfrak{p})$, then M is candidate and $\mathfrak{p}_M$ is the M-version of $\mathfrak{p}$.

2. The statements "$\mathfrak{p}_M$ is the M-version of $\mathfrak{p}$" is absolute between transitive universes. If $\mathfrak{p}_M$ is the M-version of $\mathfrak{p}$, and M thinks that M' is ord-transitive and that $\mathfrak{p}_{M'}$ is the M'-version of $\mathfrak{p}_M$, then $\mathfrak{p}_{M'}$ is the M'-version of $\mathfrak{p}$.
3. If $M[G]$ is a forcing extension of M, and $\mathfrak{p}_M$ the M-version of $\mathfrak{p}$, then $\mathfrak{p}_M$ is also the $M[G]$-version of $\mathfrak{p}$.
4. For $x \in \text{hco}$, a nep parameter $\mathfrak{p}$ and a theory T in the language $\{\in, c^x, c^{\mathfrak{p}}\}$, the existence of a candidate M containing x such that $(M, \in, x, \mathfrak{p}_M)$ satisfies T is absolute between universes containing ω_1^V (and, of course, x,$\mathfrak{p}$ and T).

This is straightforward, apart from the last item, which follows from the following modification of Shoenfield absoluteness.

Remark 2.5. Shelah's paper [9] uses another notion of nep-parameter With our definition, for every $\mathfrak{p}$ and M there is exactly one M-version $\mathfrak{p}_M$ of $\mathfrak{p}$, but this is not the case for Shelah's notion. (There, a candidate is defined as pair $(M, \mathfrak{p}_M)$ such that $\mathfrak{p}_M \in M$ is *some* M-version of $\mathfrak{p}$.) Both notions satisfy Lemma 2.4.

Lemma 2.6. Assume that

- S is a set of sentences in the first order language using the relation symbol $\in$ and the constant symbols $c^x, c^{\mathfrak{p}}$,
- $\text{ZFC}^* \subseteq \text{ZFC}$,
- L' is a transitive ZFC-model (set or class) containing ZFC^*, ω_1^V, $\mathfrak{p}$, and S,
- $x \in \text{hco}^{L'}$.

If in V there is a $(\text{ZFC}^*, \mathfrak{p})$-candidate M containing x such that $(M, \in, x, \mathfrak{p}_M) \models S$, then there is such a candidate in L'.

Proof. We call such a candidate a good candidate. So we have to show:

(2.3) If there is a good candidate in V, then there is one in L'.

Just as in the proof of Shoenfield absoluteness, we will show that a good candidate M corresponds to an infinite descending chain in a partial order T defined in L'. (Each node of T is a finite approximation to M). Then we use that the existence of such a chain is absolute.

We define for a nep-parameter y

$$\text{f-clos}(y) = \{y(a) : a \in \text{dom}(y)\} \cup \bigcup_{a \in \text{dom}(y)} \text{f-clos}(y(a)). \tag{2.4}$$

So every $z \in \text{f-clos}(y)$ is again a nep-parameter.

Fix in L' for every $y \in (\{x\} \cup \text{trans-clos}(x)) \setminus \text{ON}$ an enumeration

$$y = \{f^y(n) : n \in \omega\}. \tag{2.5}$$

Also in L', we fix some $\delta \geq \omega_1^V$ bigger than every ordinal in $\{x\} \cup \text{trans-clos}(x)$ and bigger than $\text{dom}(y)$ for every $y \in \text{f-clos}(\mathfrak{p}) \cup \{\mathfrak{p}\}$.

We can assume that S contains ZFC* as well as the sentence "$c^{\mathfrak{p}}$ is a nep-parameter". We use (in L') the following fact:

> Let S be a theory of the countable (first-order) language $\mathcal{L}_S$. Then there is a theory S' (of a countable language $\mathcal{L}_{S'} \supset \mathcal{L}_S$) such that the deductive closure of S' is a conservative extension of S, and every sentence in S' has the form $(\forall x_1)(\forall x_2)\dots(\forall x_n)(\exists y)\psi(x_1,\dots,x_n,y)$ for some quantifier free formula ψ (using new relation symbols of S').

So we fix S' and $\mathcal{L}'$, consisting of relation symbols R_i ($i \in \omega$) of arity $r_i \geq 1$, and constant symbols c_i ($i \in \omega$). We can assume that there are constant symbols for ω and for each natural number. We can further assume

- $c_0 = c^x$, $c_1 = c^{\mathfrak{p}}$,
- $R_0 = R^{\in}(x,y)$ expresses $x \in y$,
- $R_1 = R^{\text{dom}}(x,y)$ expresses "x is a function and $\text{dom}(x) = y$",
- $R_2 = R^{\text{f-clos}}(x)$ expresses $x \in \text{f-clos}(c^{\mathfrak{p}}) \cup \{c^{\mathfrak{p}}\}$,
- $R_3 = R^{\text{ON}}(x)$ expresses $x \in \text{ON}$.

We set $\mathcal{L}'_i = \{R_0 \dots R_{i-1}, c_0 \dots c_{i-1}\}$. and fix an enumeration $(\varphi_i)_{i\in\omega}$ of all sentences in S' such that φ_i is a $\mathcal{L}'_i$-sentence. We now define the partial order T as follows: A node $t \in T$ consists of the natural number n^t, the sequences $(c_i^t)_{i \leq n^t}$ and $(R_i^t)_{i \leq n^t}$, and the following functions with domain n^t: ord-val^t, $x\text{-val}^t$, $\mathfrak{p}\text{-val}^t$, and rk^t such that the following is satisfied:

- $n^t \geq 4$. We interpret $n^t = \{0,\dots,n^t - 1\}$ to be the universe of the following $\mathcal{L}'_{n^t}$-structure: $c_i^t \in n^t$ is the t-interpretation of c_i and $R_i^t \subseteq (n^t)^{r_i}$ is the t-interpretation of R_i for all $i < n^t$.
- $\text{ord-val}^t : n^t \to \delta \cup \{\text{na}\}$. If c_i is the constant symbol for some $m \leq \omega$, then $\text{ord-val}^t(c_i^t) = m$. If $\text{ord-val}^t(a) \neq \text{na}$, then we have the following: $R^{\text{ON}^t}(a)$ holds, and $R^{\in^t}(b,a)$ holds iff $\text{ord-val}^t(b) \in \text{ord-val}^t(a)$. (Where we use the notation that $\text{na} \notin y$ for all y.)
- $x\text{-val}^t : n^t \to \{x\} \cup \text{trans-clos}(x) \cup \{\text{na}\}$ such that $x\text{-val}^t(c^{x^t}) = x$. If $x\text{-val}^t(a) \notin \text{ON} \cup \{\text{na}\}$, then $R^{\in^t}(b,a)$ iff $x\text{-val}^t(b) \in x\text{-val}^t(a)$. If $x\text{-val}^t(a) \in \text{ON}$, then $x\text{-val}^t(a) = \text{ord-val}^t(a)$.
- $\mathfrak{p}\text{-val}^t : n^t \to \{\mathfrak{p}\} \cup \text{f-clos}(\mathfrak{p}) \cup \{\text{na}\}$ such that $\mathfrak{p}\text{-val}^t(c^{\mathfrak{p}^t}) = \mathfrak{p}$ and $\mathfrak{p}\text{-val}^t(a) \neq \text{na}$ iff $R^{\text{f-clos}^t}(a)$. If $R^{\text{f-clos}^t}(a)$ and $R^{\text{dom}^t}(a,b)$, then $\text{ord-val}^t(b) = \text{dom}(\mathfrak{p}\text{-val}^t(a))$.
- $\text{rk}^t : n^t \to \delta$ is a rank-function. I.e., if $R^{\in^t}(a,b)$, then $\text{rk}^t(a) < \text{rk}^t(b)$.

We set $t \geq_T t'$ if

- $n^{t'} \geq n^t$, and all the interpretations and functions in t' are extensions of the ones in t. (So we will omit the indices t and t'.)

- If $i \leq n^t$, and $\varphi_i \in S'$ is the sentence $(\forall x_1) \dots (\forall x_l)(\exists y)\psi(\vec{x}, y)$, then for all $\vec{a}$ in n^t there is a $b \in n^{t'}$ such that $t' \vDash \psi(\vec{a}, b)$.

- Assume that $i < n^t$, $a < n^t$ and x-val$(a) = y \notin \mathrm{ON} \cap \{\mathrm{na}\}$. Then there is a $b < n_{t'}$ such that x-val$(b) = f^y(i)$, cf. (2.5).

Then we get the following:

- T is a partial order.

- The definition of T can be spelled out in L', the definition is absolute, and every node of T is element of L'. So T is element of L'.

- In particular T has an infinite descending chain in L' iff T has one in V.

- T has an infinite descending chain iff there is a good candidate.

Let us show just the last item: Clearly, a suitable candidate defines an infinite descending chain: Given M, we can extend it to an S'-model (since S' is a conservative extension of S) and find a rank function rk for M. Then we can construct a chain as a subset of those nodes $t \in T$ that correspond to finite subsets of M: To every such t we just have to put enough elements into t' to witness the requirements.

On the other hand, a chain defines a candidate: The union of the structures in the chain is a $\mathcal{L}'$-structure M' and an S'-model. The function rk defines a rank on M'. So we can define by induction on this rank a function $i : M' \to V$ the following way:

$$i(x) = \begin{cases} \text{ord-val}(a) & \text{if ord-val}(x) \neq \text{na} \\ \{i(y) : y \in x\} & \text{otherwise.} \end{cases}$$

We set $M' = i[M]$. By induction, i is an isomorphism between $(M', R^{\in}, R^{\mathrm{ON}}, x^{M'}, \mathfrak{p}^{M'})$ and $(M, \in, \mathrm{ON}, x, \mathfrak{p}_M)$, i.e., that M is the required good candidate. $\square$

Remark 2.7.

- If $\mathfrak{p}$ is a real, then the transitive collapse of a candidate still is a candidate. So if x is a real and S as above, the existence of an appropriate candidate is equivalent to the existence of a transitive candidate, which is a $\undertilde{\mathbf{\Sigma}}^1_2$ statement (in the parameters $\mathfrak{p}, x, S$).

- There is also a notion of non-wellfounded non elementary (nw-nep) forcing, cf. [10], where candidates do not have to be wellfounded. Then the existence of a candidate (with a real parameter) is even a $\undertilde{\mathbf{\Sigma}}^1_1$-statement.

2.2. Non elementary proper forcing

We investigate forcing notions Q defined with a nep-parameter $\mathfrak{p}$: $Q = \{x : \varphi_{\in Q}(x, \mathfrak{p})\}$. If M is a (ZFC$^*, \mathfrak{p}$)-candidate, we assume that in M the class $\{x : \varphi_{\in Q}(x, p_M)\}$ is a set, which we will denote by Q^M. Generally such a Q^M does not have to be a subset of M, but to simplify notation (as in Assumption 1.12) we assume that Q is disjoint to ON (we can assume that this requirement is explicitly stated in the formula $\varphi_{\in Q}$). Then $Q^M \subset M$. Analogously, we assume that $q \leq p$ iff $\varphi_{\leq Q}(q, p, \mathfrak{p})$, and that in M, $\{(p, q) : \varphi_{\leq Q}(q, p, \mathfrak{p}_M)\}$ is a quasiorder on Q^M. We write $q \leq^M p$ for $M \vDash \varphi_{\leq Q}(q, p, \mathfrak{p}_M)$. Additionally we require that these formulas are upwards absolute. To summarize:

Definition 2.8.
- M_1 is a candidate in M_2 means the following: M_1 is a candidate, M_2 is either a candidate or $M_2 = V$, $M_1 \in M_2$, and M_2 knows that M_1 is countable.
- $\varphi(x)$ is upwards absolute for candidates means: If M_1 is a candidate in M_2, $a \in M_1$, and $M_1 \vDash \varphi(a)$, then $M_2 \vDash \varphi(a)$.
- A forcing Q is upwards absolutely defined by the nep-parameter $\mathfrak{p}$, if the following is satisfied:
  In V and all (ZFC$^*, \mathfrak{p}$)-candidates M, $\varphi_{\in Q}$ defines a set and $\varphi_{\leq Q}$ defines a quasiorder on this set, and $\varphi_{\in Q}$ and $\varphi_{\leq Q}$ are upwards absolute for candidates.

As usual, we define:

Definition 2.9. $q \in Q$ is Q-generic over M (or just: M-generic), if q forces that (the V-generic filter) G_Q is Q^M-generic over M.

Recall that "G is M-generic" is defined in 1.11. Of course, G_Q will generally not be a subset of Q^M.

Note that "$p \in Q$", "$q \leq p$" and therefore "$p \parallel q$" are upward absolute, but $\perp$ is not. (It will be absolute in most simple examples of nep-forcing, but typically not in nep-iterations or similar constructions using nep forcings as building blocks). This effect is specific for nep forcing, it appears neither in proper forcing (since for $N \prec H(\chi)$, incompatibility always is absolute), nor in Suslin proper (since the absoluteness of incompatibility is part of the definition).

Since $\perp$ is not absolute, "q is M-generic" is generally *not* equivalent to "q forces that all dense D in M meet G". (The V-generic G is not necessarily a Q^M-filter.)

Now we can finally define:

Definition 2.10. Q is a non elementary proper (nep) forcing for (ZFC$^*, \mathfrak{p}$), defined by formulas $\varphi_{\in Q}(x, \mathfrak{p})$, $\varphi_{\leq Q}(x, y, \mathfrak{p})$, if

- Q is upwards absolutely defined for (ZFC$^*, \mathfrak{p}$)-candidates, and
- for all (ZFC$^*, \mathfrak{p}$)-candidates M and for all $p \in Q^M$ there is an M-generic $q \leq p$.

Sometimes we will denote the $\mathfrak{p}$ and ZFC* belonging to Q by $\mathfrak{p}_Q$ and ZFC$^*_Q$ and denote a (ZFC$^*_Q, \mathfrak{p}_Q$)-candidate by "Q-candidate".

We will only be interested in normal forcings:

Definition 2.11. A nep-definition Q is normal, if

- ZFC* is normal (cf. 2.3),
- $Q \subseteq$ hco in V and in all candidates (cf. 1.9),
- "$p \in Q$" and "$q \leq p$" are absolute between V and $H(\chi)$ (for sufficiently large regular χ).

If ZFC* is normal, then the ord-collapse collapse of any $N \prec H(\chi)$ containing $\mathfrak{p}$ is a candidate. So we get:

Lemma 2.12. If Q is normal, then for any $p \in Q$ there is a candidate M such that $q \in Q^M$. If Q is normal and nep, then Q is proper.

Proof. This follows directly from Lemma 2.4 (and the fact that in the definition of proper one can assume that the elementary submodels contain an arbitrary fixed parameter, see e.g. [1, Def. 3.7]). $\square$

As already mentioned, we are only interested in normal forcings, and we will later tacitly assume normality whenever we say a forcing is nep.

Remark 2.13. However, it might sometimes make sense to investigate non-normal nep forcings. Of course such forcings do not have to be proper. An example can be found in [9, 1.19]: We assume CH in V, and define a forcing Q for which we get generic conditions not for all ZFC$^-$ models, but for all models of $2^{\aleph_0} = \aleph_2$. This forcing can collapse $\aleph_1$.

2.3. Some simple properties

Shoenfield absoluteness 2.6 immediately gives us many simple cases of absoluteness. We just give an example: If Q is upward absolutely defined and normal, then $q \leq p$ is equivalent to "there is a candidate M thinking that $q \leq p$". So in particular:

Corollary 2.14. Assume that V' is an extension of V with the same ordinals, and that Q is (normal) nep in V as well as in V'. Then $p \in Q$, $q \leq p$ and $p \parallel q$ are absolute between V and V'. (But "A is a maximal antichain" is only downwards absolute from V' to V.)

The basic theorem of forcing can be formulated as: For a transitive countable model M and P in M

$$(2.6) \quad [M \vDash p \Vdash \varphi(\underset{\sim}{\tau})] \text{ iff } [M[G] \vDash \varphi(\underset{\sim}{\tau}[G]) \text{ for every } M\text{-generic filter } G \in V \text{ containing } p].$$

(And there always is at least one M-generic filter $G \in V$ containing p.)

By 1.17 we get the following:

(2.7) If M is a countable, ord-transitive model and $P \in M$, then (2.6) holds.

With the usual abuse of notation, the essential property of proper forcing can be formulated as follows: If M is an elementary submodel of $H(\chi)$ and Q in M is proper, then

$$(2.8)\quad [M \vDash p \Vdash \varphi(\underset{\sim}{\tau})] \text{ iff } [M[G] \vDash \varphi(\underset{\sim}{\tau}[G]) \text{ for every } M\text{- and } V\text{-generic filter } G \text{ containing } p].$$

(And there always is at least one M- and V-generic filter containing p.)

For nep forcings we get exactly the same:

(2.9) If Q is nep and M a Q-candidate, then (2.8) holds.

If M_1 is a candidate in M_2, and q is Q-generic over M_1, then q does not have to be generic over M_2 (since M_2 can see more dense sets). Of course, the other direction also fails: If q is M_2-generic, then generally it is not M_1-generic (corresponding to the fact in non-ccc proper forcing that not every V-generic filter has to be N-generic): M_1 could think that D is predense, but M_2 could know that D is not, or M_1 could think that $p_1 \perp p_2$, but M_2 sees that $p_1 \parallel p_2$. Even for very simple Q satisfying that $\perp$ is absolute "$\{p_i : i \in \omega\}$ is a maximal antichain" need not be upwards absolute (in contrast to Suslin proper forcing, see example 3.10).

3. Examples

There are oodles of examples nep forcings. Actually:

Rule of Thumb 3.1. Every nicely definable forcing notion that can be proven to be proper is actually nep.

This rule does not seem to be quite true. A very partial potential counterexample is 3.17. However, the rule seems to hold in most cases, and becomes even truer if the proof of properness uses some form of pure decision and fusion, e.g., for σ-closed or Axiom A. (And in these cases, the proof of the nep property is just a trivial modification of the proof of properness.)

Overview of this section:

- Transitive nep forcing: The forcings is a set of reals, the definition uses only a real parameter. In this case it is enough to consider transitive candidates.

  3.1 Suslin proper and Suslin$^+$.

  3.2 A specific example from the theory of creature forcing.

- Non-transitive nep: The forcings are not subsets of $H(\aleph_1)$, and we have to use non-transitive candidates.

  3.3 Trival examples: σ-closed forcings.

  3.4 Products of creature forcings and similar constructions.

  Other examples of of non-transitive nep forcings are iterations of nep forcings. We will investigate countable support iterations in the Section 4.

- Additional topics:

  3.5 Nep, creature forcing, and Zapletal's idealized forcing.

  3.6 Counterexamples: forcings that are not nep.

3.1. Suslin proper forcing

Assume that $Q \subseteq \omega^\omega$ is defined using a real parameter $\mathfrak{p}$.

In this case it is enough to consider transitive candidates: Such a candidate is just a countable transitive model of ZFC* containing $\mathfrak{p}$.[11]

The first notion of this kind was the following:

Definition 3.2. A (definition of a) forcing Q is Suslin in the real parameter $\mathfrak{p}$, if $p \in Q$, $q \leq p$ and $p \perp q$ are $\Sigma^1_1(\mathfrak{p})$.

For Suslin forcings, the nep property is called "Suslin proper":

Definition 3.3.
- Q is Suslin proper, if Q is Suslin and nep. I.e., for every (transitive) candidate M and every $p \in Q^M$ there is an M-generic $q \leq p$.
- Q is Suslin ccc, if Q is Suslin and ccc.

Suslin ccc implies Suslin proper (in a very strong and absolute way, cf. [3]). It seems unlikely that Suslin plus proper implies Suslin proper, but we do not have a counterexample. Cohen, random, Hechler and Amoeba forcing are Suslin ccc. Mathias forcing is Suslin proper.

Some forcings are not Suslin proper just because incompatibility is not Borel, for example Sacks forcing. This motivated a generalization of Suslin proper, Suslin$^+$ [1, p. 357]. It is easy to see that every Suslin$^+$ forcing is nep as well, and that many popular tree-like forcings are Suslin$^+$, e.g., Laver, Sacks and Miller [4].

3.2. An example of a creature forcing

A more general framework for definable forcings is creature forcing, presented in the monograph [6] by Rosłanowski and Shelah. They introduce many ways to build basic

[11] More specifically, the straightforward proof shows that in this case "Q is nep" — i.e. "nep with respect to all ord-transitive models" — is equivalent to: "Q is nep with respect to all transitive models".

forcings out of creatures, and use such basic forcings in constructions such as products or iterations.

Typically, the creatures are finite and the basic creature forcing consist of ω-sequences (or similar hereditarily countable objects made) of creatures. The proofs that such forcings are proper actually give nep. We demonstrate this effect on a specific example (that will also be used in Subsection 3.5). This specific example is in fact Suslin proper, but other simple (and similarly defined) creature forcing notions are nep but not Suslin proper.

We fix a sufficiently fast growing[12] function $\mathbf{F} : \omega \to \omega$ and set

$$k_i^* := \prod_{j<i} \mathbf{F}(j). \tag{3.1}$$

Definition 3.4. An i-creature is a function $\phi : \mathcal{P}(a) \to \omega$ such that

- $a \subseteq \mathbf{F}(i)$ is nonempty.
- ϕ is monotonic, i.e., $b \subset c \subseteq a$ implies $\phi(b) \leq \phi(c)$.
- ϕ has bigness, i.e., $\phi(b \cup c) \leq \max(\phi(b), \phi(c)) + 1$ for all $b, c \subseteq a$.
- $\phi(\emptyset) = 0$ and $\phi(\{x\}) \leq 1$ for all $x \in a$.

We set $\mathrm{val}(\phi) := a$, $\mathrm{nor}(\phi) := \phi(a)$, and we call ϕ_1 stronger than ϕ_0, or: $\phi_1 \leq \phi_0$, if $\mathrm{val}(\phi_1) \subseteq \mathrm{val}(\phi_0)$ and $\phi_1(b) \leq \phi_0(b)$ for all $b \subseteq \mathrm{val}(\phi_1)$.

For every ϕ and $x \in \mathrm{val}(\phi)$ there is a stronger creature ϕ' with domain $\{x\}$. For each i, there are only finitely many i-creatures.

Another way to write bigness is:

$$\text{If } b = c_1 \cup c_2 \subseteq a \text{ then either } \phi(c_1) \geq \phi(b) - 1 \text{ or } \phi(c_2) \geq \phi(b) - 1. \tag{3.2}$$

Definition 3.5. A condition p of P is a sequence $(p(i))_{i\in\omega}$ such that $p(i)$ is an i-creature and $\liminf_{i\to\infty} \sqrt[k_i^*]{\mathrm{nor}(p(i))} = \infty$. A condition q is stronger than p, if $q(i)$ is stronger than $p(i)$ for all i.

Given a $p \in P$, we can define the trunk of p as follows: Let l be maximal such that $\mathrm{val}(p(i))$ is a singleton $\{x_i\}$ for all $i < l$. Then the trunk is the sequence $(x_i)_{i<l}$.

We define the P-name $\underset{\sim}{\eta}$ to be the union of all trunks of conditions in the generic filter. For every n, the set of conditions with trunk of length at least n is (open) dense. If $q \leq p$ then the trunk of q extends the trunk of p. So $\underset{\sim}{\eta}$ is the name of a real, more specifically $\underset{\sim}{\eta} \in \prod_{i\in\omega} \mathbf{F}(i)$.

P is nonempty: For example, the following is a valid condition: $\mathrm{val}(p(n)) = \mathbf{F}(n)$, and $p(n)(b) = \lfloor \log_2(|b|) \rfloor$.

[12] It is enough to assume $\mathbf{F}(i) > 2^{i^{(k_i^*)}}$.

LEMMA 3.6. *P satisfies fusion and pure decision, so P is ω^ω-bounding and nep (and in particular proper).*

Sketch of proof. This is an simple case of [7, 2.2]. We give an overview of the proof, which uses the creature-forcing concepts of bigness and halving:

Bigness: Assume that ϕ is an i-creature with $\mathrm{nor}(\phi) > 1$, and that $F : \mathrm{val}(\phi) \to 2$. Then there is a $\psi \leq \phi$ such that $\mathrm{nor}(\psi) \geq \mathrm{nor}(\phi) - 1$ and such that $F \restriction \mathrm{val}(\psi)$ is constant.

(This follows immediately from (3.2).)

Halving: Let ϕ be an i-creature. Then there is an i-creature $\mathrm{half}(\phi) \leq \phi$ such that

- $\mathrm{nor}(\mathrm{half}(\phi)) \geq \lceil \mathrm{nor}(\phi)/2 \rceil$.
- If $\psi \leq \mathrm{half}(\phi)$ and $\mathrm{nor}(\psi) > 0$, then there is a $\psi' \leq \phi$ such that $\mathrm{nor}(\psi') \geq \lceil \mathrm{nor}(\phi)/2 \rceil$ and $\mathrm{val}(\psi') \subseteq \mathrm{val}(\psi)$.

(Proof: Define $\mathrm{half}(\phi)$ by $\mathrm{val}(\mathrm{half}(\phi)) = \mathrm{val}(\phi)$ and $\mathrm{half}(\phi)(b) = \max(0, \phi(b) - \lfloor \mathrm{nor}(\phi)/2 \rfloor)$. Given ψ as above, we set $b := \mathrm{val}(\psi)$ and define ψ' by $\mathrm{val}(\psi') = b$ and $\psi'(c) = \phi(c)$ for all $c \subseteq b$. Then

$$0 < \mathrm{nor}(\psi) = \psi(b) \leq \mathrm{half}(\phi)(b) = \phi(b) - \lfloor \mathrm{nor}(\phi)/2 \rfloor,$$

so $\mathrm{nor}(\psi') = \phi(b) \geq \lceil \mathrm{nor}(\phi)/2 \rceil$.)

Fusion: We define $q \leq_m p$ by: $q \leq p$, $q \restriction m = p \restriction m$, and for all $n \geq m$ either $q(n)$ is equal to $p(n)$ or $\sqrt[k_n^*]{\mathrm{nor}(q(n))} > m$. If $(p_n)_{n \in \omega}$ is a sequence of conditions such that $p_{n+1} \leq_{n+1} p_n$, then there is a canonical limit $p_\infty < p_n$.

Set $\mathrm{pos}(p, n) = \prod_{i<n} \mathrm{val}(p(n))$. For $s \in \mathrm{pos}(p, n)$, we construct $p \wedge s \leq p$ by enlarging the stem of p to be s (or, if the stem was larger than n to begin with, then the stem extends s and we set $p \wedge s = p$). The set $\{p \wedge s : s \in \mathrm{pos}(p, n)\}$ is predense under p. Let D be an open dense set. We say that p essentially is in D, if there is an $n \in \omega$ such that $p \wedge s \in D$ for all $s \in \mathrm{pos}(p, n)$.

Pure decision: For $p \in P$, $n \in \omega$ and $D \subseteq P$ open dense there is a $q \leq_n p$ essentially in D.

Then the rest follows by a standard argument:

Nep: Note that $p \in P$ and $q \leq p$ and $q \leq_k p$ are Borel (so $p \perp q$ is absolute; actually $\perp$ is Borel as well, i.e., P is Suslin proper). Fix a transitive model M and a $p_0 \in P^M$. Enumerate all the dense sets in M as $D_1, D_2, \ldots$. Given $p_n \in M$, pick in M some $p_{n+1} \leq_{n+1} p_n$ essentially in D_{n+1}. In V, build the limit $p_\omega \leq p_0$. Then p_ω is M-generic: Let G be a P-generic filter over V containing p_ω. Fix $m \in \omega$. We have to show that $G \cap M$ meets D_m. Note that G contains p_m (since $p_\omega \leq p_m$). In M, there is an $n \in \omega$ such that $p_m \wedge s \in D_m$ for all $s \in \mathrm{pos}(p_m, n)$. The definitions of $\mathrm{pos}(p_m, n)$ as well as $p_m \wedge s$ are absolute between M and V, the set $\mathrm{pos}(p_m, n)$ is finite (and therefore subset of M), the set $\{p_m \wedge s : s \in \mathrm{pos}(p_m, n)\}$ is predense (in V), so $p_m \wedge s \in G$ for some $s \in \mathrm{pos}(p_m, n)$. So we get $G \cap M \cap D_m \neq \emptyset$.

Continuous reading of names and therefore *ω^ω-bounding* follows equally easily.

It remains to show pure decision. Fix p, n and D and set $p_0 = p$. Given p_m, we construct p_{m+1} as follows:

- Choose

  $$h_m > n+m \text{ such that } \sqrt[k_l^*]{\operatorname{nor}(p_m(l))} > n+2m \text{ for all } l \geq h_m. \tag{3.3}$$

- Enumerate $\operatorname{pos}(p_m)$ as $s_1, \dots, s_M$. Note that $M \leq k^*_{h_m}$, according to (3.1).
- Set $p_m^0 = p_m$. Given p_m^{k-1}, pick p_m^k such that
  - $\operatorname{nor}(p_m^k(l)) > (n+2m)^{k_l^*}/2^k$ for all $l \geq h_m$.
  - $p_m^k(l) = p_m(l)$ for $l < h_m$.
  - Either $p_m^k \wedge s_k$ is essentially in D (deciding case), or it is not possible to find such a condition then $p_m^k(l) = \operatorname{half}(p_m^{k-1}(l))$ for all $l \geq h_m$ (halving case).
- Set p_{m+1} to be p_m^M. In particular, $\sqrt[k_l^*]{\operatorname{nor}(p_{m+1}(l))} > (n+2m)/2$ for all $l \geq h_m$.

Let p_ω be the limit of all the p_m. For every $n \in \omega$ define by downward induction on $h = n, n-1, \dots, h_0$ the h-creatures $\phi_{n,h}$ and sets $\Lambda_{n,h} \subseteq \operatorname{pos}(p_\omega, h)$ in the following way:

- $\Lambda_{n,n}$ is the set of $s \in \operatorname{pos}(p_\omega, n)$ such that $p_\omega \wedge s$ is essentially in D.
- Assume $h_0 \leq h < n$. So for all $s \in \operatorname{pos}(p_\omega, h)$ some of the extensions in of s to $\operatorname{pos}(p_\omega, h+1)$ will be in $\Lambda_{n,h+1}$ while others will be not. By shrinking $p_\omega(h)$ at most k_h^* many times, each time using bigness, we can guarantee that the resulting h-creature $\phi_{n,h}$ satisfies: For all $s \in \operatorname{pos}(p_\omega, h)$ either all extension compatible with $\phi_{n,h}$ are in $\Lambda_{n,h+1}$ or no extension is. Set $\Lambda_{n,h}$ to be the set of those s such that the extensions all are in $\Lambda_{n,h+1}$. Note that $\sqrt[k_h^*]{\phi_{n,h}} > 1/2 \sqrt[k_h^*]{p_\omega(h)}$.

For each h, there are only finitely many possibilities for $\Lambda_{n,h}$ and $\phi_{n,h}$, so using König's lemma, we can get a sequence $(\phi_{*,h}, \Lambda_{*,h})_{h_0 \leq h < \omega}$ such that for all N there is an $n > N$ such that

$$(\phi_{*,h}, \Lambda_{*,h}) = (\phi_{n,h}, \Lambda_{n,h}) \text{ for all } h_0 \leq h \leq N. \tag{3.4}$$

We claim

$$\Lambda_{*,h_0} = \operatorname{pos}(p_\omega, h_0) \tag{3.5}$$

Then we choose any n such that $\Lambda_{n,h_0} = \Lambda_{*,h_0}$ and define q by

$$q(l) = \begin{cases} p_\omega(l) & \text{if } l < h_0 \text{ or } l \geq n \\ \phi_{l,h} & \text{otherwise.} \end{cases}$$

Then q essentially is in D, according to (3.5) and the definition of $\Lambda_{n,h}$.

So it remains to show (3.5). Assume towards a contradiction that $s \in \mathrm{pos}(p_\omega) \setminus \Lambda_{*,h_0}$. Let q' be the condition with stem s and the creatures $(\phi_{*,h})_{h_0 \le h < \omega}$. Pick some $r \le q'$ in D.

Let s' be the trunk of r. So s' extends s. Let h be the length of s'. Without loss of generality, we can assume that

$$\sqrt[k_l^*]{\mathrm{nor}(r(l))} > 2 \text{ for all } l \ge h \tag{3.6}$$

and that $h = h_m$ for some m, where h_m is the number picked in (3.3) to construct p_{m+1}. In particular, $s' = s_k$ for some k, so

$$r \le p_m^k \tag{3.7}$$

We know that $r \in D$. This implies that

$$r' \coloneqq p_m^k \wedge s_k \text{ essentially is in } D \text{ (and } r \le r'). \tag{3.8}$$

Assume otherwise. Then pick $H > h_m$ such that $\sqrt[k_l^*]{\mathrm{nor}(r(l))} > (n+2m)/2$ for all $l \ge H$. For $h_m \le l < H$, we can unhalve $r(l)$ to get some $\tilde{r}(l)$ with norm at least $\mathrm{nor}(p_m^{k-1})/2 > (n+2m)^{k_l^*}/2^k$. Then the condition consisting of trunk s', the creatures $\tilde{r}(l)$ for $h_m \le l < H$ and $r(l)$ for $l \ge H$ would be a suitable condition for the deciding case, a contradiction to the fact that we are in the halving case. This shows (3.8).

Note that $p_\omega \wedge s' \le p_m^k \wedge s'$, so by (3.8) we get that $p_\omega \wedge s'$ essentially is in D. We can now derive the desired contradiction:

$$p_\omega \wedge s' \text{ is not essentially in } D. \tag{3.9}$$

Proof: Assume otherwise, i.e., for some N every $s'' \in \mathrm{pos}(p_\omega, N)$ extending s' is in D. Pick $n > N$ as in (3.4). Then according to the definition of $\Lambda_{n,h}$, we get $s' \in \Lambda_{n,h_m}$ and therefore $s \in \Lambda_{n,h_0}$, a contradiction. This shows (3.9). □

3.3. σ-closed forcing notions

The simplest (and not very interesting) examples of non-transitive nep-forcings are the σ-closed ones. We use the following obvious fact:

Fact 3.7. Assume that Q is upwards absolutely defined, that $\perp$ is upwards absolute as well (and therefore absolute) and that Q is σ-closed (in V). Then Q is nep.

It is not enough to assume that Q is ccc (in V and all candidates) instead of σ-closed, see Example 3.17.

So the following definition of $Q = \{f : \omega_1 \to \omega_1 \text{ partial, countable}\}$ is nep:

Example 3.8. Define Q by $\mathfrak{p} = \omega_1$ and $f \in Q$ if $f : \mathfrak{p} \to \mathfrak{p}$ is a countable partial function. Then Q is nep.

Note that we cannot use ω_1 in the definition directly, since there are candidates M such that $\omega_1^M > \omega_1^V$. Neither could we use $f : \alpha \to \mathfrak{p}$, $\alpha \in \mathfrak{p}$, since such an f in a candidate M really has domain $\alpha \cap M$, which is generally not an ordinal (i.e., this definition would not be upwards absolute).

More generally, we can get the examples:

Example 3.9. Assume that $\mathfrak{p}$ codes the ordinals $\kappa^{\mathfrak{p}}$ and $\lambda^{\mathfrak{p}}$, and set $Q = \{f : \kappa^{\mathfrak{p}} \to \lambda^{\mathfrak{p}}$ partial, countable$\}$ (ordered by extension). Then Q is nep.

This example shows that a nep forcing can look completely different in different candidates: Assume $\kappa^{\mathfrak{p}} = \omega_1$ and $\lambda^{\mathfrak{p}} = \omega_2$. So in V, Q collapses ω_2 to ω_1. Let $N \prec H(\chi)$, $M = \text{ord-col}(N)$, and $M_0 \in V$ a forcing-extension of M for the collapse of ω_1 to ω. Then M_0 is a candidate, and M_0 thinks that ω_1^V is countable, so Q is trivial in M_0. If $M_1 \in V$ is a forcing-extension of M for the collapse of ω_2 to ω_1, then in M_1 Q is isomorphic to the set of countable partial functions from ω_1 to ω_1.

A slight variation (still σ-closed):

Example 3.10. Set $\mathfrak{p} = \omega_1$, $Q = \{f : \mathfrak{p} \to L \cap 2^\omega$ partial, countable$\}$ (ordered by extension). Then Q is nep, and there is a candidate M which thinks that A is a countable maximal antichain of Q^M, but A is not maximal in V.

Proof. $x \in L$ is upwards absolute, so $\in_Q$, $\leq_Q$ and $\perp_Q$ are upwards absolute. Clearly Q is σ-closed in V. So Q is nep. Assume $V = L$, and pick some $N \prec L_\kappa$ for κ regular. Set $M = \text{ord-col}(N)$. In L, construct M' as a forcing-extension of M for the collapse of ω_1 to ω. Then M' thinks $L \cap 2^\omega$ is countable, i.e., that $\{(0, x) : x \in L \cap 2^\omega\}$ is a countable maximal antichain. $\square$

Another, trivial example for a countable antichain with non-absolute maximality is the (trivial) forcing defined by $Q = \{1_Q\} \cup (L \cap 2^\omega)$ and $x \leq y$ iff $y = 1_Q$ or $x = y$.

3.4. Non-transitive creature forcing

Some creature forcing constructions use a countable support product (or a similar construction) built from basic creature forcings. In the useful cases these forcings can be shown to be proper, and the proof usually also shows nep. One would take the index set of the product to be an ordinal κ, and choose the nep parameter $\mathfrak{p}$ with domain κ such that $\mathfrak{p}(\alpha)$ is the nep-parameter (a real) for the basic creature forcings Q_α.

To give the simplest possible example:

Lemma 3.11. The countable support product (of any size) of Sacks forcings is nep.

Proof. Again, the standard proof of properness works. First some notation: A splitting node is a node that has two immediate successors. The n-th splitting front F_n^T of a perfect tree $T \subseteq 2^{<\omega}$ is the set of splitting nodes $t \in T$ such that t has exactly n splitting nodes below it. Note that F_n^T is a front (i.e., it meets every branch) and therefore finite (since T has finite splitting). Let κ be the index set of the product. So a condition p consists of a countable domain $\mathrm{dom}(p) \subseteq \kappa$ and for every $i \in \mathrm{dom}(p)$ a perfect tree $p(i)$. In particular, $q \leq p$ means $\mathrm{dom}(q) \supseteq \mathrm{dom}(p)$ and $q(i) \subseteq p(i)$ for all $i \in \mathrm{dom}(p)$.

- For $u \subseteq \kappa$ finite, $q \leq_{n,u} p$ means: $q \leq p$, and $F_n^{p(i)} = F_n^{q(i)}$ for all $i \in u$.
- Fusion: If we use some simple bookkeeping, we can guarantee that a sequence $p_{n+1} \leq_{n,u_n} p_n$ has a limit p_ω. (It is enough to make sure that the u_n are increasing and that $\bigcup_{n\in\omega} u_n$ covers $\mathrm{dom}(p_\omega)$.)
- For $u \subseteq \mathrm{dom}(p)$ finite, we set $\mathrm{pos}_u(p,n) = \prod_{i\in u} F_n^{p(i)}$ (a finite set). For $\eta \in \mathrm{pos}_u(p,n)$ there is a canonical $p \wedge \eta \leq p$ defined in the obvious way (we increase some trunks).
- Pure decision: given a condition p, some finite $u \subseteq \mathrm{dom}(p)$, some $n \in \omega$ and an open dense set D, we can strengthen p to some $q \leq_{n,u} p$ such that $q \wedge \eta \in D$ for all $\eta \in \mathrm{pos}_u(q,n)$.

  To show this, just enumerate $\mathrm{pos}_u(q,n+1)$ as $\nu_0,\dots,\nu_{M-1}$, set $p_0 = p$, given p_m find $p' \leq p \wedge \nu_m$ in D and then set p_{m+1} to be p' "above ν_m" and p_m "on the parts incompatible with ν_m". Then set $q = p_M$.
- This implies nep: Let the forcing parameter $\mathfrak{p}$ code κ (e.g., $\mathfrak{p} : \kappa \to \{0\}$). Then we can define P to consist of all countable partial functions p with domain $\mathrm{dom}(\mathfrak{p})$ such that $p(\alpha)$ is a perfect tree for all $\alpha \in \mathrm{dom}(p)$. This is an absolute definition, and compatibility is absolute.

  Fix $p = p_0 \in M$. Enumerate as $D_0, D_1, \dots$ all sets in M such that M thinks D_i is dense. Given $p_{m-1} \in M$, pick a suitable u_m and find in M some $p_m \leq_{u_m,m} p_{m-1}$ such that $p_m \wedge s \in D_m$ for all $s \in \mathrm{pos}_u(p,m)$. In V, fuse the sequence into p_ω. Then $p_\omega \leq p$ is M-generic:

  Assume that G contains p_ω an therefore p_m. We know that $p_m \wedge s$ is in G for some $s \in \mathrm{pos}_{u_m}(p_m,m)$. Then $p_m \wedge s \in D_m \cap M \cap G$.
- With similar standard arguments we get ω^ω-bounding. □

3.5. Idealized forcing

Zapletal [12] developed the theory of (proper) forcing notions of the form $P_I = \mathrm{Borel}/I$ for (definable) ideals I. (A smaller set is a stronger condition.) The generic filter G_I of such forcing notions is always determined by a canonical generic real $\utilde{\eta}_I$. How does nep and creature forcing fit into this framework?

- According to the Rule of Thumb 3.1, most P_I which can be shown to be proper, are in fact nep. But we do not know of any particular theorems or counterexamples.

- In particular, we do not know whether there is a good characterization of the (definable) ideals I such that P_I is nep. (Even assuming that P_I is proper, which is a tricky property in itself, cf. [12, 2.2].)
- Most nicely definable forcing notions with hereditarily countable conditions such that the generic object is determined by a real are equivalent to some P_I, and [12] proves several theorems in that direction. (E.g., in many ccc cases there is a natural generic real, and the ideal I can be taken to consist of those Borel sets that are forced not to contain the generic.) However, there are natural examples of creature forcings where the generic filter is determined by a generic real and yet the forcing is not of the form P_I. The next lemma gives an example.
- Many of the nice consequences that we get for (transitive) nep forcings also follow for forcings of the form P_I (not assuming nep, but sometimes other additional properties). For example the preservation Theorem [5, 9.4] mentioned in the introduction corresponds to [12, 6.3.3].

The following lemma is due to Zapletal.[13]

Lemma 3.12. Let P be the forcing of subsection 3.2.

1. The generic filter G is determined by the generic real $\underset{\sim}{\eta}$.
2. $(P,\underset{\sim}{\eta})$ is not equivalent to a forcing of the form $(P_I,\underset{\sim}{\eta}_I)$.

To make this precise, we have to specify what we mean with "equivalent". We use the following version:

Definition 3.13. A forcing notion P together with the P-name $\underset{\sim}{\eta}$ are equivalent to P_I (with the canonical generic real $\underset{\sim}{\eta}_I$), if there are P-names $\tilde{G}'_I$ and $\underset{\sim}{\eta}'_I$ and a P_I-name G' such that P forces: G'_I is the P_I-generic filter over V corresponding to the generic real $\underset{\sim}{\eta}'_I$, and $G'[G'_I]_{P_I} = G$.

I.e., we can reconstruct the P-generic filter G by evaluating the P_I-name G' with the P_I-generic filter G'_I.

In particular, this implies

$$(\forall p \in P)(\exists q \leq p)(\exists \tilde{B} \in P_I)\; q \Vdash_P \tilde{B} \in G'_I \;\&\; \tilde{B} \Vdash_{P_I} p \in G'. \tag{3.10}$$

We will need the following straightforward fact:

Lemma 3.14. Assume that $(P,\underset{\sim}{\eta})$ is equivalent to P_I, and that there is a Borel function f such that $\Vdash_P \underset{\sim}{\eta}'_I = f(\underset{\sim}{\eta})$. Then the canonical map $\varphi : P_J \to \mathrm{ro}(P)$ defined by $B \mapsto [\![\underset{\sim}{\eta} \in B]\!]_P$ is a dense embedding, where we set $J = \{B : \Vdash_P \underset{\sim}{\eta} \notin B^{V[G]}\}$.

[13] Jindřich Zapletal, personal communication, November 2007.

Proof. Given $p \in P$, we need some B such that $0 \neq [\![\underset{\sim}{\eta} \in B]\!] \leq p$. Let $q, \tilde{B}, q$ be as in (3.10), and set $B = f^{-1}\tilde{B}$. In particular $\tilde{B} \Vdash_{P_I} p \in G'$, so

$$\underset{\sim}{\eta} \in B \text{ iff } f(\underset{\sim}{\eta}) =\in \tilde{B} \text{ iff } \underset{\sim}{\eta}'_I \in \tilde{B} \text{ iff } \tilde{B} \in G'_I, \text{ which implies } p \in G'[G'_I],$$

i.e., $p \in G$. Also, $q \Vdash_P \tilde{B} \in G'_I$, so $q \leq [\![\underset{\sim}{\eta} \in B]\!] \leq p$. □

A density argument together with [12, 3.3.2] gives the following:

Lemma 3.15. Assume that P is ω^ω bounding and has Borel reading of names with respect to the P-name $\underset{\sim}{\eta}$ and that $(P, \underset{\sim}{\eta})$ is equivalent to P_I. Fix $p_0 \in P$. Then there is a $p_1 \leq p_0$ such that $P' = \{p \in P : p \leq p_1\}$ satisfies the following: For all p there is a compact set C such that $0 \neq [\![\underset{\sim}{\eta} \in C]\!]_{\mathrm{ro}(P')} \leq p$.

Borel reading means: For all P-names $\underset{\sim}{r}$ for a real and all $p \in P$ there is a Borel function f and a $q \leq p$ forcing that $\underset{\sim}{r} = f(\underset{\sim}{\eta})$.

Note that the forcing of Subsection 3.2 has Borel reading (even continuous reading) of names from the canonical generic $\underset{\sim}{\eta}$.

Proof. Given $p_0 \in P$, there is some $p_1 \leq p$ and f Borel such that p_1 forces $\underset{\sim}{\eta}'_I$ to be $f(\underset{\sim}{\eta})$. So according to Lemma 3.14, the canonical embedding $\varphi : P_J \to \mathrm{ro}(P')$ is dense for $J = \{B : p_1 \Vdash_P \underset{\sim}{\eta} \notin B^{V[G]}\}$ and $P' = \{p \leq p_1\}$. Given $p \in P'$, find some Borel-code B such that $\varphi(B) \leq p$. [12, 3.3.2] gives a J-positive compact subset of B. □

Proof of 3.12. Proof of (1).

We will use the following property of norms, cf. Definition 3.4:

(3.11) For norms ϕ_0, ϕ_1 with $\mathrm{val}(\phi_0) \cap \mathrm{val}(\phi_1) \neq \emptyset$ there is a weakest norm $\phi_0 \wedge \phi_1$ stronger than both ϕ_0 and ϕ_1.

Proof: We define $\psi = \phi_0 \wedge \phi_1$ the following way: $\mathrm{val}(\psi) = \mathrm{val}(\phi_0) \cap \mathrm{val}(\phi_1)$ and $\psi(b)$ is defined by induction on the cardinality of b: If $|b| \leq 1$, then $\psi(b) = \min(\psi(b) = \min \phi_0(b), \phi_1(b))$. Otherwise, $\psi(b) = \min(X(b))$, for

$$X(b) = \{\phi_0(b), \phi_1(b)\} \cup \{1 + \max(\psi(b_0), \psi(b_1)) : b_0 \cup b_1 = b\}.$$

We have to show that ψ is a norm: Bigness follows imediately from the definition. It remains to show monotonicity. We show by induction on b:

$$(\forall c \subseteq b)\, \psi(c) \leq \psi(b)$$

I.e., $(\forall m \in X(b))\, \psi(c) \leq m$. For $m = \phi_0(b)$, we have $\psi(c) \leq \phi_0(c) \leq \phi_0(b) = m$. The same holds for $m = \phi_1(b)$. So assume $m = 1 + \max(\psi(b_0), \psi(b_1))$, without loss of generality for nonempty and disjoint b_0, b_1. Then $b_0 \cap c \subsetneq b$ and $b_1 \cap c \subsetneq b$, so by definition $\psi(c) \leq 1 + \max(\psi(b_0 \cap c), \psi(b_1 \cap c))$ which is (by induction) at most $1 + \max(\psi(b_0), \psi(b_1)) = m$.

On the other hand it is clear that ψ is the biggest possible norm that is smaller than ϕ_0 and ϕ_1. So we get (3.11).

We will also need:

(3.12) $(\forall b \subseteq \mathrm{val}(\phi_0 \wedge \phi_1))\,(\exists b_0, b_1)\, b = b_0 \cup b_1 \ \&\ (\phi_0 \wedge \phi_1)(b) \geq \max(\phi_0(b_0), \phi_1(b_1))$

Proof: Again, write ψ for $\phi_0 \wedge \phi_1$. By induction on $|b|$: If $\psi(b) = \phi_0(b)$, we can set $b_0 = b$ and $b_1 = \emptyset$. Analogously for $\psi(b) = \phi_1(b)$. If $\psi(b) = 1 + \max(\psi(c_0), \psi(c_1))$ for $c_0 \subsetneq b$ and $c_1 \subsetneq b$, then by induction $\psi(c_0) \geq \max(\phi_0(d^0_0), \phi_1(d^1_0))$ and $\psi(c_1) \geq \max(\phi_0(d^0_1), \phi_1(d^1_1))$, so we can set $b_0 = d^0_0 \cup d^0_1$ and $b_1 = d^1_0 \cup d^1_1$. Then

$$\begin{aligned}\psi(b) =& 1 + \max(\psi(c_0), \psi(c_1)) \geq 1 + \max(\phi_0(d^0_0), \phi_0(d^0_1), \phi_1(d^1_0), \phi_1(d^1_0)) \geq \\ & \phi_0(d^0_0 \cup d^0_1)\end{aligned}$$

(because of bigness of ϕ_0), and analogously $\psi(b) \geq \phi_1(d^1_0 \cup d^1_1)$. This shows (3.12).

For compatible $p, q \in P$ we can define $p \wedge q$ by $(p \wedge q)(i) = p(i) \wedge q(i)$. This is the weakest condition stronger than both p and q. An immediate consequence of (3.11) is: $p \perp q$ is equivalent to

(3.13) $(\exists n \in \omega)\ \mathrm{val}(p(n)) \cap \mathrm{val}(q(n))) = \emptyset$ or

$$(\exists b \subseteq \omega \text{ infinite})\,(\exists M \in \omega)\,(\forall n \in b)\ \mathrm{nor}(p(n) \wedge q(n)) < M^{k^*_n}.$$

An obvious candidate for reconstructing the generic filter G from the generic real $\underset{\sim}{\eta}$ (that works with many tree-like forcings) would be the set

$$H_0 = \{p \in P : \underset{\sim}{\eta} \in \prod_{n \in \omega} \mathrm{val}(p(n))\}.$$

However, due to the halving property of P, this fails miserably: There are incompatible conditions q and r with $\mathrm{val}(q(n)) = \mathrm{val}(r(n))$ for all n. More specifically, we get the following: For all p there is an $r \leq p$ such that

(3.14) $r \perp \mathrm{half}(p)$, and $\mathrm{val}(r(n)) \subseteq \mathrm{val}(\mathrm{half}(p)(n))$ for all n.

Proof: Set $q(n) = \mathrm{half}(p)$. Pick for all sufficiently large n some $a_n \subseteq \mathrm{val}(q(n))$ such that $q(n)(a_n) = 2$. Using the halving property, we can find for all n some $\phi_n \leq p(n)$ such that $\mathrm{val}(\phi_n) \subseteq a_n$ and $\mathrm{nor}(\phi_n) > \mathrm{nor}(p(n))/2$. Set $r = (\phi_n)_{n \in \omega}$. Then r and q cannot be compatible, since $q(n)(\mathrm{val}(r(n)))$ is bounded. This shows (3.14).

Back to the proof. First note the following: Fix $p \in P$. Let $X(p)$ be the set of all sequences $\bar{b} = (b_n)_{n \in d}$ where d is an infinite subset of ω and $b_n \subseteq \mathrm{val}(p(n))$ such that $\{\sqrt[k^*_n]{p(n)(b_n)} : n \in d\}$ is bounded. Fix some $\bar{b} \in X(b)$. Then p forces that $\underset{\sim}{\eta}$ is not in the set

(3.15) $\underset{\sim}{N}_{p,\bar{b}} = \{\nu \in \prod_{n \in \omega} \mathrm{val}(p(n)) : (\exists^\infty n \in d) \nu(n) \in b_n\}.$

Proof: Assume towards a contradiction that some $p' \leq p$ forces $\underset{\sim}{\eta} \in \underset{\sim}{N}_{p,\bar{b}}$. So there is a bound M such that $p'(n)(b_n) < M^{k^*_n}$ for all $n \in d$. Fix $N(n)$ such that $\mathrm{nor}(p'(l)) > (n + 1 +$

$M)^{k_l^*} > 1+(n+M)^{k_l^*}$ for all $l > N(n)$. For all $l > N(n)$ we get $p'(l)(\text{val}(p'_l)\setminus b_l) > (n+M)^{k_l^*}$ (by bigness). Let p'' be the condition $p'(l) \restriction (\text{val}(p'_l) \setminus b_l)$ for $l \in (b \setminus N(0))$. Then p'' forces that $\underset{\sim}{\eta} \notin \underset{\sim}{N}_{p,\bar{b}}$, a contradiction. This shows (3.15).

We claim that the following defines G:

$$H = H_0 \cap \{p \in P : (\forall \overline{b} \in X(p) \cap V)\ \underset{\sim}{\eta} \notin \underset{\sim}{N}_{p,\bar{b}}\}. \tag{3.16}$$

$H \supseteq G$ by (3.15), so it is enough to show that all $p_1, p_2 \in H$ are compatible. Set $b_n = \text{val}(p_1(n)) \cap \text{val}(p_2(n))$. Note that b_n is nonempty, since $p_1, p_2 \in H_0$. So according to (3.13) we can assume towards a contradiction that the following holds (in V):

$$(\exists b \subseteq \omega \text{ infinite})(\exists M \in \omega)(\forall n \in b)\ \text{nor}(p(n) \wedge q(n)) < M^{k_n^*}.$$

According to (3.12), we get c_n^1, c_n^2 such that $c_n^1 \cup c_n^2 = b_n$ and $p_i(n)(c_n^i) < M^{k_n^*}$ for $n \in b$ and $i \in \{0,1\}$. We assumed that $\underset{\sim}{\eta} \notin \underset{\sim}{N}_{p_1,\bar{c}^1}$, i.e., $\underset{\sim}{\eta}(n) \in c_n^1$ for only finitely many n. The same is true for c_n^2, a contradiction. This shows (3.16) and therefore item (1).

Note that to construct G from $\underset{\sim}{\eta}$, we use the (complicated) set $(2^\omega)^V$; compare that with the much easier construction of H_0.

Proof of (2).

Let us assume towards a contradiction that P is equivalent to P_I. So it satisfies the assumptions of Lemma 3.15. Fix $p \in P'$, and set $q = \text{half}(p)$. Let C be compact such that

$$0 \neq [\![\underset{\sim}{\eta} \in C]\!] < q. \tag{3.17}$$

Then $\prod_{n\in\omega} \text{val}(q(n)) \subseteq C$, since C is closed. Let $r \leq p$ be incompatible to q such that $\text{val}(r(n)) \subseteq \text{val}(q(n))$ as in (3.14). Then $\prod_{n\in\omega} \text{val}(r(n)) \subseteq C$, therefore $r \Vdash \underset{\sim}{\eta} \in C$. So $r \leq^* q$ by (3.17), which contradicts $r \perp q$. □

3.6. Counterexamples

Being nep is a property of the definition, not the forcing. Of course we can find for any given proper forcing a definition which is not nep (take any definition that is not upwards absolute). For the same trivial reasons, a forcing "absolutely equivalent" to a nep forcing doesn't have to be nep itself. For example:

Example 3.16. There are upward absolute definitions of (trivial) forcings P, Q s.t. in V and all candidates, P is a dense suborder of Q, P is nep but Q is not nep.

Proof. Pick $\mathfrak{p} \in L \cap 2^\omega$ and a candidate M_0 that thinks $\mathfrak{p} \notin L$. Define $P = \{1, p_1, p_2\}$, $x \leq_P y$ iff $y = 1$ or $x = y$. Set $Q = P \cup \{q_1, q_2\}$ and define the order on Q by: $1 \leq q_i \leq p_i$, and if $\mathfrak{p} \in L$, then also $p_2 \leq q_1$ and $p_1 \leq q_2$. These definitions are upwards absolute and P is nep. However, $M_0 \vDash$ "$q_1 \perp q_2$". But every Q-generic Filter over V contains q_1 and q_2, so there cannot be a Q-generic condition over M_0. □

If Q, $\leq$ and $\perp$ are Σ^1_1 and Q is ccc, then Q is Suslin ccc, and therefore (transitive) nep. (One of the reasons is that in the Σ^1_1-case it is absolute for countable antichains to be maximal.) This is not true anymore if the definition of Q is just Σ^1_2:

Example 3.17. Let Q be random forcing in L ordered by inclusion, i.e.,

$$Q = \{r \in L : r \text{ is a Borel-code for a non-null-set}\}.$$

Then $p \in Q$ is Σ^1_2 and $q \leq p$ and $p \perp q$ are (relatively) Borel, and in V and all candidates Q is ccc. But Q is not nep.

Proof. Pick in L a (transitive) candidate M such that M thinks that ω_1^L (and therefore Q) is countable. In particular there is for each $n \in \omega$ a maximal antichain A_n in M such that $\mu(X_n) < 1/n$ for $X_n = \bigcup_{a \in A_n} a$. (Of course M thinks that X_n is not in L. But really it is, simply because $M \subseteq L$.) Take any $q \in Q^V$, and pick n such that $1/n < \mu(q)$. Then $q' = q \setminus X_n$ is positive and in L, and a generic filter containing q' does not meet the antichain A_n. ◻

It is however not clear whether Q could not have another definition that is nep, or at least whether Q is forcing-equivalent to a nep forcing. If L is very small (or very large) in V, then Q is Cohen (or random, respectively) and thus equivalent to a nep forcing notion. If V' is an extension of $V = L$ by a random real, then in V' the forcing Q (which is "random forcing in L") seems to be more complicated (it adds an unbounded real, but no Cohen). We do not know whether in this case Q is equivalent to a nep forcing.

4. Countable support iterations

This section consists of three subsections:

4.1 We introduce the basic notation and preservation theorem. We get generic conditions for the limit, but not an upwards absolute definition of the forcing notion.

4.2 We introduce an equivalent definition of the iteration which is upwards absolute. So the limit is again nep.

4.3 We modify the notions of Subsection 4.1 to subsets of the ordinals, and give a nice application.

For this section, we fix a sequence $(Q_\alpha)_{\alpha \in \epsilon}$ of forcing-definitions and a nep-parameter $\mathfrak{p}$ coding the parameters $(\mathfrak{p}_\alpha)_{\alpha \in \epsilon}$, i.e., $\mathfrak{p}$ is a nep-parameter with domain ϵ and $\mathfrak{p}(\alpha)$ is the nep-parameter used to define Q_α for each $\alpha \in \epsilon$. (So we assume that the sequence of defining formulas and parameters live in the ground model.)

To further simplify notation, we also assume that candidates are successor-absolute, i.e., "α is successor" and the function $\alpha \mapsto \alpha + 1$ are absolute for all candidates.

Remark 4.1. This assumption is not really necessary. Without it, we just have to use "M thinks that $\alpha = \zeta + 1$" instead of just "$\alpha = \zeta + 1$" in the definition of G^M_α etc., similarly to 4.20.

Also, we assume the following (which could be replaced by weaker conditions, but is satisfied in practice anyway):

- In every forcing extension of V, each Q_α is normal nep (for ZFC* candidates).
- We only start constructions with candidates M such that generic extensions $M[G]$ satisfy ZFC*.[14]

4.1. Properness without absoluteness

We use the following notation: For any forcing notion, $q \leq^* p$ means $q \Vdash p \in G$.

Definition 4.2. Let M be a candidate.

- P_β is the countable support iteration (in other terminology: the limit of) $(P_\alpha, Q_\alpha)_{\alpha\in\beta}$ (for all $\beta \leq \epsilon$). We use G_α to denote the P_α-generic filter over V, and $G(\alpha)$ for the Q_α-generic filter over $V[P_\alpha]$.
- P^M_β is the element of M so that M thinks: P^M_β is the countable support iteration of the sequence $(Q_\alpha)_{\alpha\in\beta}$ (for $\beta \in \epsilon \cap M$).

In certain P_ϵ-extensions of V the generic filter G defines a canonical P^M_ϵ-generic G^M_ϵ over M:

Definition 4.3. Given $G \subset P_\epsilon$, we define G^M_α by induction on $\alpha \in \epsilon \cap M$ by using the following definition, provided it results in a P^M_α-generic filter over M. In that case we say "G is (M, P_α)-generic". Otherwise, G^M_α (and G^M_β for all $\beta > \alpha$) are undefined.

- If $\alpha = \zeta + 1$, then G^M_α consists of all $p \in P^M_\alpha$ such that $p \restriction \zeta \in G^M_\zeta$ and $p(\zeta)[G^M_\zeta] \in G(\zeta)$.
- If α is a limit, then G^M_α is the set of all $p \in P^M_\alpha$ such that $p \restriction \zeta \in G^M_\zeta$ for all $\zeta \in \alpha \cap M$.

Definition 4.4.

- Assume that G is (M, P_α)-generic and $\zeta \in \alpha \cap M$. Then we set $G^M(\zeta) = \{\underset{\sim}{q}[G^M_\zeta] : (\exists p)\, p \cup (\zeta, \underset{\sim}{q}) \in G^M_{\zeta+1}\}$. I.e., $G^M(\zeta)$ is the usual $Q_\zeta^{M[G^M_\zeta]}$-generic filter over $M[G^M_\zeta]$ as defined in $M[G^M_\alpha]$.
- q is (M, P_α)-generic means that $q \in P_\alpha$ forces that the P_α-generic filter G is (M, P_α)-generic. If $p \in P^M_\alpha$ (or if p is just a P^M_α-name (in M) for some P^M_α-condition), then q is (M, P_α, p)-generic, if q additionally forces that $p \in G^M_\alpha$ (or that $p[G^M_\alpha] \in G^M_\alpha$, resp.).

The following is an immediate consequence of the definition:

[14] Formally we can require that M satisfies some stronger ZFC$'$ and that ZFC$'$ proves that every formula of ZFC* is forced by all countable support iterations of forcings of the form Q_α. Also, we assume that ZFC proves that $H(\chi)$ satisfies ZFC* for sufficiently large regular χ, and that ZFC proves that the defining formulas are absolute between V and $H(\chi)$.

Facts 4.5.
- If $\zeta \in M \cap \alpha$, then $G^M(\zeta) = Q_\zeta^{M[G_\zeta^M]} \cap G(\zeta)$.
- If q is (M, P_α, p)-generic and $\zeta \in M \cap \alpha$, then $q \restriction \zeta$ is $(M, P_\zeta, p \restriction \zeta)$-generic.

G_α^M is absolute in the following sense:

Lemma 4.6. Assume that M, N are candidates in V, $M \in N$, V' is an extension of V, $\alpha \in M \cap \epsilon$, and $G \subset P_\alpha$ is an element of V' which is (N, P_α)-generic.

1. G_α^M (in V') is the same as $G_\alpha^N)_\alpha^M$ (in $N[G_\alpha^N]$). In other words, the P_α^M-filter calculated in V' from G is the same as the P_α^M-filter calculated in $N[G_\alpha^N]$ from G_α^N.
2. In particular, G_α is (M, P_α)-generic iff N thinks that G_α^N is (M, P_α)-generic.
3. If G is (M, P_α)-generic and τ a P_α^M-name (in M), then "$x = \tau[G_\alpha^M]$" is absolute between $N[G_\alpha^N]$ and V'.

Proof. By induction on $\alpha \in \epsilon \cap M$: (2) follows from (1) by definition, and (3) from (1) using 1.18.

Assume $\alpha = \zeta + 1$. Then $p \in G_\alpha^M$ iff $p \restriction \zeta \in G_\zeta^M$ and $p(\zeta)[G_\zeta^M] \in G(\zeta)$ iff $N[G_\zeta^N] \vDash p \restriction \zeta \in G_\zeta^M$ (by induction hypothesis 1) and $N[G_\zeta^N] \vDash p(\zeta)[G_\zeta^M] \in G^N(\zeta)$ (by induction hypothesis 3 and the fact that $M[G_\zeta^M] \vDash q \in Q_\zeta$ implies $N[G_\zeta^N] \vDash q \in Q_\zeta$).

Now assume α is a limit. Then $p \in G_\alpha^M$ iff ($p \restriction \zeta \in G_\zeta^M$ for all ζ) iff ($N \vDash p \restriction \zeta \in G_\zeta^M$ for all ζ) (by induction hypothesis 1) iff $N \vDash p \in G_\alpha^M$. $\square$

So here we use that Q_α is upwards absolutely defined in $V[G_\alpha]$, and that $M[G_\zeta^M] \in N[G_\zeta^N]$ both are candidates.

The definitions are compatible with ord-collapses of elementary submodels:

Lemma 4.7. Let $N \prec H(\chi)$, $M = \text{ord-col}(N)$, and let G be P_α-generic over V. Then

1. G is N-generic iff it is (M, P_α)-generic.

If G is N-generic and $p, \underset{\sim}{\tau} \in N$, then

2. $p \in G$ iff $\text{ord-col}^N(p) \in G_\alpha^M$,
3. $\text{ord-col}^{N[G_\alpha]}(\underset{\sim}{\tau}[G]) = (\text{ord-col}^N(\underset{\sim}{\tau}))[G_\alpha^M]$;
4. in particular, $(M[G_\alpha^M], M, \in)$ is the ord-collapse of $(N[G_\alpha], N, \in)$.

Proof. The image of x under the ord-collapse (of the appropriate model, i.e., either N or $N[G]$) is denoted by x'. Induction on α:

(1,2 successor, $\alpha = \zeta + 1$.) Assume that $G \restriction P_\zeta =: G_\zeta$ is P_ζ-generic over N. Fix $p \in P_\alpha \cap N$. Then $p \in G$ iff { $p \restriction \zeta \in G_\zeta$ and $p(\zeta)[G_\zeta] \in G(\zeta)$ } iff {$p' \restriction \zeta \in G_\zeta^M$ (according to induction hypothesis (2)) and $p'(\zeta)[G^M_\zeta] \in G(\zeta)$ (according to induction hypothesis (3)) $\}$[15] iff $p' \in G^M_\alpha$.

(1,2 limit) Assume that G_ζ is P_ζ-generic over N for all $\zeta \in \alpha \cap N$. Fix $p \in P_\alpha \cap N$. $p \in G$ iff $\{\ p \restriction \zeta \in G_\zeta$ for all $\zeta \in \alpha \cap N\ \}$ iff $p' \in G^M_\zeta$ for all $\zeta \in \alpha \cap M\ \}$ (by hypothesis (2)) iff $p' \in G^M_\alpha$.

(3) Induction on the depth of the name $\underset{\sim}{\tau}$:
Let $A \in N$ be a maximal antichain deciding whether $\underset{\sim}{\tau} \in V$ (and if so, also the value of $\underset{\sim}{\tau}$). Assume $a \in A \cap G \cap N$. If a forces $\underset{\sim}{\tau} = \check{x}$ for $x \in V$, then M thinks that $a' \in G^M_\alpha$ forces $\underset{\sim}{\tau}'$ to be $x' \in V$, so we get $\underset{\sim}{\tau}'[G^M_\alpha] = x'$. If a forces that $\underset{\sim}{\tau} \notin V$, then

$$\underset{\sim}{\tau}'[G^M_\alpha] = \{\underset{\sim}{\sigma}'[G^M_\alpha] : (\sigma, p) \in \tau \cap N, p' \in G^M_\alpha\} = \{(\underset{\sim}{\sigma}[G])' : (\sigma, p) \in \tau \cap N, p \in G\}$$

(by induction). It remains to be shown that this is the ord-collapse of $\underset{\sim}{\tau}[G] = \{\underset{\sim}{\sigma}[G] : \sigma \in \tau\}$. For this it is enough to note that for all $\rho[G] \in \underset{\sim}{\tau}[G] \cap N[G]$ there is a $(\underset{\sim}{\sigma}, p) \in \underset{\sim}{\tau}$ such that $p \in G$ and $\underset{\sim}{\sigma}[G] = \rho[G]$. ◻

P_α satisfies (a version of) the properness condition for candidates:

Lemma 4.8. For every candidate M and $p \in P^M_\alpha$ there is an (M, P_α, p)-generic q such that $\mathrm{dom}(q) \subseteq M \cap \alpha$.

The proof is more or less the same as the iterability of properness given in [1]. Since we will later need a "canonical" version of the proof, we will introduce the following notation:

Definition 4.9. For $\alpha < \epsilon$, let gen_α be a P_α-name for a function such that the following is forced by P_α: If M is a candidate, $\sigma : \omega \to M$ surjective, and $p \in Q^M_\alpha$ then $\mathrm{gen}_\alpha(M, \sigma, p)$ is an M-generic element of $Q^{V[G_\alpha]}$ stronger than p.

(It is clear that such functions exist, since we assume that P_α forces that Q_α is nep. Later we will assume that we can pick gen_α in some absolute way, cf. 4.13).

For $\alpha \le \beta < \epsilon$, let $P_{\beta/\alpha}$ denote the set of P_α-names p for elements of P_β such that P_α forces $p \restriction \alpha \in G_\alpha$. (I.e., $P_{\beta/\alpha}$ is a P_α-name for the quotient forcing.) As usual, we can define the M-version: $p \in P^M_{\beta/\alpha}$ means that p is a P^M_α-name (in M) for a P^M_β-condition, and if G is (M, P_α)-generic, then $p[G^M_\alpha] \restriction \alpha \in G^M_\alpha$.

Lemma 4.8 is a special case of the following:

Induction Lemma/Definition 4.10. Assume that M is a candidate, $\sigma : \omega \to M$ surjective, $\alpha, \beta \in M$, $\alpha \le \beta \le \epsilon$, $p \in P^M_{\beta/\alpha}$, q is (M, P_α)-generic, and that $\mathrm{dom}(q) \subseteq \alpha \cap M$. We define the canonical (M, σ, P_β, p)-extension q^+ of q such that $q^+ \in P_\beta$ and q^+ is (M, P_β, p)-generic and $\mathrm{dom}(q^+) \subseteq M \cap \beta$.

Proof. Induction on $\beta \in M$.

[15] using the fact that that $p(\zeta)[G_\zeta] \in Q_\zeta$ is hereditarily countable modulo ordinals and therefore not changed by the collapse

Successor step $\beta = \zeta + 1$: By induction we have the canonical $(M, \sigma, P_\zeta, p \restriction \zeta)$-extension $q^+ \in P_\zeta$. In particular, q^+ forces that $M' := M[G^M_\zeta]$ is a candidate and that $p' := p(\zeta)[G^M_\zeta] \in Q^{M'}_\zeta$. By applying σ to the P^M_ζ-names in M, we get a canonical surjection $\sigma' : \omega \to M'$. We define the canonical β-extension q^{++} to be $q^+ \cup (\zeta, \mathrm{gen}_\zeta(M', \sigma', p'))$. Assume that G_β contains q^{++}. Then G^M_ζ is P^M_ζ-generic and contains $p \restriction \zeta$. If $A \subseteq P^M_\beta$ is (in M) a maximal antichain, then

$$A' := \{a(\zeta) : a \in A,\, a \restriction \zeta \in G^M_\zeta\} \subseteq Q_\zeta^{M[G^M_\zeta]}$$

is a maximal antichain in $M[G^M_\zeta]$. Since $\mathrm{gen}_\zeta(M', p')$ is in $G(\zeta)$, there is exactly one $a' \in A' \cap G(\zeta)$, i.e., there is exactly one $a \in A \cap G^M_\beta$. So q^{++} is really (M, P_β, p)-generic.

Limit step: Assume $\alpha = \alpha_0 < \alpha_1 \dots$ is cofinal in $M \cap \beta$. Set $D_0 = P^M_\beta$ and let $(D_n)_{n\in\omega}$ enumerate the P^M_β-dense subset in M. (Note that we get this enumeration canonically from σ.) First we define $(p_n)_{n\in\omega}$ such that $p_0 = p$, $p_n \in P^M_{\beta/\alpha_n}$, and ($M$ thinks that) $P^M_{\alpha_n}$ forces

- $p_n \in D_n$,
- $p_{n-1} \restriction \alpha_n \in G^M_{\alpha_n}$ implies $p_n \leq p_{n-1}$.

Then we pick $q = q_{-1} \subseteq q_0 \subseteq q_1 \dots$ such that q_n is the canonical $(M, \sigma, P_{\alpha_{n+1}}, p_n \restriction \alpha_{n+1})$-generic extension of the (M, P_{α_n})-generic q_n. We set $q^+ := \bigcup_{n\in\omega} q_n$.

By induction we get:

- q_n is $(M, P_{\alpha_{n+1}}, p_m \restriction \alpha_{n+1})$-generic for all $m \leq n$.
- q_n forces $p_l[G^M_{\alpha_{n+1}}] \leq p_m[G^M_{\alpha_{n+1}}]$ (in P^M_β) for $m \leq l \leq n+1$.

q^+ is G^M_β is (M, P_β, p)-generic: Let G be V-generic and contain q^+.

- G^M_β meets D_n: $p_n[G^M_\beta] \in G^M_\beta$, since $p_n[G^M_\beta] \restriction \alpha_{m+1} \in G^M_\beta$ for all $m \geq n$.
- Let r, s be incompatible in P^M_β. In M, the set

  $$D = \{p \in P^M_\beta : (\exists \zeta < \beta)(\exists t \in \{r, s\}) p \restriction \zeta \Vdash_\zeta t \notin G_\zeta\} \subseteq P^M_\beta$$

  is dense, and if $p \in D \cap G^M_\beta$, then $p \restriction \zeta \in G^M_\zeta$, so $t \restriction \zeta \notin G^M_\zeta$, and $t \notin G^M_\beta$.

□

We repeat Lemma 4.8 with our new notation:

Corollary 4.11. Given a candidate M, $\sigma : \omega \to M$ surjective, and $p \in P^M_\alpha$, we can define the canonical generic $q = \mathrm{gen}(M, \sigma, P_\alpha, p)$. Also, $\mathrm{dom}(q) \subseteq M \cap \alpha$.

So P_α satisfies the properness-clause of the nep-definition. However, P_α is not nep, since the statement "$p \in P_\alpha$" is not upwards absolute.

Remark 4.12. There are two obvious reasons why "$p \in P_\alpha$" is not upwards absolute: First of all, names look entirely different in various candidates. For example, if M thinks that $\underset{\sim}{\tau}$ is the standard name for ω_1, then a bigger candidate N will generally see that $\underset{\sim}{\tau}$ is not a standard name for ω_1. So if Q is the (trivial) forcing $\{\omega_1\}$, then a condition in $P * Q$ is a pair (p, q), where P is (essentially) a standard P-name for ω_1. So if M thinks that $(p, q) \in P * Q$, then N (or V) will generally not think that $(p, q) \in P * Q$. So we cannot use the formula "$p \in P_\alpha$" directly. We will use pairs (M, p) instead, where M is a candidate and $p \in P_\alpha^M$. Another way to circumvent this problem would be to use absolute names for hco-objects (inductively defined, starting with, e.g., "standard name for α", and allowing "name for union of $(x_i)_{i\in\omega}$" etc). The second reason is that forcing is generally not absolute (even when we use absolute names): M can wrongly think that p forces that $\underset{\sim}{q}_2 \leq \underset{\sim}{q}_1$, i.e., that $(p, \underset{\sim}{q}_2) \leq (p, \underset{\sim}{q}_1)$ in $P * \underset{\sim}{Q}$. We will avoid this by interpreting (M, p) to be a *canonical* (M, P_α, p)-generic.

4.2. The nep iteration: properness and absoluteness.

Now we will construct a version of P_ϵ that is forcing equivalent to the usual countable support iteration and upwards absolutely defined. We will need to construct generics in a canonical way, so we assume the following:

Assumption 4.13. There is an absolute (definition of a) function gen_α such that P_α forces: If M is a candidate, $\sigma : \omega \to M$ surjective and $p \in Q_\alpha^M$ then $\mathrm{gen}_\alpha(M, \sigma, p)$ is Q_α^M-generic over M and stronger than p.

Definition 4.14. P_α^{nep} consists of tuples (M, σ, p), where M is a candidate, $\sigma : \omega \to M$ surjective, and $p \in P_\alpha^M$.

So "$x \in P_\alpha^{\mathrm{nep}}$" obviously is upwards absolute.

We will interpret (M, σ, p) as the "canonical M-generic condition forcing that $p \in G_\alpha^M$". (Generally there are many generic conditions, and incompatible ones, so we have to single out a specific one, the canonical generic, and for this we need 4.13).

Recall the construction of gen from Definition/Lemma 4.10. If we use Assumption 4.13, we get:

Corollary 4.15. $\mathrm{gen} : P_\alpha^{\mathrm{nep}} \to P_\alpha$ is such that

1. $\mathrm{gen}(M, \sigma, p)$ is (M, P_α, p)-generic
2. If M, N are candidates, $M, \sigma \in N$, and G is (N, P_α)-generic, then $\mathrm{gen}^N(M, \sigma, p) \in G_\alpha^N$ iff $\mathrm{gen}^V(M, \sigma, p) \in G$.

(Here, gen^N is the result of the construction 4.10 carried out inside N, and analogously for V.) Of course gen cannot really be upwards absolute (i.e., we cannot have $\mathrm{gen}^N(M, \sigma, p) = \mathrm{gen}^V(M, \sigma, p)$), since $x \in P_\alpha$ is not upwards absolute. However, (2) gives us a sufficient amount of absoluteness.

Proof. (1) is clear. For (2), just go through the construction of 4.10 again and check by induction that this construction is really sufficiently "canonical", i.e., absolute. □

If $\sigma_1 \neq \sigma_2$ both enumerate M, then we do not require $\mathrm{gen}(M,\sigma_1,p)$ and $\mathrm{gen}(M,\sigma_2,p)$ to be compatible.

Let us first note that a function gen as above also satisfies the following:

Corollary 4.16. 1. If N thinks that $q \leq^* \mathrm{gen}^N(M,\sigma_M,p)$, then $\mathrm{gen}^V(N,\sigma_N,q) \leq^* \mathrm{gen}^V(M,\sigma_M,p)$.

2. If $N \prec H(\chi)$, $p \in N$, and (N',p') is the ord-collapse of (N,p), and $\sigma' : \omega \to N'$ is surjective, then $\mathrm{gen}(N',\sigma',p') \leq^* p$.

Proof. (1) Assume G_α contains $\mathrm{gen}^V(N,\sigma_N,q)$. So G_α is (N,P_α)-generic and G_α^N contains q and therefore $\mathrm{gen}^N(M,\sigma_M,p)$. So by 4.15(2), G_α contains $\mathrm{gen}^V(M,\sigma_M,p)$.

(2) Assume that the V-generic filter G contains $\mathrm{gen}(N',\sigma',p')$. Then by definition, $G^{N'}$ is N'-generic and contains p'. So G is N-generic and contains p, according to 4.7. □

Now we can define:

Definition 4.17. $(M_2,\sigma_2,p_2) \leq^{\mathrm{nep}} (M_1,\sigma_1,p_1)$ means: $M_1,\sigma_1 \in M_2$, and M_2 thinks that (M_1 is a candidate and that) $p_2 \leq^* \mathrm{gen}^{M_2}(M_1,\sigma_1,p_1)$.

By Corollary 4.16(1) $\leq^{\mathrm{nep}}$ is transitive. It follows:

Theorem 4.18. 1. $\mathrm{gen} : (P_\alpha^{\mathrm{nep}},\leq^{\mathrm{nep}}) \to (P_\alpha,\leq^*)$ is a dense embedding.

2. $(P_\alpha^{\mathrm{nep}},\leq^{\mathrm{nep}})$ is nep.

Proof. (1)

- If $(M_2,\sigma_2,p_2) \leq^{\mathrm{nep}} (M_1,\sigma_1,p_1)$, then $\mathrm{gen}(M_2,\sigma_2,p_2) \leq^* \mathrm{gen}(M_1,\sigma_1,p_1)$ by 4.16(1).
- If $(M_2,\sigma_2,p_2) \perp^{\mathrm{nep}} (M_1,\sigma_1,p_1)$, then $\mathrm{gen}(M_2,\sigma_2,p_2) \perp^* \mathrm{gen}(M_1,\sigma_1,p_1)$: Assume that $q \leq^* \mathrm{gen}(M_i,\sigma_i,p_i)$ $(i \in \{1,2\})$. Let $N \prec H(\chi)$ contain q and M_i,σ_i,p_i, and let (M_3,p_3) be the ord-collapse of (N,q) and $\sigma_3 : \omega \to M_3$ surjective. Then $(M_3,\sigma_3,p_3) \leq^{\mathrm{nep}} (M_i,\sigma_i,p_i)$.
- gen is dense: For $p \in P_\alpha$ pick an $N \prec H(\chi)$ containing p and let (N',p') be the ord-collapse of (N,p). Then $\mathrm{gen}(N',p') \leq^* p$, according to 4.16.2.

(2): The definitions of P^{nep} and $\leq^{\mathrm{nep}}$ are clearly upwards absolute. If N is a candidate and N thinks $(M,p) \in P^{\mathrm{nep}}$, then $(N,\mathrm{gen}^N(M,p)) \leq^{\mathrm{nep}} (M,p)$ is N-generic:

Assume G^{nep} is a P^{nep}-generic filter over V containing (N,q) (for some q). Since gen is a dense embedding, G^{nep} defines a P_α-generic filter G_α over V, and G_α contains $\mathrm{gen}(N,q)$. This implies that G_α is (N,P_α)-generic.

We have to show that $G^{\mathrm{neq}} \cap P^{\mathrm{nep},N}$ is $P^{\mathrm{nep},N}$-generic over N. In N, the mapping $\mathrm{gen}^N : P^{\mathrm{nep},N} \to P^N_\alpha$ is dense, and G^N_α is P^N_α-generic over N. So the set $G' = \{(M,p) : \mathrm{gen}^N(M,p) \in G^N_\alpha\}$ is $P^{\mathrm{nep},N}$-generic over N. But $(M,p) \in G'$ iff $(M,p) \in G$, according to 4.15(2). $\square$

Remark 4.19. So the properties 4.15(1) and 4.16(1) are enough to show that P^{nep} can be densely embedded into P_α. But 4.15(2) is needed to show that P^{nep} actually is nep: Otherwise P^{nep} still is an upwards absolute forcing definition, and for every $p \in (P^{\mathrm{nep}})^M$ there is a $q \leq p$ in P^{nep} forcing that there is an $(P^{\mathrm{nep}})^M$-generic filter over M, namely the reverse image of G^M_α under gen^M but this filter doesn't have to be the same as $G^{\mathrm{nep}} \cap (P^{\mathrm{nep}})^M$.

4.3. Iterations along subsets of ϵ

As before we assume that $(Q_\alpha)_{\alpha\in\epsilon}$ is a sequence of forcing-definitions.

We can of course define a countable support iteration along every subset w of ϵ:

P_w, the c.s.-iteration of $(Q_\alpha)_{\alpha\in w}$ along w, is defined by induction on $\alpha \in w$: $P_{w\cap\alpha}$ consists of functions p with countable domain $\subseteq w \cap \alpha$. If α is the w-successor of ζ, then $p \in P_{w\cap\alpha}$ iff $p \restriction \zeta \in P_{w\cap\zeta}$ and $\zeta \notin \mathrm{dom}(p)$ or $p(\zeta)$ is a $P_{w\cap\zeta}$-name for for an object in Q_α. If α is a w-limit, then $p \in P_{w\cap\alpha}$ iff $p \restriction \zeta \in P_{w\cap\zeta}$ for all $\zeta \in \alpha \cap w$.

Of course this notion does not bring anything new: Assume $\beta \leq \epsilon$ is the order type of w, and let $i : \beta \to w$ be the isomorphism. Then P_w is isomorphic to the c.s.-iteration $(R_\alpha, Q_{i(\alpha)})_{\alpha<\beta}$.

We can calculate P_w inside M and extend our notation to that case:

Definition 4.20. Let M be a candidate, $w \subseteq \epsilon$, $w \in M$.

- P_w is the countable support iteration along the order w.
- P^M_w is the forcing P_w as constructed in M.
- v covers w (with respect to M) if $\epsilon \supseteq v \supseteq w \cap M$.
  (If $w \notin \mathrm{ON}$, then this is independent of M, since $w \subseteq M$ for each candidate M.)
- If v covers w, and $G_v \subseteq P_v$, then we define $G^M_{v\to w}$ by the following induction on $\alpha \in \epsilon \cap M$, provided this results in a P^M_w-generic filter over M. Otherwise, $G^M_{v\to w}$ is undefined. Let $p \in P^M_{\cap w\cap\alpha}$. If M thinks that $w \cap \alpha$ has no last element, then $p \in G^M_{v\to w\cap\alpha}$ iff $p \restriction \beta \in G^M_{v\to w\cap\beta}$ for all $\beta \in \alpha \cap M$. If $w \cap \alpha$ has the last element β, then $p \in G^M_{v\to w\cap\alpha}$ iff $p \restriction \beta \in G^M_{v\to w\cap\beta}$ and $p(\beta)[G^M_{v\to w\cap\beta}] \in G_v(\beta)$.
- A G_v such that $G^M_{v\to w}$ is defined is called (M,P_w)-generic.
- Assume that G_v is (M,P_w)-generic and $\zeta \in w \cap M$. Then we set

$$G^M_{v\to w}(\zeta) = \{\underset{\sim}{q}[G^M_{v\to w\restriction\zeta}] : \exists p\, p \cup (\zeta, \underset{\sim}{q}) \in G^M_{v\to w}\}.$$

- If v covers w, $q \in P_v$, and p is a P^M_w-condition (or p is just in M a P^M_w-name for a P^M_w-condition), then q is $(M, P_{v\to w}, p)$-generic if q forces that G_v is (M, P_w)-generic and $p \in G^M_{v\to w}$ (or $p[G^M_{v\to w}] \in G^M_{v\to w}$, resp.).

The same proofs as 4.6, 4.7 and 4.8 give us the according results for P_w:

Lemma 4.21. Assume that V' is an extension of V

- M and N are candidates in V, $M \in N$,
- $v \in N$ and u covers v with respect to N,
- $w \in M$, $M \in N$ and N thinks that M is a candidate and that v covers w with respect to M,
- $G_u \in V'$ is (N, P_v)-generic.

Then we get:

1. In V, v covers w with respect to M.
2. If $\zeta \in v \cap N$, then $G^N_{u\to v}(\zeta) = Q_\zeta^{N[G^N_{u\to v}]} \cap G_u(\zeta)$.
3. $(G^M_{u\to w})^{V'} = (G^M_{v\to w})^{N[G^N_{u\to v}]}$.
4. In particular, G is $(M, P_{u\to w})$-generic iff $N[G^N_{u\to v}]$ thinks that $G^N_{u\to v}$ is $(M, P_{v\to w})$-generic.
5. If G_u is (M, P_w)-generic and $\underset{\sim}{\tau}$ a P^M_w-name in M, then $\underset{\sim}{\tau}[G^M_{u\to w}]$ (calculated in $V[G_u]$) is the same as $\underset{\sim}{\tau}[G^M_{v\to w}]^M$ (calculated in $N[G^N_{u\to v}]$).
6. If q is $(N, P_{u\to v}, p)$-generic and $\alpha \in \epsilon \cap N$, then $q \restriction \alpha$ is $(N, P_{u\restriction\alpha\to v\cap\alpha}, p \restriction \alpha)$-generic.

Lemma 4.22. Let $N \prec H(\chi)$, $v \in N$, $(M, w) = \text{ord-col}(N, v)$,[16] and let G_v be P_v-generic over V. Then

1. G_v is N-generic iff it is $(M, P_{v\to w})$-generic.

If G_v is N-generic and $p, \underset{\sim}{\tau} \in N$, then

2. $p \in G_v$ iff $\text{ord-col}^N(p) \in G^M_{v\to w}$, and
3. $\text{ord-col}^{N[G_v]}(\underset{\sim}{\tau}[G_v]) = (\text{ord-col}^N(\underset{\sim}{\tau}))[G^M_{v\to w}]$.

Lemma 4.23. If M is a candidate, $w \in M$, v covers w, and $p \in P^M_w$, then there is a $(M, P_{v\to w}, p)$-generic $q \in P_v$.

[16] so either $w = v \cap N$ or $w = v \in \mathrm{ON}$

We give the following Lemma (used for Q =Mathias in [11]) as an example for how we can use this iteration:

LEMMA 4.24. Let $\bar{B} = (B_i)_{i\in I}$ be a sequence (in V) of Borel codes. Let $Q_\alpha = Q$ be the same nep forcing (definition) for all $\alpha < \epsilon$. If P_{ω_1} forces $\bigcap \bar{B} = \emptyset$, then P_ϵ forces $\bigcap \bar{B} = \emptyset$.

Proof. We assume that $\bigcap \bar{B} = \emptyset$ is forced by P_{ω_1} and therefore by all P_α for $\alpha \in \omega_1$. We additionally assume towards a contradiction that

$$p_0 \Vdash_\epsilon \underset{\sim}{\eta}_0 \in \bigcap B_i. \tag{4.1}$$

We fix a "countable version" of the name $\underset{\sim}{\eta}_0$: Let $N_0 \prec H(\chi)$ contain $\underset{\sim}{\eta}_0$ and p_0. Let $(M_0, \underset{\sim}{\eta}'_0, p'_0)$ be the ord-collapse of $(N_0, \underset{\sim}{\eta}_0, p_0)$. Set $w = \epsilon \cap N_0 = \epsilon \cap M_0$. In particular, w is countable.

Since w covers ϵ with respect to M_0, we can find an $(M_0, P_{w\to\epsilon}, p'_0)$-generic condition q_0 in P_w. Under q_0 we can define the P_w-name

$$\underset{\sim}{\tau} := \underset{\sim}{\eta}'_0[G^{M_0}_{w\to\epsilon}]. \tag{4.2}$$

So whenever q is in a G_w-generic filter, then $\underset{\sim}{\tau}[G_w]$ is the same as $\underset{\sim}{\eta}'_0[G^{M_0}_{w\to\epsilon}]$.

P_w is isomorphic to P_α for some countable α, so we know that P_w forces $\underset{\sim}{\tau} \notin \bigcap \bar{B}$. In particular, we can find a $\tilde{q} \le q_0$ and an $i_0 \in I$ such that

$$\tilde{q} \Vdash \underset{\sim}{\tau} \notin B_{i_0}. \tag{4.3}$$

Let $N_1 \prec H(\chi)$ contain the previously mentioned objects. In particular $w \subset N_1$. Let $(M_1, P', \tilde{q}')$ be the ord-collapses of $(N_1, P_w, \tilde{q})$. By elementarity, $P' = P^{M_1}_w$. Since ϵ covers w, we can find an $(M_1, P_{\epsilon\to w}, q'_0)$-generic condition q_1 in P_ϵ.

Let G be a P_ϵ-generic filter over V containing q_1. Set $r = \underset{\sim}{\eta}_0[G]$. So $r \in \bigcap \bar{B}$ by (4.1). On the other hand, $\tilde{r} := \underset{\sim}{\tau}'[\tilde{G}]$ is not in B_{i_0} for $\tilde{G} := G^{M_1}_{\epsilon\to w}$. Also, $\tilde{r} = \underset{\sim}{\eta}'_0[\tilde{G}^{M_0}_{w\to\epsilon}]$. It remains to show that $r = \tilde{r}$. This follows from transitivity (see Lemma 4.21), i.e., $\tilde{G}^{M_0}_{w\to\epsilon} = G^{M_0}_{\epsilon\to\epsilon}$, and the fact that $G^{M_0}_{\epsilon\to\epsilon} = G^{M_0}_\epsilon$, and from elementarity (see Lemma 4.7), i.e., $(M_0[G^{M_0}_\epsilon], M_0)$ is the ord-collapse of $(N_0[G], N_0)$. ◻

References

[1] GOLDSTERN, M. Tools for your forcing construction. In *Set theory of the reals (Ramat Gan, 1991)*, vol. 6 of *Israel Math. Conf. Proc.* Bar-Ilan Univ., Ramat Gan, 1993, pp. 305–360.

[2] GOLDSTERN, M., KELLNER, J., SHELAH, S., AND WOHOFSKY, W. Borel Conjecture and Dual Borel Conjecture. *Trans. Amer. Math. Soc. 366*, 1 (2014), 245–307. Dedicated to the memory of Richard Laver (1942–2012). `http://arxiv.org/abs/1105.0823`.

[3] IHODA, J. I., AND SHELAH, S. Souslin forcing. *J. Symbolic Logic 53*, 4 (1988), 1188–1207.

[4] Kellner, J. Preserving non-null with Suslin$^+$ forcings. *Arch. Math. Logic 45*, 6 (2006), 649–664.

[5] Kellner, J., and Shelah, S. Preserving preservation. *J. Symbolic Logic 70*, 3 (2005), 914–945.

[6] Rosłanowski, A., and Shelah, S. Norms on possibilities. I. Forcing with trees and creatures. *Mem. Amer. Math. Soc. 141*, 671 (1999), xii+167.

[7] Rosłanowski, A., and Shelah, S. Measured creatures. *Israel J. Math. 151* (2006), 61–110.

[8] Shelah, S. *Proper and improper forcing*, second ed. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1998.

[9] Shelah, S. Properness without elementaricity. *J. Appl. Anal. 10*, 2 (2004), 169–289.

[10] Shelah, S. On nicely definable forcing notions. *J. Appl. Anal. 11*, 1 (2005), 1–17.

[11] Shelah, S., and Steprāns, J. Possible cardinalities of maximal abelian subgroups of quotients of permutation groups of the integers. *Fund. Math. 196*, 3 (2007), 197–235.

[12] Zapletal, J. *Forcing idealized*, vol. 174 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2008.

AMS Subject Classification: 03E35, 03E40

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Lavoro pervenuto in redazione il 07.11.2011