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A SACKS REAL OUT OF NOWHERE

JAKOB KELLNER† AND SAHARON SHELAH‡

Abstract. There is a proper countable support iteration of length ω adding no new reals at finite stages and adding a Sacks real in the limit.

§1. Introduction.

Preservation theorems are a central tool in forcing theory:

> Let $(P_\alpha, \underset{\sim}{Q}_\alpha)_{\alpha<\epsilon}$ be a forcing iteration. Assume that Q_α is (forced to be) nice for all $\alpha < \epsilon$. Then P_ϵ is nice.[1]

A niceness (or preservation) property usually implies that the forcing does not change the universe too much. Among the most important preservation theorems are:

> The finite support iteration of ccc forcings is ccc. [8]

and

> The countable support iteration of proper forcings is proper. [6]

In this paper we investigate proper countable support iterations, so the limits are always proper. Many additional preservation properties are preserved as well, for example ω^ω-bounding (i.e., not adding an unbounded real). This is a special instance of a general preservation theorem by the second author ("Case A" of [7, XVIII §3]) which is also known as "first preservation theorem" [1, Section 6.1.B] or "tools-preservation" [4, Section 5], see also [5, Theorem 2.4]. Many additional preservation theorems for proper countable support iterations can be found in [7], or, from the point of view of large cardinals, in [9].

We investigate iterations where all iterands are NNR, which means that they do not add new reals. So the iterands (and therefore the limit as well) satisfy all instances of tools-preservation. However, it turns out that the limit can add a new real r. The first example was given by Jensen [3], and the phenomenon was further investigated in [7, V]. So what do we know about the real r? We know that it has to be bounded by an old real (i.e., a real in the ground model), corresponding to

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[1]Often preservation theorems also have the (weaker) form "If ϵ is a limit, and all P_α are nice for $\alpha < \epsilon$, then P_ϵ is nice."

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the iterable preservation property "ω^ω-bounding". r will even satisfy the stronger Sacks property. In particular r cannot be, e.g., a Cohen, random, Laver or Mathias real. In the previously known examples, the proof that a new real r is added is rather indirect and does not give much "positive" information about r. So it is natural to ask which kind of reals *can* appear in proper NNR limits. Todd Eisworth asked this question for the simplest and best understood real that satisfies the Sacks property, the Sacks real. In this paper, we show that Sacks reals indeed can appear in this way:

THEOREM 1. *There is an iteration $(P_n, \underaccent{\tilde}{Q}_n)_{n<\omega}$ such that each Q_n is forced to be proper and NNR and such that the countable support limit P_ω adds a Sacks real. Moreover, P_ω is equivalent to $S * P'$, where S is Sacks forcing and P' is NNR.*[2]

The Theorem can be interpreted in two ways:

On the one hand, it indicates limitations of possible preservation theorems: "Not adding a Sacks real" is obviously not iterable (even with rather strong additional assumptions).

On the other hand, it shows that Sacks forcing is exceptionally "harmless": It satisfies every usual iterable preservation property.[3] So the Sacks model (the model constructed by starting with CH and iterating ω_2 Sacks forcings in a countable support iteration) has all the corresponding properties as well.[4]

In a continuation of this work we will say more about the kind of reals that can be added in limits of NNR iterations (e.g., generics for other finite splitting lim sup tree forcings). It turns out that many of these reals can appear at limit stages, but some of them not at stage ω, but only at later stages, e.g., ω^2.

We thank a referee for suggesting several improvements in the presentation.

§2. Sacks conditions as squares of terms. In this section, we introduce the forcing notion Q_*, which is forcing equivalent to Sacks forcing. We will later work with Q_* in the proof of Theorem 1.

A Sacks condition (or Sacks tree) is a perfect tree $T \subseteq 2^{<\omega}$. Given T, we call a node t a splitting node if t has two immediate successors in T.

Let F_n be the set of the n-th splitting nodes, cf. Figure 1. So $t \in F_n$ means that t is a splitting node and that there are n splitting nodes below t. Since T is perfect, F_n is a front, which means that every branch through T meets F_n exactly once. Being a front is stronger than just being a maximal antichain, and due to König's Lemma every front is finite.

A branch b through T is an element of 2^ω and therefore a sequence $(b_0, b_1, \dots)$ for some $b_n \in \{0, 1\}$. Intuitively speaking, we can describe "the arbitrary branch" b of T by interpreting each b_n to be a term $t_n(x_0, \dots, x_n)$, where the value of t_n (0 or 1) depends on x_l for $l \le n$, and x_l is a variable with values in $\{0, 1\}$ that tells us whether we choose the left (0) or right (1) path at the front F_l.

[2]We do not claim that P' is proper.

[3]More exactly: Sacks forcing satisfies every property X such that: Every proper NNR forcing satisfies X, X is preserved under proper countable support iterations, and if P does *not* satisfy X, then $P * \underaccent{\tilde}{Q}$ does not satisfy X either for any NNR Q.

[4]Of course this is already known for many of the popular properties, cf. [2] or [9], which shows that in some respect Sacks forcing is the "most tame" forcing possible. This corresponds to the fact that all of the usual cardinal characteristics (apart from the continuum) are $\aleph_1$ in the Sacks model.

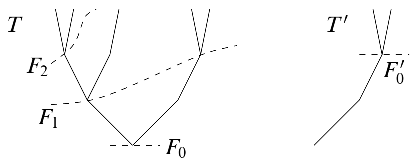

FIGURE 1. F_n is the front of n-th splitting nodes.

A more formal description of terms can be found in Definition 2.4, but a simple example is much more instructive: In the tree T of Figure 1, the sequence of terms begins as follows:

$$t_0 = x_0, \quad t_1 = \begin{cases} x_1 & \text{if } x_0 = 0, \\ 1 & \text{otherwise,} \end{cases} \quad t_2 = \begin{cases} x_2 & \text{if } x_0 = 0 \text{ and } x_1 = 0, \\ 1 & \text{if } x_0 = 0 \text{ and } x_1 = 1, \\ x_1 & \text{otherwise.} \end{cases}$$

We will use the following notation:

Given a Sacks tree T, the sequence $\bar{t}$ of terms defined as above is called the canonical term sequence for T. (2.1)

Let a be an assignment, that is a map that assigns each variable a value in $\{0, 1\}$. Then a can be extended to evaluate terms t to $t \circ a \in \{0, 1\}$, so we can evaluate the term sequence $\bar{t} = (t_0, t_1, \dots)$ to

$$\bar{t} \circ a := (t_0 \circ a, t_1 \circ a, \dots) \in 2^\omega.$$

If $\bar{t}$ is the canonical term sequence for T and a an assignment, then $\bar{t} \circ a$ is a branch through T. Moreover, every branch can be obtained this way:

$$T = \{\bar{t} \circ a \restriction n : n \in \omega,\ a \text{ an assignment}\}. \tag{2.2}$$

The following property is trivial, but important: Fix n. Then there is a finite set I such that we can determine the value that is assigned to x_n by an assignment a provided we know the values $(t_i \circ a)_{i \in I}$. We denote this by the following: For a canonical term sequence $\bar{t}$,

$$\text{each } x_n \text{ is determined by finitely many } t_i. \tag{2.3}$$

(Proof: Let l be the maximum of the heights of the nodes in F_n. Set $I = \{0, \dots, l\}$.)

In the example above, x_0 is determined by t_0, and x_1 by (t_0, t_1, t_2), but not by (t_0, t_1).

Let $T' \subseteq T$ be a perfect subtree, and call the canonical term sequence $(t'_0, t'_1, \dots)$, written as terms in the variables $x'_0, x'_1, \dots$. In the example of Figure 1, we get:

$$t'_0 = 1, \quad t'_1 = 1, \quad t'_2 = x'_0.$$

The fronts F'_n "refine" F_n: If $t \in F'_n$, then $t \geq s$ for a unique $s \in F_n$. So the variables $(x'_0, \dots, x'_l)$ give at least as much information (about the branch) as $(x_0, \dots, x_l)$. In other words, we can calculate the value of x_n given the values $(x'_0, \dots, x'_n)$, and we write this dependence as a term $\phi_n(x'_0, \dots, x'_n)$. This defines a function (or: term sequence) ϕ that assigns to each variable x_n a term $\phi_n(x'_0, \dots, x'_n)$. We will call ϕ a substitution. So for every assignment a of the variables x', we get the same result when we apply a to the term sequence $\bar{t}'$ as we get when we apply $\phi \circ a$ to $\bar{t}$. In other notation, $\bar{t}' = \bar{t} \circ \phi$.

In the example, the substitution ϕ has the following values:

$$x_0 = \phi_0(x'_0) = 1, \quad x_1 = \phi_1(x'_0, x'_1) = x'_0, \quad \dots$$

It is easy to check that, e.g., $t'_2(x'_0, x'_1, x'_2) = x'_0$ is indeed the same as $t_2(x_0, x_1, x_2)$ after applying the substitution ϕ, i.e., $t'_2 = t_2 \circ \phi$:

$$t_2 = \begin{cases} x_2 & \text{if } x_0 = 0 \text{ and } x_1 = 0, \\ 1 & \text{if } x_0 = 0 \text{ and } x_1 = 1, \\ x_1 & \text{otherwise.} \end{cases} \qquad t_2 \circ \phi = \left\{ \begin{array}{ll} \phi_2 & \text{if } 1 = 0 \text{ and } x'_0 = 0, \\ 1 & \text{if } 1 = 0 \text{ and } x'_0 = 1, \\ x'_0 & \text{otherwise} \end{array} \right\} = x'_0.$$

Also, each x'_j is determined by finitely many ϕ_i. This means: For each j there is a finite set I such that the following holds: If a, b are assignments of $\{x'_0, x'_1, \dots\}$ that map x'_j to different values, then $(\phi_i \circ a)_{i \in I} \neq (\phi_i \circ b)_{i \in I}$. (Proof: Pick $l \in \omega$ such that each node in F_l is longer than every node in F'_{j+1}, and set $I = \{0, \dots, l\}$.)

So far, we used different variable symbols (x_i and x'_i) for variables used in $\bar{t}$ and $\bar{t}'$ (in the hope to make the concept of substitution a bit clearer). Of course this is not necessary, and we will only use x_i in the following. We will see that the following partial order S^* is equivalent to Sacks forcing: S^* consists of sequences of terms $(t_i)_{i \in \omega}$ using the variables x_j ($j \in \omega$) such that

$$\begin{array}{rl} \text{(i)} & t_i \text{ depends only on } x_j \text{ with } j \leq i, \text{ and} \\ \text{(ii)} & \text{each } x_j \text{ is determined by finitely many } t_i. \end{array} \tag{2.4}$$

The order is defined as follows: $\bar{t}'$ is stronger than $\bar{t}$, if there is a substitution ϕ such that $\bar{t}' = \bar{t} \circ \phi$ and

$$\begin{array}{rl} \text{(i)} & \phi_i \text{ only depends on } x_j \text{ with } j \leq i, \text{ and} \\ \text{(ii)} & \text{each } x_j \text{ is determined by finitely many } \phi_i. \end{array} \tag{2.5}$$

It is easy to check that $\leq$ is reflexive and transitive; and that $\circ$ is associative: The identity substitution witnesses $\bar{t} \leq \bar{t}$; and if $\bar{t}' = \bar{t} \circ \phi'$ and $\bar{t}'' = \bar{t}' \circ \phi'$ then $\bar{t}'' = (\bar{t} \circ \phi) \circ \phi' = \bar{t} \circ (\phi \circ \phi')$.

We could omit (2.5)(ii): If ϕ is any substitution, and if $\bar{t}$ and $\bar{t} \circ \phi$ both are in S^*, then ϕ satisfies (ii) anyway.

Substitutions (as defined in (2.5)) are obviously exactly the same as conditions in S^* (as defines in (2.4)). This fact is not deep or of any real importance, but it will simplify our notation. So let us describe this effect once more:

Assume that $\bar{s}$ and $\bar{t}$ both are conditions in S^*. We can interpret $\bar{t}$ as a substitution ϕ such that $\phi_n = t_n$. (I.e., $t_n(x'_0, \dots, x'_n)$ calculates the value of x_n.) Then $\bar{s} \circ \bar{t}$ is again element of S^* (and stronger than $\bar{s}$). On the other hand, if $\bar{t}'$ is stronger than $\bar{s}$, then this is witnessed by a substitution ϕ, which we can in turn interpret as element of S^*.

We can interpret a $\bar{t} \in S^*$ as continuous function from 2^ω to 2^ω, and map $\bar{t}$ to its image, or to the associated tree:

LEMMA 2.1. *Let Ψ map $\bar{t} \in S^*$ to $\{(\bar{t} \circ a) \restriction n : n \in \omega, a \text{ an assignment}\}$. Then Ψ is a surjective complete embedding (in particular order preserving) from S^* into Sacks forcing.*

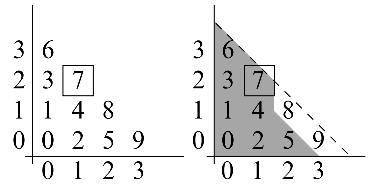

FIGURE 2. We use a canonical ordering of $\omega \times \omega$. The node $(1, 2)$ corresponds to 7. The nodes (n, m) smaller than $(1, 2)$ all satisfy $n + m \leq 1 + 2 < 4$, as in (2.8).

PROOF. $\Psi(\bar{t})$ is a perfect tree: Pick any $s = (\bar{t} \circ a) \restriction n \in \Psi(\bar{t})$. Note that $t_0, \ldots, t_{n-1}$ use a finite set A of variables. Pick $x_j \notin A$. Then x_j is determined by $t_0, \ldots, t_{l-1}$ for some l. Pick assignments b, c extending $a \restriction A$ such that $x_j \circ b \neq x_j \circ c$. Then $(\bar{t} \circ b) \restriction l \neq (\bar{t} \circ c) \restriction l$ (otherwise they would determine the same value for x_j), so we get two incomparable nodes in $\Psi(\bar{t})$ both extending s.

We see from (2.2) that Ψ is surjective. It is clear that Ψ preserves $\leq$.

Ψ preserves $\perp$: Assume that $\Psi(\bar{t})$ and $\Psi(\bar{s})$ both contain the perfect tree T. By thinning out T, we can assume the following: If l is the length of a node in F_n, then $t_1(\bar{x}), \ldots, t_l(\bar{x})$ determine x_n, and the same holds for $\bar{s}$. Let $\bar{r}(\bar{x}') \in S^*$ be the canonical sequence of T. So $x'_0 \ldots x'_{n-1}$ determine a node in F_n, and therefore sufficiently many $t_1, \ldots, t_{l-1}$ to determine x_n. This defines a substitution ϕ witnessing that $\bar{r}$ is stronger than $\bar{t}$. The same applies to $\bar{s}$. $\dashv$

Of course Ψ is not injective. For example, if we simply interchange x_0 and x_1 in a suitable sequence $\bar{t}$, then we can still get a valid term sequence (different from the original one), but the image under Ψ will be the same. In S^*, the index set of the term sequence is ω. We will later need $\omega \times \omega$-sequences; so we will just identify ω with $\omega \times \omega$, using a canonical order. See Figure 2.

$$\tau \colon \omega \times \omega \to \omega \text{ is defined by } \tau(n, m) = n + \frac{1}{2}(n + m)(n + m + 1). \tag{2.6}$$

The bijection τ defines a linear order of $\omega \times \omega$ of order type ω:

$$(i, j) \trianglelefteq (n, m) \text{ iff } \tau(i, j) \leq \tau(n, m). \tag{2.7}$$

We will later use the following trivial fact:

$$\text{If } i + j < n \text{ and } (i', j') \trianglelefteq (i, j) \text{ then } i' + j' < n. \tag{2.8}$$

We now rewrite S^* in the form of $\omega \times \omega$-sequences:

DEFINITION 2.2. Q_* consists of squares of terms $(t_{n,m})_{n,m \in \omega}$ using the variables $x_{i,j}$ $(i, j \in \omega)$ such that

(i) $t_{n,m}$ depends only on $x_{i,j}$ with $(i, j) \trianglelefteq (n, m)$, and
(ii) each $x_{i,j}$ is determined by finitely many $t_{n,m}$.

The order is defined as follows: $\bar{t}$ is stronger than $\bar{s}$, if there is a condition $\phi \in Q_*$ such that $\bar{t} = \bar{s} \circ \phi$.

Since Q_* is isomorphic to S^*, Lemma 2.1 gives us:

COROLLARY 2.3. *Q_* is forcing equivalent to Sacks forcing.*

We now add the a formal definition of term, assignment and substitution:

Definition 2.4. • Let X be a set. We will call an element $v \in X$ a variable (or: variable symbol). We will interpret v as a "binary variable", i.e., the value of v is 0 or 1. In this paper, we will use $X = \{x_i : i \in \omega\}$ and $X = \{x_{i,j} : i, j \in \omega\}$.

- An X-term t consists of[5] a sequence $(v_0, \dots, v_{l-1})$ for some $0 \leq l < \omega$ and $v_i \in X$, together with a function $f : 2^l \to 2$. (So for $l = 0$, the sequence of variables is empty and the term is a constant.) We usually write terms as $t(v_0, \dots, v_{l-1})$. Abusing notation, we identify the variable v with the "identity term" corresponding to (v), Id.
- An assignment a is a function $X \to 2$. Assignments extend to all X-terms in the natural way. In other words, given an assignment a, we can apply a to a term t to get an element of 2. We denote the result of applying a to a term (or variable) t by $t \circ a \in 2$.
- Similarly, a substitution ϕ maps X to X-terms. Equivalently, a substitution is a sequence $(\phi_v)_{v \in X}$ of X-terms indexed by X. Again, we can extend a substitution to act on all X-terms, and we write $t \circ \phi$ for the result. We can also apply substitutions to sequences $\bar{t} = (t_v)_{v \in X}$ of terms (indexed by X), the result $\bar{t} \circ \phi$ is another sequences of terms indexed by X. The application of substitutions is associative: For term sequences $\bar{r}$, $\bar{s}$ and $\bar{t}$, all indexed by X, we get $\bar{r} \circ (\bar{s} \circ \bar{t}) = (\bar{r} \circ \bar{s}) \circ \bar{t}$.
- The variable (or term) s "is determined by the terms $t_0, \dots, t_n$" means that
$$(t_0 \circ a, \dots, t_n \circ a) = (t_0 \circ b, \dots, t_n \circ b) \text{ implies } s \circ a = s \circ b$$
for all assignments a and b. In other words, if we know the value of $t_0, \dots, t_n$, we can infer the value of s.
- According to our formal definition, two terms that depend on different variables are distinct (even if these variables are not relevant). However, we will only be interested in terms "as functions", i.e., modulo the following equivalence relation: $t =^* s$ means that $t \circ a = s \circ a$ for all assignments a. In particular, the last $=$ sign in Definition 2.2 really means $=^*$ etc.

§3. A simple case.

In the rest of the paper, δ always denotes a countable limit ordinal.

In this section, we construct a proper, NNR countable support iteration and argue that the limit adds a real that it is similar to a Sacks real (i.e., it adds a generic object for a forcing that looks in some way similar to the Q_* defined in the previous section). In the rest of the paper, we deal with an analog (but notationally more complicated) construction that actually adds a Sacks real.

So the purpose of this section is to give some idea of the constructions we use to prove Theorem 1, using a somewhat simplified notation. The reader who does not feel the need of such an introduction can safely continue with the next section.

We do not give any proofs in this section, but refer to the proofs of the more general statements. *Caution*: We use the same symbols for the simpler objects in this section and for the analog constructions in the rest of the paper.

[5] Formally we could let t be a triple $(X, (v_0, \dots, v_{l-1}), f)$, to guarantee that X is disjoint to the terms built from it.

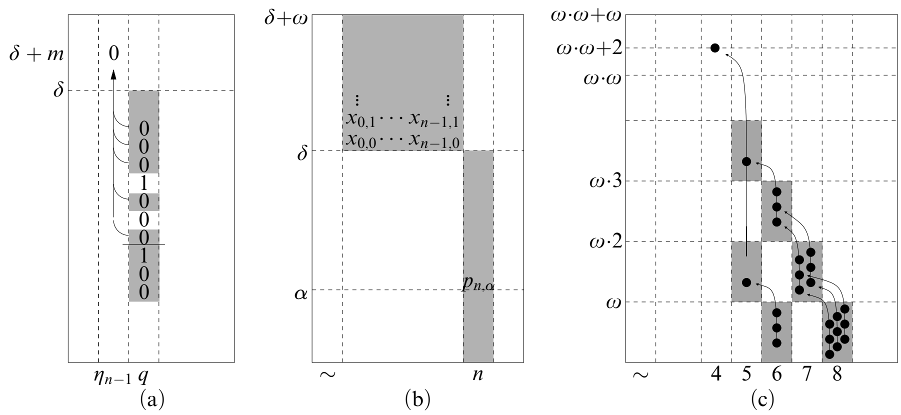

FIGURE 3. (a) q coheres with $\eta_{n-1}(\delta+m)$. The gray area indicates $v_{\delta,n-1,m}$. (b) An element of R: $p_{n,\delta+m} = x_{n,m}$. The term $p_{n,\alpha}$ only depends on $x_{i,j}$ with $i < n$. (c) Filling in the term $x_{4,2}$ at at various positions: In the bottom row, $I_{4,2}$ contains 6 and 8; the $1 \times \omega$-blocks where the terms $x_{4,2}$ are added (indicated by the gray area) propagates up-left.

The forcing iteration will start with a preparatory forcing $\tilde{P}$, followed by $\underset{\sim}{Q}_0$, $\underset{\sim}{Q}_1, \ldots$. $\tilde{P} * P_n$ stands for $\tilde{P} * Q_0 * \cdots * Q_{n-1}$. We will also use the countable support limit of $\tilde{P} * P_n$. Since all forcings are proper, this countable support limit is the same as $\tilde{P} * P_\omega$, where P_ω is the $\tilde{P}$-name for the countable support limit of the P_n.

The preparatory forcing adds cofinal subsets $v_{\delta,n,m} \subseteq \delta$ of of order type ω for every limit ordinal $\delta < \omega_1$ and $n, m \in \omega$. In more detail:

DEFINITION 3.1. A condition $\tilde{p}$ in $\tilde{P}$ consists of a limit ordinal $\mathrm{ht}(\tilde{p}) \in \omega_1$ and a sequence $(v_{\delta,n,m})_{0<\delta<\mathrm{ht}(\tilde{p}),n,m\in\omega}$, such that $v_{\delta,n,m} \subseteq \delta$ is cofinal and has order type ω, and v_{δ,n,m_1} and v_{δ,n,m_2} are disjoint for $m_1 \neq m_2$. $\tilde{P}$ is ordered by extension.

So $\tilde{P}$ is σ-closed.

DEFINITION 3.2. Q_0 is (the $\tilde{P}$-name) for $2^{<\omega_1}$, ordered by extension.

So Q_0 is σ-closed as well, and adds the generic sequence $\eta_0 \in 2^{\omega_1}$.

Given $\tilde{P} * P_n = \tilde{P} * Q_0 * \cdots * Q_{n-1}$ such that Q_{n-1} adds the generic sequence $\eta_{n-1} \in 2^{\omega_1}$, we define the $\tilde{P} * P_n$-name Q_n (see also Figure 3(a)):

DEFINITION 3.3. Let q be a partial function from ω_1 to 2, $\delta \subseteq \mathrm{dom}(q)$. q and η_{n-1} cohere at $\delta + m$, if $\eta_{n-1}(\delta + m) = q(\alpha)$ for all but finitely many $\alpha \in v_{\delta,n-1,m}$. Abusing notation, we just say q coheres with $\eta_{n-1}(\delta + m)$.

We set $q \in Q_n$, if $q \in 2^{<\omega_1}$ and q coheres with $\eta_{n-1}(\delta + m)$ for all $\delta \leq \mathrm{dom}(q)$, $m \in \omega$.

LEMMA 3.4. *The following is forced by P_n:*

(i) *If $q \in Q_n$, $q' \in 2^{\mathrm{dom}(q)}$, and $q'(\alpha) = q(\alpha)$ for all but finitely many $\alpha \in \mathrm{dom}(q)$, then $q' \in Q_n$.*

(ii) *Q_n is separative,*[6] *and adds a generic sequence $\eta_n \in 2^{\omega_1}$ defined by $\bigcup_{q \in G(n)} q$.*
(iii) *If $n > 0$, then Q_n is not σ-closed.*
(iv) *Q_n is proper and NNR, i.e., Q_n adds no new real.*

For a proof, see Lemmas 4.7, 4.9 and 4.11. Note that (i)–(iii) are very easy, and (iv) is straightforward (but a bit cumbersome).

Lemma 3.5. *$\tilde{P} * P_\omega$ adds a new real. In particular, the $\tilde{P}$-generic element $\bar{v}$ together with $(\eta_n(m))_{n,m\in\omega}$ determines the generic filter.*

Proof. If we know $v_{\omega,n-1,m}$ and $\eta_n(l)$ for all $l \in \omega$, then we can determine $\eta_{n-1}(\omega + m)$. So if we know all $v_{\delta,n,m}$ and all $\eta_n(m)$ for $\delta < \omega_1, n, m \in \omega$, then we can by induction on $\alpha < \omega_1$ calculate all $\eta_n(\alpha)$ for $n \in \omega$. ⊣

We now define a dense subforcing R of $\tilde{P} * P_\omega$. See Figure 3(b). We use the notion of variable, term, assignment and substitution, as in Definition 2.4, for the set of variables $X = \{x_{i,j} : i, j \in \omega\}$.

Definition 3.6. $R = \bigcup_{\delta<\omega_1} R_{\delta+\omega}$. A condition p in $R_{\delta+\omega}$ consists of $\tilde{p}$ and $\bar{p}$ such that

- $\tilde{p} \in \tilde{P}$, $\mathrm{ht}(\tilde{p}) = \delta + \omega$.
- $\bar{p} = (p_{n,\alpha})_{n\in\omega,\alpha\in\delta+\omega}$.
- $p_{n,\delta+m}$ is the term $x_{n,m}$.
- For $\alpha < \delta$, $p_{n,\alpha}$ is a term using only variables $x_{i,j}$ with $i < n$.
- For $\alpha \leq \delta$ limit and $n, m \in \omega$, $p_{n,\alpha+m} = p_{n+1,\zeta}$ for all but finitely many $\zeta \in v_{\alpha,n,m}$.

We identify two conditions p and q if $\tilde{p} = \tilde{q}$ and $p_{n,\alpha} =^* q_{n,\alpha}$ for all $n \in \omega$, $\alpha < \delta$.

We can interpret $p \in R$ as a condition $(\tilde{p}, p(0), p(1), \dots)$ in $\tilde{P} * P_\omega$: After forcing with $\tilde{P} * P_n$, we have the generic sequences $(\eta_i)_{i<n}$. This defines a canonical assignment of $x_{i,j}$ for $i < n$, namely $x_{i,j} := \eta_i(\delta + j)$. This assignment evaluates $(p_{n,\alpha})_{\alpha<\delta}$ to a condition in Q_n (assuming that $\tilde{p}$ is element of the $\tilde{P}$-generic filter), and we define $p(n)$ to be that condition. Using this identification, we get:

Lemma 3.7. *R is a dense subset of $\tilde{P} * P_\omega$.*

For a proof, see Lemma 5.8. The proof is again a bit cumbersome, and uses similar arguments (chains of countable elementary submodels) as the proof of 3.4(iv).

Note the following simple properties for $p \in R_{\delta+\omega}$:

- If $\delta = \omega$, we get $(\forall n, m)(\exists^\infty k) p_{n+1,k} = x_{n,m}$.
- If $\delta = \omega + \omega$, we get $(\forall n, m)(\exists^\infty k) p_{n+2,k} = x_{n,m}$.
- If $\delta = \omega \cdot \omega$, then we get $(\forall n, m)(\exists^\infty n')(\exists^\infty m') p_{n',m'} = x_{n,m}$.

Actually, the last item holds for all $\delta \geq \omega \cdot \omega$, which can easily be seen by induction; and we get some kind of converse as well:

Lemma 3.8. *Assume that $(r_{i,j})_{i,j\in\omega}$ is a matrix of terms such that*
(i) *$r_{i,j}$ depends only on $x_{n,m}$ with $n < i$,*
(ii) *$(\forall n, m)(\exists^\infty n')(\exists^\infty m') r_{n',m'} = x_{n,m}$*
(iii) *$(\forall n)(\exists^\infty k) r_{n,k} = 0$.*

Then there is a $p \in R_{\omega\cdot\omega+\omega}$ such that $r_{i,j} = p_{i,j}$ for all $i, j \in \omega$.

[6] That is, for every $p \in Q$ there are $q_1, q_2 \leq p$ such that $q_1 \perp q_2$.

PROOF (SKETCH). We have to define a suitable $\tilde{p}$ (i.e., the sequence $(v_{\alpha,n,m})_{\alpha\le\omega,n,m\in\omega})$ as well as $p_{i,\alpha}$ for $i\in\omega$ and $\omega\le\alpha<\omega\cdot\omega$.

We deal with one variable after the other, see Figure 3(c). Assume we are dealing with $x_{n,m}$. Set

$$I_{n,m}=\{n'\ge n+2\colon(\exists^\infty k)r_{n',k}=x_{n,m}\}.$$

According to (ii), $I_{n,m}$ is infinite. We define $v_{\omega\cdot\omega,n,m}\subseteq[\omega,\omega\cdot\omega[$ such that

$$v_{\omega\cdot\omega,n,m}\cap[i\cdot\omega,(i+1)\cdot\omega[$$

contains a single element (not used so far) if $i+m+1\in I_{n,m}$ and is empty otherwise. We set $r_{n+1,\alpha}=x_{n,m}$ for all $\alpha\in v_{\omega\cdot\omega,n,m}$; and propagate the $x_{n,m}$ diagonally down.

We repeat the same construction for all the other $x_{n,m}$, and then set all the remaining terms $r_{n,\alpha}=0$. To get coherence for these points as well, we just define the remaining v's in a way so that they only point to r's that are 0. At height ω, we use (iii) to do this, at other heights we just have to make sure to leave enough space when choosing the elements of $v_{\delta,n,m}$. ⊣

We now describe how to "stack" a condition on top of another one to get a stronger condition. See Figure 5(b) for a graphical illustration.

- If we "cut away the bottom part" of a condition $q\in R_{\delta+\delta'+\omega}$ at height δ, then we get a condition $q'\in R_{\delta'+\omega}$. Formally we can define q' as follows:
  - $\beta\in\tilde{q}'_{\alpha,n,m}$ iff $\delta+\beta\in\tilde{q}_{\delta+\alpha,n,m}$.
  - $q'_{n,\alpha}=q_{n,\delta+\alpha}$.

  We denote this q' by $q\restriction[\delta,\delta+\delta'+\omega]$.
- We can stack any condition $q'\in R_{\delta'+\omega}$ on top of some condition $p\in R_\delta+\omega$, resulting in some $q\in R_{\delta+\delta'+\omega}$ such that $q\restriction[\delta,\delta+\delta'+\omega]=q'$. Formally, q is defined as follows:
  - $\tilde{q}\restriction\delta=\tilde{p}$.
  - For $\alpha<\delta'$, we set $\delta+\beta\in\tilde{q}_{\delta+\alpha,n,m}$ iff $\beta\in\tilde{q}'_{\alpha,n,m}$.
  - For $\alpha<\delta'$, we set $q_{n,\delta+\alpha}=q'_{n,\alpha}$.
  - We define the substitution ϕ by $\phi_{n,m}=q'_{n,m}$, and set $q_{n,\alpha}=p_{n,\alpha}\circ\phi$ for all $\alpha<\delta$.

  We denote this q by $p\Lsh q'$.

It is clear that $p\Lsh q'\le p$ (interpreted as element of $\tilde{P}*P_\omega$). The converse is true as well:

$$\begin{aligned}&\text{If } q\le p\text{, then either } q=p \text{ or } q=p\Lsh q',\\ &\text{where } q'=q\restriction[\mathrm{ht}(p),\mathrm{ht}(q)].\end{aligned}\tag{3.1}$$

The proof of (3.1) uses the following simple fact: For every $p\in R$,

$$\begin{aligned}&\text{every partial assignment of every finite } A\subseteq\{x_{i,j}\colon i,j\in\omega\}\\ &\text{is compatible with } p.\end{aligned}\tag{3.2}$$

In other words, if $f\colon\omega\times\omega\to2$ is a finite partial function, then it is compatible with p (interpreted as element of $\tilde{P}*P_\omega$) that $\eta_n(\delta+m)=f(n,m)$ for all $(n,m)\in\mathrm{dom}(f)$.

Given a $p\in R_\delta$ (we assume $\delta\ge\omega\cdot\omega$), we can map p to the square of terms $\sigma(p)=(p_{n,m})_{n,m\in\omega}$. Then σ maps R to Q_{**} in an order preserving way, where Q_{**} is defined as follows:

DEFINITION 3.9. Q_{**} is the set of all sequences $\bar{t} = (t_{i,j})_{i,j\in\omega}$ of terms such that

(i) $t_{n,m}$ depends only on $x_{i,j}$ with $i < n$,
(ii) $(\forall n, m)(\exists^\infty n')(\exists^\infty m')t_{n',m'} = x_{n,m}$,

$\bar{t} \leq \bar{s}$, if there is a substitution ϕ such that:

(iii) $t_{i,j} = s_{i,j} \circ \phi$ for all $i, j \in \omega$.
(iv) $\phi_{n,m}$ only depends on $x_{i,j}$ with $i \leq n$.

LEMMA 3.10. *R (or equivalently: $\tilde{P} * \tilde{P}_\omega$) adds a generic filter for Q_{**}.*

Note that Q_{**} looks somewhat similar to the Q_* defined in the previous section. For Q_* instead of Q_{**}, the Theorem is the main part of Theorem 1. In the rest of the paper, we will modify the constructions so that we actually end up with Q_* instead of Q_{**}.

PROOF (SKETCH). We already mentioned that $\sigma\colon R \to Q_{**}$ preserves $\leq$. Assume that G is R-generic over V, and define

$$G_{**} = \{\bar{t} \in Q_{**}\colon (\exists p \in G)\, \sigma(p) \leq \bar{t}\}.$$

It is enough to show the following:

$$\text{For } p \in R \text{ there is a } \bar{s} \leq_{Q_{**}} \sigma(p) \text{ such that for all } \bar{t} \leq \bar{s} \text{ then there is an } q \leq_R p \text{ such that } \sigma(q) \leq \bar{t}. \tag{3.3}$$

Then the Lemma follows: First note that G_{**} does not contain incompatible elements, since σ is order preserving. Now assume that $D \subseteq Q_{**}$ is dense, and (towards a contradiction) that p forces that G_{**} does not meet D. Then pick some $\bar{t} \leq \bar{s}$ in D and some q as above, contradiction.

To show (3.3), we define $\bar{s} \in Q_{**}$ via the substitution ϕ witnessing $\bar{s} \leq \sigma(p)$, defined as follows: For each n, let $(\phi_{n,m})_{m\in\omega}$ enumerate (with infinite repetitions) the constant term 0 and all variables $x_{i,j}$ with $i < n$. So ϕ maps $x_{n,m}$ to the term $\phi_{n,m}$.

Now pick any $\bar{t}$ that is stronger than $\bar{s}$, witnessed by some substitution ψ. Note that $\phi \circ \psi$ satisfies the requirements of Lemma 3.8. So there is a $q' \in R$ such that $\sigma(q') = \phi \circ \psi$. Then $p \wedge q'$ is as required. ⊣

In the rest of the paper, we will modify the constructions of this section in such a way that we end up with Q_* instead of Q_{**}. It turns out that this does not require any new concepts, just a more awkward notation.

§4. The NNR iteration.

In the rest of the paper, δ always denotes a countable limit ordinal.

First we define a σ-closed preparatory forcing $\tilde{P}$, which gives us for every limit $\alpha \in \omega_1$ a subset of α of order type ω and some simple coding sequences.

DEFINITION 4.1. $\tilde{p} \in \tilde{P}$ if for some $\mathrm{ht}(\tilde{p}) < \omega_1$, $\tilde{p}$ consists of sequences

$$v_{\delta,n,m},\ j_{\delta,n,m},\ \text{and } f_{\delta,n,m,k} \text{ for } \delta < \mathrm{ht}(\tilde{p}),\ m, n, k \in \omega,$$

such that

- each $v_{\delta,n,m}$ is a cofinal, unbounded subset of δ of order type ω.
- $m_1 \neq m_2$ implies that v_{δ,n,m_1} and v_{δ,n,m_2} are disjoint.
- $j_{\delta,n,m}$ is an increasing function from ω to ω.
- $f_{\delta,n,m,k}$ is a surjective function from $2^{[j_{\delta,n,m}(k), j_{\delta,n,m}(k+1)-1]}$ to 2.

$\tilde{P}$ is ordered by extension.

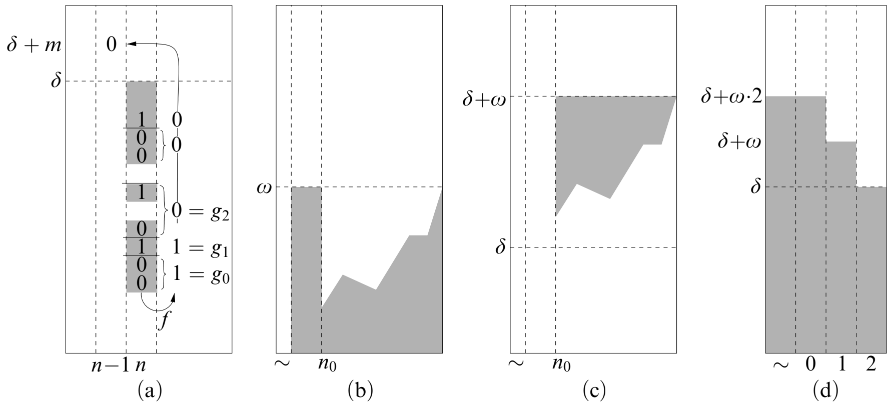

FIGURE 4. (a) η_n coheres with $\eta_{n-1}(\delta+m)$ above $k_0 = 2$. $v_{\delta,n-1,m}$ is indicated by the gray area, $j_{\delta,n-1,m}$ corresponds to the partition of this area, $f_{\delta,n-1,m,k}$ calculates $g_{\delta,n-1,m,k}$. (b) There are conditions which determine the gray values, but these conditions are not dense. (c) No condition can determine all gray values. (d) A typical condition in $\tilde{P} * P_3$: The gray area indicates the domain, all the values are "constants".

LEMMA 4.2. *$\tilde{P}$ is σ-closed, and forces that $2^{\aleph_0} = \aleph_1$.*

PROOF. In the $\tilde{P}$-extension V' define A_δ by $l \in A_\delta$ iff $\delta+2l \in v_{\delta+\omega,0,0}$. By a simple density argument, $\{A_\delta : \delta < \omega_1\}$ contains all old reals and therefore all reals. ⊣

Fix δ, n, m, k. Given $v_{\delta,n-1,m}$, $j_{\delta,n-1,m}$ and $f_{\delta,n-1,m,k}$, we define the function $g_{\delta,n-1,m,k} : 2^\delta \to \{0,1\}$ the following way, cf. Figure 4(a):

Fix $\eta_n \in 2^\delta$. For $i \in \omega$, let ζ_i be the i-th element of $v_{\delta,n-1,m}$. Set $b_i = \eta_n(\zeta_i)$. So $\bar{b} = (b_i)_{i\in\omega} \in 2^\omega$. Look at $\bar{b} \restriction [j_{\delta,n-1,m}(k), j_{\delta,n-1,m}(k+1)-1]$. This is a 0-1-sequence of appropriate length, so we can apply $f_{\delta,n-1,m,k}$. We call the result $g_{\delta,n-1,m,k}(\eta_n)$. To summarize: Let ζ_i be the i-th element of $v_{\delta,n-1,m}$. Then we define

$$g_{\delta,n-1,m,k}(\eta) = f_{\delta,n-1,m,k}((\eta(\zeta_i))_{j_{\delta,n-1,m}(k)\le i<j_{\delta,n-1,m}(k+1)}).$$

We will be interested in sequences $(\eta_n)_{n\in\omega}$ that cohere with respect to $g_{\delta,n,m,k}$. We again refer to Figure 4(a):

DEFINITION 4.3. Let η_{n-1} and η_n be partial functions from ω_1 to 2, $\delta+m \in \mathrm{dom}(\eta_{n-1})$, $\delta \subseteq \mathrm{dom}(\eta_n)$, $k_0 \in \omega$. We say that η_{n-1} and η_n cohere at $\delta+m$ above k_0, if $\eta_{n-1}(\delta+m) = g_{\delta,n-1,m,k}(\eta_n)$ for all $k \ge k_0$. We say that η_{n-1} and η_n cohere at $\delta+m$, if they cohere above some k_0. Abusing notation, we also say that η_n coheres with $\eta_{n-1}(\delta+m)$.

Let $\tilde{G}$ be $\tilde{P}$-generic over V, and define in $V[\tilde{G}]$ the forcing notion Q_0:

DEFINITION 4.4. $p \in Q_0$ iff $p \in 2^{\mathrm{ht}(p)}$ for some $\mathrm{ht}(p) < \omega_1$. Q_0 is ordered by extension.

So Q_0 is σ-closed and adds the generic $\eta_0 \in 2^{\omega_1}$. Assume that $n \ge 1$, $\tilde{P} * P_n = \tilde{P} * Q_0 * \cdots * Q_{n-1}$, and Q_{n-1} adds the generic sequence $\eta_{n-1} \in 2^{\omega_1}$. Let $\tilde{G} * G_n$ be $\tilde{P} * P_n$-generic over V. In $V[\tilde{G} * G_n]$, we define Q_n the following way:

Definition 4.5. $p \in Q_n$ iff $p \in 2^{\mathrm{ht}(p)}$ for some limit ordinal $\mathrm{ht}(p) < \omega_1$ and η_{n-1} and p cohere everywhere, i.e., at all $\delta + m < \mathrm{ht}(p) + \omega$. Q_n is ordered by extension.

Notation 4.6. We denote the $\tilde{P}$-generic filter by $\tilde{G}$, we set $\tilde{P} * P_n = \tilde{P} * Q_0 * \cdots * Q_{n-1}$, with generic filter $\tilde{G} * G_n$. Since $\tilde{P}$ is proper, the countable support limit of $\tilde{P} * P_n$ is the same as $\tilde{P} * P_\omega$, where P_ω is the $\tilde{P}$-name for the countable support limit of the P_n. The generic filter of $\tilde{P} * P_\omega$ is denoted by $\tilde{G} * G_\omega$; and $G(n)$ is the Q_n-generic filter (a $\tilde{P} * P_{n+1}$-name, or equivalently a $\tilde{P} * P_n$-name for a Q_n-name).

Lemma 4.7. *The following is forced by* $\tilde{P} * P_n$:

(i) *Conditions can be finitely modified*: *If* $q \in Q_n$, $q' \in 2^{\mathrm{dom}(q)}$ *and* $q'(\alpha) = q(\alpha)$ *for all but finitely many* $\alpha \in \mathrm{dom}(q)$, *then* $q' \in Q_n$.

(ii) *If* $p \in Q_n$ *and* $\beta > \mathrm{ht}(p)$ *is a limit ordinal, then there is a* $q \leq p$ *with* $\mathrm{ht}(q) = \beta$. *In particular,* Q_n *adds the generic object* $\eta_n = \bigcup G(n) \in 2^{\omega_1}$ (*which in turn determines the generic filter* $G(n)$).

(iii) Q_n *is separative* (*and in particular nontrivial*), *and not* σ*-closed for* $n \geq 1$.

Proof. (i) is trivial.

(ii) Let $(\delta_i, m_i)_{i\in\omega}$ enumerate all pairs (δ, m) such that $\mathrm{ht}(p) < \delta \leq \beta$ and $m \in \omega$. Define an increasing sequence p_i of partial functions from β to $\{0, 1\}$:

Set $p_0 = p$. For $i > 0$, assume that $\mathrm{dom}(p_{i-1}) = \mathrm{ht}(p) \cup \bigcup_{j<i} v_{\delta_j,n-1,m_j}$. Then $v_{\delta_i,n-1,m_i} \cap \mathrm{dom}(p_{i-1})$ is finite:

- If $j < i$ and $\delta_j = \delta_i$, then $m_j \neq m_i$ and $v_{\delta_i,n-1,m_i}$ and $v_{\delta_j,n-1,m_j}$ are disjoint.
- If $j < i$ and $\delta_j \neq \delta_i$, then $v_{\delta_i,n-1,m_i} \cap v_{\delta_j,n-1,m_j}$ are finite (since $v_{\delta,n-1,m}$ is a cofinal subset of δ of order type ω).
- For the same reason, $v_{\delta_i,n-1,m_i} \cap \mathrm{ht}(p)$ is finite.

Therefore we can extend p_{i-1} to some p_i by adding values at $v_{\delta_i,n,m_i} \setminus \mathrm{dom}(p_i)$ that cohere with $\eta_{n-1}(\delta_i + m_i)$. (Recall that $f_{\delta_i,n-1,m_i,k}$ is onto.) Set $p_\omega = \bigcup p_n$, and fill in arbitrary values (e.g., 0) at $\beta \setminus \mathrm{dom}(p_\omega)$. This gives a $q \leq p$ with $\mathrm{ht}(q) = \beta$.

(iii) follows from (i) and (ii). ⊣

Remark 4.8. For this proof, as well as for most of the following, the preparatory forcing $\tilde{P}$ is not necessary: The definition of Q_n works for any reasonably defined sequences $\bar{v}$, $\bar{j}$, $\bar{f}$. Only in Section 6 we need that these sequences are generic. (Guessing with, e.g., a $\diamondsuit$-sequence is not enough, as discussed in Section 7.)

Lemma 4.9. $(\eta_n)_{n\in\omega}$ *is determined by* $\tilde{G}$ *and* $(\eta_n(m))_{n,m\in\omega}$. *In particular,* $\tilde{P} * P_\omega$ *adds a new real.*

Of course, we do not use any particular property of the countable support limit here. More generally, we get:

Assume V' is an extension of V that contains some $\tilde{G}$ and a sequence $(G(n))_{n\in\omega}$ such that $\tilde{G}$ is $\tilde{P}$-generic over V and $G(n)$ is Q_n-generic over $V[\tilde{G} * G_n]$. Fix $\delta < \omega_1$, $n_0 \in \omega$ and $f\colon \omega \to \omega$ and set

$$x = (\eta_n(\alpha))_{n \geq n_0, \delta + f(n) < \alpha < \delta + \omega}.$$

Then $(G_n)_{n\in\omega}$ is in $V[\tilde{G}, x]$.

See Figure 4(c).

PROOF. By induction on $1 \leq h \leq n_0$, each $\eta_n(\alpha)$ is determined for $n \geq n_0 - h$, $\delta + \omega \cdot h \leq \alpha < \delta + \omega \cdot (h+1)$. By induction on limit ordinals $\delta + \omega \cdot n_0 < \delta' < \omega_1$, each $\eta_n(\alpha)$ is determined for $\delta + \omega \cdot n_0 < \alpha < \delta'$. $\dashv$

Remark 4.10. For all $n_0 \in \omega$ and $f \colon \omega \to \omega$, there are conditions in $\tilde{P} * P_\omega$ that determine all $\eta_n(m)$ for $n < n_0$ or $m < f(n)$, cf. Figure 4(b). (The reason is that $\tilde{P} * P_{n_0}$ does not add new reals, as we will see in the next lemma, and that each Q_n-condition can be modified at finitely many places.) However, these conditions are *not* dense. (For exactly the same reason: There is a condition p_0 stating that $(\eta_n(0))_{n\in\omega}$ codes $(\eta_n \restriction \omega)_{n\in\omega}$, via a simple injection from $\omega \times \omega$ to ω. Then according to the last Lemma, no $p' \leq p_0$ can determine all $\eta_n(0)$.)

LEMMA 4.11. *$\tilde{P} * P_n$ forces that Q_n is proper and does not add a new ω-sequence of ordinals.*

PROOF. Work in $V' = V[\tilde{G} * G_n]$ and fix some large regular cardinal χ^*.

Let $N^* \prec H^{V'}(\chi^*)$ be a countable elementary submodel containing $\tilde{G}, \eta_{n-1}$ and $p_0 \in Q_n$. Set $\delta^* = N^* \cap \omega_1$. Let $(D_i)_{i\in\omega}$ list all dense subsets of Q_n that are in N^*, and assume $D_0 = Q_n$. It is enough to show the following:

$$\text{There is a } q \leq p_0 \text{ with } \mathrm{ht}(q) = \delta^* \text{ such that } q \text{ is stronger than some } p_i \in D_i \cap N^* \text{ for every } i \in \omega. \tag{4.1}$$

Then q is in particular N^*-generic, which shows that Q_n is proper. And if $\underset{\sim}{f} \in N^*$ is a name for a function from ω to the ordinals, then the value of $\underset{\sim}{f}(n)$ is determined in the dense set $D_{i(n)}$ for some $i(n) \in \omega$ an therefore by q. This shows that no new $\underset{\sim}{f}$ is added by Q_n.

So let us prove 4.1. Pick (in V') a sequence $(N_i)_{i\in\omega}$ and a large, regular $\chi \in N^*$ such that:

- $N_i \in N^*$.
- $N_i \prec H(\chi)$ is countable.
- N_0 contains η_{n-1} and p_0, N_{i+1} contains N_i and D_{i+1}.

Set $\beta_i = N_i \cap \omega_1$. So $\sup_{i\in\omega}(\beta_i) = \delta^*$.

Fix (in V') any $\eta^* \in Q_n$ of height δ^*. In particular η^* coheres with $\eta_{n-1}(\delta^* + m)$ for all m. Set

$$u_i = \{\alpha \in v_{\delta^*, n-1, m} \colon m < i, \beta_i \leq \alpha < \beta_{i+1}\}.$$

Each u_i is finite. We will construct q such that $q \supseteq \eta^* \restriction u_i$ for all $i \geq 1$. This guarantees that q coheres with $\eta_{n-1}(\delta^* + m)$ for all $m \in \omega$.

Assume that $p_i \in N_i \cap D_i$ is already defined. We extend it to $p_{i+1} \in N_{i+1} \cap D_{i+1}$: The finite sequence $\eta^* \restriction u_{i+1}$ is in N_{i+1}, so we can[7] (in N_{i+1}) extend p_i first to some $p' \supseteq \eta^* \restriction u_{i+1}$ in Q_n. Then extend p' to $p_{i+1} \in D_{i+1} \cap N_{i+1}$.

Set $q = \bigcup_{i\in\omega} p_i$. Then q is in Q_n: We already know that q coheres with $\eta_{n-1}(\delta^* + m)$. For $\alpha < \delta^*$, let i be such that $\alpha < \beta_i$. Then q extends p_{i+1} which coheres with $\eta_{n-1}(\alpha + m)$. $\dashv$

As an immediate consequence we get the following fact, illustrated in Figure 4(d):

[7] by using 4.7(i,ii).

COROLLARY 4.12. *The conditions* $(\tilde{p}, p_0, \dots, p_{n-1})$ *of the following form are dense in* $\tilde{P} * P_n$: $\mathrm{ht}(\tilde{p}) = \delta + \omega \cdot (n-1)$ *for some* δ, *and in* V *there is a sequence* $(p'_0, \dots, p'_{n-1})$ *such that* $p'_i \in 2^{\delta+\omega\cdot(n-1-i)}$ *and* p_i *is the standard name*[8] *for* p'_i.

§5. A dense subset.

We will now use the notions of variable, term and substitution as defined in Definition 2.4. The set of variables we use is $\{x_{n,m} : n, m \in \omega\}$.

Assume that $\bar{p}$ is a sequence of terms $(p_{n,\alpha})_{n\in\omega,\alpha<\delta}$. In $V[\tilde{G}]$, $\bar{p}$ can be interpreted as a promise that the generic sequence $(\eta_n)_{n\in\omega}$ is compatible with $\bar{p}$, i.e., that there is an assignment a such that $p_{n,\alpha} \circ a = \eta_n(\alpha)$ for all $n \in \omega, \alpha < \delta$. Of course such a promise can be inconsistent, for example if $\delta = \omega$ and each $p_{n,m}$ is (the constant term) 0.

DEFINITION 5.1. $R = \bigcup_{\delta<\omega_1} R_{\delta+\omega}$. A condition p in $R_{\delta+\omega}$ consists of $\tilde{p}$ and $\bar{p}$ such that:

- $\tilde{p} \in \tilde{P}$, $\mathrm{ht}(\tilde{p}) = \delta + 1$ (or equivalently $\delta + \omega$).
- $\bar{p} = (p_{n,\alpha})_{n\in\omega,\alpha<\delta+\omega}$.
- $p_{n,\delta+m}$ is the term $x_{n,m}$.
- If $\alpha < \delta$, then $p_{n,\alpha}$ is a term that only depends on $x_{l,k}$ with $l < n$.
- For every $m, n \in \omega$ and $\alpha \le \delta$ limit there is a $k_0 < \omega$ such that for all assignments a, we get that $(p_{n+1,\zeta} \circ a)_{\zeta<\alpha}$ coheres with $p_{n,\alpha+m} \circ a$ above k_0.

We interpret terms are functions, not syntactical objects, so we identify two elements p, q of $R_{\delta+\omega}$ if they satisfy $\tilde{p} = \tilde{q}$ and $p_{n,\alpha} =^* q_{n,\alpha}$ for all n, α; see Definition 2.4.

Elements of R can be interpreted as statements about the generic sequence:

DEFINITION 5.2. The canonical assignment a^c_δ assigns the value $\eta_n(\delta + m)$ to the variable $x_{n,m}$. (So a^c_δ is a $\tilde{P} * P_\omega$-name.) We also use a^c_δ as a $\tilde{P} * P_n$-name for the partial assignment that maps $\eta_l(\delta + m)$ to the variable $x_{l,m}$ for all $l < n$.

DEFINITION 5.3. Let $i : R \to \tilde{P} * P_\omega$ map $p \in R_{\delta+\omega}$ to $(\tilde{p}, q(0), q(1), \dots)$ defined as follows: For each n, $q(n)$ is the $\tilde{P} * P_n$-name for the sequence $(p_{n,\alpha} \circ a^c_\delta)_{\alpha<\delta}$.

LEMMA 5.4. (i) *$i(p)$ actually is a condition in* $\tilde{P} * P_\omega$.

(ii) *$i(p)$ is the truth value (in* $\mathrm{ro}(\tilde{P} * P_\omega)$*) of the following statement:* $\tilde{p} \in \tilde{G}$, *and* $\bar{p}$ *is compatible with the generic sequence* $\bar{\eta}$.

(iii) *In particular, this truth value is positive. Moreover, the truth value remains positive if we additionally assign specific values for finitely many of the variables* $x_{n,m}$.

Here, "$\bar{p}$ is compatible with the generic sequence $\bar{\eta}$" means: There is some assignment a such that $p_{n,\alpha} \circ a = \eta_n(\alpha)$ for all $\alpha < \delta + \omega$. Since $p_{n,\delta+m} = x_{n,m}$, the only assignment that can ever witness compatibility is the canonical assignment a^c_δ.

More formally, and slightly stronger, we can formulate the last item as: Given $f : \omega \to \omega$ and $(b_{n,i})_{n\in\omega,i<f(n)}$ with $b_{n,i} \in \{0, 1\}$, the truth value of the following statement is non-zero:

[8] With "standard name for x" (x in the ground model) we mean the (canonical) name $\check{x}$ that evaluates to x for all generic filters.

- $\tilde{p} \in \tilde{G}$,
- $p_{n,\alpha} \circ a^c_\delta = \eta_n(\alpha)$ for all $\alpha < \delta + \omega$,
- and additionally $\eta_n(\delta + i)$ (or equivalently $x_{n,i} \circ a^c_\delta$) is $b_{n,i}$ for all $n \in \omega$ and $i < f(n)$.

Proof. (i) it follows from the definition of R that each $q(n)$ is a valid condition in Q_n. (ii) The canonical assignment is the only assignment that can possibly witness compatibility. (iii) Given f and $b_{n,i}$ as above, we can just extend $q(n)$ to be the name of some condition q' in Q_n of height $\delta + \omega$ (instead of just δ) such that $q'(\delta + i) = b_{n,i}$ for all $i < f(n)$. For this we need, as usual, just Lemma 4.7(i,ii). ⊣

Remark 5.5.
- It is easy to see (similarly to 4.7) that $R_{\delta+\omega}$ is nonempty for all δ. We will only prove this (implicitly) for "stationary many" δ, in Lemma 5.8: $\sigma'' R$ is dense in $\tilde{P} * P_\omega$.
- In view of this Lemma, the proof of (iii) can be compared to Remark 4.10: While we cannot densely determine the gray area of Figure 4(b), we can densely determine such an area shifted up to some δ.

If $\alpha < \delta$, then $p_{n,\alpha}$ can be calculated from finitely many $p_{l,\delta+m}$ with $l < n$ (since $p_{n,\alpha}$ is a term using variables $x_{l,m}$, $l < m$, and $p_{l,\delta+m} = x_{l,m}$). We can also calculate values in the other direction:

Lemma 5.6. (i) *$p_{n,\alpha}$ is determined by finitely many $x_{l,k}$ with $l < n$.*
(ii) *$x_{n,m}$ can be determined by finitely many $p_{l,k}$ with $l > n, k \in \omega$.*

More generally, we get (cf. Figure 5(a)): If $p \in R_{\delta+\omega}$ and $\beta < \delta$ (not necessary a limit), then every $p_{n,\alpha}$ with $\beta + \omega \leq \alpha < \delta + \omega$ can be determined by finitely many $p_{l,\zeta}$ with $l > n, \beta < \zeta < \beta + \omega$. More precisely: There is a $k \in \omega$ and a sequence $(l_i, \zeta_i)_{i<k}$ such that $l_i > n$, $\beta < \zeta_i < \beta + \omega$ and for all assignments a, b the following holds: If $p_{n,\alpha} \circ a \neq p_{n,\alpha} \circ b$, then $(p_{l_i,\zeta_i} \circ a)_{i<k} \neq (p_{l_i,\zeta_i} \circ b)_{i<k}$.

Proof. By induction on α: Assume $\alpha = \beta + \omega + m$. Then $(p_{n+1,\zeta})_{\zeta<\beta+\omega}$ coheres with $p_{n,\alpha}$ above some k_0, so we can use $f_{\beta+\omega,n,m}$ to get $p_{n,\alpha}$. Now assume that the statement is true for all $\alpha < \nu$, ν limit. If $\alpha = \nu + m$, then $p_{n,\alpha}$ again is determined by the values of certain $p_{l,\zeta}$ with $l > n, \beta < \zeta < \nu$, each of which in turn is determined (by induction) by finitely many $p_{l',\zeta'}$ with $\beta < \zeta' < \beta + \omega$. ⊣

We can identify R with a subset of $P * P_\omega$:

Lemma 5.7. *$i: R \to \tilde{P} * P_\omega$ is injective.*

Proof. Fix $p \in R_{\delta+\omega}, q \in R_{\delta'+\omega}$, $p \neq q$. If $\tilde{p} \neq \tilde{q}$, then $i(q) \neq i(p)$. So assume that $\tilde{p} = \tilde{q}$ (in particular $\delta' = \delta$). Since $p \neq q$, there is an (n, α) and a (finite, partial) assignment a such that $q_{n,\alpha} \circ a \neq p_{n,\alpha} \circ a$. According to 5.4(iii), $i(q)$ is compatible with a. Let $r \leq i(q)$ force that the generic sequences are compatible with a. Then r forces that $i(p)$ is not in the generic filter, since it determines a different value for $\eta_n(\alpha)$ than $i(q)$. ⊣

So we can interpret R as a subset of $P * P_\omega$; and we usually do so, that is, we will may just write p instead of $i(p)$ and R instead of $i'' R$, as in the following:

Lemma 5.8. *$R \subseteq P * P_\omega$ is dense.*

The proof is a bit cumbersome, but really just a modification of the proof of Lemma 4.11.

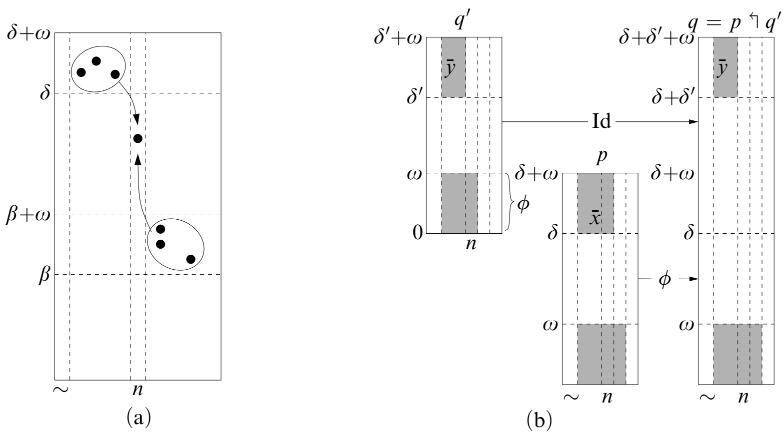

FIGURE 5. Elements of R. (a) Dependence in both directions, according to Lemma 5.6. (b) Conditions can be stacked to get stronger conditions. If on the other hand q is stronger than p, then it can be split accordingly.

PROOF. Fix $(\tilde{p}, p(0), p(1), \dots) \in \tilde{P} * P_\omega$, and a countable $N^* \prec H(\chi^*)$ containing $(\tilde{p}, p(0), p(1), \dots)$. Set $\delta^* = N^* \cap \omega_1$. It is enough to show:

$$\text{There is a } q \in R_{\delta^*+\omega} \text{ such that } i(q) \leq (\tilde{p}, p(0), p(1), \dots). \tag{5.1}$$

The $\tilde{P} * P_n$-condition $(\tilde{p}, p(0), \dots, p(n))$ will be denoted by $p \restriction n$. For $q \in R$ and $n \in \omega$, we set

$$q(n) = (q_{n,\alpha})_{\alpha<\delta^*+\omega} \quad \text{and} \quad q \restriction n := (\tilde{q}, q(0), q(1), \dots, q(n-1)).$$

Just as q can be interpreted as a condition in $\tilde{P} * P_\omega$ in a canonical way (cf. 5.4), we can interpret $q \restriction n$ as a condition in $\tilde{P} * P_n$. In particular, "$q \restriction n$ forces φ" means the following:

If $\tilde{G} * G_n$ is $\tilde{P} * P_n$-generic over V, if $\tilde{G}$ contains $\tilde{q}$ and if $\eta_0, \dots, \eta_{n-1}$ are compatible with $(q_{l,\alpha})_{l<n,\alpha<\delta^*+\omega}$, then φ holds in $V[\tilde{G} * G_n]$.

Let us call an antichain E in $\tilde{P} * P_n$ nice, if every condition e in E has the form of Corollary 4.12. These conditions are dense, so we get:

For all $n \in \omega$, $X \in V$ and all $\underset{\sim}{\tau}$ such that $\tilde{P} * P_n$ forces that $\underset{\sim}{\tau} \in \check{X}$ there is a nice maximal antichain B deciding $\underset{\sim}{\tau}$. I.e., for each $b \in B$ there is an $x^b \in X$ such that b forces $\underset{\sim}{\tau} = x^b$. (5.2)

The induction hypothesis. We will construct $\tilde{q}$ in $\tilde{Q}$ of height $\delta^* + \omega$ and, by induction on $n \geq 0$, the condition $q(n)$—i.e., the terms $(q_{n,\alpha})_{\alpha<\delta^*}$ depending on variables $x_{i,j}$ with $i < n$—such that the following holds:

(i) $q \restriction (n+1)$ satisfies the conditions on elements of $R_{\delta^*+\omega}$.
(ii) $q \restriction (n+1)$ is $\tilde{P} * P_{n+1}$-generic over N^*.
(iii) $q \restriction (n+1)$ is stronger than $(\tilde{p}, p(0), \dots, p(n))$.
(iv) $q \restriction (n+1)$ decides every nice maximal antichain E of $\tilde{P} * P_{n+1}$ in N^* by finite case distinction.

More formally: Item (i) means

(i)′ for all $m \in \omega$ and $\alpha \leq \delta^*$ limit there is a $k_0 < \omega$ such that for all assignments a, we have that $(q_{n,\zeta} \circ a)_{\zeta<\alpha}$ coheres with $q_{n-1,\alpha+m} \circ a$ above k_0.

And item (iv) means: For every nice maximal antichain E of $\tilde{P} * P_{n+1}$ in N^* there is an $l^E \in \omega$, a sequence $(e_0^E, \dots, e_{l^E-1}^E)$ of elements of $E \cap N^*$ and a sequence $(t_0^E, \dots, t_{l^E-1}^E)$ of terms using only variables $x_{i,j}$ with $i < n+1$ such that $q \restriction (n+1)$ forces the following:

(iv)′ There is exactly one $k < l^E$ such that $t_k^E \circ a_{\delta^*}^c = 1$ (cf. 5.2), and $e_k^E \in \tilde{G} * G_n$ for this k.

This implies the following (where we apply Lemma 5.4(iii)):

(v) For all partial assignments b of the (finitely many) variables used in any of the t_k^E there is exactly one $k < l^E$ such that $t_k^E \circ b = 1$.

Note that (iii) (for all n) implies (5.1).

Step 1: *Finding* $\tilde{q}$. First extend $\tilde{p}$ to $\tilde{p}'$ such that $\mathrm{ht}(\tilde{p}') = \delta^*$ and such that for every dense subset D of $\tilde{P}$ in N^* there is an $d \in D \cap N^*$ weaker that $\tilde{p}'$ (this is possible since $\tilde{P}$ is σ-closed). In particular, $\tilde{p}'$ is $\tilde{P}$-generic over N^*, and if $E \subseteq \tilde{P}$ is a maximal antichain in N^*, then $\tilde{p}'$ decides the $e \in E$ that will be in the generic filter (and $e \in N^*$).[9]

We further extend $\tilde{p}'$ to $\tilde{q}$ by adding some arbitrary value at δ^*. So $\mathrm{ht}(\tilde{q}) = \delta^* + 1$ (or equivalently $\delta^* + \omega$).

Step 2: *Finding* $q(0)$. This case, $n = 1$, is simple since Q_0 is σ-closed.

We have to define the (constant) terms $(q_{0,\alpha})_{\alpha<\delta^*}$. Let $(D_i)_{i\in\omega}$ enumerate all $\tilde{P}$-names in N^* for open dense subsets of Q_0, such that $D_0 = Q_0$

We now define r_n and s_n for $n \in \omega$ such that:

(a) r_n is a $\tilde{P}$-name in N^*, forced by $\tilde{p}$ to be a Q_0 condition and element of D_n.
(b) s_n is a 0-1-sequence in N^*, forced by $\tilde{q}$ to be r_n.
(c) r_{n+1} is forced to extend s_n.

Set $r_0 = p(0)$. This satisfies (a). Given an r_n satisfying (a), note that $\tilde{P}$ does not add new countable sequences of ordinals. So every condition in $Q_0^{N^*[\tilde{G}]}$, in particular r_n, already exists in the ground model N^*. So r_n is decided by a maximal antichain, and therefore by $\tilde{q}$, to be some sequence $s_n \in N^*$; satisfying (b). Also, since $s_n \in 2^{<\omega_1} \cap N^*$, we can find in N^* a $\tilde{P}$-name r_{n+1} for an element of D_{n+1} extending s_n.

Fix $\alpha < \delta^*$, and set $q_{0,\alpha}$ to be the term with constant value $s_n(\alpha)$ (for sufficiently large n). This defines $q(0)$. So $\tilde{q}$ forces that that $q(0)$ is Q_0-generic over $N^*[\tilde{G}]$, i.e., $(\tilde{q}, q(0))$ is $\tilde{P} * P_1$-generic over N^* and forces that $(\tilde{p}, p(0)) \in \tilde{G} * G_1$. So (i)–(iii) are satisfied. Now fix some nice, maximal antichain $E \subset \tilde{P} * P_1$ such that $E \in N^*$. Every $e \in E$ is of the form $(\tilde{e}, e(0))$ for a 0-1-sequence $e(0)$ in V. If $e \in N^*$, then $\tilde{e}$ and $e(0)$ have height less than δ^*. In particular, every $e \in E \cap N^*$ is either extended

[9] So $\tilde{p}'$ decides "everything" about $N^*[\tilde{G}]$. Of course, $N^*[\tilde{G}]$ is not an element of V (since it contains, e.g., $\tilde{G}$). But every formula about $N^*[\tilde{G}]$ (with parameters in N^*) is already decided in V "modulo $\tilde{p}'$", since every such formula is decided by an antichain. We can find such a strong $\tilde{p}'$ since $\tilde{P}$ is σ-complete, and we can do the same for $\tilde{P} * P_1$. However, for $n \geq 1$, Q_n is not σ-complete, and we will not be able to decide everything with the generic condition $q(n)$; but we will still be able to decide "modulo finite case distinction".

by $(\tilde{q}, q(0))$ or is incompatible with it. Since $E \in N^*$ is a maximal antichain, and since $(\tilde{q}, q(0))$ is $\tilde{P} * P_1$-generic over N^*, we know that there has to be exactly one $e^* \in E$ compatible with $(\tilde{q}, q(0))$, and $e^* \in N^*$. In other words, $(\tilde{q}, q(0))$ decides the element $e^* \in E \cap N^*$ that is going to be in $\tilde{G} * G_1$. So to satisfy (iv)$'$, we can set $l^E = 1$, $t_0^E = 1$, $e_0^E = e^*$.

Step 3: *The successor step.* Now things get a bit more complicated, since Q_n is not σ-closed any more. We assume that the induction hypothesis (i)–(iv) holds for $n-1$. So we already have $q \restriction n$ want to find $q_{n,\alpha}$ for $0 \leq \alpha < \delta^*$. As previously, we let $(D_i)_{i \in \omega}$ enumerate all $\tilde{P} * P_n$-names in N^* for open dense subsets of Q_n (and we set $D_0 = Q_n$).

First we fix (in V) a term-sequence $(t^*_\alpha)_{\alpha \in \bigcup_{m \in \omega} v_{\delta^*, n-1, m}}$ such that:

- If $\alpha \in v_{\delta^*, n-1, m}$, then t^*_α only depends on $x_{n-1,m}$.
- For all $m \in \omega$, the sequence $\bar{t}^*$ coheres with q_{n-1, δ^*+m} (which is just $x_{n-1,m}$) above some k_0.
- For every $\beta < \delta^*$, the partial sequence $\bar{t}^* \restriction \beta$ uses only finitely many variables.

We can find such a sequence since the $f_{\delta^*, n-1, m, k}$ defined by $\tilde{q}$ are surjective and the $v_{\delta^*, n-1, m}$ are disjoint (for different m) cofinal subsets of δ^* of order type ω.

We will construct in V by induction on $i \in \omega$

- a finite set v_i of variables $x_{l,j}$ with $l < n$,
- for every (partial) assignment a of v_i a $\tilde{P} * P_n$-name r_i^a in N^*,
- a finite set w_i of variables $x_{l,j}$ with $l < n$,
- for every assignment b of w_i a 0-1-sequence s_i^b in N^*,
- an ordinal $\beta_i < \delta^*$,

such that the following holds:

(a) $v_{i+1} \supseteq w_i \supseteq v_i$.
(b) If a is an assignment of v_i, then r_i^a is a name (in N^*) for an element of D_i.
(c) If b is an assignment of w_i and a its restriction to v_i, then $(q \restriction n) \& b$ forces[10] $s_i^b = r_i^a$.

Set $w_0 = \emptyset$. So there is only one assignment, the empty one, of w_0. We set $r_0^\emptyset = p(n)$.[11] Assume that for some $i \geq 0$ we already have w_i, and r_i^a for all assignments a of w_i. Fix a. Note that $\tilde{P} * P_n$ does not add any new countable sequences of ordinals, so according to (5.2) r_i^a is decided by a nice maximal antichain E of $\tilde{P} * P_n$ in N^*. Using item (iv) of the induction hypothesis, we choose the sequences $(e_0^E, \dots, e_{l^E}^E)$ of and $(t_0^E, \dots, t_{l^E}^E)$. Let v' be the (finite) set of variables used in any of the t_k^E. Set $v_i^a = w_i \cup v'$. Let b be an assignment of v_i^a extending a. According to (v), there is a unique $k \leq l^E$ such that $t_k^E \circ b = 1$. We call this element $k(b)$. The element $e_{k(b)}^E$ determines r_i^a to be a specific 0-1-sequence of V, and we

[10] For $x \in \tilde{P} * P_n$, $x \& b$ is the truth value (in $\mathrm{ro}(\tilde{P} * P_n)$) of the following statement: $x \in \tilde{G} * G_n$, and b is compatible with $\eta_0, \dots, \eta_{n-1}$, i.e., for every $x_{l,k} \in w_i$, we have $\eta_{l, \delta^*+k} = x_{l,k} \circ b$. For this notation we can use $x = p \restriction n$, and also $x = q \restriction n$, since we can canonically interpret $q \restriction n$ as element of $\tilde{P} * P_n$.

[11] More formally, we should set

$$r_0^\emptyset = \begin{cases} p(n) & \text{if } p(n) \in Q_n, \\ \emptyset & \text{otherwise,} \end{cases}$$

since $p(n)$ is forced to be in Q_n by $p \restriction n$, not by the empty condition.

call this sequence s_i^b. Note that $s_i^b \in N^*$. We can do this for all assignments a of w_i, and set $v_i = \bigcup v_i^a$.

We still have to construct β_i, w_{i+1} and r_{i+1}^b. We pick in N^* a $\tilde{P} * P_n$-name N for a countable elementary submodel of $H^{V[\tilde{G}*G_n]}(\chi)$ containing D_{i+1} and all the (finitely many) s_i^b. Since $N \cap \omega_1$ is an $\tilde{P} * P_n$-name for an ordinal, there are only finitely many possibilities modulo $q \restriction n$, and we can choose $\beta_i \in N^* \cap \omega_1$ larger than every possibility for $N \cap \omega_1$.

The terms t_α^* for $\alpha < \delta_i$ use only a finite set w' of variables (of the form $x_{n-1,m}$). Set $w_{i+1} = v_i \cup w'$. Fix an assignment a of w_{i+1} and let b be the restriction to v_i. Fix the index set

$$I = \{\alpha < \beta_i : (\exists j \leq i)\alpha \in v_{\delta^*,n-1,j}\}.$$

The finite set I is in N^*. Set

$$\bar{x} = (\bar{t}^* \circ b) \restriction I.$$

This is a finite partial function in N^* from I to $\{0, 1\}$. We define the $\tilde{P} * P_n$-name r_{i+1}^a in N^* by the following construction in $N^*[\tilde{G} * G_n]$: (Let d be some fixed element of D_{i+1}.)

- Assume that $\beta' = N \cap \omega_1 < \beta_i$. (Otherwise set $r_{i+1}^a = d$.)
- Assume that s_i^b is a Q_n-condition. (Otherwise set $r_{i+1}^a = d$.)
- In N, extend s_i^b to some Q_n-condition containing $\bar{x} \restriction (\beta' \setminus \mathrm{ht}(s_i^b))$. (As usual, use 4.7 inside N.)
- Again in N, pick some condition r_{i+1}^a in D_{i+1} extending s'. In particular, r_{i+1} has height less than β_i.

This ends the construction. We can summarize all the possibilities of $s_i^a(\alpha)$ into the term $q_{n,\alpha}$ (depending on the variables in v_i). This defines $q(n)$.

It remains to be shown that $q \restriction n + 1$ satisfies the induction hypothesis.

For (i)$'$, first assume $\alpha < \delta^*$. Let D_i be the set of conditions of length $\geq \alpha$. Let $q_{n-1,\alpha+m}$ be determined by the finite set v of variables, and set $v' = v \cup v_i$. Fix an assignment b of v'. In particular b determines $q_{n-1,\alpha+m}$ as well as $q(n) \restriction \alpha$, since $q(n)$ "extends" $s_i^{b'} \in D_i$ (where b' is the restriction of b to v_i). Since $q \restriction n$ is compatible with the finite assignment b, we know that $q(n) \circ b \restriction \alpha$ coheres with $q_{n-1,\alpha+m} \circ b$ above some k_0^b. So we can set k_0 to be the maximum of all the k_0^b for all assignments b of v'.

Now assume $\alpha = \delta^*$ and $m \in \omega$. Pick $\gamma \in v_{\delta^*,n-1,m} \setminus \beta_m$. Look at the term $q_{n,\gamma}$. According to the construction,

$$q_{n,\gamma} \circ b = s_i^{b'}(\gamma) \circ b = t_\alpha^* \circ b$$

for all assignments b, and therefore $q(n)$ coheres with $x_{n-1,m}$.

Let us now show (iv). Let $E \in N^*$ be a nice, maximal antichain of $\tilde{P} * P_{n+1}$. Let D be the $\tilde{P} * P_n$-name for the following open dense subset of Q_n

$$D = \{q \leq e(n) : e \in E, e \restriction n \in \tilde{G} * G_n\}.$$

We know that D appears as some D_i in the list of dense sets in N^*. Fixing an assignment a of v_i, we get s_i^a in N^* such that $s_i^a \in D_i[\tilde{G} * G_n]$. We set

$$A^a = \{e \restriction n : e \in E, e(n) \subseteq s_i^a\}.$$

This is a nice $\tilde{P} * P_n$-antichain and maximal under $(q \restriction n)\&a$. We can extend it to a nice maximal antichain B^a. By induction hypothesis, we can determine modulo $q \restriction n$ the element b of B^a chosen by $\tilde{G} * G_n$ filter by finite case distinction. Then $b^\frown s_i^a$ is the element of E chosen by $\tilde{G} * G_{n+1}$. Combining the finite case distinction for the s_i^a with the finite case distinctions for the according B^a gives the desired result. ⊣

Since R is a subset of $\tilde{P} * P_\omega$, it is also a partial order (and since it is dense, it is forcing equivalent to $\tilde{P} * P_\omega$). We now show that we can interpret the order on R in a different way, using substitutions of terms:

Definition 5.9. Let $p \in R_{\delta+\omega}$ and $q \in R_{\delta'+\omega}$. We call q term-stronger than p, if either $p = q$ or if the following holds: $\tilde{q} \leq \tilde{p}$ (in particular $\delta' \geq \delta$), and $p_{n,\alpha} \circ \phi =^* q_{n,\alpha}$ for all $\alpha < \delta$ and for the substitution ϕ defined by $\phi_{n,m} = q_{n,\delta+m}$.

(Again, recall that we interpret terms as functions, so we use $=^*$ as defined in 2.4.)

Lemma 5.10. *The condition $q \in R$ is term-stronger than $p \in R$ iff $i(q) \leq i(p)$.*

Proof. Assume that q is not term-stronger than p. If $\tilde{q}$ is not stronger than $\tilde{p}$ in $\tilde{P}$, then $i(q)$ cannot be stronger than $i(p)$. So assume $\tilde{q} \leq \tilde{p}$. According to the definition of term-stronger, $q_{l,\alpha} =^* p_{l,\alpha} \circ \phi$ fails for some l, α. These terms depend on finitely many variables $x_{n,m}$, and there is a partial assignment a of these variables such that $q_{l,\alpha} \circ a \neq p_{l,\alpha} \circ \phi \circ a$. According to Lemma 5.4(iii), we can force the generic sequence to be compatible with q and a. Then $i(q)$ is in the generic filter, but $i(p)$ is not, contradicting $i(q) \leq i(q)$. ⊣

If q' is a condition, then $(q'_{n,m})_{n,m\in\omega}$ can be interpreted as substitution: For $p \in R_{\delta+\omega}$ and $q' \in R_{\delta'+\omega}$, we can stack q' on top of p—overlapping at $[\delta, \delta+\omega[$—to get a condition $q \in R_{\delta+\delta'+\omega}$ stronger than p, cf. Figure 5(b). We write $q = p \Lsh q'$. More precisely:

Definition 5.11. For $p \in R_{\delta+\omega}$ and $q' \in R_{\delta'+\omega}$, we define the condition $q = p \Lsh q'$ in $R_{\delta+\delta'+\omega}$ as follows

- $\tilde{q} \restriction (\delta+1) = \tilde{p}$.
- $\tilde{q}(\alpha)$ for $\alpha \geq \delta+\omega$ is defined the following way:
  $v^q_{\delta+\alpha,n,m} = \{\delta+\beta : \beta \in v^{q'}_{\alpha,n,m}\}$,
  $j^q_{\delta+\alpha,n,m} = j^{q'}_{\alpha,n,m}$,
  $f^q_{\delta+\alpha,n,m,k} = f^{q'}_{\alpha,n,m,k}$.
- $q_{n,\delta+\alpha} = q'_{n,\alpha}$.
- If $\alpha < \delta$, then $q_{n,\alpha} = p_{n,\alpha} \circ \phi$ for the substitution ϕ defined by $\phi_{n,m} = q'_{n,m}$.

Fact 5.12. (i) If $p \in R_{\delta+\omega}$ and $q' \in R_{\delta'+\omega}$, then $p \Lsh q'$ is stronger than p.
(ii) If $q \in R_{\delta+\delta'+\omega}$ is stronger than $p \in R_{\delta+\omega}$, then we can "split" q into $p \in R_{\delta+\omega}$ and $q' \in R_{\delta'+\omega}$ such that $q = p \Lsh q'$.

Remarks 5.13.
- Of course we generally cannot split a condition at every level: If $q \in R_{\delta''+\omega}$ and $\delta' < \delta''$, then we generally do not get $q = p \Lsh q'$ for some $p \in R_{\delta'+\omega}$.
- The Fact shows that for all $q \in R$ there are only finitely many $p \geq q$, see Figure 6(a).

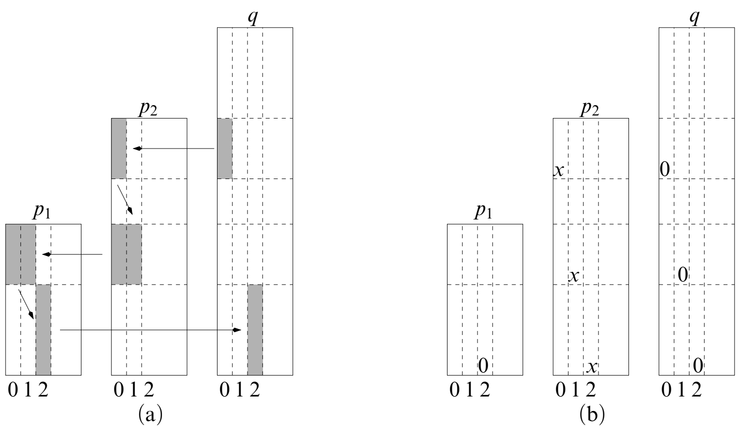

FIGURE 6. (a) If q is stronger than $p1$ and p_2, then $q_{2,m}$ is constant for all m. (The gray area indicates constant terms; the $\tilde{P}$ parts are not displayed.) (b) If p_1, p_2 are compatible (i.e., weaker than some q), they do not have to be comparable.

- Note that two compatible conditions generally are not comparable, see Figure 6(b). (Otherwise, according to the previous item, R would be isomorphic to a tree of height ω and therefore collapse the continuum.)
- The situation is similar to Q_* defined in Section 2: The conditions that are stronger than some $p \in R$ are exactly those with another condition $q' \in R$ stacked on top.

§6. Sacks reals as squares of terms again. We will now investigate the relation of R and Q_*. Given a $p \in R$, we can restrict p to an $\omega \times \omega$-matrix of terms:

DEFINITION 6.1. For $p \in R$, set $\sigma(p) = (p_{n,m})_{n,m \in \omega}$.

Note that

$$\sigma(p \mathbin{\Lsh} q') = \sigma(p) \circ \sigma(q'). \tag{6.1}$$

So stacking q' on top of p translates to applying $\sigma(q')$ (as substitution) to $\sigma(p)$.

Generally $\sigma(p)$ will not be element of Q_*, and for a $\bar{t} \in Q_*$ there generally is no $p \in R$ such that $\sigma(p) = \bar{t}$. The reason is that some obvious conditions on the term-matrix are incomparable: In Q_*, we require

$t_{i,j}$ only depends on $x_{n,m}$ such that $(n, m) \trianglelefteq (i, j)$,

whereas every $\sigma(p)$ obviously satisfies

$p_{i,j}$ only depends on $x_{n,m}$ such that $n < i$.

We will now define a dense subset $R' \subseteq R$ such that $\sigma'' R' \subseteq Q_*$, and such that R' adds a Q_*-generic object. This proves the first part of Theorem 1, since Q_* is forcing equivalent to Sacks forcing and R' is (as a dense subset) equivalent to R, which in turn is dense in $P * P_\omega$. So $P * P_\omega$ adds a Sacks real.

LEMMA 6.2. (i) *There is an $r^\Delta \in R_{\omega\cdot\omega+\omega}$ such that $\sigma(r^\Delta) \in Q_*$ and $\sigma(p \mathbin{\Lsh} r^\Delta) \in Q_*$ for all $p \in R$.*

(ii) *There is an* $r^{\mathrm{mult}} \in R_{\omega\cdot\omega+\omega}$ *such that for all* $\phi \in Q_*$ *there is an* $r \in R_{\omega\cdot\omega+\omega}$ *with* $\sigma(r^{\mathrm{mult}}) \circ \phi = \sigma(r)$.

We postpone the proof to the end of the section. We set

$$R' = \{p \mathbin{\text{\textdaggerdbl}} r^\Delta \colon p \in R\}.$$

This is a dense subset of R, since $p \mathbin{\text{\textdaggerdbl}} r^\Delta \leq p$ for all p. As a consequence of the previous Lemma, we get:

Corollary 6.3. (a) *If* $l \in R'$ *and* $p \leq_R l$, *then* $\sigma(p \mathbin{\text{\textdaggerdbl}} r^\Delta) \leq_{Q_*} \sigma(l)$.

(b) *If* $p \in R'$ *then there is an* $\bar{s} \leq \sigma(p)$ *such that for all* $\bar{t} \leq \bar{s}$ *then there is an* $q \leq p$ *in* R' *such that* $\sigma(q) \leq \bar{t}$.

(c) *The forcing notion* R' *adds a generic for* Q_*. *So* $\tilde{P} * P_\omega$ *adds a Sacks real.*

Proof of the Corollary. (a) Assume that $p = l \mathbin{\text{\textdaggerdbl}} q'$. Then $\sigma(p \mathbin{\text{\textdaggerdbl}} r^\Delta) = \sigma(l) \circ \sigma(q' \mathbin{\text{\textdaggerdbl}} r^\Delta)$; and $\sigma(q' \mathbin{\text{\textdaggerdbl}} r^\Delta)$ is an element of Q_* and therefore witnesses that $\sigma(p \mathbin{\text{\textdaggerdbl}} r^\Delta)$ is stronger than $\sigma(l)$.

(b) Set $\bar{s} = \sigma(p \mathbin{\text{\textdaggerdbl}} r^{\mathrm{mult}} \mathbin{\text{\textdaggerdbl}} r^\Delta)$, and let ϕ witnesses $\bar{t} \leq \bar{s}$, i.e., $\phi \in Q_*$ and

$$\bar{t} = \bar{s} \circ \phi = \sigma(p) \circ \sigma(r^{\mathrm{mult}}) \circ \sigma(r^\Delta) \circ \phi.$$

$\sigma(r^\Delta) \circ \phi \in Q_*$, so by Lemma 6.2(ii), there is an $r \in R$ such that

$$\sigma(r) = \sigma(r^{\mathrm{mult}}) \circ \sigma(r^\Delta) \circ \phi, \text{ so}$$
$$\sigma(p \mathbin{\text{\textdaggerdbl}} r) = \sigma(p) \circ \sigma(r) = \bar{s} \circ \phi = \bar{t}.$$

Set $q = p \mathbin{\text{\textdaggerdbl}} r \mathbin{\text{\textdaggerdbl}} r^\Delta$. Then $q \in R'$ and $q \leq p$. Furthermore,

$$\sigma(q) = \sigma(p \mathbin{\text{\textdaggerdbl}} r) \circ \sigma(r^\Delta) = \bar{t} \circ \sigma(r^\Delta) \leq \bar{t}.$$

(c) Let G' be R'-generic over V. We show that the following set is Q_*-generic filter over V:

$$G_* = \{\bar{r} \in Q_* \colon (\exists q \in G')\, \sigma(q) \leq \bar{r}\} \tag{6.2}$$

First note that G_* does not contain incompatible elements: Assume that $\bar{r}_1$ and $\bar{r}_2$ are in G_*. Then there are $l_1, l_2 \in G'$ such that $\sigma(l_i) \leq r_i$. Since G' is a filter, there is some $p \leq l_1, l_2$ in G'. The set

$$\{p' \mathbin{\text{\textdaggerdbl}} r^\Delta \colon p' \leq p\}$$

is dense below p, so there is some $q = p' \mathbin{\text{\textdaggerdbl}} r^\Delta$ in G'. According to (a), the q satisfies $\sigma(q) \leq \sigma(l_1), \sigma(l_2)$.

Now assume that $D \subseteq Q_*$ is dense, and (towards a contradiction) that p forces that G_* does not meet D. Then pick $\bar{s}$ as in (b), pick $\bar{t} \leq \bar{s}$ in D and pick q again as in (b). So q forces that $\sigma(p) \leq t$ is in G_*, a contradiction. So we know that (6.2) is generic. $\dashv$

It remains to prove Lemma 6.2. All these facts are easy to *see*, but a bit cumbersome to write down formally. So the reader might be better off drawing a picture than reading the proof. Fix an injective function from $\omega^{<\omega}$ to ω, $(a_1, \dots, a_l) \mapsto \ulcorner a_1, \dots, a_l \urcorner$, with coinfinite range.

The construction of r^Δ. All we need is a $r^\Delta \in R_{\omega\cdot\omega+\omega}$ satisfying the following:

$$r^\Delta_{n,m} \text{ only depends on variables } x_{i,j} \text{ such that } i + j < n. \tag{6.3}$$

Then, if we stack r^Δ on top of any $p \in R_{\delta+\omega}$, the resulting $q = p \upharpoonleft r^\Delta$ will satisfy (6.3) as well. Also, every element q of R satisfies that each $x_{i,j}$ depends on finitely many $q_{n,m}$ for $n, m \in \omega$, according to Lemma 5.6(ii). Therefore $\sigma(p \upharpoonleft r^\Delta)$ will satisfy all requirements for an element of Q_*, which proves Lemma 6.2(i).

We now construct r^Δ.

- When defining $\tilde{r}^\Delta$, only the v part is nontrivial; we set each $j_{\alpha,n,m}\colon \omega \to \omega$ and $f_{\alpha,n,m,k}\colon 2 \to 2$ to be the identity function for all α, n, m, k.[12]
- We deal with one variable at a time. Assume that we deal with x_{n_0,m_0}.
- Set $v_{\omega\cdot\omega,n_0,m_0} = \{\omega\cdot k + \ulcorner n_0, m_0 \urcorner \colon k > m_0\}$.
  For $\alpha \in v_{\omega\cdot\omega,n_0,m_0}$, we set $r^\Delta_{n_0+1,\alpha} = x_{n_0,m_0}$.
- If $n = n_0 + l$ for some $l \geq 1$, if $m = \ulcorner a_0, a_1, \dots, a_l \urcorner$ with $a_0 = n_0, a_1 = m_0$, and if $k > m_0 - l$, then set $v_{\omega\cdot(k+1),n,m} = \{\omega \cdot k + \ulcorner a_0, \dots, a_l, j \urcorner \colon j \in \omega\}$, and for $\alpha \in v_{\omega\cdot(k+1),n,m}$, we set $r^\Delta_{n+1,\alpha} = x_{n_0,m_0}$.
- We repeat this for all $x_{i,j}$. (Note that the $v_{\alpha,n,m}$ defined for different m will be disjoint).
- So far, whenever we have defined some $v_{\beta,n,m}$ to contain α, we also guaranteed that $r^\Delta_{n+1,\alpha}$ and $r^\Delta_{n,\beta+m}$ are the same variable.
- We now set all $r^\Delta_{n,\alpha}$ that are undefined so far to be the constant term 0, and define every $v_{\beta,n,m}$ that is undefined so far in a way such that every member α of $v_{\beta,n,m}$ satisfies $r^\Delta_{n+1,\alpha} = 0$. (Here, we use that the coding function has coinfinite range.)

It is easy to see that the object r^Δ defines this way is element of R. Each $r^\Delta_{n,\alpha}$ is either an x_{n_0,m_0} or 0. If $r^\Delta_{n,m} = x_{n_0,m_0}$, then $n > m_0 + n_0$. Given n, $r^\Delta_{n,m} = 0$ for infinitely many m.

The construction of r^{mult}. We will first show the following:

LEMMA 6.4. *If $(p_{n,m})_{n,m\in\omega}$ satisfies*

1. *$p_{n,m}$ is a term depending only on $x_{i,j}$ with $i < n$,*
2. *$(\forall n)\,(\exists^\infty m)\; p_{n,m} = 0$,*
3. *$(\forall i,j)\,(\exists^\infty n)\,(\forall M)\,(\exists m_0 \dots m_k > M)\; x_{i,j}$ is determined by $p_{n,m_0}, \dots, p_{n,m_k}$,*

then there is a $q \in R_{\omega\cdot\omega+\omega}$ such that $\sigma(q) = \bar{p}$.

PROOF. The proof is very similar to the preceding construction. The reader might just consult Figure 3(c).

Assume we have such a sequence $\bar{p}$. We have to define $q \in R_{\omega\cdot\omega+\omega}$. We already know that $q_{n,m} = p_{n,m}$ for $n, m \in \omega$.

We more or less repeat the construction above, to get all $q_{n,\alpha}$ and all $v_{\beta,n,m}$, but only for $\beta \geq \omega + \omega$, and we deal with $v_{\omega,n,m}$ later. Assume we are dealing with x_{n_0,m_0}. Set

$$I_{n_0,m_0} = \{n' \geq n_0 + 2 \colon (\forall M)\,(\exists m'_0 \dots m'_k > M)\; x_{n_0,m_0} \text{ is determined by } p_{n',m'_0}, \dots, p_{n',m'_k}\}.$$

[12]This corresponds to the simpler version of $\tilde{P}$ in Section 3.

According to assumption (3), I_{n_0,m_0} is infinite.

For all $\alpha > \omega$ limit and all n, m, k we will set $j_{\alpha,n,m}$ and $f_{\alpha,n,m,k}$ to be the identity functions.

We set

$$v_{\omega\cdot\omega,n_0,m_0} = \{i\cdot\omega + \ulcorner n_0, m_0\urcorner \colon i + m_0 + 1 \in I_{n_0,m_0}\}.$$

(So $v_{\omega\cdot\omega,n_0,m_0} \cap [i\cdot\omega, (i+1)\cdot\omega[$ contains a singleton if $i + m_0 + 1 \in I_{n_0,m_0}$, and is empty otherwise.) We set $q_{n_0+1,\alpha} = x_{n_0,m_0}$ for all $\alpha \in v_{\omega\cdot\omega,n_0,m_0}$; and "propagate the x_{n_0,m_0} diagonally down": If $n = n_0 + l$ for some $l \geq 1$, if $i > 0$ and $i + m + l \in I_{n_0,m_0}$, and if $m = \ulcorner a_0, a_1, \dots, a_l\urcorner$ such that $a_0 = n_0$, $a_1 = m_0$, then set

$$v_{(i+1)\cdot\omega,n,m} = \{i\cdot\omega + \ulcorner a_0, \dots, a_l, j\urcorner \colon j \in \omega\}.$$

We iterate this for all x_{n_0,m_0}, and set all $q_{n,\alpha}$ that have not been defined in this process to be the constant term 0. Also we set the $v_{\beta,n,m}$ for $\beta > \omega$ that have not been defined yet to contain only $\alpha > \omega$ such that $q_{n+1,\alpha} = 0$. (Remember that the coding had coinfinite range.)

So we have all $q_{n,\alpha}$ and all $v_{\beta,n,m}$, $j_{\beta,n,m}$ and $f_{\beta,n,m,k}$ for $\beta > \omega$.

We still have to define $v_{\omega,n,m}$, $j_{\omega,n,m}$ and $f_{\omega,n,m,k}$. For this, we use a simple book-keeping: At stage i, there are only finitely many pairs (n, m) for which any of these objects are already partially defined. For all of these (n, m), we also have: $v_{\omega,n,m}$ is defined up to height $M_{n,m}$, $j_{\omega,n,m}$ is defined up to some $h_{n,m}$ such that $j_{\omega,n,m}(h_{n,m} - 1) = M_{n,m} - 1$ $f_{\omega,n,m,k}$ is defined for exactly the $k < h_{n,m}$. Let M be the maximum of all $M_{n,m}$ for a given stage.

The book-keeping gives us an (n_0, m_0) and an (n, m) such that $q_{n,\omega+m} = x_{n_0,m_0}$. By our construction, we know that x_{n_0,m_0} can be determined by finitely many and arbitrary large $q_{n+1,m'}$. Fix $m'_0, \dots, m'_{l-1}$ bigger than M such that $q_{n+1,m'_0}, \dots, q_{n+1,m'_{l-1}}$ determines x_{n_0,m_0}. Extend $v_{\omega,n,m'}$ to contain exactly $\{m'_0, \dots, m'_l\}$, continue $j_{\omega,n,m'}$ by setting $j_{\omega,n,m'}(h_{n,m}) = M_{n,m} + l - 1$ and define $f_{\omega,n,m',h_{n,m}}$ so that it calculates x_{n_0,m_0}.

At the end, again set the $v_{\omega,n,m}$ that have not been defined in this process to contain only m' such that $q_{n+1,m'} = 0$. To be able to do this, we use at height ω assumption (2). ⊣

We can now define r^{mult}: We can take any condition in R satisfying

- $r^{\text{mult}}_{n,m}$ only depends on $x_{i,j}$ with $i + j < n$.
- Every $x_{i,j}$ with $i + j < n$, as well as the constant 0 term, occurs infinitely often in $\{r^{\text{mult}}_{n,m} \colon m \in \omega\}$.

If we set $\bar{p} = r^{\text{mult}} \circ \phi$ for some $\phi \in Q_*$, we get:

- $p_{n,m}$ only depends on $x_{i,j}$ with $i + j < n$. (Due to (2.8).) So we satisfy (1).
- For all n, infinitely many $p_{n,m}$ are 0. So we satisfy (2).
- $x_{i,j}$ is determined by $(\phi_{l_i,k_i})_{i\in I}$. Fix any n bigger than $\max(l_i + k_i \colon i \in I)$. Then $x_{i,j}$ is determined by finitely many $p_{n,m}$ (where we can pick the m's arbitrarily large). So we satisfy (3).

So $r^{\text{mult}} \circ \phi$ satisfies all assumptions of the previous Lemma, and we get a q as desired.

§7. The quotient forcing. It might look tempting to assume $\diamond$ to construct the coding sequences $\bar{v}, \bar{j}, \bar{f}$ instead of using the preparatory forcing. (We just have to "guess" correctly sufficiently often for the proofs to work.) However, this is not possible: Otherwise, Sacks forcing would be *equivalent* to P_ω (since P_ω adds a Sacks real s which in turn determines the P_ω-generic filter G_ω). But Sacks reals are minimal, and the Q_0 generic $\eta_0 \in 2^{\omega_1}$ is not in the ground model V. Therefore $V[s] = V[\eta_0]$, a contradiction to the fact that $V[\eta_0]$ does not add new reals.

In particular, if we look at P_ω in $V[\tilde{G}]$, then P_ω does not add a Sacks real (over $V[\tilde{G}]$), just a Sacks real over V.

So $\tilde{P} * P_\omega$ adds a Sacks real s but is not equivalent to Sacks forcing, and s does not determine the $\tilde{P}$-generic object $\tilde{G}$. However, every new ω-sequence is already added by s:

Lemma 7.1. *If $\tilde{G} * G_\omega$ is $\tilde{P} * P_\omega$-generic over V, and if $r \in V[\tilde{G} * G_\omega]$ is an ω-sequence of ordinals, then $r \in V[s]$. Here we set $s = (\bar{\eta}_n(m))_{n,m\in\omega}$, the Sacks real over V.*

Proof. If $\tilde{q} \in \tilde{G}$ has height δ, then s together with $\tilde{q}$ determines G_n up to height δ for all n (just as in Lemma 4.9). So if $N \prec H(\chi)$ and $\tilde{q} \in \tilde{G}$ has height $N \cap \omega_1$, then s together with $\tilde{q}$ determines whether $r \in \tilde{G} * G_\omega$ for any $r \in R \cap N$.

Assume towards a contradiction that $p \in R$ forces that $\underset{\sim}{f}$ is an ω-sequence of ordinals not added by s. Choose an $N \prec H(\chi)$ containing $p, \underset{\sim}{f}$, and an N-generic $q \leq p$. Each $\underset{\sim}{f}(n)$ is decided by some maximal antichain $A \in N$. But for each $a \in A \cap N$, s together with $\tilde{q}$ determines whether a is in G. In particular, $\underset{\sim}{f}[G] \in V[s]$. $\dashv$

This proves the second part of Theorem 1: Since R forces that there is some Sacks real over V and since Sacks forcing is homogeneous, R can be factored as Sacks composed with some P'. Since the Sacks real already adds all new ω-sequences, P' is NNR.

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KURT GÖDEL RESEARCH CENTER FOR MATHEMATICAL LOGIC
UNIVERSITY OF VIENNA
WÄHRINGER STRAßE 25
1090 WIEN, AUSTRIA
E-mail: kellner@fsmat.at
URL: http://www.logic.univie.ac.at/~kellner

EINSTEIN INSTITUTE OF MATHEMATICS
EDMOND J. SAFRA CAMPUS, GIVAT RAM
THE HEBREW UNIVERSITY OF JERUSALEM
JERUSALEM, 91904, ISRAEL
and
DEPARTMENT OF MATHEMATICS
RUTGERS UNIVERSITY
NEW BRUNSWICK, NJ 08854, USA
E-mail: shelah@math.huji.ac.il
URL: http://www.math.rutgers.edu/~shelah