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WINNING THE PRESSING DOWN GAME BUT NOT BANACH-MAZUR

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Abstract. Let S be the set of those $\alpha \in \omega_2$ that have cofinality ω_1. It is consistent relative to a measurable that the nonempty player wins the pressing down game of length ω_1, but not the Banach-Mazur game of length $\omega + 1$ (both games starting with S).

§1. Introduction. We set $E^{\kappa}_{\theta} = \{\alpha \in \kappa : \mathrm{cf}(\alpha) = \theta\}$. Let S be a stationary set. We investigate two games, each played by players called "empty" and "nonempty". Empty has the first move.

In the Banach-Mazur game $\mathrm{BM}(S)$ of length θ, the players choose decreasing stationary subsets of S. Empty wins, if at some $\alpha < \theta$ the intersection of these sets is nonstationary. (Exact definitions are give in the next section.)

In the pressing down game $\mathrm{PD}(S)$, empty cannot choose a stationary subset of the moves so far, but only a regressive function. Nonempty chooses a homogeneous stationary subset.

So it is at least as hard for nonempty to win BM as to win PD.

BM can be really harder than PD. This follows from well known facts about precipitous ideals (cf. 2.4 for a more detailed explanation): Nonempty can never win $\mathrm{BM}_{\leq\omega}(\omega_2)$, but it is consistent (relative to a measurable) that nonempty wins $\mathrm{PD}_{<\omega_1}(\omega_2)$. The reason is the following: In BM, empty can first choose $E^{\omega_2}_{\omega}$, and empty always wins on this set. However in PD, it is enough for nonempty to win on $E^{\omega_2}_{\omega_1}$, which is consistent. In a certain way this is "cheating", since nonempty wins PD on $E^{\omega_2}_{\omega_1}$ but looses BM on the disjoint set $E^{\omega_2}_{\omega}$, and the difference arises because empty has the first move in BM.

So a better question is: Can nonempty win $\mathrm{PD}(S)$ but loose $\mathrm{BM}(S)$ even if nonempty gets the first move,[1] for example[2] on $S = E^{\omega_2}_{\omega_1}$?

We show that this is indeed the case:

THEOREM 1.1. *It is consistent relative to a measurable that for $\theta = \aleph_1$ and $S = E^{\theta^+}_{\theta}$, nonempty wins $PD_{<\omega_1}(S)$ but not $BM_{\leq\omega}(S)$, even if nonempty gets the first move.*

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[1] This is equivalent to: nonempty does not win $\mathrm{BM}_{\leq\omega}(S')$ for any stationary $S' \subseteq S$.

[2] $S = E^{\omega_2}_{\omega_1}$ is the simplest possible example, since empty always wins PD if every element of S has cofinality ω, cf. 2.3.2.

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The same holds for $\theta = \aleph_n$ (*for* $n \in \omega$) *etc.*

Various aspects of these and related games have been studied for a long time.

Note that in this paper we consider the games on sets, i.e., a move is an element of the powerset of κ minus the (nonstationary) ideal. A popular (closely related but not always equivalent) variant is to consider games on a Boolean algebra B: Moves are elements of B, in our case B would be the powerset of κ *modulo* the ideal.

Also note that in Banach-Mazur games of length greater than ω, it is relevant which player moves first at limit stages (in our definition this is the empty player). Of course it is also important who moves first at stage 0 (in this paper again the empty player), but the difference here comes down to a simple density effect (cf. 2.1.4).

The Banach-Mazur BM game has been investigated e.g., in [5] and [15]. It is closely related to the so-called "ideal game" and to precipitous ideals, cf. Theorem 2.3 and [9], [1], or [4]. BM is also related to the "cut & choose game" of [6].

The pressing down game is related to the Ehrenfeucht-Fraïssé game in model theory, cf. [13] or [3], and has applications in set theory as well [12].

Other related games have been studied e.g., in [7] and [14].

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§2. Banach-Mazur, pressing down, and precipitous ideals.

Let κ and θ be regular, $\theta < \kappa$.

We set $E^\kappa_\theta = \{\alpha \in \kappa : \operatorname{cf}(\alpha) = \theta\}$. $\mathscr{E}^\kappa_\theta$ is the family of stationary subsets of E^κ_θ. Analogously for $E^\kappa_{>\theta}$ etc.

Instead of "the empty player has a winning strategy for the game G" we just say "empty wins G" (as opposed to: empty wins a specific run of the game).

$\mathscr{I}$ denotes a fine, normal ideal on κ. (I.e., every $\alpha \in \kappa$ is in $\mathscr{I}$. Together with normality this implies that $\mathscr{I}$ is $<\kappa$-complete.)

A set $S \subseteq \kappa$ is called $\mathscr{I}$-positive if $S \notin \mathscr{I}$.

Definition 2.1. Let κ be regular, and $S \subseteq \kappa$ an $\mathscr{I}$-positive set.

- $\mathrm{BM}_{<\zeta}(\mathscr{I}, S)$, the *Banach-Mazur game* of length ζ starting with S, is played as follows:

  At stage 0, empty plays an $\mathscr{I}$-positive $S_0 \subseteq S$, nonempty plays $T_0 \subseteq S_0$. At stage $\alpha < \zeta$, empty plays an $\mathscr{I}$-positive $S_\alpha \subseteq \bigcap_{\beta<\alpha} S_\beta$ (if possible), and nonempty plays some $T_\alpha \subseteq S_\alpha$.

  Empty wins the run, if $\bigcap_{\beta<\alpha} S_\beta \in \mathscr{I}$ at any stage $\alpha < \zeta$. Otherwise nonempty wins.

  (For nonempty to win a run, it is not necessary that $\bigcap_{\beta<\zeta} S_\beta$ is $\mathscr{I}$-positive or even just nonempty.)
- $\mathrm{BM}_{\leq\omega}(\mathscr{I}, S)$ is $\mathrm{BM}_{<\omega+1}(\mathscr{I}, S)$. (So empty wins the run iff $\bigcap_{n<\omega} S_n \in \mathscr{I}$, i.e., the game is naturally equivalent to one of length ω.)
- $\mathrm{PD}_{<\zeta}(\mathscr{I}, S)$, the *pressing down game* of length ζ starting with S, is played as follows:

  At stage $\alpha < \zeta$, empty plays a regressive function $f_\alpha : \kappa \to \kappa$, and nonempty plays some f_α-homogeneous $T_\alpha \subseteq S \cap \bigcap_{\beta<\alpha} T_\beta$.

  Empty wins the run, if $T_\alpha \in \mathscr{I}$ for any $\alpha < \zeta$. Otherwise, nonempty wins.

- $\mathrm{PD}_{\leq\omega}(\mathscr{I}, S)$ is $\mathrm{PD}_{<\omega+1}(\mathscr{I}, S)$. (I.e., empty wins the run iff $S \cap \bigcap_{n\in\omega} T_n \in \mathscr{I}$.)
- $\mathrm{BM}_{<\zeta}(S)$ is $\mathrm{BM}_{<\zeta}(\mathrm{NS}, S)$, and $\mathrm{PD}_{<\zeta}(S)$ is $\mathrm{PD}_{<\zeta}(\mathrm{NS}, S)$ (where NS denotes the nonstationary ideal).

$\mathrm{PD}_{<\theta}$ could equivalently be defined such that nonempty chooses at stage α some $\beta_\alpha \in \kappa$, and empty wins the run if $S \cap \bigcap_{\zeta<\alpha} f^{-1}(\beta_\zeta) \in \mathscr{I}$ for some $\alpha < \theta$.

The following is trivial:

FACTS 2.1. 1. *Assume $S \subseteq T$.*
- *If empty wins $BM_{<\zeta}(\mathscr{I}, S)$, then empty wins $BM_{<\zeta}(\mathscr{I}, T)$.*
- *If nonempty wins $BM_{<\zeta}(\mathscr{I}, T)$, then nonempty wins $BM_{<\zeta}(\mathscr{I}, S)$.*
- *If empty wins $PD_{<\zeta}(\mathscr{I}, T)$, then empty wins $PD_{<\zeta}(\mathscr{I}, S)$.*
- *If nonempty wins $PD_{<\zeta}(\mathscr{I}, S)$, then nonempty wins $PD_{<\zeta}(\mathscr{I}, T)$.*

2. *Assume that $\mathscr{I} \subseteq \mathscr{J}$, and that $\mathscr{J}$ is also fine and normal.*
   - *If empty wins $PD_{<\zeta}(\mathscr{I}, S)$, then empty wins $PD_{<\zeta}(\mathscr{J}, S)$.*
   - *If nonempty wins $PD_{<\zeta}(\mathscr{J}, S)$, then nonempty wins $PD_{<\zeta}(\mathscr{I}, S)$.*
3. *In particular, if nonempty wins $PD_{<\zeta}(\mathscr{I}, S)$, then nonempty wins $PD_{<\zeta}(S)$.*
4. *Let BM' be the variant of BM where nonempty gets the first move (at stage 0 only). The difference between BM and BM' is a simple density effect:*
   - *Empty wins $BM'_{<\zeta}(\mathscr{I}, S)$ iff empty wins $BM_{<\zeta}(\mathscr{I}, S')$ for all positive $S' \subseteq S$ iff empty has a winning strategy for BM with S as first move.*
   - *Empty wins $BM_{<\zeta}(\mathscr{I}, S)$ iff empty wins $BM'_{<\zeta}(\mathscr{I}, S')$ for some positive $S' \subseteq S$.*
   - *Nonempty wins $BM'_{<\zeta}(\mathscr{I}, S)$ iff nonempty wins $BM_{<\zeta}(\mathscr{I}, S')$ for some positive $S' \subseteq S$.*
5. *Assume that S is $\mathscr{I}$-positive, and let $\mathscr{I}_S$ be generated by $\mathscr{I} \cup \{\kappa \setminus S\}$. Then $A \in \mathscr{I}_S$ iff $A \cap S \in \mathscr{I}$, and empty wins $BM_{<\theta}(\mathscr{I}, S)$ iff empty wins $BM_{<\theta}(\mathscr{I}_S, \kappa)$. The same holds for PD or the ideal game (defined below), and for player nonempty instead of player empty.*

(For 3, use that $\mathscr{I}$ is normal, which implies $\mathrm{NS} \subseteq \mathscr{I}$.)

We will use the following definitions and facts concerning precipitous ideals, as introduced by Jech and Prikry [9]. We will usually refer to Jech's *Millennium Edition* [8] for details.

DEFINITION 2.2. Let $\mathscr{I}$ be a fine, normal ideal on κ.

- Let V be an inner model of W. $U \in W$ is called a normal V-ultrafilter if the following holds:
  - If $A \in U$, then $A \in V$ and A is a subset of κ.
  - $\alpha \notin U$ for all $\alpha \in \kappa$, and $\kappa \in U$.
  - If $A, B \in V$ are subsets of κ, $A \subseteq B$ and $A \in U$, then $B \in U$.
  - If $A \in V$ is a subset of κ, then either $A \in U$ or $\kappa \setminus A \in U$.
  - If $f \in V$ is a regressive function on $A \in U$, then f is constant on some $B \in U$.

  (Note that we do not require iterability or amenability.)
- A normal V-ultrafilter U is wellfounded, if the ultrapower of V modulo U is wellfounded. In this case the transitive collapse of the ultrapower is denoted by $\mathrm{Ult}_U(V)$.

- For a $<\kappa$-complete ideal $\mathscr{I}$, let $P_{\mathscr{I}}$ be the family of $\mathscr{I}$-positive sets ordered by inclusion. $P_{\mathscr{I}}$ forces that the generic filter G is a V-ultrafilter (cf. [8, 22.13]). An ideal $\mathscr{I}$ is called precipitous, if it is κ-complete and $P_{\mathscr{I}}$ forces that G is wellfounded.
- The ideal game on $\mathscr{I}$ is played just like $\mathrm{BM}_{\leq\omega}(\mathscr{I},\kappa)$, but empty wins iff $\bigcap_{n\in\omega} S_n$ is empty (as opposed to "in $\mathscr{I}$").

So if empty wins the ideal game, then empty wins $\mathrm{BM}_{\leq\omega}(\mathscr{I},\kappa)$. And if nonempty wins $\mathrm{BM}_{\leq\omega}(\mathscr{I},\kappa)$, then nonempty wins the ideal game.

THEOREM 2.3. *Let $\mathscr{I}$ be a fine, normal ideal on κ.*

1. (Jech, cf. [8, 22.21]). *$\mathscr{I}$ is not precipitous iff empty wins the ideal game. So in this case empty also wins $BM_{\leq\omega}(\mathscr{I},\kappa)$.*
2. (cf. [1]). *If $\mathscr{I}$ is such that E^κ_ω is $\mathscr{I}$-positive, then nonempty cannot win the ideal game, and empty wins*[3] *$PD_{\leq\omega}(\mathscr{I},E^\kappa_\omega)$ and therefore also $BM_{\leq\omega}(\mathscr{I},\kappa)$.*
3. (Jech, Prikry [4], cf. [8, 22.33]). *If $\mathscr{I}$ is precipitous, then κ is measurable in an inner model.*
4. (Laver, cf. [1] or [8, 22.33]). *Assume that U is a normal measure on κ. Let $\aleph_1 \leq \theta < \kappa$ be regular and let $Q = Levy(\theta, <\kappa)$ be the Levy collapse* (cf. Lemma 6.1). *In $V[G_Q]$, let $\mathscr{F}$ be the filter generated by U and $\mathscr{I}$ the corresponding ideal. Then $\mathscr{I}$ is fine and normal, and the family of $\mathscr{I}$-positive sets has a $<\theta$-closed dense subfamily.*

   So in particular in $V[G_Q]$ nonempty wins $BM_{<\theta}(\mathscr{I},S)$ for all $\mathscr{I}$-positive sets S (nonempty just has to pick sets from the dense subfamily), and therefore nonempty wins $PD_{<\theta}(S)$ (cf. 2.1.3).
5. (Magidor [4], penultimate paragraph). *One can modify this forcing to get a $<\theta$-closed dense subset of $\mathscr{E}^{\theta^+}_\theta$.*

   So in particular, $\mathscr{E}^{\theta^+}_\theta$ can be precipitous.

Mitchell [4] showed that even for $\theta=\aleph_0$, $\mathrm{Levy}(\theta,<\kappa)$ gives a precipitous ideal on $\theta^+=\omega_1$ (and with Magidor's extension, NS_{ω_1} can be made precipitous). So the ideal game is interesting on ω_1, but our games are not:

COROLLARY 2.4. 1. *Empty always wins $PD_{\leq\omega}(S)$ and $BM_{\leq\omega}(S)$ for $S\subseteq\omega_1$.*
2. *It is equiconsistent with a measurable that nonempty wins $BM_{<\theta}(E^{\theta^+}_\theta)$ for e.g., $\theta=\aleph_1$, $\theta=\aleph_2$, $\theta=\aleph^+_{\aleph_7}$ etc.*
3. *The following is consistent relative to a measurable: Nonempty wins $PD_{<\theta}(\theta^+)$ but not $BM_{\leq\omega}(\theta^+)$ for e.g., $\theta=\omega_1$.*

PROOF. (1) is just 2.3.2, and (2) follows from 2.3.3–4.

(3) Let κ be measurable, and Levy-collapse κ to θ^+. According to 2.3.2, nonempty wins $\mathrm{PD}_{<\omega_1}(S)$ for all $S\in U$, in particular for $S=\theta^+$. However, empty wins $\mathrm{BM}_{\leq\omega}(\theta^+)$ (by playing $E^{\theta^+}_\omega$). ⊣

In the rest of the paper we will deal with the proof of Theorem 1.1.

[3]There is even a fixed sequence of winning moves for empty: For every $\alpha\in E^\kappa_\omega$ let $(\alpha_n)_{n\in\omega}$ be a normal sequence in α. As move n, empty plays the function that maps α to α_n. If β and β' are both in $\bigcap_{n\in\omega} T_n$, then $\beta_n=\beta'_n$ for all n and therefore $\beta=\beta'$.

§**3. Overview of the proof.** We assume that κ is measurable, and that $\omega < \theta < \kappa$ is regular.

STEP 1. We construct models M satisfying:

$(*)$ κ is measurable and player empty wins $\mathrm{BM}_{\leq\omega}(S)$ for every stationary S.

We present two constructions, showing that $(*)$ is true in $L[U]$ as well as compatible with larger cardinals:

(i) The inner model $L[U]$, Section 4:
Let D be a normal measure on κ, and set $U = D \cap L[D]$. Then in $L[U]$, (the dual ideal of) U is the only normal precipitous ideal on κ. In particular, $L[U]$ satisfies $(*)$.

(ii) Forcing $(*)$, Section 5:
(α) We construct a partial order $R(\kappa)$ forcing that empty wins $\mathrm{BM}_{\leq\omega}(S)$ for all S. However, $R(\kappa)$ does not preserve measurability of κ.
(β) We use $R(\kappa)$ to force $(*)$ while preserving e.g., supercompactness.

STEP 2. Now we look at the Levy-collapse Q that collapses κ to θ^+.

In Section 6 we will see: If in $V[G_Q]$, nonempty wins $\mathrm{BM}_{\leq\omega}(\dot{S})$ for some $\dot{S} \in \mathscr{E}^\kappa_\theta$, then in V nonempty wins $\mathrm{BM}_{\leq\omega}(\tilde{S})$ for some $\tilde{S} \in \mathscr{E}^\kappa_{\geq\theta}$.

So if we start with V satisfying $(*)$ of Step 1, then Q forces:

- Nonempty does not win $\mathrm{BM}_{\leq\omega}(\dot{S})$ for any stationary $\dot{S} \subseteq E^\kappa_\theta$. Equivalently: Nonempty does not win $\mathrm{BM}_{\leq\omega}(E^\kappa_\theta)$, even if nonempty gets the first move.
- Nonempty wins $\mathrm{PD}_{<\theta}(E^\kappa_\theta)$. This follows from 2.3.4: Nonempty wins $\mathrm{PD}_{<\theta}(S)$ for all $S \in U$, and $E^\kappa_\theta = (E^\kappa_{\geq\theta})^V \in U$.

§**4.** U **is the only normal, precipitous ideal in** $L[U]$**.** If $V = L$, then there are no normal, precipitous ideals (recall that a precipitous ideal implies a measurable in an inner model). Using Kunen's results on iterated ultrapowers, it is easy to relativize this to $L[U]$:

THEOREM 4.1. *Assume $V = L[U]$, where U is a normal measure on κ. Then the dual ideal of U is the only normal, precipitous ideal on κ.*

In particular, NS_κ is nowhere precipitous, and empty wins $\mathrm{BM}_{\leq\omega}(S)$ for any stationary $S \subseteq \kappa$.

Remark: Much deeper results by Jech and later Gitik show that, for example,

$(\star)$ κ is measurable and either E^κ_λ or $\mathrm{NS}_\kappa \restriction \mathrm{Reg}$ is precipitous

implies more than a measurable (in an inner model) [2, Sect. 5], so $(\star)$ fails not only in $L[U]$ but also in any other universe without "larger inner-model-cardinals". However, it is not clear to us whether the same holds e.g., for

$(\star')$ κ is measurable and $\mathrm{NS}_\kappa \restriction S$ is precipitous for some S.

Back to the proof of Theorem 4.1:

If empty does not win $\mathrm{BM}_{\leq\omega}(S)$, then empty does not win the ideal game starting with S, and empty does not win the ideal game on the ideal NS_S defined in 2.1.5. That means that NS_S is precipitous. But NS_S can never be equal to the dual of U, a contradiction. (S can be partitioned into disjoint positive subsets, but U is an ultrafilter). So it is enough to show that the dual ideal of U is the only normal, precipitous ideal.

If $\mathscr{I}$ is a normal, precipitous ideal, then $P_{\mathscr{I}}$ forces that the generic filter G is a normal, wellfounded V-ultrafilter (cf. [8, 22.13]). So it is enough to show that in any forcing extension, U is the only normal, wellfounded V-ultrafilter on κ. We will do this in Lemma 4.3.

If $U \in L[U]$ and $L[U]$ thinks that U is a normal ultrafilter on κ, then we call the pair $(L[U], U)$ a κ-model.

If D is a normal ultrafilter on κ, and $U = D \cap L[D]$, then $(L[U], U)$ is a κ-model.

We will use the following results of Kunen [10], cited as Theorem 19.14 and Lemma 19.16 in [8]:

LEMMA 4.2. 1. *For every ordinal κ there is at most one κ-model.*

2. *Assume $\kappa < \lambda$ are ordinals, $(L[U], U)$ is the κ-model and $(L[W], W)$ the λ-model. Then $(L[W], W)$ is an iterated ultrapower of $(L[U], U)$, in particular: There is an elementary embedding $i : L[U] \to L[W]$ definable in $L[U]$ such that $W = i(U)$.*
3. *Assume that*
   - *$(L[U], U)$ is the κ-model,*
   - *A is a set of ordinals of size at least κ^+,*
   - *θ is a cardinal such that $A \cup \{U\} \subset L_\theta[U]$, and*
   - *$X \subseteq \kappa$ is in $L[U]$.*

   Then there is a formula φ, ordinals $\alpha_i < \kappa$ and $\gamma_i \in A$ such that in $L_\theta[U]$, X is defined by $\varphi(X, \alpha_1, \dots, \alpha_n, \gamma_1, \dots, \gamma_m, U)$.

(That means that in $L[U]$ there is exactly one y satisfying $\varphi(y, \alpha_1, \dots)$, and $y = X$.)

LEMMA 4.3. *Assume $V = L[U]$, where U is a normal ultrafilter on κ. Let V' be a forcing extension of V, and $G \in V'$ a normal, wellfounded V-ultrafilter on κ. Then $G = U$.*

PROOF. In V', let $j : V \to \mathrm{Ult}_G(V)$ be elementary. Set $\lambda = j(\kappa) > \kappa$ and $W = j[U]$. So $\mathrm{Ult}_G(V)$ is the λ-model $L[W]$.

In V, we can define a function $J : \mathrm{ON} \to \mathrm{ON}$ such that in V', $J(\alpha)$ is a cardinal greater than $(\alpha^\kappa)^{+V'}$. (After all, V' is just a forcing extension of V.) So $J(\alpha)$ is greater than both $i(\alpha)$ and $j(\alpha)$. In V, let $\mathscr{C}$ be the class of ordinals that are ω-limits of iterations of J, i.e., $\alpha \in \mathscr{C}$ if $\alpha = \sup(\alpha_0, J(\alpha_0), J(J(\alpha_0)), \dots)$. If $\alpha \in C$, then $i(\alpha) = j(\alpha) = \alpha$, since

$$\begin{aligned} i(\alpha) &= \sup(i(\alpha_0), i(J(\alpha_0)), i(J(J(\alpha_0))), \dots) \\ &\leq \sup(J(\alpha_0), J(J(\alpha_0)), J(J(J(\alpha_0))), \dots) = \alpha. \end{aligned}$$

Also, each $\alpha \in \mathscr{C}$ is a cardinal in V', since it is a supremum of cardinals.

In V', pick a set A of κ^+ many members of $\mathscr{C}$, and $\theta \in \mathscr{C}$ such that $A \cup \{U\} \subseteq L_\theta[U]$. Pick any $X \subseteq \kappa$. Then in $L[U]$, X is defined by

$$L_\theta[U] \vDash \varphi(X, \vec{\alpha}, \vec{\gamma}, U).$$

Let k be either i or j. Then by elementarity, in $L[W]$, $k(X)$ is the set Y such that

$$L_\theta[W] \vDash \varphi(Y, \vec{\alpha}, \vec{\gamma}, W),$$

since $W = k(U)$ and $k(\beta) = \beta$ for all $\beta \in \kappa \cup A \cup \{\theta\}$.

Therefore $i(X) = j(X) = Y$. So $X \in G$ iff $\kappa \in j(X) = i(X)$ iff $X \in U$, since both G and U are normal. $\dashv$

§5. Forcing empty to win. As in the last section, we construct a universe in which empty wins $\mathrm{BM}_{\leq\omega}(S)$ for every stationary $S \subseteq \kappa$, this time using forcing. This shows that the assumption is also compatible with e.g., κ supercompact.

5.1. The basic forcing.

ASSUMPTION 5.1. κ is inaccessible and $2^\kappa = \kappa^+$.

We will define the $<\kappa$-support iteration $(P_\alpha, Q_\alpha)_{\alpha<\kappa^+}$ and show:

LEMMA 5.2. *P_{κ^+} forces: Empty has a winning strategy for $BM_{\leq\omega}(\kappa)$ where empty's first move is κ. P_{κ^+} is κ^+-cc and has a dense subforcing P'_{κ^+} which is $<\kappa$-directed-closed and of size κ^+.*

We use two basic forcings (more precisely: forcing-definitions) in the iteration:

- If $S \subseteq \kappa$ is stationary, then Cohen(S) adds a Cohen subset of S. Conditions are functions $f : \zeta \to \{0, 1\}$ with $\zeta < \kappa$ successor such that $\{\xi < \zeta : f(\xi) = 1\}$ is a subset of S. ζ is called height of f. Cohen(S) is ordered by inclusion. This forcing adds the generic set $S' = \{\zeta < \kappa : (\exists f \in G) f(\zeta) = 1\} \subset S$.
- If $\lambda \leq \kappa^+$, and $(S_i)_{i<\lambda}$ is a family of stationary sets, then Club$((S_i)_{i<\lambda})$ consists of $f : (\zeta \times u) \to \{0, 1\}$, $\zeta < \kappa$ successor, $u \subseteq \lambda$, $|u| < \kappa$ such that $\{\xi < \zeta : f(\xi, i) = 1\}$ is a closed subset of S_i. ζ is called height of f, u domain of f. Club$((S_i)_{i<\lambda})$ is ordered by inclusion.

The following is well known:

LEMMA 5.3. *Cohen(S) is $<\kappa$-closed and forces that the generic Cohen subset $S' \subseteq S$ is stationary.*

So Cohen(S) is a well-behaved forcing, adding a generic stationary subset of S. Club$((S_i)_{i<\lambda})$ adds unbounded closed subsets of each S_i. Other than that it is not clear why this forcing should e.g., preserve the regularity of κ (and it will generally not be σ-closed). However, we will shoot clubs only through complements of Cohen-generics which we added previously, and this will simplify matters considerably.

The P_α will add more and more moves to our winning strategy.

Set $D = \{\delta < \kappa^+ : \delta \text{ limit}\}$ (D stands for "destroy").

Set $\mathcal{T} = (\kappa^+)^{<\omega}$, a tree ordered by inclusion. (Let us call the order $\preceq_{\mathcal{T}}$.) Find a bijection $i : \mathcal{T} \to \kappa^+ \setminus D$ so that $s \preceq_{\mathcal{T}} t$ implies $i(s) \leq i(t)$. Let M be the image of i, i.e., $\kappa^+ = D \cup M$. (M stands for "moves".) i defines a tree-order $\preceq_M$ on M such that $\alpha \preceq_M \beta$ implies $\alpha \leq \beta$. Tree-order means that for $\alpha \in M$, the set of $\preceq_M$-predecessors of M is finite and totally ordered by $\preceq_M$. This defines for $\alpha \in M$ the sequence $\alpha_0 \preceq_M \alpha_1 \preceq_M \cdots \preceq_M \alpha_m \preceq_M \alpha$ of predecessors.

For $\delta \in D$, we can look at all infinite branches through $M \cap \delta$. Some of them will be "new", i.e., not in $M \cap \gamma$ for any $\gamma \in D \cap \delta$. Let λ_δ be the number of these new branches, i.e., $0 \leq \lambda_\delta \leq 2^\kappa = \kappa^+$.

We define Q_α by induction on α, and assume that at stage α (i.e., after forcing with P_α) we have already defined a partial strategy. (For increasing α, the partial strategy will increase, i.e., it will know responses to more initial segments of runs of the game.) We will see that P_α forces $2^\kappa = \kappa^+$. This allows us to use some simple bookkeeping to pick at stage α some $T_\alpha \subseteq \kappa$ such that every $T \subseteq \kappa$ in $\bigcup_{\beta<\kappa^+} V[G_\beta]$ appears as some T_ζ. In more detail:

Fix an enumeration $(\tilde{T}_{\alpha,\gamma})_{\gamma\in\kappa^+}$ of all (P_α-names for) subsets of κ. Fix a $\psi : M \to \kappa^+ \times \kappa^+$ such that $\psi(\alpha) = (\beta, \gamma)$ implies $\beta \leq \alpha$, and such that for all $\alpha \in M$ and $\beta, \gamma \in \kappa^+$ there is an immediate $\preceq_M$-successor α' of α such that $\psi(\alpha') = (\beta, \gamma)$. For $\psi(\alpha) = (\beta, \gamma)$, set $T_\alpha = \tilde{T}_{\beta,i}$ if it satisfies some additional assumption $(*)$ (see below); otherwise pick some arbitrary T_α satisfying $(*)$.

We work in $V[G_\alpha]$ to define Q_α:

- $\alpha \in M$, with the predecessors $0 = \alpha_0 < \alpha_1 \cdots < \alpha_m < \alpha$. By induction we know that at stage α_m
  - we dealt with the sequence $x_{\alpha_m} = (\kappa, T_{\alpha_1}, S_{\alpha_1}, T_{\alpha_2}, \dots, S_{\alpha_{m-1}}, T_{\alpha_m})$, which is played according to empty's partial strategy (at stage α_m),
  - we defined Q_{α_m} to be Cohen(T_{α_m}), adding the generic set S_{α_m},
  - this S_{α_m} was added to the partial strategy as response to x_{α_m}.

  Now we use the bookkeeping described above to pick T_α satisfying:

  $(*)$ $T_\alpha \subset S_{\alpha_m}$ is stationary, and the partial strategy is not (at stage α) already defined on $x_\alpha = x_{\alpha_m}{}^\frown(S_{\alpha_m}, T_\alpha)$.

  Then we set $Q_\alpha =$ Cohen(T_α), and add the Q_α-generic $S_\alpha \in V[G_{\alpha+1}]$ to the partial strategy as response to x_α.
- $\alpha \in D$. In V, there are $0 \leq \lambda_\alpha \leq \kappa^+$ many new branches b_i. (All old branches have already been dealt with in the previous D-stages.) For each new branch $b_i = (\alpha_0^i < \alpha_1^i < \dots)$, we set $S^i = \bigcap_{n\in\omega} S_{\alpha_n^i}$, and we set $Q_\alpha = \text{Club}((\kappa \setminus S^i)_{i\in\lambda_\alpha})$.

So empty always responds to nonempty's move T with a Cohen subset of T, and the intersection of an ω-sequence of moves according to the strategy is made non-stationary.

We will show:

Lemma 5.4. *P_{κ^+} does not add any new countable sequences of ordinals, forces that κ is regular and that the Q_α-generic S_α (i.e., empty's move) is stationary for all $\alpha \in M$.*

We will prove this Lemma later. Then the rest follows easily:

Lemma 5.5. *P_{κ^+} forces that the partial strategy is a winning strategy for player empty in the game $BM_{\leq\omega}(\kappa)$, using κ as first move.*

Proof. At the final limit stage, P_{κ^+} does not add any new subsets of κ, nor any countable sequences of such subsets. (In particular, there are only κ^+ many.) Work in $V[G_{\kappa^+}]$.

We first show that the partial strategy is a strategy: Assume towards a contradiction that there is some minimal $m \geq 0$ and a sequence $x = (\kappa, T_1', S_1', T_2', S_2', \dots, S_m', T_{m+1}')$ such that x is a valid initial sequence of a run played according to the partial strategy, but we do not have a response to x. So S_m' was added as response to $x \restriction 2m$, at some stage $\alpha \in M$, i.e., α has predecessors $\alpha_0 < \cdots < \alpha_m$, and $T_i' = T_{\alpha_i}$ and $S_i' = S_{\alpha_i}$ for $i < m$, and $S_m' = S_\alpha$. T_{m+1}' appears in some V_β for $\beta < \kappa^+$, i.e., $T_{m+1}' = \tilde{T}_{\beta,i}$ for some $i < \kappa^+$. Then there is some $\alpha' \in M$ such that $\alpha' > \beta$ is immediate $\preceq_M$-successor of α and such that $\psi(\alpha') = (\beta, i)$. So at stage α' we add to the partial strategy $S_{\alpha'}$ as response to x (unless we already added a response at an earlier stage), a contradiction.

Now we show that the strategy is actually a winning strategy: Let $y = (\kappa, T'_1, S'_1, T'_2, S'_2, \dots)$ be an infinite run of the game such that nonempty uses the partial strategy. Then $x \restriction 2n$ corresponds to an element of M for every n, and x defines a branch b through M. $b \in V$, since P_{κ^+} does not add new countable sequences of ordinals. Let $\alpha \in D$ be minimal so that $x \restriction 2n < \alpha$ for all n. Then in the D-stage α, the stationarity of $\bigcap_{n\in\omega} S'_n$ was destroyed, i.e., empty wins the run x. ⊣

We now define the dense subset of P_α:

Definition 5.6. $p \in P'_\alpha$ if $p \in P_\alpha$ and there are (in V) a successor ordinal $\varepsilon(p) < \kappa$, $(f_\alpha)_{\alpha\in\mathrm{dom}(p)}$ and $(u_\alpha)_{\alpha\in\mathrm{dom}(p)\cap D}$ such that:

- If $\alpha \in M$, then $f_\alpha : \varepsilon(p) \to \{0, 1\}$.
- If $\alpha \in D$, then $u_\alpha \subseteq \lambda_\alpha$, $|u_\alpha| < \kappa$, and $f_\alpha : \varepsilon(p) \times u_\alpha \to \{0, 1\}$.
- Moreover, for $\alpha \in D$, u_α consists exactly of the new branches through $\mathrm{dom}(p) \cap \alpha \cap M$.
- $p \restriction \alpha \Vdash p(\alpha) = f_\alpha$.

So a $p \in P'_\alpha$ corresponds to a "rectangular" matrix with entries in $\{0, 1\}$. Of course only some of these matrices are conditions of P_α and therefore in P'_α.

Lemma 5.7. 1. *P'_α is ordered by extension. (I.e., if $p, q \in P'_\alpha$, then $q \le p$ iff q (as matrix) extends p.)*
2. *$P'_\alpha \subseteq P_\alpha$ is a dense subset.*
3. *P'_α is $<\kappa$-directed-closed, in particular P_α does not add any new sequences of length $< \kappa$ nor does it destroy stationarity of any subset of κ.*

Proof. (1) should be clear.

(3) Assume all p_i are pairwise compatible. We construct a condition q by putting an additional row on top of $\bigcup p_i$ (and filling up at indices where new branches might have to be added). So we set

- $\mathrm{dom}(q) = \bigcup \mathrm{dom}(p_i)$.
- $\varepsilon(q) = \bigcup \varepsilon(p_i) + 1$.
- For $\alpha \in \mathrm{dom}(q) \cap M$, we put 0 on top, i.e., $q_\alpha(\varepsilon(q) - 1) = 0$.
- For $\alpha \in \mathrm{dom}(q) \cap D$, and $i \in \bigcup \mathrm{dom}(p_i(\alpha))$, set $q_\alpha(\varepsilon(q) - 1, i) = 1$.
- For $\alpha \in \mathrm{dom}(q) \cap D$, if i is a new branch through $M \cap \mathrm{dom}(q) \cap \alpha$ and not in $\bigcup \mathrm{dom}(p_i(\alpha))$, set $q_\alpha(\xi, i) = 0$ for all $\xi < \varepsilon(q)$.

Why can we do that? If $\alpha \in M$, whether the bookkeeping says that $\varepsilon(q) - 1 \in T_\alpha$ or not, we can of course always choose to not put it into S_α (i.e., set $q_\alpha(\varepsilon(q) - 1) = 0$). Then for $\alpha \in D$, $\varepsilon(q) - 1$ will definitely not be in the intersection along the branch i, so we can put it into the complement.

(2) By induction on α. Assume $p \in P_\alpha$.

$\alpha = \beta + 1$ is a successor. We know that P_β does not add any new $< \kappa$ sequences of ordinals, so we can strengthen $p \restriction \beta$ to a $q \in P'_\beta$ which decides $f = p(\beta) \in V$. Without loss of generality $\varepsilon(q) \ge \mathrm{height}(f)$, and we can enlarge f up to $\varepsilon(q)$ by adding values 0 (note that $\mathrm{height}(f) < \kappa$ is a successor, so we do not get problems with closedness when adding 0). And again, we also add values for the required "new branches" if necessary.

If α is a limit of cofinality $\geq \kappa$, then $p \in P_\beta$ for some $\beta < \alpha$, so there is nothing to do.

Let α be a limit of cofinality $< \kappa$, i.e., $(\alpha_i)_{i\in\lambda}$ is an increasing cofinal sequence in α, $\lambda < \kappa$. Using (2), define a sequence $p_i \in P'_{\alpha_i}$ such that $p_i < p_j \wedge p \restriction \alpha_i$ for all $j < i$, then use (3). ⊣

How does the quotient forcing $P^\alpha_{\kappa^+}$ (i.e., P_{κ^+}/G_α) behave compared to P_{κ^+}?

- Assume $\alpha \in D$. In $V[G_\alpha]$, Q_α shoots a club through the complement of the (probably) stationary set $\bigcap_{i\in\omega} S^i$. In particular, Q_α cannot have a $<\kappa$-closed subset.
- Nevertheless, $P_\alpha * Q_\alpha$ has a $<\kappa$-closed subset (and preserves stationarity).
- So if we factor P_{κ^+} at some $\alpha \in D$, the remaining $P^\alpha_{\kappa^+}$ will look very different from P_{κ^+}.
- However, if we factor P_{κ^+} at $\alpha \in M$, $P^\alpha_{\kappa^+}$ will be more or less the same as $P^\alpha_{\kappa^+}$ (just with a slightly different bookkeeping).

In particular, we get:

LEMMA 5.8. *If $\alpha \in M$, then the quotient $P^\alpha_{\kappa^+}$ will have a dense $<\kappa$-closed subset* (*and therefore it will not collapse stationary sets*).

(The proof is the same as for the last lemma.)

Note that for this result it was necessary to collapse the new branches as soon as they appear. If we wait with that, then (looking at the rest of the forcing from some stage $\alpha \in M$) we shoot clubs through stationary sets that already exist in the ground model, and things get more complicated.

Now we can easily prove Lemma 5.4:

PROOF OF LEMMA 5.4. In stage $\alpha \in M$, nonempty's previous move S_{α_m} is still stationary (by induction), the bookkeeping chooses a stationary subset T_{α_m} of this move, and we add S_α as Cohen-generic subset of T_{α_m}. So according to Lemma 5.3, S_α is stationary at stage $\alpha + 1$, i.e., in $V[G_{\alpha+1}]$. But since $\alpha + 1 \in M$, the rest of the forcing, $P^{\alpha+1}_{\kappa^+}$, is $<\kappa$-closed and does not destroy stationarity of S_α. ⊣

5.2. Preserving measurability. We can use the following theorem of Laver [11], generalizing an idea of Silver: If κ is supercompact, then there is a forcing extension in which κ is supercompact and every $<\kappa$-directed closed forcing preserves the supercompactness. Note that we can also get $2^\kappa = \kappa^+$ with such a forcing.

COROLLARY 5.9. *If κ is supercompact, we can force that κ remains supercompact and that empty wins $BM_{\leq\omega}(S)$ for all stationary $S \subseteq \kappa$.*

Remark: It is possible, but not obvious, that we can also start with κ just measurable and preserve measurability. It is at least likely that it is enough to start with strong to get measurable. Much has been published on such constructions, starting with Silver's proof for violating GCH at a measurable (as outlined in [8, 21.4]).

§**6. The Levy collapse.** We show that after collapsing κ to θ^+, nonempty still has no winning strategy in BM.

Assume that κ is inaccessible, $\theta < \kappa$ regular, and let $Q = \mathrm{Levy}(\theta, <\kappa)$ be the Levy collapse of κ to θ^+: A condition $q \in Q$ is a function defined on a subset of

$\kappa \times \theta$, such that $|\operatorname{dom}(q)| < \theta$ and $q(\alpha, \xi) < \alpha$ for $\alpha > 1, (\alpha, \xi) \in \operatorname{dom}(q)$ and $q(\alpha, \xi) = 0$ for $\alpha \in \{0, 1\}$.

Given $\alpha < \kappa$, define $Q_\alpha = \{q : \operatorname{dom}(q) \subseteq \alpha \times \theta\}$ and $\pi_\alpha : Q \to Q_\alpha$ by $q \mapsto q \restriction (\alpha \times \theta)$.

The following is well known (see e.g., [8, 15.22] for a proof):

LEMMA 6.1.
- *Q is κ-cc and $<\theta$-closed.*
- *In particular, Q preserves stationarity of subsets of κ: If p forces that $\dot{C} \subseteq \kappa$ is club, then there is a $C' \subseteq \kappa$ club and a $q \leq p$ forcing that $C' \subseteq \dot{C}$.*
- *If $q \Vdash p \in G$, then $q \leq p$ (i.e., $\leq^*$ is the same as $\leq$).*

We will use the following simple consequence of Fodor's lemma (similar to a Δ-system lemma):

LEMMA 6.2. *Assume that $p \in Q$ and $S \in \mathscr{E}^\kappa_{\geq\theta}$. If $\{q_\alpha \mid \alpha \in S\}$ is a sequence of conditions in Q, $q_\alpha < p$, then there is a $\beta < \kappa$, a $q \in Q_\beta$ and a stationary $S' \subseteq S$, such that $q \leq p$ and $\pi_\alpha(q_\alpha) = q$ for all $\alpha \in S'$.*

PROOF. For $q \in Q$ set $\operatorname{dom}^\kappa(q) = \{\alpha \in \kappa : (\exists \zeta \in \theta)\,(\alpha, \zeta) \in \operatorname{dom}(q)\}$. For $\alpha \in S$ set $f(\alpha) = \sup(\operatorname{dom}^\kappa(q_\alpha) \cap \alpha)$. f is regressive, since $|\operatorname{dom}^\kappa(q_\alpha)| < \theta$ and $\operatorname{cf}(\alpha) \geq \theta$. By the pressing down lemma there is a $\beta < \kappa$ such that $T = f^{-1}(\beta) \subseteq S$ is stationary.

For $\alpha \in T$, set $h(\alpha) = \pi_{\beta+1}(q_\alpha)$. The range of h is of size at most $|\beta \times \theta|^{<\theta} < \kappa$. So there is a stationary $S' \subseteq T$ such that h is constant on S', say q. If $\alpha \in S'$, then $\sup(\operatorname{dom}^\kappa(q_\alpha) \cap \alpha) = \beta$, therefore $\pi_\alpha(q_\alpha) = \pi_{\beta+1}(q_\alpha) = q$.

Pick $\alpha \in S'$ such that $\alpha > \sup(\operatorname{dom}^\kappa(p))$. $q_\alpha \leq p$, so $q = \pi_\alpha(q_\alpha) \leq \pi_\alpha(p) = p$. $\dashv$

LEMMA 6.3. *Assume that*
- *κ is strongly inaccessible, $\theta < \kappa$ regular, $\mu \leq \theta$,*
- *$Q = Levy(\theta, < \kappa)$,*
- *$\dot{S}$ is a Q-name for an element of $\mathscr{E}^\kappa_\theta$,*
- *$\tilde{p} \in Q$ forces that $\dot{F}$ is a winning strategy of nonempty in $BM_{<\mu}(\dot{S})$.*

Then in V, nonempty wins $BM_{<\mu}(\tilde{S})$ for some $\tilde{S} \in E^\kappa_{\geq\theta}$.

If $\dot{S}$ is a standard name for $T \in (E^\kappa_{\geq\theta})^V$, then we can set $S = T$.

PROOF. First assume that $\dot{S}$ is a standard name.

For a run of $\mathrm{BM}_{<\mu}(S)$, we let A_ε and B_ε denote the εth moves of empty and nonempty. We will construct by induction on $\varepsilon < \mu$ a strategy for empty, including not only the moves B_ε, but also Q-names $\dot{A}'_\varepsilon, \dot{B}'_\varepsilon$, and Q-conditions $p_\varepsilon, \langle p^\varepsilon_\alpha \mid \alpha \in B_\varepsilon\rangle$, such that the following hold:
- $p_\varepsilon \leq p_\xi$ and $p^\varepsilon_\alpha \leq p^\xi_\alpha$ for $\xi < \varepsilon$.
- p_ε forces that $(\dot{A}'_\xi, \dot{B}'_\xi)_{\xi \leq \varepsilon}$ is an initial segment of a run of $\mathrm{BM}_{<\mu}(\dot{S})$ in which nonempty uses the strategy $\dot{F}$.
- $p_\varepsilon \Vdash \dot{A}'_\varepsilon \subseteq A_\varepsilon$.
- For $\alpha \in B_\varepsilon$, $\pi_\alpha(p^\varepsilon_\alpha) = p_\varepsilon$ (in particular $p^\varepsilon_\alpha \leq p_\varepsilon$), and $p^\varepsilon_\alpha \Vdash$ "$\alpha \in \dot{B}'_\varepsilon$".

Assume that we have already constructed these objects for all $\xi < \varepsilon$.

In limit stages ε, we first have to make sure that $\bigcap_{\xi<\varepsilon} B_\xi$ is stationary (otherwise nonempty has already lost). Pick a q stronger than each p_ξ for $\xi < \varepsilon$. (This is possible since Q is $<\theta$-closed.) Then q forces that $\bigcap_{\xi<\varepsilon} B_\xi = \bigcap_{\xi<\varepsilon} A_\xi \supseteq \bigcap_{\xi<\varepsilon} \dot{A}'_\xi$ and that $(\dot{A}'_\xi, \dot{B}'_\xi)_{\xi\leq\varepsilon}$ is a valid initial segment of a run where nonempty uses the strategy, in particular $\bigcap_{\xi<\varepsilon} \dot{A}'_\xi$ is stationary.

So now ε can be a successor or a limit, and empty plays the stationary set $A_\varepsilon \subseteq \bigcap_{\xi<\varepsilon} B_\xi$. (That implies that p^ξ_α is defined for all $\alpha \in A_\varepsilon$ and $\xi < \varepsilon$.)

- Define the εth move of empty in $V[G_Q]$ to be

$$\dot{A}'_\varepsilon = \{\alpha \in A_\varepsilon : (\forall \xi < \varepsilon)\, p^\xi_\alpha \in G_Q\},$$

  and pick $\tilde{p}_\varepsilon \leq p_\xi$ for $\xi < \varepsilon$ (for $\varepsilon = 0$, pick $\tilde{p}_0 = \tilde{p}$).

  $\tilde{p}_\varepsilon$ forces that $\dot{A}'_\varepsilon \subseteq \bigcap_{\xi<\varepsilon} \dot{B}'_\xi$, since p^ξ_α forces that $\alpha \in \dot{B}'_\xi$. $\tilde{p}_\varepsilon$ also forces that $\dot{A}'_\varepsilon$ is stationary:
  Otherwise there is a $C \subseteq \kappa$ club and a $q \leq \tilde{p}_\varepsilon$ forcing that $C \cap \dot{A}_\varepsilon$ is empty (cf. 6.1). $q \in Q_\beta$ for some $\beta < \kappa$. Pick $\alpha \in (C \cap A_\varepsilon) \setminus (\beta + 1)$. For $\xi < \varepsilon$, $\pi_\alpha(p^\xi_\alpha) = p_\xi \geq q$, and $q \in Q_\beta$, so q and p^ξ_α are compatible. Moreover, the conditions $(q \cup p^\xi_\alpha)_{\xi\in\varepsilon}$ are decreasing, so there is a common lower bound q' forcing that $p^\xi_\alpha \in G_Q$ for all ξ, i.e., that $\alpha \in \dot{A}'_\varepsilon$, a contradiction.
- Given $\dot{A}'_\varepsilon$, we define $\dot{B}'_\varepsilon$ as the response according to the strategy $\dot{F}$.
- Now we show how to obtain the next move of nonempty, B_ε, (in the ground model), as well as p^ε_α for $\alpha \in B_\varepsilon$. B_ε of course has to be a subset of the stationary set S defined by

$$S = \{\alpha \in A_\varepsilon \mid \tilde{p}_\varepsilon \not\Vdash \alpha \notin \dot{B}'_\varepsilon\}.$$

  For each $\alpha \in S$, pick some $p^\varepsilon_\alpha \leq \tilde{p}_\varepsilon$ forcing that $\alpha \in \dot{B}'_\varepsilon$. By the definition of $\dot{A}'_\varepsilon$ and since $\tilde{p}_\varepsilon \Vdash \dot{B}'_\varepsilon \subseteq \dot{A}'_\varepsilon$, we get

$$p^\varepsilon_\alpha \Vdash (\forall \xi < \varepsilon)\, p^\xi_\alpha \in G_Q,$$

  which means that for $\alpha \in S$ and $\xi < \varepsilon$, $p^\varepsilon_\alpha \leq p^\xi_\alpha$.

  Now we apply Lemma 6.2 (for $p = \tilde{p}_\varepsilon$). This gives us $S' \subseteq S$ and $q \leq \tilde{p}_\varepsilon$. We set $B_\varepsilon = S'$ and $p_\varepsilon = q$.

If $\dot{S}$ is not a standard name, set

$$S^0 = \{\alpha \in E^\kappa_{\geq\theta} : \tilde{p} \not\Vdash \alpha \notin \dot{S}\}$$

As above, for each $\alpha \in S_0$, pick a $\tilde{p}^{-1}_\alpha \leq \tilde{p}$ forcing that $\alpha \in \dot{S}$, and choose a stationary $\tilde{S} \subseteq S^0$ according to Lemma 6.2. Now repeat the proof, starting the sequence (p_ε) and (p^ε_α) already at $\varepsilon = -1$. ⊣

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