Math. Log. Quart. **50**, No. 1, 18 – 28 (2004) / **DOI** 10.1002/malq.200310071

F-products and nonstandard hulls for semigroups

Jakob Kellner* and **Hans Ploss**

Institut für Mathematik der Universität Wien, Boltzmanngasse 9, 1090 Wien, Österreich

Received 13 March 2003, revised 6 July 2003, accepted 15 July 2003
Published online 1 December 2003

Key words Nonstandard functional analysis, one-parameter semigroup of linear operators, F-product.
MSC (2000) 46S20, 47D03, 46B08, 03H05

Derndinger [2] and Krupa [5] defined the F-product of a (strongly continuous one-parameter) semigroup (of linear operators) and presented some applications (e. g. to spectral theory of positive operators, cf. [3]). Wolff (in [7] and [8]) investigated some kind of nonstandard analogon and applied it to spectral theory of group representations. The question arises in which way these constructions are related. In this paper we show that the classical and the nonstandard F-product are isomorphic (Theorem 2.6). We also prove a little "classical" corollary (2.7.).

1 Basic notation

1.1 Semigroups

Let E be a (real or complex) Banach space. We denote the norm of an element f of E by $|f|$. $\mathcal{L}(E)$ is the set of bounded linear functions from E to E. The elements of $\mathcal{L}(E)$ are called bounded linear *operators*, and the norm of an operator A is is denoted by $\|A\|$. A one-parameter semigroup of bounded linear operators (or *semigroup*, for short) is a function $T : \mathbb{R}_0^+ \longrightarrow \mathcal{L}(E)$ such that $T(0) = \mathrm{Id}_E$ and $T(t_1 + t_2) = T(t_1) \circ T(t_2)$. A semigroup T is called strongly continuous (or *continuous*, for short), if for all $f \in E$, $\lim_{t\to 0} |T(t)f - f| = 0$ (i. e. $\lim_{t\to 0} T(t) = \mathrm{Id}_E$ in the strong topology). T is called *uniformly continuous*, if $\lim_{t\to 0} \|T(t) - \mathrm{Id}_E\| = 0$ (i. e. $\lim_{t\to 0} T(t) = \mathrm{Id}_E$ in the topology induced by the operator-norm).

For a continuous semigroup T and $f \in E$ we define $A(f) := \lim_{h\to 0}(1/h)(T(h)f - f)$, if this limit exists (in E). $D(A)$ is the set of $f \in E$ such that $A(f)$ exists. A is called the *generator of* T.

More about the (classical) theory of continuous semigroups can be found in the textbook [3]. We will only need the following results:

Lemma 1.1

1. *For every continuous semigroup T there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that for all $t \in \mathbb{R}_0^+$, $\|T(t)\| \leq M e^{\omega t}$.*

2. *$D(A)$ is a dense linear subspace of E, and A is a closed linear operator (i. e. the graph of A is a closed subset of $E \times E$).*

3. *$D(A) = E$ if and only if $T(t)$ is uniformly continuous.*

4. *If $f \in D(A)$ and $h > 0$, then $|T(h)f - f| \leq h \cdot |A(f)| \cdot \sup_{s\leq h} \|T(s)\|$.*

5. *If $f \in D(A)$ and $h > 0$, then $|(1/h)(T(h)f - f) - A(f)| \leq \sup_{s\leq h}(|T(s)A(f) - A(f)|)$.*

6. *For any $\bar{f}$ and $h > 0$ there is $f \in D(A)$ such that*

$$A(f) = \frac{T(h)\bar{f} - \bar{f}}{h} \quad \textit{and} \quad |\bar{f} - f| \leq \sup_{s\leq h}(|T(s)\bar{f} - \bar{f}|).$$

* Corresponding author: e-mail: kellner@logic.univie.ac.at

Remark The last three items of the lemma follow from the following two facts:
If $f \in D(A)$ and $t > 0$, then $T(t)f - f = \int_0^t T(s)Af\,\mathrm{d}s$.
For all $f \in E$ and $t > 0$, $\int_0^t T(s)f\,\mathrm{d}s$ is in $D(A)$, and $A(\int_0^t T(s)f\,\mathrm{d}s) = T(t)f - f$.
(The integral is defined as the limit of the Riemann-sums.)
Using $|\int f(s)\,\mathrm{d}s| \le \int |f(s)|\,\mathrm{d}s$, the items 4 and 5 follow directly from $T(t)f - f = \int_0^t T(s)Af\,\mathrm{d}s$. We get item 6 by defining $f := (1/h)\int_0^h T(s)\bar{f}\,\mathrm{d}s$.

The famous theorem of Hille-Yosida states that uniformly continuous semigroups are exactly the semigroups of the form $T(t) = \mathrm{e}^{tA}$, where A is in $\mathcal{L}(E)$. If $T(t) = \mathrm{e}^{tA}$, then A is the generator of T, and $D(A) = E$. In this case, the constructions in this paper do not result in anything new. So we will mainly be interested in semigroups that are not uniformly continuous.

Our basic example for such a semigroup is the following:

Example 1.2 C^{b} is the (real) Banach space of uniformly continuous, bounded functions from $\mathbb{R}$ to $\mathbb{R}$ (with the sup-norm, denoted by $\|\cdot\|_\infty$). T is the translation semigroup, defined by $T(t)f(x) := f(x+t)$. Then T is continuous (since each $f \in C^{\mathrm{b}}$ is uniformly continuous), but T is not uniformly continuous: for example, let $f_k(x) = \sin(kx)$. Then $f_k \in C^{\mathrm{b}}$ and $|f_k| = 1$. For all $t > 0$ there exists a k such that $|T(t)f_k - f_k| > 1$, i.e. $\|T(t) - \mathrm{Id}_E\| > 1$. f is in $D(A)$ if and only if f' exists and belongs to E. In this case, $A(f) = f'$.

1.2 Robinsonian (nonstandard) analysis

We present the concept of nonstandard extensions following [1].

Let X be a set. Then $P(X)$ denotes the set of all subsets of X and

$$V_0(X) = X, \quad V_{n+1}(X) = V_n(X) \cup P(V_n(X)), \quad V(X) = \bigcup_{n\in\omega} V_n(X).$$

We assume that (a suitable copy of) $\mathbb{N}$ exists in X, and that all vector spaces we are interested in are subsets of X. So all sets of numbers, subspaces of E etc. are in $V(X)$. For technical reasons, we want to treat the elements of X as urelements, i.e. we want to avoid that for some $y \in V(X)$ and $x \in X$, $y \in x$. We can do that by assuming (without loss of generality) that X is a base-set, i.e. X does not contain the empty set and $x \cap V(X)$ is empty for all $x \in X$.

A *nonstandard universe* is a triple $\langle V(X), V(Y), {}^* : V(X) \longrightarrow V(Y)\rangle$ such that:

- X and Y are infinite base-sets, $X \subset Y$, ${}^*X = Y$, for all $x \in X$, ${}^*x = x$,
- (non-triviality) for every infinite subset A of X: $A \subsetneq {}^*A$,
- (transfer principle) if $\varphi(x_1, \dots, x_n)$ is a Σ_0-formula and $a_1, \dots, a_n \in V(X)$, then $V(X) \models \varphi(a_1, \dots, a_n)$ iff $V(Y) \models \varphi({}^*a_1, \dots, {}^*a_n)$.

Σ_0-formulas are first order formulas φ (in the Language $\{\in\}$) such that only quantifiers of the form $\forall x \in y$ and $\exists x \in y$ occur in φ.

Note that the transfer principle is similar to the well-known Łoś' theorem for ultraproduct-constructions. However, $V(Y)$ cannot be an ultrapower of $V(X)$, since otherwise Łoś' theorem would apply to *all* first order formulas, including the formula "for all natural numbers n, there is a decreasing $\in$-chain of length n". This sentence cannot be true in $(V(Y), \in)$, since $\in$ is well-founded and $V(Y)$ has nonstandard natural numbers. However, Y can be an ultrapower of X, and this specific kind of nonstandard extension is the most important one in our context:

Let $\mathcal{U}$ be a countably incomplete ultrafilter over a set I (i.e. there is a countable family A_n of elements of $\mathcal{U}$ such that $\bigcap A_n \notin \mathcal{U}$). Let Y be the ultrapower of X with respect to $\mathcal{U}$. (Without loss of generality we can assume that X and Y are base-sets). Then $V(Y)$ can be made a nonstandard extension of $V(X)$ in a way that the map * restricted to X is the usual ultrapower injection, i.e. ${}^*x$ is the equivalence class of the constant function c_x. Non-triviality follows from the fact that $\mathcal{U}$ is countably incomplete, and the transfer principle is proved similar to Łoś' theorem.

Nonstandard extension obtained in this way are called *bounded ultrapowers*.

In any nonstandard extension there are infinite numbers: The sets ${}^*\mathbb{N} \supsetneq \mathbb{N}$ and ${}^*\mathbb{R} \supsetneq \mathbb{R}$ are called the *nonstandard natural numbers* and the *nonstandard real numbers*. Sometimes we will call $\mathbb{N}$ and $\mathbb{R}$ the *standard natural numbers* and the *standard real numbers*, respectively, to emphasis the difference.

A nonstandard number r is called *finite* if there is a standard natural number n such that $|r| < n$ (in $V({}^*X)$). So a natural number is finite iff it is standard. A nonstandard number that is not finite is called infinite. A nonstandard real number r is called *infinitesimal* if $r = 0$ or $r \neq 0$ and $1/r$ is infinite.

Let r, $s \in {}^*\mathbb{R}$, then $r \approx s$ stands for "$r - s$ is infinitesimal". For each finite $r \in {}^*\mathbb{R}$ there is a unique real number r' such that $r' \approx r$. This r' is denoted by $\mathrm{std}(r)$

Similar notation will be introduced for elements of nonstandard normed vector spaces.

The following notions are central in nonstandard analysis:

$y \in V(Y)$ is called *standard* if there is a $x \in V(X)$ such that $y = {}^*x$.

$y \in V(Y)$ is called *internal* if there is a $x \in V(X)$ such that $y \in {}^*x$.

$y \in V(Y)$ is called *external* if y is not internal.

Note that for $y \in Y$, y is standard iff $y \in X$, so the notation is compatible with our use of standard natural or standard real numbers.

An example of an internal set that is not standard is the set of all nonstandard natural numbers less than some m, where m is infinite.

If A, B are internal (or standard), then so are $A \cup B$, $A \setminus B$, etc.

The set of all infinite natural numbers in not internal. This is a special case of the so-called spillover principle:

Theorem 1.3 (Spillover Principle) *Let A be internal.*

1. *A contains arbitrary large finite (i. e. standard) natural numbers iff A contains arbitrary small infinite natural numbers.*

2. *A contains arbitrary small positive standard reals iff A contains arbitrary large positive infinitesimals.*

It is important to note that A just has to be internal, it is not necessary that A is internal.

If A is a set in $V(X)$, then $P^{\text{fin}}(A)$, the set of all finite subsets of A, is in $V(X)$ as well, and is mapped to ${}^*P^{\text{fin}}(A) \in V(Y)$. If A was infinite, this set will contain more elements than just the sets ${}^*B$, where $B \subset A$ is finite. These new elements are called *hyperfinite* (note that they do not have to be finite). So a set B in $V(Y)$ is called *hyperfinite*, if there is some $A \in V(X)$ such that $B \in {}^*P^{\text{fin}}(A)$.

The example above, $\{n \in {}^*\mathbb{N} : n < m\}$, is an example of an infinite, hyperfinite, internal set. It is easy to see that all hyperfinite sets are internal.

For the examples 2.9 and 3.3, we use a nonstandard extension such that the standard reals are a subset of a hyperfinite set.

More general, we call a nonstandard extension $V(Y)$ an *enlargement* if for every $A \in V(X) \setminus X$ there is a hyperfinite $B \in V(Y)$ such that $\{{}^*a : a \in A\} \subset B$.

It is provable (using the Axiom of Choice, of course) that for any base-set X there is a index set I and an ultrafilter $\mathcal{U}$ over I such that the bounded ultrapower with respect to $\mathcal{U}$ is an enlargement (for details, see e. g. [1]). The outline of the proof is as follows:

1. For every κ, there are κ-good filters.
2. κ-good filters result in κ-saturated extensions.
3. Given X, let κ_X be the cardinality of $V(X)$. Then κ_X-saturated extensions are enlargements.

The only consequence of the concept of enlargement that we are going to use is the following

Lemma 1.4 *Let $V({}^*X)$ be a nonstandard enlargement of the universe. Then there is a (infinite) nonstandard natural number k such that $\sin(kt)$ is infinitesimal for all standard reals t.*

Proof. The set $\mathbb{R}$ of standard reals is a subset of some hyperfinite set A of nonstandard reals. It is a well–known (classical) fact that for every finite set of reals A, and every positive ε there is a natural number k such that for all $t \in A$, $\sin(kt) < \varepsilon$. So by transfer of the classical fact, setting ε infinitesimal, we get a $k \in {}^*\mathbb{N}$ such that for all $t \in \mathbb{R}$, $\sin(kt)$ is infinitesimal. $\square$

When we compare nonstandard constructions with classical constructions, we will assume that our nonstandard extension is a bounded ultrapower (generated by the same ultrafilter) – otherwise it isn't clear what the classical analogon should be. It should be noted that there are nonstandard universes which are not bounded ultrapowers: An ultrapower is always $\aleph_1$-saturated (since all filters are $\aleph_1$-good), but the union of a countable elementary chain of ultrapowers is not $\aleph_1$-saturated (see [1, p. 290, 4.4.29]). However, every extension can be seen as limit of ultrapowers (see [1]), and in practice, authors often restrict their attention to the case of ultrapowers (see e. g. [4, p. 88]).

2 Ultrapowers and nonstandard-hulls

2.1 Constructions without semigroups

Let E be a normed vector space, I an arbitrary set. We define $l^\infty(E)$ to be the space of all bounded E-valued I-sequences with the sup-norm (denoted by $\|\cdot\|_\infty$). Then $l^\infty(E)$ is a normed vector space. Note that I does not have to be countable, it can be of any cardinality.

Assume $\mathcal{U}$ is a filter over I (not necessarily an ultrafilter). As in [2], we define $c_\mathcal{U}$ to be the set of sequences $\langle f_i\rangle$ such that for all ε there is a set $J \subseteq I$ such that $J \in \mathcal{U}$ and $|f_i| < \varepsilon$ for all $i \in J$.

Lemma 2.1 *$|y + c_\mathcal{U}| := \inf\{|x| : x \in y + c_\mathcal{U}\}$ is a norm on $l^\infty(E)/c_\mathcal{U}$, and E is a Banach space if and only if $l^\infty(E)$ is a Banach space. If E is a Banach space, then $l^\infty(E)/c_\mathcal{U}$ is a Banach space as well.*

Proof. If E is a Banach space, then so is $l^\infty(E)$ (since Cauchy-sequences converge pointwise). Clearly, $c_\mathcal{U}$ is a closed subspace of $l^\infty(E)$. And any normed (and complete) vector space can be factored by a closed subspace, resulting in a normed (and complete, respectively) vector space. If $l^\infty(E)$ is complete, then clearly so is E (otherwise take any non-converging Cauchy-sequence f_i in E, and map it to the sequence $g_i = (f_i, 0, 0, \dots)$ in $l^\infty(E)$). □

Note that $l^\infty(E)/c_\mathcal{U}$ can be a Banach space although E (and therefore $l^\infty(E)$) is not, see the remark after Theorem 2.3.

The corresponding nonstandard construction is the following: Let $V({}^*X)$ be a nonstandard universe, and $E \subset X$ a normed vector space. Then ${}^*E$ is a nonstandard normed vector space, and a standard vector space (without canonical norm).

We define the finite part of ${}^*E$, denoted by fin^*E, to consist of all elements f of ${}^*E$ such that $|f|$ is finite. Then $f \approx g$ means that $|f - g|$ is infinitesimal. The infinitesimal part of ${}^*E$, denoted by E_0, consists of all f such that $|f|$ is infinitesimal, i. e. $f \approx 0$. Clearly, E_0 is a (standard) vector subspace of fin^*E. By $\widehat{E}$ we denote the quotient of fin^*E and E_0, and the canonical quotient map $\mathrm{fin}^*E \longrightarrow \widehat{E}$ is denoted by $\widehat{\cdot}$, i. e. $\widehat{f}$ is the quotient-class of f, and f is called a *representant of* $\widehat{f}$.

$\widehat{E}$ is a normed vector space with the norm $|\widehat{f}\,| := \mathrm{std}(|f|)$, where f is any representant of $\widehat{f}$ and $\mathrm{std}(r)$ is the standard part of a finite nonstandard real r. Of course, the vector space operations are defined by $\widehat{f} + \widehat{g} := \widehat{f + g}$ and $\alpha\widehat{f} := \widehat{\alpha f}$.

Lemma 2.2 *If $V({}^*X)$ is a bounded ultrapower, then $\widehat{E}$ is a Banach space.*

A proof can be found e. g. in [6, p. 59], noting that a bounded ultrapower is $\aleph_1$-good and therefore the extension will be $\aleph_1$-saturated.

A straightforward calculation proves that for an ultrafilter $\mathcal{U}$, the F-product and the nonstandard hull are the same:

Theorem 2.3 *Let $\mathcal{U}$ be an ultrafilter and $V({}^*X)$ the corresponding bounded ultrapower. Then $\iota : l^\infty(E)/c_\mathcal{U} \longrightarrow \widehat{E}$ defined by $\iota(f + c_\mathcal{U}) = \widehat{f}$ is an isomorphism.*

So if $\mathcal{U}$ is a countably incomplete ultrafilter, then $l^\infty(E)/c_\mathcal{U}$ is a Banach space (according to Lemma 1.1), regardless of whether E was a Banach space or not.

2.2 Classical constructions for semigroups

2.2.1 The maximal continuous subspace

Assume, $T(t)$ is a semigroup on E (not necessarily continuous), and assume M and ω are such that $\|T(t)\| \leq M\mathrm{e}^{\omega t}$ (for continuous semigroups, such M and ω always exist, according to Lemma 1.1).

Define $(E)^{T\text{-}\max} := \{f \in E : \lim_{t\to 0} |T(t)f - f| = 0\}$. A subspace F of E is called T-invariant, if $T(t)(f) \in F$ for all $f \in F$ and $t \in \mathbb{R}^+$.

Lemma 2.4 *$(E)^{T\text{-}\max}$ is a closed T-invariant subspace of E, and it is maximal in the family of subspaces F of E such that $T(t)$ restricted to F is continuous.*

Proof.

(a) $(E)^{T\text{-}\max}$ is closed. Assume f_i, $f \in E$ such that $f_n \to f$, and choose an $\varepsilon > 0$. Let n be such that $|f_n - f| < \min(\varepsilon/6M, \varepsilon/3)$. Then

$$|T(t)f - f| \leq |T(t)f - T(t)f_n| + |T(t)f_n - f_n| + |f_n - f|.$$

If $t < \delta$ (for a suitable δ), then $\|T(t)\| < 2M$ and $|T(t)f_n - f_n| < \varepsilon/3$ (since $f_n \in (E)^{T\text{-}\max}$), so

$$|T(t)f - f| \leq 2M\frac{\varepsilon}{6M} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \leq \varepsilon.$$

(b) $(E)^{T\text{-}\max}$ is invariant. Assume $f \in (E)^{T\text{-}\max}$ and $s \in \mathbb{R}_0^+$. Then

$$|T(t)(T(s)(f)) - T(s)(f)| = |T(s)(T(t)(f) - f)| \leq \|T(s)\|\,|T(t)f - f|.$$

(c) $(E)^{T\text{-}\max}$ is maximal. If T is continuous on F and $f \in F$, then by definition f is an element of $(E)^{T\text{-}\max}$. □

Remark This definition and lemma just isolate a part of the proof that is given e. g. in [2] for Corollary 2.5. This way we don't have to repeat the same argument again (e. g. for Lemma 2.8).

2.2.2 m^T

Let $T(t)$ be continuous, $\|T(t)\| \leq M\mathrm{e}^{\omega t}$, and let $\langle f_i \rangle$ be an element of $l^\infty(E)$. Define $\tilde{T}(t)(\langle f_i \rangle) := \langle T(t)f_i \rangle$. Since $T(t)$ is bounded, $\tilde{T}(t)(\langle f_i \rangle) \in l^\infty(E)$ for all $\langle f_i \rangle \in l^\infty(E)$ and $t \geq 0$. It is clear that $\tilde{T}$ defines a semigroup on $l^\infty(E)$. Also, $\|\tilde{T}(t)\| \leq M\mathrm{e}^{\omega t}$. But in general $\tilde{T}$ will not be continuous on $l^\infty(E)$: Let T be the translation semigroup on C^{b} as in Example 1.2, $I = \omega$, $\mathcal{U}$ a free ultrafilter over ω, and $f_k(x) = \sin(kx) \in C^{\mathrm{b}}$. Then $f = \langle f_k \rangle \in l^\infty(E)$, but $\tilde{T}(t)f - f$ does not converge.

One can show rather easily that $\tilde{T}$ is continuous on $l^\infty(E)$ if and only if T is uniformly continuous on E (assuming of course that the index set I is infinite).

We define m^T to be the maximal continuous subspace of $l^\infty(E)$ with respect to T.

Corollary 2.5 *m^T is a closed, $\tilde{T}$-invariant subspace of $l^\infty(E)$.*

2.2.3 The F-product of a semigroup

As we have seen, if T is a semigroup defined on E, under certain assumptions T can be extended to a (not necessarily continuous) semigroup $\tilde{T}$ on $l^\infty(E)$. Assume $\langle f_i \rangle \in c_{\mathcal{U}}$ (i. e. for all $\varepsilon \in \mathbb{R}^+$, $|f_i| < \varepsilon$ holds on a filter-set). $\tilde{T}(t)(\langle f_i \rangle) \in c_{\mathcal{U}}$ for all $t \geq 0$, since $\tilde{T}(t)$ is a bounded operator. This means that $c_{\mathcal{U}}$ is $\tilde{T}$-invariant, and that we can define $\tilde{T}$ on $l^\infty(E)/c_{\mathcal{U}}$. Of course, $\tilde{T}$ is a semigroup on $l^\infty(E)/c_{\mathcal{U}}$ (and generally not continuous).

$c_{\mathcal{U}}$ is not a subspace of m^T. (For example, any $\langle f_i \rangle$ such that $f_i = 0$ on a filter-set is in $c_{\mathcal{U}}$, but not necessarily in m^T.) However, it is easy to see that $c_{\mathcal{U}} \cap m^T$ is a closed subset of m^T, and $m^T/(c_{\mathcal{U}} \cap m^T)$ is a subspace of $l^\infty(E)/c_{\mathcal{U}}$.

To be more exact:

Lemma 2.6 *$\phi : m^T/(c_{\mathcal{U}} \cap m^T) \longrightarrow l^\infty(E)/c_{\mathcal{U}}$ defined by $\phi(\langle f_i \rangle + c_{\mathcal{U}} \cap m^T) = \langle f_i \rangle + c_{\mathcal{U}}$ is a (well-defined) injective isometry.*

P r o o f. See [5, p. 163]. □

Lemma 2.7 *$m^T/(c_{\mathcal{U}} \cap m^T)$ is a closed, $\tilde{T}$-invariant subspace of $l^\infty(E)/c_{\mathcal{U}}$, and the restriction of the semigrpoup $\tilde{T}$ to $m^T/(c_{\mathcal{U}} \cap m^T)$ is continuous.*

P r o o f. The quotient $m^T/(c_{\mathcal{U}} \cap m^T)$ of the Banach space m^T is a Banach space, and so is its isometric image. The rest is clear. □

In general, $m^T/(c_{\mathcal{U}} \cap m^T)$ is *not* the maximal continuous subspace of $l^\infty(E)/c_{\mathcal{U}}$. This is shown in Example 2.9 (using that $m^T/(c_{\mathcal{U}} \cap m^T)$ corresponds to $\widehat{E_T}$, see Lemma 2.11).

2.3 Nonstandard constructions for semigroups

Let $V({}^*X)$ be a nonstandard universe and $E \subset X$ a Banach space.

Assume $A : E \longrightarrow E$ is a continuous linear operator, and f an element of fin^*E and g is an element of E_0 (the finite and infinitesimal parts of the nonstandard vector space ${}^*E$, respectively). Then ${}^*Af$ (and ${}^*Ag$) are again finite (and infinitesimal, respectively). Therefore A defines a bounded linear operator $\widehat{A} : \widehat{E} \longrightarrow \widehat{E}$ by $\widehat{A}(\widehat{f}\,) = \widehat{Af}$. (If A is not bounded, then generally $\widehat{A}$ cannot be defined in a canonical way.)

If T is a semigroup, then for all positive reals t, $T(t)$ is continuous, and $\widehat{T}(t) := \widehat{T(t)}$ is a semigroup on $\widehat{E}$.

If $\|T(t)\| < M\mathrm{e}^{\omega t}$ (this is always the case if T is continuous), then an alternative way to define $\widehat{T}$ is the following: Let ${}^*T$ be a nonstandard semigroup on ${}^*E$. If t is a positive, finite nonstandard real, then $\|{}^*T(t)\| < M\mathrm{e}^{\omega t}$, and so we get: If $f \in {}^*E$ is finite (or infinitesimal), then so is ${}^*T(t)f$. Thus $\widehat{T}(t) : \widehat{E} \longrightarrow \widehat{E}$ is well-defined by $\widehat{T}(t)(\widehat{f}\,) = {}^*\widehat{T(t)(f)}$, even for finite nonstandard t. If t is standard, then the two definitions are equivalent.

Assume T is continuous, i. e. for all f in E and all $\varepsilon > 0$ there is a $\delta > 0$ such that $|T(t)f - f| < \varepsilon$ for all $t < \delta$. We apply the transfer principle to this sentence, and get: For all $f \in {}^*E$ and all $\varepsilon \in {}^*\mathbb{R}^+$, there is $\delta \in {}^*\mathbb{R}^+$ such that $|{}^*T(t)f - f| < \varepsilon$ for all positive reals $t < \delta$. However, if $\varepsilon > 0$ is a standard real, then it is not guaranteed that $\delta > 0$ can be chosen to be standard as well (δ could be infinitesimal). We define two subspaces of fin^*E (the finite part of ${}^*E$): E_T consists of all the vectors f with the following property: If $\varepsilon > 0$ is standard, then there is a standard $\delta > 0$ such that $|{}^*T(t)f - f| < \varepsilon$ for all nonstandard $t < \delta$. $E_{\max}$ consists of all f satisfying the same condition for standard $t < \delta$ only. I. e. a finite $f \in {}^*E$ is in $E_{\max}$ iff for all $\varepsilon \in \mathbb{R}^+$ there is $\delta \in \mathbb{R}^+$ such that $|{}^*T(t)f - f| < \varepsilon$ for all positive standard reals $t < \delta$.

It is easy to see that a finite $f \in {}^*E$ is in E_T iff for all positive infinitesimal t, $|{}^*T(t)f - f|$ is infinitesimal.

Clearly, E_0 is a subspace of E_T, which is in turn a subspace of $E_{\max}$. So, if $f \in E_T$ (or $E_{\max}$) and $f \approx g$, then $g \in E_T$ (or $E_{\max}$, respectively), and we can define the quotient spaces $\widehat{E_T} = E_T/E_0$ and $\widehat{E}_{\max} = E_{\max}/E_0$. Remember that (other than fin^*E), E_T is a (canonically) normed vector space.

Lemma 2.8

1. *$\widehat{E}_{\max}$ is the maximal continuous subspace of $\widehat{E}$ with respect to $\widehat{T}$.*
2. *$\widehat{E}_{\max}$ is a closed, $\widehat{T}$-invariant subspace of $\widehat{E}$.*
3. *$\widehat{E_T}$ is a closed, $\widehat{T}$-invariant subspace of $\widehat{E}_{\max}$.*

P r o o f.

1. Assume, $\lim_{t\to 0} |\widehat{T}(t)\widehat{f} - \widehat{f}| = 0$, and fix $\varepsilon \in \mathbb{R}^+$. Then there exists $\delta \in \mathbb{R}^+$ such that for all positive reals $t < \delta$, $|\widehat{T}(t)\widehat{f} - \widehat{f}| < \varepsilon/2$. Assume that f is a representant of the quotient $\widehat{f}$, and that $t < \delta$. Then $\mathrm{std}(|{}^*T(t)f - f|) < \varepsilon/2$, and therefore $|{}^*T(t)f - f| < \varepsilon$ in ${}^*\mathbb{R}$, so $f \in E_{\max}$ and $\widehat{f} \in \widehat{E}_{\max}$.

2. follows from 1. and Lemma 2.4.

3. Assume that $\widehat{f}_n \in \widehat{E_T}$ and $\widehat{f}_n \to \widehat{f}$ in E_T. Let f_n and f be representants of $\widehat{f}_n$ and $\widehat{f}$, respectively. Assume h is a positive infinitesimal. Then $|T(h)f - f| \leq |T(h)f - T(h)f_n| + |T(h)f_n - f_n| + |f_n - f|$, which is smaller than every positive standard ε, as $|f - f_n|$ gets arbitrary small, $|T(h)f - f| \leq \|T(h)\|\,|f - f_n|$, and $|T(h)f_n - f_n|$ is infinitesimal. To show the invariance, assume that $\widehat{f} \in \widehat{E_T}$, and let $f \in E_T$ be a representant. Let $t \in bRp$ and let h be a positive infinitesimal. Then $|T(h)T(t)f - T(t)f| \leq \|T(t)\|\,|T(h)f - f|$, which is infinitesimal. So $T(t)f \in E_T$, and therefore $\widehat{T}(t)\widehat{f} \in \widehat{E_T}$. □

Remark In [8], an alternative way to construct E_T is presented. Theorem 4.6.1 there corresponds to our Theorem 3.4.

In general, $\widehat{E_T}$ is a proper subset of $\widehat{E}_{\max}$ (or equivalently: E_T is a proper subset of $E_{\max}$). To see this, we bring an example similar to [5, p. 165] or [7, p. 209]:

Example 2.9 Let T be the translation semigroup on C^{b} (as in Example 1.2), $V({}^*X)$ an enlargement, k such that $\sin(kt)$ is infinitesimal for all $t \in \mathbb{R}$ (see Lemma 1.4). Define the element f of ${}^*C^{\mathrm{b}}$ by $f(x) = \sin(kx)$. Then $|f| = 1$, and f is an element of $E_{\max}$, but not of E_T.

Proof. To see that $f \in E_{\max}$, it is enough to show that for all positive (standard) reals t, $|T(t)f - f|$ is infinitesimal. For any standard reals t, $\sin(k(x+t)) - \sin(kx) = \sin(kx)(\cos(kt) - 1) + \sin(kt)\cos(kx)$. $\sin(kx)$ and $\cos(kx)$ are finite, and $\sin(kt)$ and $(\cos(kt) - 1) = \sqrt{1 - \sin^2(kt)} - 1$ are infinitesimal. Therefore $|T(t)f - f| = \|\sin(k(x+t)) - \sin(kx)\|_\infty$ is infinitesimal. To see that $f \notin E_T$, it is enough to note that $h := 1/k$ is infinitesimal, but $|T(h)f - f| = \|\sin(k(x+h)) - \sin(kx)\|_\infty = \|\sin(x+1) - \sin(x)\|_\infty$ is not. □

2.4 Relation of classical and nonstandard constructions

Assume $V({}^*X)$ is the bounded ultrapower of an ultrafilter $\mathcal{U}$ over I. Then the isomorphism ι of Theorem 2.3 allows us to identify $l^\infty(E)/c_{\mathcal{U}}$ and $\widehat{E}$. We can also consider $m^T/(c_{\mathcal{U}} \cap m^T)$ to be a subspace of $l^\infty(E)/c_{\mathcal{U}}$ (via the injective isometry ϕ of Lemma 2.6).

Lemma 2.10 *$\widehat{E}_{\max}$ is the maximal continuous subspace of $l^\infty(E)/c_{\mathcal{U}}$ with respect to $\tilde{T}$*

(More formally, one would have to write $\widehat{E}_{\max} = \iota((l^\infty(E)/c_{\mathcal{U}})^{\tilde{T}\text{-}\max})$.)

Proof. This follows from Lemma 2.8 and the fact that $\iota(\tilde{T}(t)f) = \widehat{T}(t)(\iota(f))$ for all $f \in l^\infty(E)/c_{\mathcal{U}}$. □

Lemma 2.11 $m^T/(c_{\mathcal{U}} \cap m^T) \subset \widehat{E_T}$.

(Again, more formally this should be written as $\iota(\phi(m^T/(c_{\mathcal{U}} \cap m^T))) \subset \widehat{E_T}$.)

Proof. By definition, $\langle f_i \rangle \in m^T$ iff for all reals $\varepsilon > 0$ there is a real $\delta > 0$ such that $\|T(t)f_i - f_i\|_\infty < \varepsilon$ for all reals $t < \delta$. Let $\widehat{f} \in \widehat{E}$ correspond to $\langle f_i \rangle$. Assume that h is an infinitesimal nonstandard real. We want to show that for an arbitrary fixed positive real ε, $|{}^*T(h)\widehat{f} - \widehat{f}| < \varepsilon$. Let δ be the real corresponding to ε as above. h corresponds to a sequence of reals $\langle h_i \rangle$ $(i \in I)$ such that $U = \{i \in I : h_i < \delta\}$ is in the ultrafilter. ${}^*T(h)\widehat{f} - \widehat{f}$ corresponds to the sequence $T(h_i)f_i - f_i$. If i is in U, then $|T(h_i)f_i - f_i| < \varepsilon$, and U is a filter-set, therefore $|{}^*T(h)\widehat{f} - \widehat{f}| < \varepsilon$. □

It is not immediately clear whether $m^T/(c_{\mathcal{U}} \cap m^T)$ is always *identical* to $\widehat{E_T}$.

One could construct a counterexample under the following assumption:

Assumption 2.12 Fix ω, M, η, ε_1, $\varepsilon_2, \ldots$, δ_1, $\delta_2, \ldots$. Assume that for all n, $m \in \mathbb{N}^+$ there is a Banach space E^n_m, a continuous semigroup T^n_m with $T^n_m(t) < M\mathrm{e}^{\omega t}$ and an $f^n_m \in E^n_m$ such that

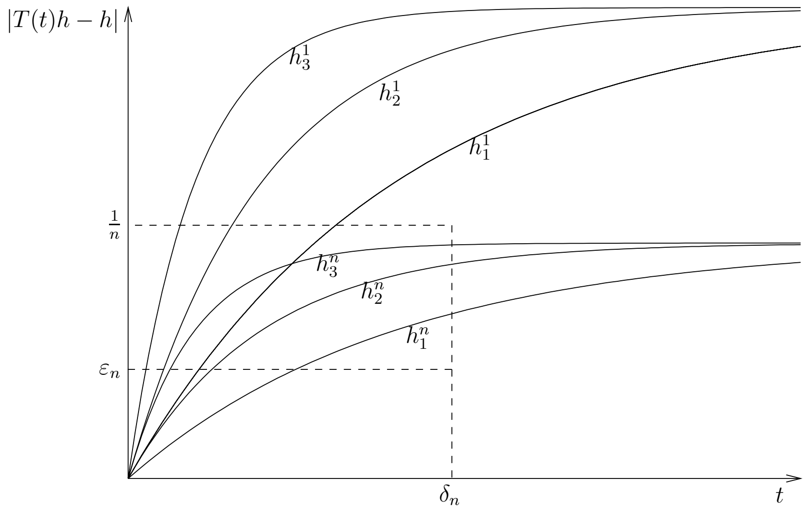
(a) $|f^n_m| = 1$,

(b) $|T^n_m(t)f^n_m - f^n_m| < 1/n$ for all $1 \leq i \leq n$ and $t < \delta_i$,

(c) for all $\bar{f} \in E^n_m$ such that $|\bar{f} - f^n_m| < \eta$ there exists $t < 1/m$ such that $|T^n_m(t)\bar{f} - \bar{f}| > \varepsilon_n$.

If this assumption holds, then let the index set I be $\mathbb{N}^+ \times \mathbb{N}^+$, and define $\bar{E} = l^\infty(E^i_j)$ with the sup-norm (i. e. $\langle f_{i,j}\rangle \in \bar{E}$ iff $f_{i,j} \in E^i_j$ and $|f_{i,j}|$ is bounded). $T(t)\langle g_{i,j}\rangle := \langle T^i_j(t) g_{i,j}\rangle$ is a semigroup on $\bar{E}$. Let E be the maximal continuous subspace of $\bar{E}$, and define $h^n_m = \langle h^n_m(i,j)\rangle \in E$ by

$$h^n_m(i,j) = \begin{cases} f^n_m & \text{if } i = n \text{ and } j = m,\\ 0 & \text{otherwise.}\end{cases}$$



$|T(t)h - h|$ for different h's

Assume that $\mathcal{U}$ is an ultrafilter over $I = \mathbb{N}^+ \times \mathbb{N}^+$, and that there is a partition $\{\sigma_{i,j}\}$ $(i, j \in \mathbb{N})$ of I such that $\bigcup_{i>n,\, j>f(i)} \sigma_{i,j} \in \mathcal{U}$ for all $n \in \mathbb{N}$ and all functions $f : \mathbb{N} \longrightarrow \mathbb{N}$, (it is easy to see that such a countably incomplete filter exists, e. g. apply a suitable filter-basis to [6, Theorem A.4]). Then define $k = \langle k_{i,j}\rangle \in \bar{E}$ by $k_{i,j} = h^n_m$, where n, m is the (unique) pair of natural numbers such that (i, j) is an element of $\sigma_{n,m}$. Now consider the nonstandard extension defined by $\mathcal{U}$, and interpret k as nonstandard element of ${}^*E$. Clearly, $|k| = 1$, since $|k_{i,j}| = 1$ for all (i, j). For all natural numbers n, $0 < t < \varepsilon_n$ and $(i, j) \in \bigcup_{r>n,\, s\in\omega} \sigma_{r,s}$ we have $|T(t)k_{i,j} - k_{i,j}| < 1/n$. So, if $\langle r_{i,j}\rangle$ is infinitesimal, then so is $|{}^*T(r)k - k|$, i. e. $k \in E_T$. Assume that the equivalence class of k is in $m^T/(c_{\mathcal{U}} \cap m^T)$. Pick a representant $\langle \bar{k}_{i,j}\rangle \in m^T$, and $U \in \mathcal{U}$ such that $|\bar{k}_{i,j} - k_{i,j}| < \eta$ for all $(i, j) \in U$. Since $U \in \mathcal{U}$, there is n_0 such that for all m there is m' such that $\sigma_{n_0,m'} \cap U \neq \emptyset$. Since $\langle \bar{k}_i\rangle \in m^T$, there is $m_0 \in \mathbb{N}$ such that $|T(t)\bar{k}_i - \bar{k}_i| < \delta_{n_0}$ for all $(i, j) \in I$ and $0 < t < 1/m_0$. If $m_0{}' > m_0$ and $(i_0, j_0) \in \sigma_{n_0,m_0{}'} \cap U$, then there is $t \in \mathbb{R}^+$ such that $t < 1/m_0{}' < 1/m_0$ and $|T(t)\bar{k}_i - \bar{k}_i| \geq \delta_{n_0}$, a contradiction.

However, our investigation of the nonstandard generator of a semigroup will show that $m^T/(c_{\mathcal{U}} \cap m^T) = \widehat{E_T}$.

3 The Generator

3.1 The generator in ${}^*E$

If T is a continuous semigroup, then ${}^*T(t)$ maps the finite (or infinitesimal) elements of ${}^*E$ to finite (or infinitesimal, respectively) elements for all finite $t \geq 0$. That is not true for an (unbounded) linear map A, as the following trivial example shows:

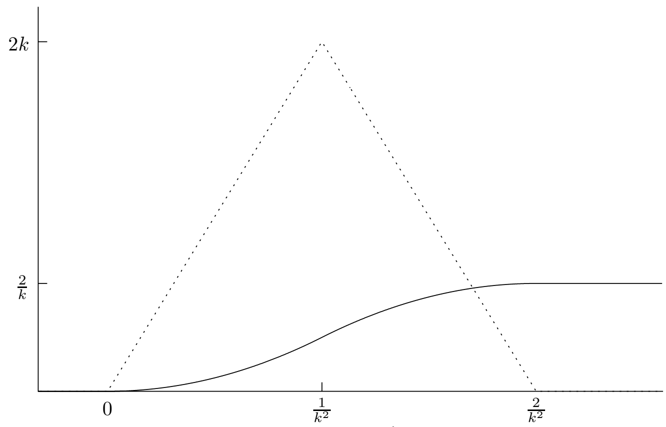
Example 3.1 Let T be the translation semigroup on C^{b} (as in Example 1.2), $V({}^*X)$ a nonstandard extension, k an infinite natural number. Then there is an infinitesimal $f_1 \in {}^*E$ such that $f_1 \in {}^*D(A)$ and ${}^*A(f)$ is infinite.

P r o o f . Define f_1 by case distinction:

$$f_1 = \begin{cases} 0 & \text{if } x < 0, \\ k^3x^2 & \text{if } 0 \le x < 1/k^2, \\ -k^3x^2 + 4kx - 2/k & \text{if } 1/k^2 \le x < 2/k^2, \\ 2/k & \text{if } 2/k^2 \le x. \end{cases}$$

Then

$$^*Af_1 = \begin{cases} 0 & \text{if } x < 0, \\ 2k^3x & \text{if } 0 \le x < 1/k^2, \\ 2k^3x + 4k & \text{if } 1/k^2 \le x < 2/k^2, \\ 0 & \text{if } 2/k^2 \le x. \end{cases}$$ □



Example 3.1. f_1 and $f_1' = {}^*A(f_1)$

Let $V({}^*X)$ be a nonstandard universe. Then the subspace $D(A)$ of E is mapped to a subspace ${}^*D(A)$ of ${}^*E$.

Lemma 3.2 *Assume f and g are elements of ${}^*D(A)$.*

1. *If f and ${}^*A(f)$ are both finite, then $f \in E_T$.*
2. *If f is finite and ${}^*A(f) \in E_T$, then $(1/h)({}^*T(h)f - f) \approx {}^*A(f)$ for all positive infinitesimal reals h.*
3. *If $k \in {}^*E$ and $(1/h)({}^*T(h)f - f) \approx k$ for all positive infinitesimal reals h, then $k \approx {}^*A(f)$.*
4. *If f, g, ${}^*A(f)$ and ${}^*A(g)$ all are in E_T, and $f \approx g$, then ${}^*A(f) \approx {}^*A(g)$.*

P r o o f.

1. By transfer of Lemma 1.1.4 we get:

$$|{}^*T(t)f - f| \le t \cdot \sup_{s \le t} \|{}^*T(s)\| \cdot |{}^*A(f)| \le tM(\mathrm{e}^{\omega t} + 1) \cdot |{}^*A(f)|,$$

which is infinitesimal for infinitesimal t.

2. For every standard $\varepsilon > 0$ and infinitesimal s, we have $|{}^*T(s){}^*A(f) - {}^*A(f)| < \varepsilon$. Therefore transfer of Lemma 1.1.5 implies that $|(1/h)({}^*T(h)f - f) - {}^*A(f)| \le \varepsilon$ for infinitesimal h. Since ε was arbitrary, $|(1/h)({}^*T(h)f - f) - {}^*A(f)|$ is infinitesimal.

3. By transfer of the definitions of A and $D(A)$, we get the following: For all positive nonstandard reals ε there is a positive nonstandard real δ such that $|(1/h)({}^*T(h)f - f) - {}^*Af| < \varepsilon$ for all $h \le \delta$. Let ε be a fixed positive infinitesimal, and choose an infinitesimal $h \le \delta$ for the appropriate δ. Then $(1/h)({}^*T(h)f - f) \approx {}^*Af$, and by assumption $(1/h)({}^*T(h)f - f) \approx k$, therefore $k \approx {}^*A$.

4. Since ${}^*Af \in E_T$, $(1/h)({}^*T(h)f - f) \approx {}^*Af$ for all infinitesimal h, and the same applies to g and ${}^*Ag$. So we have

$$|{}^*Af - {}^*Ag| \le \left|{}^*Af - \frac{{}^*T(t)f - f}{t}\right| + \left|\frac{{}^*T(t)f - f}{t} - \frac{{}^*T(t)g - g}{t}\right| + \left|\frac{{}^*T(t)g - g}{t} - {}^*Ag\right| \le \varepsilon.$$

So we just have to show that $|(1/t)({}^*T(t)f - f) - (1/t)({}^*T(t)g - g)| = |(1/h)({}^*T(t)(f - g) - (f - g))|$ is infinitesimal for infinitesimal h. Transfer of Lemma 1.1.4 shows that

$$\Big|\frac{{}^*T(t)(k) - (k)}{h}\Big| \leq \sup_{s\leq h}\|{}^*T(s)\| \cdot |{}^*A(k)|$$

for all positive h and $k \in {}^*D(A)$. Now apply this to $k = f - g$. □

As we have already seen in Example 3.1, $f \approx g$ does not imply ${}^*A(f) \approx {}^*A(g)$ (set $f := f_1$, $g := 0$). Also, it is not enough to assume that f, g, ${}^*A(f)$ and ${}^*A(g)$ are all in $E_{\max} \cap {}^*D(A)$ for the implication to hold:

Example 3.3 Assume T, k and f are as in Example 2.9. Define $g(x) := \cos(kx)/k$. Then $g \approx 0$. Clearly $g \in E_{\max}$ (it is even in E_T), and ${}^*A(g) = f$, i. e. ${}^*A(g) \in E_{\max}$, but ${}^*A(g)$ is not infinitesimal.

The same example shows that $g \in E_T$ does not imply ${}^*A(g) \in E_T$.

3.2 The generator in $\widehat{E_T}$

As we have already seen in Lemma 2.8, $\widehat{T}$ is a continuous semigroup on $\widehat{E_T}$ (since $\widehat{E_T}$ is a subspace of $\widehat{E}_{\max}$). The generator is called $\widehat{A}$.

Theorem 3.4 *$\widehat{f}$ is element of $D(\widehat{A})$ if and only if there is a representant f of $\widehat{f}$ such that f is in D^*A and ${}^*Af$ is in E_T. In this case, $\widehat{A}(\widehat{f}) = \widehat{{}^*A(f)}$.*

Proof. Assume $\widehat{f}$ has a representant f as in the theorem. We must show that $\widehat{f} \in D(\widehat{A})$ and $\widehat{A}(\widehat{f}) = \widehat{{}^*A(f)}$, i. e. we fix a (standard) real $\varepsilon > 0$ and have to find $\delta > 0$ such that $|(1/h)(\widehat{T}(h)\widehat{f} - \widehat{f}) - \widehat{{}^*A(f)}| < \varepsilon$ for all $h < \delta$. By Lemma 3.2.2, $n(h) := |(1/h)({}^*T(h)f - f) - {}^*A(f)|$ is infinitesimal for infinitesimal h. Note that n is an internal function, since it is element of ${}^*\{h \longmapsto |(1/h)(T(h)f - f) - A(f)| : f \in D(A)\}$. Therefore the set $X = \{h : n(h) \geq \varepsilon\}$ is internal as well, and we can apply the spillover principle 1.3: Assume toward a contradiction that for all standard $\delta > 0$ there was a standard $h < \delta$ such that $n(h) \geq \varepsilon$, i. e. there are arbitrary small standard reals in X. Then there is a infinitesimal real in X as well, a contradiction. Therefore $\widehat{f} \in D(\widehat{A})$.

Assume on the other hand that $\widehat{f}$, $\widehat{g}$ are elements of $\widehat{E_T}$ such that $\widehat{A}(\widehat{f}) = \widehat{g}$. We want to show that $f \in D^*A$ and ${}^*Af \in E_T$ for some representant f of $\widehat{f}$. Let $\bar{f}$, $\bar{g}$ be arbitrary representants of f and g, respectively (according to Example 3.1, we cannot hope that $\bar{f}$ already has the required properties). For all nonstandard natural numbers m define c_m to be the set of nonstandard natural numbers $n > m$ such that

$$\Big|\frac{{}^*T(1/n)f' - f'}{1/n} - g'\Big| < 1/m.$$

For all standard natural numbers m, the set c_m is not empty, since $\widehat{A}(\widehat{f}) = \widehat{g}$. The set $X = \{m : c_m \neq \emptyset\}$ is internal and contains all standard natural number, therefore it contains an infinite natural number as well (according to the spillover principle). Let m be such an infinite number, and n an element of c_m. Now we apply the transfer of Lemma 1.1.6, setting $h := 1/n$. So we get f in $D({}^*A)$ such that

$$|\bar{f} - f| \leq \sup_{s\leq h}(|{}^*T(s)\bar{f} - \bar{f}|),$$

which is infinitesimal, since $\bar{f} \in E_T$ and s is infinitesimal. Therefore f is a representant of $\widehat{f}$ as well. Also, ${}^*A(f) = (1/h)({}^*T(h)\bar{f} - \bar{f})$, so $|{}^*A(f) - {}^*A(\bar{g})| \leq 1/m$ is infinitesimal, therefore ${}^*A(f) \in E_T$ as well. □

3.3 The equivalence of F-product and nonstandard hull

Let $V({}^*X)$ be a bounded ultrapower, T continuous with generator A.

Lemma 3.5 $\widehat{E_T} = m^T/(c_{\mathcal{U}} \cap m^T)$.

P r o o f. Assume $m^T/(c_{\mathcal{U}} \cap m^T) \subsetneq \widehat{E_T}$. Since $m^T/(c_{\mathcal{U}} \cap m^T)$ is a closed subspace of $\widehat{E_T}$ and $D(\widehat{A})$ is dense in $\widehat{E_T}$, there must be an $\widehat{f}$ which is element of $D(\widehat{A})$ but not of $m^T/(c_{\mathcal{U}} \cap m^T)$. We chose a representant f of $\widehat{f}$ such that $f \in D({}^*A)$ and ${}^*Af \in E_T$. Since f is an element of the ultraproduct, f is an equivalence class of a sequence f_i $(i \in I)$ such that for all i, $f_i \in D(A)$ and $\langle A(f_i) \rangle \in l^\infty(E)$. But then

$$|\tilde{T}(h)f - f| = \sup_{i \in I}(|T(h)f_i - f_i|) \leq h \cdot \sup_{i \in I}(|A(f_i)|) \cdot \sup_{s \leq h}(\|T(s)\|)$$

(according to Lemma 1.1.4), so $f \in m^T$, a contradiction. □

Combining this with Lemmas 2.11 and 2.8, using the isometry ι of Lemma 2.3, we get:

Theorem 3.6 *$\widehat{E}$ is isomorphic to $l^\infty(E)/c_{\mathcal{U}}$, and $\widehat{E_T}$ is isomorphic to $m^T/(c_{\mathcal{U}} \cap m^T)$.*

So we get the following picture. Here p are the canonical projections from a space to its quotient, and $\widehat{\cdot}$ maps a finite element f of ${}^*E$ to the equivalence class $\widehat{f}$. The maps labeled with p and $\widehat{\cdot}$ are surjective, the ones labeled with $\cong$ are isometries. $>$ denotes the subspace relation, for $>^1$ the isometry ϕ of Lemma 2.6 is used:

$$\begin{array}{ccccc}
l^\infty(E) & & > & & m^T \\
\downarrow p & & & & \downarrow p \\
l^\infty(E)/c_{\mathcal{U}} & > & (l^\infty(E)/c_{\mathcal{U}})^{T\text{-}\max} & >^1 & m^T/(c_{\mathcal{U}} \cap m^T) \\
\cong\updownarrow \iota & & \cong\updownarrow \iota & & \cong\updownarrow \iota \\
\widehat{E} & > & \widehat{E}_{\max} & > & \widehat{E_T} \\
\uparrow \widehat{\cdot} & & \uparrow \widehat{\cdot} & & \uparrow \widehat{\cdot} \\
\mathrm{fin}^*E & > & E_{\max} & > & E_T
\end{array}$$

Remark The entries of the first three rows are Banach spaces, the last row consists of vector spaces.

So we know that Assumption 2.12 cannot hold. This proves

Corollary 3.7 *Assume that T_m^n are contraction semigroups on E (i. e. for all $t \in \mathbb{R}^+$, $\|T(t)\| \leq 1$), and $f_m^n \in E$ are such that $|f_m^n| = 1$ and $|T_m^n(t)f_m^n - f_m^n| < 1/n$ for all $t < 1/n$. Then there are $f' \in E$ and n, m, such that $|f' - f_m^n| < 1/2$ and $|T_m^n(t)f' - f'| < 1/n$ for all $t < 1/m$.*

References

[1] C. C. Chang and H. J. Keisler, Model Theory, 3. ed. (North-Holland, Amsterdam 1990).
[2] R. Derndinger, Über das Spektrum positiver Generatoren. Mathematische Z. **172**, 281 – 293 (1980).
[3] K.-J. Engel and R. Nagel, One-parameter Semigroups for Linear Evolution Equations (Springer, New York 2000).
[4] A. E. Hurd and P. A. Loeb, An Introduction to Nonstandard Real Analysis (Academic Press, Orlando 1985).
[5] A. Krupa, On various generalizations of the notion of an $\mathcal{F}$-power to the case of unbounded operators. Bull. Pol. Acad. Sci. Math. **38**, 159 – 166 (1990).
[6] T. Lindstrøm, An invitation to nonstandard analysis. In: Nonstandard Analysis and its Applications (N. Cutland, ed.), pp. 1 – 105 (Cambridge Univ. Press, Cambridge 1988).
[7] M. P. H. Wolff, Spectral theory of group representations and their nonstandard hull. Israel J. Math. **48**, 205 – 224 (1984).
[8] M. P. H. Wolff, Functional analysis. In: Nonstandard Analysis for the Working Mathematician (P. A. Loeb and M. P. H. Wolff, eds.), pp. 235 – 257 (Kluwer, Dordrecht 2000).