Adding the evasion number to Cichoń's Maximum

Jakob Kellner

TU WIEN

2021-09-16

joint work with M. Goldstern, D. Mejía and S. Shelah

supported by FWF Der Wissenschaftsfonds.

Evasion and prediction

Definition (Blass 1994 [6], Brendle 1995 [7])

- A predictor is a pair (D, π) with $D \in [\omega]^{\aleph_0}$, $\mathrm{dom}(\pi) = D$, and $\pi(n) : \omega^{<n} \to \omega$ for $n \in D$.
- (D, π) predicts ("covers") an $f \in \omega^\omega$, if $\pi(n)(f \restriction n) = f(n)$ for all but finitely many $n \in D$. (An F_σ relation.)
- $\mathfrak{e}$, the evasion number, is the (un)bounding number of this covering relation: The smallest size of a set of functions not covered by a single predictor.
- $\mathfrak{e}^\perp$ the dominating number: The smallest size of a set of predictors covering all functions.
- $\mathfrak{e}_2$ is the analoguous characteristic for prediction in 2^ω instead of ω^ω.

Note that $\mathfrak{e} \leq \mathfrak{e}_2$ (as it is easier to predict for 2^ω).

Evasion and prediction (ctd.)

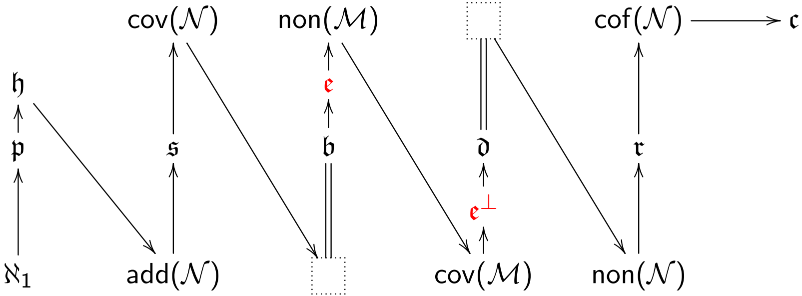
A natural forcing to increase $\mathfrak{e}$ is the following:

Definition (Brendle, Shelah 1996 [9]; following Brendle 1995 [7])

- x is a "single prediction on n" if $x \in \omega^{n+1}$;
  the intention is that $x \restriction n$ is predicted to be extended by $x(n)$.
- A condition of $p \in \mathbb{Br}$ consists of
  - $n_p \in \omega$,
  - $d_p \in 2^n$, ($d_p^{-1}\{1\}$ is meant to approx. $\underset{\sim}{D}$),
  - a finite set P_d of single predictions on ℓ for $\ell \in d^{-1}\{1\}$.
  - a finite set F_d of functions, all splitting below n_p.
- A stronger condition q has larger n, F, P, a d extending the old one, and for all new $\ell \in d_q^{-1}\{1\}$ and all old $f \in F_p$, $f \restriction (\ell + 1)$ has to be in the new P_q.

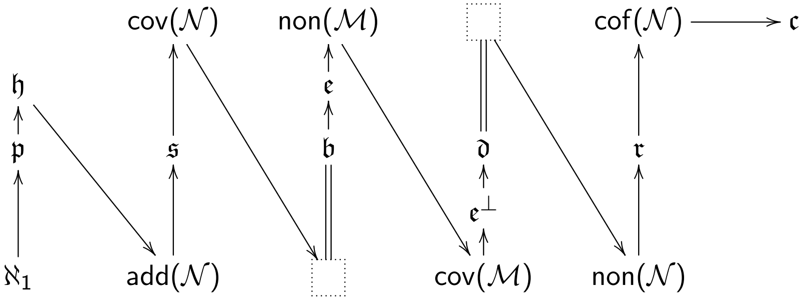
$\mathbb{Br}$ is σ-centred.
(The centering parameter of p is $(n_p, d_p, P_p, \{f \restriction n_p : \ f \in F_p\})$.)

The result



- (Known [1, 3, 4, 5]) It is consistent that the characteristics in Cichoń's diagram, plus $\mathfrak{p}$, $\mathfrak{h}$, $\mathfrak{s}$, $\mathfrak{r}$ are ordered as in the diagram.
- $\mathfrak{s}$ can be anywhere (regular) between $\mathfrak{p}$ and $\mathfrak{b}$, with $\mathfrak{r}$ dual.
- Instead of $\mathfrak{s}$, $\mathfrak{r}$, we can get $\mathfrak{m}$ (between $\aleph_1$ and $\mathfrak{p}$.)
- New (in preparation): We can add the evasion number $\mathfrak{e}$ and its dual $\mathfrak{e}^\perp$ as indicated.

A mockery of an overview



All these proofs, old and new, have basically the following steps:

Step 1 Left side only, with $\mathrm{cov}(\mathcal{M}) = \mathfrak{c} > \mathrm{non}(\mathcal{M})$ (but with "strong witnesses" for the Cichoń characteristics).

Step 2 Boolean Ultrapowers (using LCs) or intersection with sequences of submodels (without LCs).

Questions

Obviously you can add lots of questions of varying difficulty and interest:

Cardinal characteristics:

- More characteristics?
- Other positions of the additional characteristics?
- Other orders in Cichoń's diagram?
- Singulars?

If you have a hammer. . .

- Apply "magic" to other constructions (in particular, non-ccc).

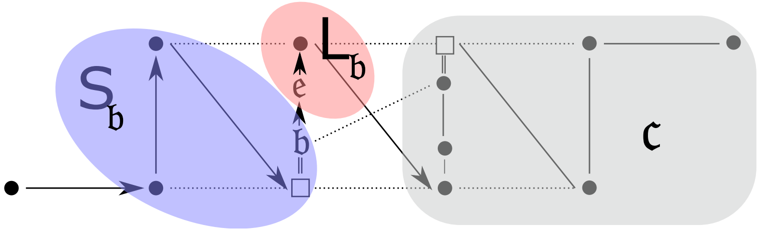
The left side

From now on we only talk about getting the left hand side.

- We use a finite support ccc iteration P_α of length $\mathfrak{c}+\mathfrak{c}$; the first $\mathfrak{c}$ iterands are Cohen. ($\mathfrak{c}$: desired target value for $2^{\aleph_0}$)
- $\mathfrak{p}$, $\mathfrak{h}$, and $\mathfrak{m}$ (or $\mathfrak{s}$, $\mathfrak{r}$) are dealt with differently; we ignore them from now on.
- At each coordinate $\alpha \in \mathfrak{c}+\mathfrak{c}$ above $\mathfrak{c}$, we deal with a characteristic $\mathfrak{x} = \mathrm{add}(\mathcal{N})$, $\mathrm{cov}(\mathcal{N})$, $\mathfrak{b}$, $\mathrm{non}(\mathcal{M})$ (and now also $\mathfrak{e}$).
- We want $\mathfrak{x}$ to become some (regular) $\lambda_\mathfrak{x}$, $\aleph_1 \leq \lambda_\mathfrak{x} < \mathfrak{c}$.
- The iterands at $\mathfrak{x}$-coordinates will be $<\lambda_\mathfrak{x}$-sized versions of the forcing increasing $\mathfrak{x}$ (e.g., for $\mathfrak{b}$ Hechler, for $\mathfrak{e}$ $\mathbb{B}\mathbb{r}$ and for $\mathrm{non}(\mathcal{M})$ E.D.). With book-keeping, this will force (a strong version of) $\mathfrak{x} \geq \lambda_\mathfrak{x}$.

(For the Cichoń characteristics, the left hand side was done 2016 by Goldstern, Mejía, Shelah [10].)

Large and small



- For $\mathfrak{x} = \mathfrak{b}$, the large ($\geq\lambda_\mathfrak{b}$) iterands are partial $\mathbb{B}\mathbb{r}$ and partial E.D., all other are $<\lambda_\mathfrak{b}$.
- For $\mathfrak{x} = \mathfrak{e}$, the large ($\geq\lambda_\mathfrak{e}$) forcings are partial E.D., all other are $<\lambda_\mathfrak{e}$.

Now what?

So far the construction, and the observation $\mathfrak{x} \geq \lambda_{\mathfrak{x}}$, is straightforward.

The obvious problem:

How do we prove (a strong version of) $\mathfrak{x} \leq \lambda_{\mathfrak{x}}$?

Strong witnesses for $\mathfrak{r}$ small

(Recall: P_α a FS ccc iteration of length $\mathfrak{c}+\mathfrak{c}$; first $\mathfrak{c}$ iterands Cohen; $\aleph_1 \leq \lambda_\mathfrak{r} < \mathfrak{c}$ regular is the target value for $\mathfrak{r}$.)

Definition

- Let $\bar{c} = (c_\alpha)_{\alpha\in\lambda_\mathfrak{r}}$ be the first $\lambda_\mathfrak{r}$ many Cohen reals added.
- $\bar{c}$ is a strong witness := every real covers only boundedly many c_α.
- P_α forces a strong witness, if P_α forces that $\bar{c}$ is a strong witness.

(For $\mathfrak{b}$, covers means "eventually dominates", for $\mathfrak{e}$ "predicts".)

- $P_{\lambda_\mathfrak{r}}$ forces a strong witness.
  ($\underset{\sim}{r}$ depends on boundedly many coordinates, every later c_α escapes.)
- A strong witness implies $\mathfrak{r} \leq \lambda_\mathfrak{r}$.
  (As $\bar{c}$ is unbounded.)
- (And: "Strong witness" is suitable for right-hand-side magic.)

The left side: keep it small

So we have: $P_{\lambda_{\mathfrak{r}}}$ forces a strong witness ($\lambda_{\mathfrak{r}} < \mathfrak{c}$);
and we want to show: $P_{\mathfrak{c}+\mathfrak{c}}$ preserves the strong witness.

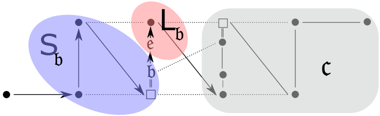
- For add($\mathcal{N}$), cov($\mathcal{N}$) and non($\mathcal{M}$) we can use "goodness" (Judah, Shelah 1990 [11], Brendle 1991 [8]):
  - All iterands are $\lambda_{\mathfrak{r}}$-good (for the $\mathfrak{r}$-relation),
  - goodness is preserved under limits,
  - goodness implies preservation of strong witnesses.

  The old arguments work without change, as the new iterands (partial $\mathbb{B}\mathbb{r}$) are σ-centered.

  We will not say more about this.
- For $\mathfrak{b}$ use UF-limits.
  - Need: All large forcings have UF-limits.
  - Old proofs used that E.D. has UF-limits.
  - Now we additionally need that $\mathbb{B}\mathbb{r}$ has UF-limits (easy).
- (New) For $\mathfrak{e}$, use FAM-limits.

Indirect proof of $\mathfrak{b} \leq \lambda_\mathfrak{b}$ using UF-limits.

- Assume towards a contradiction:
  p forces that the $P_{\mathfrak{c}+\mathfrak{c}}$-name $\underset{\sim}{r}$ dominates unboundedly many c_α.
- So: $E \in [\lambda_\mathfrak{b}]^{\lambda_\mathfrak{b}}$, and for $\alpha \in E$ there is $p_\alpha \leq p$ forcing that $c_\alpha(k) < \underset{\sim}{r}(k)$ for all $k > n_\alpha$.
- Make a Δ system of the domains of p_α, and "homogenize":
  - All $n_\alpha := n^*$,
  - $p_\alpha(\alpha)$ same Cohen condition c^* in V (of length $n^{**} > n^*$),
  - For a small heart-coordinate β, we identify Q_β with μ for some $\mu < \lambda_\mathfrak{b}$, all $p_n(\beta)$ are the same element of μ (determined in V).
  - For a large heart-coordinate β, Q_β (partial E.D. or $\mathbb{B}\mathrm{r}$) all $p_n(\beta)$ have the same "centering parameter" (determined in V).



- Pick any ω many of these conditions, call them p_ℓ, and the Cohen-coordinate they refer to α_ℓ (for $\ell \in \omega$).

Indirect proof of $\mathfrak{b} \leq \lambda_{\mathfrak{b}}$ (ctd.)

	domain in heart						domain outside							
	large			small			α_0	α_1	α_2	$\dots$	?	?	?	$\dots$
p_0	$\underset{\sim}{l}^0_0$	$\underset{\sim}{l}^1_0$	$\dots$	s^0	s^1	$\dots$	$c^{*\frown}0$			$\dots$	?			
p_1	$\underset{\sim}{l}^0_1$	$\underset{\sim}{l}^1_1$	$\dots$	s^0	s^1	$\dots$		$c^{*\frown}1$		$\dots$		?		
p_2	$\underset{\sim}{l}^0_2$	$\underset{\sim}{l}^1_2$	$\dots$	s^0	s^1	$\dots$			$c^{*\frown}2$	$\dots$			?	
$\dots$		$\dots$			$\dots$			$\dots$					$\dots$	
q	$\underset{\sim}{q}^0_2$	$\underset{\sim}{q}^1_2$	$\dots$	s^0	s^1	$\dots$								

- Modify each p_ℓ by extending $p_\ell(\alpha_\ell)$ by ℓ.
- Crucial requirement:
  There is a "limit" q forcing that infinitely many p'_n are in G.
- This gives the desired contradiction:
  $p_\ell \Vdash \underset{\sim}{r}(n^{**}) > c_{\alpha_\ell}(n^{**})$, so $p'_\ell \Vdash \underset{\sim}{r}(n^{**}) > \ell$. $\square$

UF limits

	domain in heart						domain outside							
	large			small			α_0	α_1	α_2	$\dots$	?	?	?	$\dots$
p_0	$\underset{\sim}{l}{}_0^0$	$\underset{\sim}{l}{}_0^1$	$\dots$	s^0	s^1	$\dots$	$c^* {}^\frown 0$			$\dots$	?			
p_1	$\underset{\sim}{l}{}_1^0$	$\underset{\sim}{l}{}_1^1$	$\dots$	s^0	s^1	$\dots$		$c^* {}^\frown 1$		$\dots$		?		
p_2	$\underset{\sim}{l}{}_2^0$	$\underset{\sim}{l}{}_2^1$	$\dots$	s^0	s^1	$\dots$			$c^* {}^\frown 2$	$\dots$			?	
$\dots$		$\dots$			$\dots$			$\dots$				$\dots$		
q	$\underset{\sim}{q}{}_2^0$	$\underset{\sim}{q}{}_2^1$	$\dots$	s^0	s^1	$\dots$								

How do we get such a limit q?

- The domain of q will be the heart.
- On the small indices, we use the constant conditions.
- So we only have to deal with the large indices.
  We can do this as E.D. and $\mathbb{B}\mathrm{r}$ have UF-limits (will not say more about this either, sorry).

UF limits

- Let $\mathbb{Q}$ be Eventually Different forcing or the $\mathbb{Br}$ forcing (or any other definable forcing).
- We define for a "homogeneous" sequence $\bar{p}$ of $\mathbb{Q}$-conditions and an ultrafilter U a limit $\lim_U(\bar{p}) \in \mathbb{Q}$ and require that it forces that "U-many p_ℓ are in G."
- More correctly:
  - Let $\underset{\sim}{A}_{\bar{p}}$ be the Q-name $\{n \in \omega : p_n \in G\}$.
  - It is sufficient:
    Q forces that $U \cup \{\underset{\sim}{A}_{\bar{p}} : q = \lim_U(\bar{p}) \in G\}$ has FIP.
- We then have to choose the **partial** forcing Q_β "closed enough":
  - For all ground-model sequences $\underset{\sim}{\bar{r}}$ of P_β-names for Q_β-elements,
  - and for "enough" P_β-names $\underset{\sim}{U}$ for ultrafilters,

  $\lim_{\underset{\sim}{U}}(\underset{\sim}{\bar{r}})$ is again in Q_β.
- This property is "iterable", and in particular we can then show that the "crucial requirement" above is satisfied.

Indirect proof of $\mathfrak{e} \leq \lambda_{\mathfrak{e}}$ using FAM-limits.

	domain in heart	$\dots$	α_2	α_3	α_4	α_5	$\dots$
$\dots$	$\dots$				$\dots$		
p_2	$\underset{\sim}{l}_0^2 \cdots s^0 \cdots$		c^*				$\dots$
p_3	$\underset{\sim}{l}_0^3 \cdots s^0 \cdots$			c^*			$\dots$
p_4	$\underset{\sim}{l}_0^4 \cdots s^0 \cdots$				c^*		$\dots$
p_5	$\underset{\sim}{l}_0^5 \cdots s^0 \cdots$					c^*	$\dots$
$\dots$	$\dots$				$\dots$		

- As before, an indirect argument gives us a homogeneous Δ-system, indexed by a subset of $E \in [\lambda_{\mathfrak{e}}^{\lambda_{\mathfrak{e}}}]$, take ω many elements.
- So p_n forces that $(\underset{\sim}{\pi}, \underset{\sim}{D})$ predicts c_{α_n} above n^*;
  and at coordinate α_n the Cohen condition $p_n(\alpha_n)$ is c^*, of length $n^{**} > n^*$.

Indirect proof of $\mathfrak{e} \leq \lambda_{\mathfrak{e}}$ using FAM-limits.

	domain in heart	…	α_2	α_3	α_4	α_5	…
…	…			…			
p_2	$\underset{\sim}{l}_0^2 \cdots s^0 \cdots$		$c^{*\frown}00$				…
p_3	$\underset{\sim}{l}_0^3 \cdots s^0 \cdots$			$c^{*\frown}01$			…
p_4	$\underset{\sim}{l}_0^4 \cdots s^0 \cdots$				$c^{*\frown}10$		…
p_5	$\underset{\sim}{l}_0^5 \cdots s^0 \cdots$					$c^{*\frown}11$	…
…	…			…			

- Fix a partition of ω with $|I_\ell| = 2^\ell$: $I_1 = \{0,1\}$, $I_2 = \{2,3,4,5\}$, …
- Modify $p_n \in I_\ell$ by extending $p_n(\alpha_n) = c^*$ by all possibilities for 2^ℓ.
- If $k \in [n^{**}, n^{**}+\ell-1] \cap \underset{\sim}{D}$, then $\underset{\sim}{\pi}(k)$ excludes 50% of the $c^{*\frown}s$; so $\leq$50% of the p_n can be in G.
- Crucial requirement: There are infinitely many ℓ such that more than half of the p_n (for $n \in I_\ell$) are in G.
  (I.e.: we want a limit forcing this.)

Indirect proof of $\mathfrak{e} \leq \lambda_{\mathfrak{e}}$ using FAM-limits.

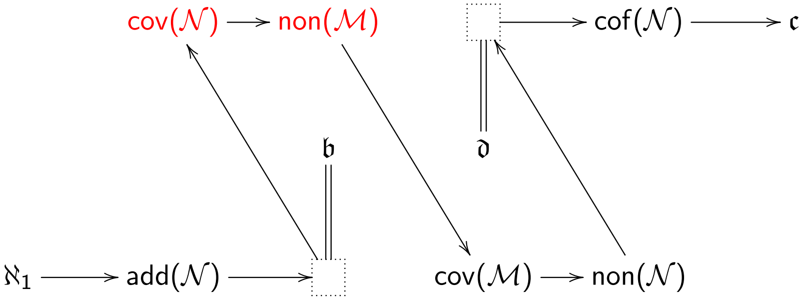
	domain in heart	…	α_2	α_3	α_4	α_5	…
…	…			…			
p_2	$\underset{\sim}{I}^2_0 \cdots s^0 \cdots$		$c^{*\frown}00$				…
p_3	$\underset{\sim}{I}^3_0 \cdots s^0 \cdots$			$c^{*\frown}01$			…
p_4	$\underset{\sim}{I}^4_0 \cdots s^0 \cdots$				$c^{*\frown}10$		…
p_5	$\underset{\sim}{I}^5_0 \cdots s^0 \cdots$					$c^{*\frown}11$	…
…	…			…			

- Again, only the large forcings (here: E.D.) are relevant.
- It is sufficient that large forcings have FAM (finitely additive measure) limits.
  (Something like: We can force arbitrary FAM-large set of indices ℓ satisfy that an arbitrary large percentage of $n \in I_\ell$ satisfies $p_n \in G$. Let us not say more about this either.)

Well...

- Problem: E.D. does not have FAM limits.
- Luckily this problem is already solved (2019 joint with Latif and Shelah [2]): We used FAM limits (based on Shelah 2000 [12]) to force an alternative ordering in Cichoń's Maximum:
  We replace E.D. with a suitable creature forcing E' which has FAM limits.
  (But E' is not σ-centered. But close enough...)
- Also luckily, E' does not only have FAM-limits, but also UF-limits.
- So we are done:
  $\mathfrak{b} \leq \lambda_{\mathfrak{b}}$ as all large forcings (partial $\mathbb{B}\mathbb{r}$ and E') have UF-limits;
  $\mathfrak{e} \leq \lambda_{\mathfrak{e}}$ as all large forcings (partial E') have FAM-limits.

Another old result: The other ordering



The (only known) alternative ordering (2019, with Latif and Shelah [2].)
(Partial) random is $\mathfrak{b}$-large, but does not have UF-limits. But it has FAM-limits, so all $\mathfrak{b}$-large forcings (random and E') have FAM-limits, which also implies $\mathfrak{b} \leq \lambda_{\mathfrak{b}}$.

Bibliography: The presented results

[1] M. Goldstern, J. K., and S. Shelah. "Cichoń's maximum". In: *Ann. of Math.* 190.1 (2019). arXiv:1708.03691, pp. 113–143.

[2] J. K., S. Shelah, and A. Tănasie. "Another ordering of the ten cardinal characteristics in Cichoń's Diagram". In: *Comment. Math. Univ. Carolin.* 60.1 (2019), pp. 61–95.

[3] M. Goldstern, J. K., D. A. Mejía, and S. Shelah. "Cichoń's maximum without large cardinals". In: *J. Eur. Math. Soc. (JEMS)* (to appear). arXiv:1906.06608.

[4] M. Goldstern, J. K., D. A. Mejía, and S. Shelah. "Controlling cardinal characteristics without adding reals". In: *J. Math. Log.* (to appear). arXiv:1904.02617.

[5] M. Goldstern, J. K., D. A. Mejía, and S. Shelah. "Preservation of splitting families and cardinal characteristics of the continuum". In: *Israel J. Math.* (to appear). arXiv:2007.13500.

Bibliography: Some classics

[6] A. Blass. "Cardinal characteristics and the product of countably many infinite cyclic groups". In: *J. Algebra* 169.2 (1994), pp. 512–540.

[7] J. Brendle. "Evasion and prediction—the Specker phenomenon and Gross spaces". In: *Forum Math.* 7.5 (1995), pp. 513–541.

[8] J. Brendle. "Larger cardinals in Cichoń's diagram". In: *J. Symbolic Logic* 56.3 (1991), pp. 795–810.

[9] J. Brendle and S. Shelah. "Evasion and prediction. II". In: *J. London Math. Soc. (2)* 53.1 (1996), pp. 19–27.

[10] M. Goldstern, D. A. Mejía, and S. Shelah. "The left side of Cichoń's diagram". In: *Proc. Amer. Math. Soc.* 144.9 (2016), pp. 4025–4042.

[11] H. Judah and S. Shelah. "The Kunen-Miller chart (Lebesgue measure, the Baire property, Laver reals and preservation theorems for forcing)". In: *J. Symbolic Logic* 55.3 (1990), pp. 909–927.

[12] S. Shelah. "Covering of the null ideal may have countable cofinality". In: *Fund. Math.* 166.1-2 (2000). Saharon Shelah's anniversary issue, pp. 109–136.