INTERESTING THIN SETS AND NASH-WILLIAMS CARDINALS

Abstract. We consider a generalization of the Nash-Williams theorem (which concerns ω) to uncountable cardinals.

1. introduction

This is a collection of small results and open questions. I haven't really given this note its final polish, and while I think its earlier parts are well readable, its later parts may be somewhat less so. This topic was suggested by Thilo Weinert, and parts of this note are the outcome of discussions with a numer of set theorists, mostly at the Technical University of Vienna, including in particular Thilo Weinert, Lukas Koschat and Wolfgang Wohofsky.

Definition 1.1. If κ is a cardinal, a subset $T \subseteq [\kappa]^{<\omega}$ is *thin* if no $a \prec b$ are both elements of T, where $a \prec b$ means that a is a proper initial segment of b.

Definition 1.2. An uncountable cardinal κ is a *Nash-Williams cardinal* if any two-colouring c of a thin subset T of $[\kappa]^{<\omega}$ has a homogeneous set H that is an unbounded subset of κ, that is, $c\restriction(T \cap [H]^{<\omega})$ is constant. For any cardinal λ, we say that κ is *Nash-Williams for λ-many colours* if the same holds true with respect to λ-colourings c.

Observation 1.3. *Every Ramsey cardinal is Nash-Williams for ω-many colours, and every Nash-Williams cardinal is weakly compact.*

Proof. Let κ be a Ramsey cardinal, let T be a thin subset of $[\kappa]^{<\omega}$, and let $c\colon T \to \omega$ be a colouring of T. We extend c to an ω-colouring C of $[\kappa]^{<\omega}$ by letting, for $x \in [\kappa]^{<\omega}$, $C(x) = c(y)$ in case there is $y \prec x$ in T, and we let $C(x) = 0$ otherwise. Note that C is well-defined since T is thin. Using that κ is a Ramsey cardinal, we find a homogeneous set H for C that is unbounded in κ, that is for every $n \in \omega$, $C\restriction[H]^n$ is constant. Let $T\restriction H$ denote $T \cap [H]^{<\omega}$. We will show that there is $i \in \omega$ such that $C[T\restriction H] = \{i\}$. Let t be of minimal length n in $T\restriction H$, and let

2020 *Mathematics Subject Classification.* 03E02,03E55.

Key words and phrases. Nash-Williams.

...

$i = C(t) = c(t)$. If $m > n$, then there is an end-extension u of t in H of length m, using that H is an unbounded subset of κ. By the definition of C, we obtain $C(u) = c(t) = i$, thus $C[[H]^m] = \{i\}$, showing that $C[T{\restriction}H] = \{i\}$.

The second statement follows trivially, for $[\kappa]^2 \subseteq [\kappa]^{<\omega}$ is thin. $\square$

2. Interesting thin sets

The question arises as to whether any thin subsets of $[\kappa]^{<\omega}$ are substantially different from $[\kappa]^n$ for some fixed $n \in \omega$. We thus make the following definition.

Definition 2.1. A subset $T \subseteq [\kappa]^{<\omega}$ is *interesting* if for every unbounded $H \subseteq \kappa$ and any $n \in \omega$, $T \cap [H]^{<\omega}$ is not contained in $[H]^n$.

Observation 2.2. *If κ is Nash-Williams for ω colours, then there are no interesting thin subsets of $[\kappa]^{<\omega}$.*

Proof. Let T be a thin subset of $[\kappa]^{<\omega}$, and let $c \colon T \to \omega$ be defined by letting $c(a) = |a|$. Let H be an unbounded subset of κ such that $c{\restriction}(T \cap [H]^{<\omega})$ is constant with value n. This means that $T \cap [H]^{<\omega} \subseteq [H]^n$, i.e., that T is not interesting. $\square$

With Lukas, we also observed the following:

Observation 2.3. *If κ is a cardinal that is not weakly compact, and $i \in \omega$, then there is an interesting thin subset T of $[\kappa]^{<\omega}$ which is in fact a subset of $[\kappa]^i \cup [\kappa]^{i+1}$. The complement of T in $[\kappa]^i \cup [\kappa]^{i+1}$ is also an interesting thin set.*

Proof. Since κ is not weakly compact, we may pick a 2-colouring

$$c \colon [\kappa]^i \to 2$$

which has no homogeneous set that is unbounded in κ. Let

$$T = c^{-1}(0) \cup \{t^\frown\alpha \mid t \in c^{-1}(1) \wedge \alpha < \kappa\}.$$

$T \subseteq [\kappa]^i \cup [\kappa]^{i+1}$ is clearly thin. Assume for a contradiction that T were not interesting, that is there is an unbounded $H \subseteq \kappa$ and $n \in \omega$ such that $T \cap [H]^{<\omega} \subseteq [H]^n$. Clearly, n could only possibly be either i or $i+1$. If n were i, then H is homogeneous for c with colour 0: if any $x \in c^{-1}(1)$ were in H, then because H is an unbounded subset of κ, some $i+1$-tuple extending x is in H as well, contradicting our assumption on H. If n were 3, then H is homogeneous for c with colour 1, for it cannot contain any element of $c^{-1}(0)$ in this case. But this, in either case, contradicts our assumption on c, for we have found a homogeneous set H for c that is an unbounded subset of κ. Now clearly, if we modify c to $d\colon [\kappa]^i \to 2$ by letting $d(x) = 1 - c(x)$ for any $x \in [\kappa]^2$, then we may obtain an interesting thin set S from d in the same way as we obtained T from c, and this set S is clearly the complement of T in $[\kappa]^i \cup [\kappa]^{i+1}$. $\square$

Using this, Observation 2.2 can be reversed.

Observation 2.4. *If there are no interesting thin subsets of $[\kappa]^{<\omega}$, then κ is Nash-Williams for ω colours.*

Proof. Pick a thin subset of $[\kappa]^{<\omega}$, which by assumption is not interesting. That is, there is an unbounded $H \subseteq \kappa$ and $n \in \omega$ such that $T \cap [H]^{<\omega} \subseteq [H]^n$. Pick $c\colon T \to \omega$. By Observation 2.3, κ is weakly compact. We may thus obtain an unbounded $H' \subseteq H$ that is homogeneous for $c\restriction(T \cap [H]^{<\omega}) = c\restriction(T \cap [H]^n)$. The existence of such H' shows that κ is Nash-Williams for ω colours, as desired. $\square$

We thus make the following definition:

Definition 2.5. We say that a cardinal κ is *thin* if there are no interesting thin subsets of $[\kappa]^{<\omega}$.

We have shown above that a cardinal κ is thin if and only if it is Nash-Williams for ω-many colours.

We now present a very useful property of interesting thin subsets of $[\kappa]^{<\kappa}$ when κ is weakly compact.

Observation 2.6. *If κ is weakly compact, $T \subseteq [\kappa]^{<\omega}$ is thin and interesting, $H \subseteq \kappa$ is unbounded, and $n \in \omega$, then there is an unbounded $H' \subseteq H$ such that $T \cap [H']^n = \emptyset$.*

Proof. We may view the characteristic function of $T\restriction[H]^n$ as a 2-colouring of $[H]^n$, and thus, using that κ is weakly compact, we may obtain a homogeneous unbounded subset $H' \subseteq H$ of κ for this 2-colouring. That is either $[H']^n$ is contained in T or is disjoint from it. But if $[H']^n \subseteq T$, since T was supposed to be thin, it follows that $T\restriction[H']^{<\omega} \subseteq [H']^n$, contradicting that T is interesting. This means that $[H']^n \cap T = \emptyset$, as desired. $\square$

Corollary 2.7. *If κ is weakly compact, and $T \subseteq [\kappa]^{<\omega}$ is thin and interesting, then the following hold.*

1. *For every unbounded $H \subseteq \kappa$, there is a $\subseteq$-decreasing sequence $\langle H_i \mid i < \omega\rangle$ of unbounded subsets of H such that for every $i < \omega$, $T \cap [H_i]^i = \emptyset$ and $\bigcap_{i<\omega} H_i = \emptyset$.*
2. *For every $\subseteq$-decreasing sequence $\langle H_i \mid i < \omega\rangle$ of unbounded subsets of κ such that for every $i < \omega$, $T \cap [H_i]^i = \emptyset$, $\bigcap_{i<\omega} H_i$ is bounded in κ.*

Proof. (1) Pick the H_i's using Observation 2.6, additionally removing all elements of limit ordinal index in the enumeration of H_i when passing to H_{i+1}. This ensures that $\bigcap_{i<\omega} H_i = \emptyset$.

(2) If for such a sequence of H_i's, $H = \bigcap_{i<\omega} H_i$ were unbounded in κ, then $T \cap [H]^{<\omega} = \emptyset$, contradicting that T was interesting.

□

We can obtain a different characterization of Nash-Williams cardinals in terms of interesting sets as follows.

Proposition 2.8. *An infinite cardinal κ is a Nash-Williams cardinal if and only if the disjoint union of any two interesting subsets of $[\kappa]^{<\omega}$ is never thin.*

Proof. First, assume that κ is a Nash-Williams cardinal, and that A and B are disjoint interesting thin subsets of $[\kappa]^{<\omega}$. Let $T = A \cup B$, and let $c\colon T \to 2$ such that $A = c^{-1}(\{0\})$ and $B = c^{-1}(\{1\})$. Let H be homogeneous for c. Then, H witnesses that either A or B is not interesting.

Now assume that κ is not a Nash-Williams cardinal, let $T \subseteq [\kappa]^{<\omega}$ be thin and interesting, and let $c\colon T \to 2$ be a colouring of T with no homogeneous unbounded subset of κ for c. Let $A = c^{-1}(\{0\})$ and $B = c^{-1}(\{1\})$. We want to show that both A and B are interesting. Clearly for no unbounded $H \subseteq \kappa$ is $A \cap [H]^{<\omega} = \emptyset$ or $B \cap [H]^{<\omega} = \emptyset$, for either would yield H to be homogeneous for c. But there cannot be $n < \omega$ such that for some unbounded $H \subseteq \kappa$ and every unbounded $H' \subseteq H$, we have $A \cap [H']^n \neq \emptyset$, for then the same would be true for T. Analogously, the same holds for B. But this already shows that both A and B are interesting, as desired. □

We thus make the following definition:

Definition 2.9. Let λ be a cardinal (which in particular includes finite cardinals). We say that a cardinal κ is λ-*thin* if the union of λ-many disjoint interesting subsets of $[\kappa]^{<\omega}$ is never thin. We say that κ is λ-*slim* if there are no λ-many pairwise disjoint interesting thin subsets of $[\kappa]^{<\omega}$.

Let us make some trivial observations. A cardinal κ is 1-thin if and only if it is 1-slim if and only if it is thin if and only if it is a Nash-Willliams cardinal for ω colours. It is 2-thin if and only if it is a Nash-Williams cardinal. The properties λ-thin and λ-slim both become weaker as λ increases. For any λ, if κ is λ-slim then it is also λ-thin: For assuming the former and given λ-many disjoint interesting subsets of $[\kappa]^{<\omega}$, we know that one of them is not thin, so their union cannot be thin either, showing that κ is λ-thin. We will show in Corollary 3.8 below that in fact, the notions of being thin and ω-slim are equivalent.

Question 2.10.
(1) If κ is 3-thin, does it follow that κ is weakly compact?
(2) If κ is weakly compact, does it follow that κ is thin? 2-thin? 3-thin? ...

3. Creating interesting thin sets

Definition 3.1. Let $T \subseteq [\kappa]^{<\omega}$ and let H be an unbounded subset of κ. Letting π denote the transitive collapsing map of H, and also the induced collapsing map on T, taking any $t \in T \cap [H]^{<\omega}$ to $\pi[t]$, we denote $\pi[T] \subseteq [\kappa]^{<\omega}$ as $T \restriction H$.

Lemma 3.2. *Assume that $T \subseteq [\kappa]^{<\omega}$ is thin, and that H is an unbounded subset of κ. Then, $T \restriction H$ is thin.*

Proof. Obvious. □

Lemma 3.3. *If κ is a Nash-Williams cardinal, $T \subseteq [\kappa]^{<\omega}$ is thin and $c\colon T \to 2$, then whenever $H \subseteq \kappa$ is unbounded, there is an unbounded $H' \subseteq H$ that is homogeneous for c.*

Proof. Let $T \subseteq [\kappa]^{<\omega}$ be thin, and $c\colon T \to 2$. Let π denote the transitive collapsing map of H. Let $d\colon \pi[T] \to 2$ be defined by $d(\pi(x)) = c(x)$ whenever $x \in T$. Since $\pi[T] = T \restriction H$ is thin by Lemma 3.2, we let $I \subseteq \kappa$ be unbounded and homogeneous for d, using that κ is a Nash-Williams cardinal. But then, $H' = \pi^{-1}[I] \subseteq H$ is an unbounded subset of κ that is homogeneous for c, as desired. □

Lemma 3.4. *Assume that $T \subseteq [\kappa]^{<\omega}$ is interesting, and that H is an unbounded subset of κ. Then, $T \restriction H$ is interesting.*

Proof. Assume that $\pi[T]$ is not interesting. Then, there is an unbounded $I \subseteq \kappa$ and $n \in \omega$ such that $\pi[T] \cap [I]^{<\omega} \subseteq [I]^n$. But then, $H' = \pi^{-1}[I] \subseteq H$ is an unbounded subset of κ, and

$$T \cap [H']^{<\omega} = \pi^{-1}[\pi[T] \cap [I]^{<\omega}] \subseteq [H']^n.$$

□

Lemma 3.5. *If $\kappa = A \dot\cup B$ is a disjoint union of unbounded subsets of κ, and $T \subseteq [\kappa]^{<\omega}$ is such that both $T \restriction A$ and $T \restriction B$ are interesting, then T is interesting.*

Proof. Let H be an unbounded subset of κ. Assume without loss of generality that $H \cap A$ is an unbounded subset of κ. Let π denote the transitive collapsing map of A. Then, $\pi[H]$ is an unbounded subset of κ, and because $T{\restriction}A$ is interesting, it follows that for no $n \in \omega$ is $(T{\restriction}A) \cap (\pi[H])^{<\omega} \subseteq (\pi[H])^n$. By the definition of $T{\restriction}A$, this implies that for no $n \in \omega$ is $T \cap [H]^{<\omega} \subseteq [H]^n$, showing that T is interesting, as desired. $\square$

Note that the above also allows us to start with interesting thin sets C and D and construct yet another interesting thin set E such that C and D are restrictions of E to sets A and B as above respectively. Another way of producing new interesting thin sets from given interesting thin sets was observed with Wolfgang and Lukas, and is essentially used already in the proof of Observation 2.3.

Lemma 3.6. *Let $T \subseteq [\kappa]^{<\omega}$ be interesting and thin. Then, both*

$$\kappa^\frown T = \{\alpha^\frown t \mid \alpha \in \kappa \wedge t \in T\} \text{ and}$$
$$T^\frown \kappa = \{t^\frown \alpha \mid \alpha \in \kappa \wedge t \in T\}$$

are interesting and thin.

Proof. Let us first consider thinness: say that both $\alpha^\frown t$ and $\beta^\frown s$ were elements of $\kappa^\frown T$, such that $\alpha^\frown t$ is a proper initial segment of $\beta^\frown s$. Then, $\alpha = \beta$ and t is a proper initial segment of s, contradicting that T is thin. If both $t^\frown \alpha$ and $s^\frown \beta$ were elements of $T^\frown \kappa$, such that $t^\frown \alpha$ is a proper initial segment of $s^\frown \beta$, then t is also a proper initial segment of s, again contradicting that T is thin.

If S is either $\kappa^\frown T$ or $T^\frown \kappa$, and S were not interesting, let $H \subseteq \kappa$ be unbounded and let $n \in \omega$ such that $S \cap [H]^{<\omega} \subseteq [H]^n$. If $n > 1$, it follows that $T \cap [H]^{<\omega} \subseteq [H]^{n-1}$. If $n = 1$, it follows that $S \cap [H]^{<\omega} = \emptyset$, for S does not contain any singletons. But then clearly, also $T \cap [H]^{<\omega} = \emptyset$, showing that T is not interesting in each case, as desired. $\square$

An interesting fact about the above construction is that if T is thin, then T and $T^\frown \kappa$ are disjoint, and in fact this holds true also for iterations of this process. For $n \in \omega$, we define $T^{\frown^n} \kappa$ as follows: $T^{\frown^0} \kappa = T$, and given $T^{\frown^{n-1}} \kappa$, we let $T^{\frown^n} \kappa = (T^{\frown^{n-1}} \kappa)^\frown \kappa$. Clearly $T^{\frown^1} \kappa = T^\frown \kappa$, and by iterated application of Lemma 3.6, it follows that if T is interesting and thin, then each $T^{\frown^n} \kappa$ is interesting and thin as well.

Observation 3.7. *Let $T \subseteq [\kappa]^{<\omega}$ be thin. If $m < n \in \omega$, then $T^{\frown^m} \kappa$ and $T^{\frown^n} \kappa$ are disjoint.*

Proof. Assume for a contradiction that t is in the intersection of the above two sets. Since $t \in T^{\frown^n} \kappa$, it follows that $t{\restriction}m \in T^{\frown^m} \kappa$. But $t{\restriction}m$ is a proper initial segment of t, and both are elements of $T^{\frown^m} \kappa$, contradicting that this set was thin. $\square$

We can conclude that the notions of being thin and ω-slim are equivalent.

Corollary 3.8. *If κ is not thin, then κ not ω-slim, i.e., if there is an interesting thin subset of $[\kappa]^{<\omega}$, then there are ω-many disjoint ones.*

Proof. Given an interesting thin subset T of $[\kappa]^{<\omega}$, by Observation 3.7, this is witnessed by $\{T^\frown{}^n \kappa \mid n \in \omega\}$. $\square$

The above shows that in particular, interesting thin sets are not closed under taking intersections. We easily also obtain the complementary result, showing that non-interesting sets are not closed under taking unions (when interesting thin sets exist, also not in case they are thin):

Lemma 3.9. *Any interesting subset of $[\kappa]^{<\omega}$ is the disjoint union of two sets which are not interesting.*

Proof. Take any interesting thin set T, and any disjoint unbounded subsets G and H of κ. Now we may divide T into T_0 and T_1 as follows: If $t \in T$ and $t \subseteq G$, put t into T_1. If $t \subseteq H$, put t into T_0. If t satisfies neither condition, put it into either set (for definiteness, let's say we put it into T_0). Now G witnesses that T_0 is not interesting, while H witnesses that T_1 is not interesting. $\square$

Question 3.10. Can we show that if there is an interesting thin subset of $[\kappa]^{<\omega}$ then there are two disjoint ones with thin union? If so, this would show that Nash-Williams cardinals are thin, i.e., that κ is Nash-Williams if and only if there are no interesting thin subsets of $[\kappa]^{<\omega}$.

4. Very interesting sets

Given a set H, and $N \subseteq \omega$, we let $[H]^N = \bigcup\{[H]^n \mid n \in N\}$.

Definition 4.1. A set $T \subseteq [\kappa]^{<\omega}$ is *very interesting* if for no unbounded $H \subseteq \kappa$ there is a finite set $N \subseteq \omega$ such that $T \cap [H]^{<\omega} \subseteq [H]^N$.

Lemma 4.2. *If κ is weakly compact, then every interesting thin subset of $[\kappa]^{<\omega}$ is very interesting.*

Proof. Assume that κ is weakly compact, and that $T \subseteq [\kappa]^{<\omega}$ is thin and interesting, however not very interesting. Then there is a finite set $N \subseteq \omega$ and an unbounded set $H \subseteq \kappa$ such that $T \cap [H]^{<\omega} \subseteq [H]^N$. By Observation 2.6 (applied $|N|$ many times), we find $H' \subseteq H$ such that $T \cap [H']^{<\omega} = \emptyset$, contradicting that T is interesting. $\square$

Question 4.3. If κ is not weakly compact, are there very interesting thin subsets of $[\kappa]^{<\omega}$? If there were a weakly compact cardinal κ with an interesting (and hence very interesting) thin subset T of $[\kappa]^{<\omega}$, then this is a Π^1_1-statement about T and would thus reflect to unboundedly many cardinals below κ. This would also yield many non-weakly compact cardinals $\bar{\kappa}$ with very interesting thin subsets of $[\bar{\kappa}]^{<\omega}$.

Observation 4.4. *There are very interesting thin subsets of $[\omega]^{<\omega}$.*

Proof. It is easy to observe that the thin subsets of $[\omega]^{<\omega}$ that are investigated in [1, Lemma 3.11] are very interesting: Whenever I is such a thin set, H is an unbounded subset of ω, and $n \in \omega$, then there is $k > n$ such that $[H]^{<\omega} \cap I$ contains a k-tuple. $\square$

Note that we could do almost the same construction at any cardinal κ, however the resulting sets I would not be interesting, as would be witnessed by any unbounded subset H of κ satisfying $H \cap \omega = \emptyset$: a key idea in the construction of these very interesting thin subsets I of ω is that lengths of certain sequences are also used as elements of sequences in I, and so those sequences crucially contain elements of ω; if $H \cap \omega = \emptyset$, $[H]^{<\omega} \cap I = \emptyset$ will follow.

5. Forcing results

Lemma 5.1. *If κ is regular, T is a thin subset of $[\kappa]^{<\omega}$ and P is $<\kappa$-strategically closed, then T is interesting if and only if it is interesting in P-generic forcing extensions. The same holds true for very interesting thin sets if κ is uncountable.*

Proof. Clearly, being interesting is a Π_1-property and hence downward absolute. Thus, assume that T is interesting, and let P be a $<\kappa$-strategically closed notion of forcing. We have to show that T is interesting in its generic extensions. Let $\dot{H}$ be a P-name for an unbounded subset of κ and let $p \in P$. We want to find $q \leq p$ forcing that $T \cap [\dot{H}]^{<\omega}$ contains at least two distinct arities (meaning two tuples of distinct arities). In the ground model, we construct a decreasing κ-sequence $\langle p_i \mid i < \kappa \rangle$ of conditions below p forcing increasingly larger α_i's to be in $\dot{H}$, which we can do by the strategic closure of P and the assumption that $\dot{H}$ is forced to be an unbounded subset of κ. Let $K = \{\alpha_i \mid i < \kappa\}$. Since K is unbounded in κ, $T \cap [K]^{<\omega}$ contains at least two distinct arities. This then holds true already for a proper initial segment of K. But there is some p_i forcing this proper initial segment of K to be contained in $\dot{H}$, yielding p_i to be our desired condition q. The argument for very interesting thin sets is essentially the same, except that we need to assume κ to be uncountable so that the union of countably many finite tuples from κ will still be bounded in κ. $\square$

Corollary 5.2. *If $n \in \omega$ and κ is not n-thin (that includes Nash-Williams and Nash-Williams for ω colours), then this is still the case in all $<\kappa$-strategically closed forcing extensions.*

Proof. Since κ not being n-thin means that there are n disjoint interesting subsets of $[\kappa]^{<\omega}$ with thin union, this is preserved by such forcing by the above lemma. $\square$

6. On arities

Definition 6.1. If $A \subseteq [\kappa]^{<\omega}$, then the set of arities of A is $\{n \in \omega \mid A \cap [\kappa]^n \neq \emptyset\}$.

Given $T \subseteq [\kappa]^{<\omega}$, let $T \restriction n = T \cap [\kappa]^{<n}$, and let $T \restriction [n, m)$ have the obvious meaning, where we also allow for $m = \omega$. If $T \subseteq [\kappa]^{<\omega}$ and $n \in \omega$, let

$$T \wr n = T \restriction n \cup (T \restriction [n, \omega))^\frown \kappa.$$

Lemma 6.2. *If A and B are disjoint interesting subsets of $[\kappa]^{<\omega}$ with thin union and $n < \omega$, then $A \wr n$ is still interesting and disjoint from B, and $(A \wr n) \cup B$ is still thin.*

Proof. It is pretty obvious that $A \wr n$ is still interesting. If $a \in A \wr n$ and a were an initial segment of some $b \in B$, then some element of A was an initial segment of b as well, contradicting our initial assumption on A and on B. If some element b of B were a proper initial segment of some $a \in A \wr n$, then it was an initial segment of some element of A as well, again contradicting our initial assumptions on A and on B. $\square$

Proposition 6.3. *If κ is not Nash-Williams, that is, there are two disjoint interesting subsets of $[\kappa]^{<\omega}$ with thin union, then there are two such sets A and B with disjoint sets of arities.*

Proof. Let A and B be disjoint interesting subsets of $[\kappa]^{<\omega}$ with thin union. We may apply the above lemma infinitely many times (with some care) to shift A and B using $\wr$ in order to ensure that they have disjoint sets of arities. ADD DETAILS!! (THEY SHOULD BE EASY ENOUGH THOUGH) $\square$

7. Interesting Filters

The following should be compared to Corollary 3.8:

Proposition 7.1. *κ is a Nash-Williams cardinal if and only if for any interesting and thin $T \subseteq [\kappa]^{<\omega}$,*

$$F(T) = \{S \subseteq T \mid S \textit{ is interesting and thin}\}$$

is a filter on $\mathcal{P}(T)$.

Proof. If κ is not Nash-Williams, then by Proposition 2.8, there is an interesting thin $T \subseteq [\kappa]^{<\omega}$ that is the disjoint union of two of its interesting subsets, showing that $F(T)$ is not a filter on $\mathcal{P}(T)$. Let us now assume that κ is Nash-Williams, and that T is interesting and thin. Clearly, $T \in F(T)$, $\emptyset \notin F(T)$, and $F(T)$ is closed under the taking of supersets. Assume both R and S are in $F(T)$, however $Q = R \cap S$ is not. This means that there is $H \subseteq \kappa$ unbounded in κ and $n \in \omega$ such that $Q \cap [H]^{<\omega} \subseteq [H]^n$. Using Lemma 3.2 and Lemma 3.4, we obtain that both $R{\restriction}H$ and $S{\restriction}H$ are interesting thin subsets of $[\kappa]^{<\omega}$. Using Observation 2.6 twice, let $H' \subseteq H$ be unbounded such that both $R{\restriction}H'$ and $S{\restriction}H'$ contain no n-tuples. Using Lemma 3.4, both $R{\restriction}H'$ and $S{\restriction}H'$ are interesting. Now, $(R{\restriction}H') \cup (S{\restriction}H')$ is a subset of $T{\restriction}H'$, and hence thin by Lemma 3.2. If $t \in (R{\restriction}H') \cap (S{\restriction}H')$, and π is the transitive collapse of H', then $t = \pi[s]$ for some $s \in Q \cap [H']^{<\omega} \subseteq [H']^n$. But then, also t is an n-tuple, contradicting what we showed above, and thus showing that $(R{\restriction}H') \cap (S{\restriction}H') = \emptyset$, which in turn, by Proposition 2.8, contradicts that κ is a Nash-Williams cardinal. □

Essentially another way of phrasing the above is the following *variant* of Proposition 2.8:

Corollary 7.2. *κ is Nash-Williams if and only if whenever S and T are interesting subsets of $[\kappa]^{<\omega}$ with thin union then $S \cap T$ is interesting as well.*

If $T \subseteq [\kappa]^{<\omega}$ is interesting and thin, we can also use T to define a collection of subsets of ω, the *interesting arities for T*, which we denote as $S(T)$, as follows:

Definition 7.3. Given $X \subseteq \omega$, let $[\kappa]^{[X]} = \bigcup_{n \in X} [\kappa]^n$. If $T \subseteq [\kappa]^{<\omega}$ is interesting and thin, we let

$$S(T) = \{X \subseteq \omega \mid T \cap [\kappa]^{[X]}\} \text{ is interesting.}$$

The following proposition and its proof are highly analogous to Proposition 7.1 and its proof.

Proposition 7.4. *κ is a Nash-Williams cardinal if and only if for any interesting and thin $T \subseteq [\kappa]^{<\omega}$, $S(T)$ is a proper filter on ω.*

Proof. If κ is not Nash-Williams, let A and B be the interesting subsets of $[\kappa]^{<\omega}$ with disjoint sets of arities and with thin (and interesting) union T provided by Proposition 6.3. Then $S(T)$ is not a filter on ω, for both the arities of A and of B are in $S(T)$, however their intersection is empty and thus not in $S(T)$.

On the other hand, let κ be Nash-Williams, and let $T \subseteq [\kappa]^{<\omega}$ be interesting and thin. We have to show that $S(T)$ is a filter on ω. Now clearly, $\omega \in S(T)$ and $S(T)$ is closed under the taking of supersets. Also, $\emptyset \notin S(T)$. It remains to show that $S(T)$ is closed under the taking of intersections. Assume both X and Y are in $S(T)$, however $Z = X \cap Y$ is not. This means that there is $H \subseteq \kappa$ unbounded in κ and $n \in \omega$ such that $T \cap [\kappa]^{[Z]} \cap [H]^{<\omega} \subseteq [H]^n$. Using Lemma 3.2 and Lemma 3.4, we obtain that both $(T \cap [\kappa]^{[X]}) \restriction H$ and $(T \cap [\kappa]^{[Y]}) \restriction H$ are interesting thin subsets of $[\kappa]^{<\omega}$. Using Observation 2.6 twice, let $H' \subseteq H$ be unbounded such that both $(T \cap [\kappa]^{[X]}) \restriction H'$ and $(T \cap [\kappa]^{[Y]}) \restriction H'$ contain no n-tuples. Using Lemma 3.4, both $(T \cap [\kappa]^{[X]}) \restriction H'$ and $(T \cap [\kappa]^{[Y]}) \restriction H'$ are interesting. Now, $(T \cap [\kappa]^{[X]}) \restriction H' \cup (T \cap [\kappa]^{[Y]}) \restriction H'$ is a subset of $T \restriction H'$ and hence thin by Lemma 3.2. If $t \in (T \cap [\kappa]^{[X]}) \restriction H' \cap (T \cap [\kappa]^{[Y]}) \restriction H'$, and π is the transitive collapse of H', then $t = \pi[s]$ for some $s \in T \cap [\kappa]^{[Z]} \cap [H']^{<\omega} \subseteq [H']^n$. But then, also t is an n-tuple, contradicting what we showed above, and thus showing that $(T \cap [\kappa]^{[X]}) \restriction H' \cap (T \cap [\kappa]^{[Y]}) \restriction H' = \emptyset$, which in turn, by Proposition 2.8, contradicts that κ is a Nash-Williams cardinal. $\square$

8. A colouring property characterization of n-thin cardinals

We show that the property of being a weakly compact n-thin cardinal is equivalent to a certain colouring property, that we call n-thin-reducibility.

Definition 8.1. Let κ be a cardinal, and let $n < \omega$. We say that κ is *n-thin-reducible* if whenever $T \subseteq [\kappa]^{<\omega}$ is thin and $c \colon T \to n$, then there is $H \subseteq \kappa$ unbounded so that $|c[T \cap [H]^{<\omega}]| < n$.

Lemma 8.2. *If κ is n-thin and weakly compact, then it is n-thin-reducible.*

Proof. Assume that T is a thin subset of $[\kappa]^{<\omega}$, and $c\colon\ T \to n$. c yields n-many disjoint thin subsets $T_i = c^{-1}[i]$ of T. As κ is n-thin, there has to be $i < n$ for which T_i is not interesting. That is, there is an unbounded $H \subseteq \kappa$ and $n \in \omega$ such that $[H]^{<\omega} \cap T_i \subseteq [H]^n$. Viewing the indicator function of $T_i \cap [\kappa]^n$ as a 2-colouring of $[\kappa]^n$, and using that κ is weakly compact, we find an unbounded $H' \subseteq H$ such that either $[H']^n \subseteq T_i$ or $[H']^n \cap T_i = [H']^{<\omega} \cap T_i = \emptyset$. In the first case, this contradicts that T_i is thin, so we are in the second case. Thus, $i \notin c[T \cap [H']^{<\omega}]$, yielding that κ is n-thin-reducible. □

Lemma 8.3. *If κ is n-thin-reducible, then it is n-thin.*

Proof. Assume for a contradiction that $(T_i)_{i<n}$ are disjoint interesting sets with thin union T. They induce a coloring $c\colon T \to n$, letting $c(t) = i$ if $t \in T_i$. Since κ is n-thin-reducible, there is $H \subseteq \kappa$ unbounded so that $|c[T \cap [H]^{<\omega}]| < n$. But this means that H witnesses one of the T_i's to not be interesting, contradicting our assumption. □

In particular, if κ is weakly compact and $n \in \omega$, then κ is n-thin if and only if it is n-thin-reducible.

9. A decreasing hierarchy from weak compactness

Probably this has been investigated? Search the literature!

Definition 9.1. Let κ be a cardinal, and let $n < \omega$. We say that κ is *n-reducible* if whenever $m \in \omega$ and $c\colon [\kappa]^m \to n$, then there is $H \subseteq \kappa$ unbounded so that $|c[[H]^m]| < n$.

Note that clearly, κ is 2-reducible if and only if κ is weakly compact, and that the property of being n-reducible becomes weaker as n increases. Furthermore, if κ is n-thin-reducible, then it is n-reducible, and the property of being n-thin-reducible also becomes weaker as n increases.

It seems unclear which of the above properties imply weak compactness.

Question 9.2.
- If κ is 3-reducible, does it follow that κ is weakly compact?
- If κ is 3-thin-reducible (this is stronger), does it follow that κ is weakly compact?

There seem to be results in the literature (see Todorcevic - partitioning pairs of countable ordinals) that imply at least that ω_1 is not n-reducible for any $n < \omega$ (and more).

10. Some small random stuff

For $T \subseteq [\kappa]^{<\kappa}$ and $\alpha < \kappa$, let $T\restriction\alpha = \{t \in T \mid t \subseteq \alpha\}$.

Observation 10.1.
- *If $T \subseteq [\kappa]^{<\kappa}$ is thin, and $\alpha < \kappa$, then $T\restriction\alpha$ is thin.*
- *If $T \subseteq [\kappa]^{<\kappa}$ is not thin, then there is $\alpha < \kappa$ such that for all $\beta \geq \alpha$, $T\restriction\beta$ is not thin.*

- *If $T \subseteq [\kappa]^{<\kappa}$ is not interesting, then there is a club $C \subseteq \kappa$ such that for every $\alpha \in C$, $T \restriction \alpha$ is not interesting.*

One thing to try is to see what happens with an interesting thin set $T \subseteq [\kappa]^{<\kappa}$ when it is mapped by an elementary embedding j with critical point κ?

11. Variants of being Nash-Williams

Definition 11.1. If κ is a cardinal and $\alpha \leq \kappa$, we say that κ is α-Nash Williams if whenever $T \subseteq [\kappa]^{<\omega}$ is thin and $c\colon T \to 2$, then there is $H \subseteq \kappa$ of order-type α which is homogeneous for c, i.e., $c \restriction (T \cap [H]^{<\omega})$ is constant.

As for Ramseyness, we should have that every α-Erdős cardinal is α-Nash Williams.

Definition 11.2. A cardinal κ is weakly Nash-Williams if whenever $T \subseteq [\kappa]^{<\omega}$ is thin and $c\colon T \to 2$, then there is $H \subseteq \kappa$ unbounded such that for every $n \in \omega$, c restricted to $T \cap [H]^n$ is constant.

Now is weakly Nash-Williams the same as weakly compact? It clearly still implies weak compactness.

References

[1] Jean A. Larson. A short proof of a partition theorem for the ordinal ω^ω. *Annals of Mathematical Logic*, 6:129–145, 1973.