Linear orders with some (but not too much) Choice

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presenting joint work with Emma Palmer and Jonathan Schilhan

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How it all began

Question (Joel Hamkins on MathOverflow, 2012)

Can there be a linear order of the universe of sets, but no wellorder of it?

In joint work with Rodrigo Freire, we showed the somewhat related:

Theorem (Freire, Holy, 2022)

It is consistent that the universe of sets can be represented as a union of levels K_α of size $|\alpha|$ such that $K_\kappa = H_\kappa$ for regular infinite κ, but there is no wellorder of the universe.

I gave a two part talk about this result at the University of Vienna in January 2023, and Jonathan Schilhan happened to be there for the second part, where I also mentioned Hamkins' question. This eventually led to the work in progress that I will talk about today.

Two results on linear orders

Theorem (Holy-Palmer-Schilhan, 2025)

It is consistent with GBc *that there is a linear order, but no wellorder of the universe of sets.*

GBc is second order Gödel-Bernays set theory (basically, ZFC with classes as objects), without the requirement of a wellorder of the universe. Classes in GBc need *not* be definable (it is still open whether the above result can be achieved using definable classes). This theorem is verified in a symmetric submodel of a class generic extension.

Theorem (Pincus, 1977)

It is consistent that every set has a linear order while DC *holds and* AC *fails. Supposedly, the same is true wrt* DC_κ.

Pincus' paper is written in a way that makes it (at least for us) almost impossible to understand. In joint work with Schilhan, we devised a modern proof using iterated symmetric extensions for the case of DC.

The basic Cohen model (1963)

It's about producing a model of $\mathrm{ZF} + \neg\mathrm{AC}$ over Gödel's constructible universe L. The idea is to first add an ω-sequence of new (Cohen) subsets of ω (with a finite forcing support product). We look at the group of permutations of their indices (that is, of ω). Symmetric objects are (informally) ones for which there is a finite set of indices (also called a *support*) such that permutations fixing those indices do not affect the object (or rather, a name for this object, letting permutations π act recursively on names): $\pi(\dot{x}) = \{(\pi(\dot{y}), \pi(p)) \mid (\dot{y}, p) \in \dot{x}\}$. We look at the extension that is made up of symmetric objects only.

- Each particular Cohen real is symmetric – permutations fixing its index do not affect it.
- The set of the added Cohen reals is symmetric – whatever way we permute them doesn't affect the set of all of them.
- BUT: No wellordering of the Cohen reals is symmetric – vague idea: if we swap two of their indices, we swap the actual Cohen reals in the ordering. **Slogan**: We add new subsets of ω, but we *forget* about their ordering.

No wellordering

Easy fact: If $p \Vdash \varphi(\dot{x}, \ldots)$, then $\pi(p) \Vdash \varphi(\pi(\dot{x}), \ldots)$.

Given a name $\dot{x}$ in the above forcing, we say that $s \subseteq \omega$ is a *support* of $\dot{x}$ if $\dot{x}$ is fixed by all permutations fixing s. We let $\mathrm{sym}(\dot{x})$ be the *least support* of a symmetric name $\dot{x}$ – can be shown to exist, for the intersection of two supports of $\dot{x}$ is also a support of $\dot{x}$. Let $\dot{c}_i$ with $i < \omega$ be the canonical names for the Cohen reals added, and let $\dot{C}$ be the canonical name for the set of all of them.

Now assume for a contradiction that $p \Vdash \dot{<}$ is a wellorder of $\dot{C}$, that $\dot{<}$ is symmetric with $\mathrm{sym}(\dot{<}) = s \subseteq \omega$, and that $p \Vdash \dot{c}_i$ is least among the Cohen reals with indices not in s. Let π be a permutation of ω that swaps i with some $j \neq i$ outside the forcing support of p (which is finite). Then $\pi(p) \parallel p$ (because p has no information on the j^{th} Cohen real) and $\pi(p)$ forces that $\dot{c}_j$ is least among the Cohen reals with indices not in s, by the above easy fact. This is clearly a contradiction – at any $r \leq p, \pi(p)$ has contradicting opinions about $\dot{<}$.

Further properties of the model

Consisting of subsets of ω, $\dot{C}$ is (forced to be) linearly ordered, by the lexicographic ordering. This can be used to show that in fact, there is a definable linear order of the universe (for it is built up by the ground model L together with $\dot{C}$) – this is a nontrivial argument due to Halpern and L'evy (1964).

In particular, every set can be linearly ordered.

A minor adaption of the above argument shows that there is (forced to be) no injection from ω to $\dot{C}$, and thus that DC is forced to fail.

Dependent Choice (DC): If R is a relation on a nonempty set X satisfying $\forall x \exists y \ R(x, y)$, then there exists $(x_i \mid i < \omega)$ such that

$$\forall i \ R(x_i, x_{i+1}).$$

A classical fact

Fact

DC implies that ω injects into every infinite set.

Proof: Let A be an infinite set. Let R be the (proper) extension relation on the set X of injective functions from natural numbers into A. Since A is infinite (not in bijection to a natural number), it follows that $\forall x \in X \, \exists y \in X \; R(x, y)$. Thus, DC yields an increasing sequence of injections $n \to A$. Taking their union yields an injection $\omega \to A$. $\square$

Obtaining the above with DC

This is Pincus' theorem from 1977. While the details in Pincus' paper are hard to grasp, he gave a very nice overview of his proof idea. Our basic proof structure is strongly based on this. The idea is kind of simple:

- After adding ω-many Cohen sets (let's call the set of all of them $\dot{C}_0$) and forgetting about their order, DC fails.
- Let's just add (again with finite forcing support) ω-many well-orderings of $\dot{C}_0$ (the set of Cohen reals added), each of length ω, let $\dot{C}_1$ be the set of all of them, and let's forget about their ordering again.
- This forcing is very similar to the basic Cohen forcing. We again obtain a failure of DC, but this time witnessed *on a higher level*.
- We continue like this for ω_1-many steps (at limit stages j, we add well-orderings of the union of the $\dot{C}_i$ for $i < j$) with finite forcing support.
- In the end, we still have a failure of AC: in our final model, we can't simultaneously wellorder all the $\dot{C}_i$, by essentially the same argument for why AC fails in the basic Cohen model.

DC holds in the final model

The idea here is that any instance of a failure of DC in the final model already appears at some intermediate stage of our symmetric iteration (because such a failure is in some sense countable) of length ω_1, and this failure will be repaired by the next stage of our iteration. The details here are still work in progress... (and in fact, we actually use a somewhat different forcing construction...)

We first thought we need to build on this result (and its generalization to higher κ) in order to produce, by a class length iteration of symmetric extensions, a model with a global linear order and no global wellorder. We were wrong. (That's why we haven't yet finished the above argument...) This can in fact be achieved more easily by first performing a single class forcing and then passing to a symmetric submodel of the extension (rather than iterating class many symmetric extensions for set forcing):

A class symmetric extension

Let $\mathbb{P}$ be the (reverse Easton) class length iteration of the forcings $\mathrm{Add}(\kappa, \kappa)$ adding κ-many Cohen subset to every regular infinite cardinal κ, starting over a model $\mathcal{M}$ of $\mathrm{GBC} + \mathrm{GCH}$. Let $\mathcal{G}$ be the group of permutations of the indices $\prod_\kappa \{\kappa\} \times \kappa$, that fixes first components (we can't swap a Cohen subset of ω_3 with a Cohen subset of ω_{17}, of course), and which may only move a set (rather than a proper class) of indices. Let $\mathcal{F}$ be the filter of subgroups of $\mathcal{G}$ generated by the subgroups $\mathrm{fix}(e)$ of permutations fixing finite sets e of indices. We say that a set or class name $\dot{X}$ is symmetric if $\mathrm{sym}(\dot{X}) = \{\pi \in \mathcal{G} \mid \pi(\dot{X}) = \dot{X}\} \in \mathcal{F}$. Pass to the submodel $\mathcal{N}$ of the full class generic extension $\mathcal{M}[G]$ which only contains objects of the form $\dot{X}^G$ for symmetric $\dot{X}$ (both in terms of sets and classes).

The forcing theorem...

The forcing theorem, ZFC version

- For every first order formula φ, the relation $p \Vdash \varphi(\dot{x}, ...)$ is definable.
- Whatever holds true in a generic extension is forced by some condition in the generic filter.

The forcing theorem, GB version

- For every first order formula (with class parameters) φ, there is a class consisting of all $(p, \dot{x}, ...)$ for which $p \Vdash \varphi(\dot{x}, ...)$.
- Whatever holds true in a generic extension is forced by some condition in the generic filter.

In both cases, the latter item (called the truth lemma) is an easy consequence of the former (the *definability* of the forcing relations).

...can fail for class forcing

Theorem (Holy, Krapf, Lücke, Njegomir, Schlicht, 2016)

There is a notion of class forcing that fails to satisfy the truth lemma for a formula of the form $\dot{x} \in \dot{y}$, that is, $\dot{x}^G \in \dot{y}^G$ holds true in a generic extension with generic filter G, but it isn't forced by any condition in G. In particular, it doesn't have definable forcing relations.

Also, class forcing need not preserve the axioms. For example, we may:

- For every ordinal α, add a new subset of ω. After doing so $\mathcal{P}(\omega)$ cannot exist.
- Add a cofinal function from ω to Ord, using functions from natural numbers n to Ord as conditions, ordered by reverse inclusion. Clearly, Replacement cannot hold with respect to this class.
- Start over $V = L$ and code this function into the GCH pattern to make the above a failure of Replacement using a definable function.

Pretameness and Tameness

There are combinatorial conditions (due to Maurice Stanley, and Sy Friedman) which ensure that class forcing works nicely: Pretameness and Tameness.

- A notion of class forcing P is pretame if and only if it satisfies the forcing theorem and preserves GBc^-, that is GBc without the powerset axiom.

  (see also [Holy, Krapf, Schlicht: Characterizations of Pretameness and the Ord-cc, 2018])

- A notion of class forcing is tame if and only if it is pretame and preserves the powerset axiom.

Back to symmetric submodels of class forcing extensions

- The forcing theorem follows from a weak version of *pretameness*: For every $p \in P$ and every *symmetric* (set length) sequence $(D_i \mid i \in I)$ with $I \in M$ of dense subclasses of $\mathbb{P}$, there is $q \leq p$ and $(d_i \mid i \in I)$ of predense below q subsets $d_i \subseteq D_i$.
- For the preservation of GBC^- to a symmetric submodel $\mathcal{N}$ of a generic extension $\mathcal{M}[G]$, we need an extra assumption: That P can be written in the form $P = \bigcup_{\alpha \in \mathrm{Ord}} P_\alpha$ with each P_α symmetric – $\mathrm{sym}(P_\alpha) = \{\pi \in \mathcal{G} \mid \pi[P_\alpha] = P_\alpha\} \in \mathcal{F}$ (easy in our particular case).
- There's also a version of tameness (seemingly incompatible to the one for normal class forcing, for it has a symmetry assumption both in its hypothesis and its conclusion) which ensures preservation of the powerset axiom.
- We can't reverse the above implications from pretameness.
- Modulo pretameness, we can reverse the implication from tameness if $\mathrm{sym}(P_\alpha)$ is the same for every α.

GBc in $\mathcal{N}$

In our case, we can easily argue for the powerset axiom by using a simple (but interesting) property of our symmetric extension:

Observation

$\mathcal{M}[G]$ and its symmetric submodel $\mathcal{N}$ have the same sets.

Proof: We show that they have the same subsets of ordinals, which is enough as $M[G]$ satisfies AC, thus every set is coded by a subset of an ordinal, and the decoding can be done in $\mathcal{N}$, for AC is not needed for this. But a standard easy density argument shows that any subset of an ordinal is coded as an initial segment of a Cohen subset of some large enough cardinal κ. Since each Cohen subset that we add is symmetric (with its index as support), it ends up in $\mathcal{N}$, and so do its initial segments. □

Note that this argument uses the forcing theorem and (small fragments of) GB^- in $\mathcal{N}$. But it yields AC and the powerset axiom in $\mathcal{N}$.

The core of our argument

Naively, it may look like essentially by the same argument as for the basic Cohen model, we can't wellorder all Cohen subsets of all regular infinite cardinals that we have added simultaneously in a symmetric way, and that thus, there is no global wellorder in $\mathcal{N}$.

And also, that since we can linearly order all Cohen subsets that we have added by using the lexicographic order, we obtain a linear order of the universe of $\mathcal{N}$ as a class of $\mathcal{N}$ (one can show that in general, classes are closed under definability).

- The latter is true, by an adaptation of the Halpern-Lévy argument for the basic Cohen model.
- The former is in fact somewhat more complicated, since permutations act on iterations in a more complicated way. This is in fact the reason why we had to allow our permutations to move set-many (as opposed to just finitely many like for the Cohen model) indices.

Permutation groups on iterations

Say we have a two-step iteration $P = C_0 * \dot{C}_1$: First, by C_0, we add ω-many Cohen subsets of ω, and then, by C_1, we add ω_1-many Cohen subsets of ω_1. Conditions are of the form $(p, \dot{q})$. Say π_0 permutes ω, and π_1 permutes ω_1. How do they act on $(p, \dot{q})$ – what is $(r, \dot{s}) = (\pi_0, \pi_1)(p, \dot{q})$? r should just be $\pi_0(p)$, but what about $\dot{s}$? Since the name $\dot{q}$ involves conditions from C_0, it should likely be influenced by π_0. It turns out that taking $\pi_1(\pi_0(\dot{q}))$ works to provide a reasonable group of permutations – however, while π_0 is literally applied to the name $\dot{q}$, with regards to π_1, $\dot{s}$ is a canonical name for a condition that results from application of π_1 to $\pi_0(\dot{q})$. This can then be extended to arbitrary length iterations in a natural (recursive) way.

No global wellorder...

- Suppose for a contradiction that $\dot{F}\colon N \to \mathrm{Ord}$ is forced to be an injection, and that $\dot{F}$ is symmetric with $\mathrm{sym}(\dot{F}) = e \subseteq \prod_\kappa \{\kappa\} \times \kappa$ finite.

  Let $\dot{c}_{\kappa,i}$ denote the canonical name for the i^{th} Cohen subset of κ that we added.
- Let $\beta > \max \mathrm{dom}(e)$ and $p \Vdash \dot{F}(\dot{c}_{\beta,0}) = \check{\gamma}$.
- Think of conditions in $\mathrm{Add}(\kappa,\kappa)$ as functions $a \mapsto 2$ with $a \in [\kappa \times \kappa]^{<\kappa}$.
- Observation: There is $\bar{p} \leq p$ such that $\forall\kappa\,\exists\delta_\kappa < \kappa\ \bar{p} \restriction \kappa \Vdash \mathrm{dom}(p(\kappa)) \subseteq \delta_\kappa \times \delta_\kappa$.
- Let $\theta > \sup(\mathrm{supp}(p))$.
- For each $\kappa \in [\beta, \theta)$, let π_κ be a permutation of κ with $\delta_\kappa \cap \pi_\kappa''\delta_\kappa = \emptyset$.
- Let π_κ be the identity otherwise.
- Let $\pi = \prod_\kappa \pi_\kappa$. Then, $\pi \in \mathrm{fix}(e)$ hence $\pi(\dot{F}) = \dot{F}$, and $\pi(\bar{p}) \parallel \bar{p} \leq p$.

Finishing the argument

- Remember: $\dot{F}\colon N \to \mathrm{Ord}$ is forced to be an injection, and $\pi(\dot{F}) = \dot{F}$.
- Remember also: $\bar{p} \leq p \Vdash \dot{F}(\dot{c}_{\beta,0}) = \check{\gamma}$, and $\pi(\bar{p}) \parallel p$.
- But then: $\pi(\bar{p}) \Vdash \dot{F}(\dot{c}_{\beta,\pi_\beta(0)}) = \check{\gamma}$.
- Since $\pi_\beta(0) \neq 0$, this contradicts the injectivity of $\dot{F}$.

Thank you for your attention!