AN AXIOMATIC APPROACH TO SYMMETRIC EXTENSIONS

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Abstract. We provide a collection of natural axioms centered around the symmetric forcing theorem, which yield the concept of symmetric extensions, avoiding the technicalities involved in any standard presentation.

1. Introduction

Symmetric extensions are an important concept in set theory, originating from Paul Cohen's famous proof of the independence of the axiom of choice from ZF, and they have since proven to be the key tool to obtain independence and consistency results over ZF, in the absence of the axiom of choice. In their usual presentation, they are based on technicalities like the concepts of genericity, forcing names and their evaluations, and on the recursively defined forcing predicates, the definition of which is particularly intricate for the basic case of atomic first order formulas.

In his [1], Rodrigo Freire has provided an axiomatic framework for set forcing over models of ZFC that is a collection of guiding principles for extensions over which one still has *control* from the ground model, and has shown that his axioms necessarily lead to the usual concepts of genericity and of forcing extensions, and also that one can infer from them the usual recursive definition of the forcing predicates. In [2], this was extended to class forcing by Freire and the author. Building on some of the basic ideas of Freire, we introduce an axiomatic framework for symmetric extensions over models of ZF, that also avoids the technicalities connected with any usual standard setup for symmetric extensions, in particular the concepts of genericity and, perhaps most importantly, the recursively defined forcing predicates. Instead, we provide a natural collection of axioms centered around the symmetric forcing theorem, that is the conjunction of the definability of the symmetric forcing relations and the truth lemma, stating that anything

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that holds true in a symmetric extension is forced by a condition in the relevant (symmetrically) generic filter, and show that this collection of axioms essentially induces the common standard concepts: that is, we derive the relevant concept of genericity, the usual recursive definitions of forcing predicates, an analogue of the structure of names for elements of symmetric extensions and their evaluations, thus exactly the same symmetric extensions, and also the preservation of the axioms of ZF to symmetric extensions. The aim of this paper is essentially twofold. First, it is to provide a new viewpoint on a central technical tool in modern set theory: Within a suitable basic setup, requiring the symmetric forcing theorem is sufficient to yield exactly the concept of symmetric extensions. Second, it is supposed to provide a self-contained way of introducing symmetric extensions axiomatically. The only point in the paper where it is strictly necessary to refer to some sort of standard setup is when we briefly argue for our axioms to actually be consistent in Section 5. In this introductory section, we want to provide a rough description of our axiomatic framework, which will be followed with formal definitions in Sections 2 and 3.

In the standard setup for symmetric extensions, they are based on so-called *symmetric systems*, that is, triples $\mathbb{S} = (\mathbb{P}, \mathcal{G}, \mathcal{F})$ where $\mathbb{P}$ is a forcing notion (i.e., a preorder[1]), $\mathcal{G}$ is a group of automorphisms of $\mathbb{P}$, and $\mathcal{F}$ is a filter on the set of subgroups of $\mathcal{G}$. Models of set theory have a large variety of symmetric systems, and these symmetric systems usually give rise to a vast number of different symmetric extensions. Symmetric systems themselves may already seem like a fairly technical notion, but in order to capture the magnitude of complex possibilities offered by the technique of symmetric extensions, it seems necessary for our basic setup to contain such notions offering a rich variety of options. Thus, just like usual (class) forcing notions were the basis of the axiomatic description of (class) forcing in [1] (and [2]), we will make the usual notion of symmetric system the basis of our symmetric extensions.

Let us fix a transitive ground model $\mathcal{M} \in V$ for this discussion, and a symmetric system $\mathbb{S} = (\mathbb{P}, \mathcal{G}, \mathcal{F}) \in M$.[2] For the sake of simplicity, we require that $\mathcal{M} \models \mathrm{ZF}$.[3] We think of conditions (that is, elements of the domain P) of $\mathbb{P}$ as having partial information on properties

[1]A preorder is a reflexive and transitive binary relation.

[2]As is common pratice, we will use M for the domain of $\mathcal{M}$, P for the domain of $\mathbb{P}$ etc.

[3]The usual ways of avoiding this extra consistency assumption apply, see for example [5].

of our extensions. If $q \leq p$ in $\mathbb{P}$, we say that q is stronger than p, and we think of stronger conditions as having more information. The automorphisms $\pi \in \mathcal{G}$ will naturally extend to maps $\Omega(\pi)$ on M, and we consider $x \in M$ to be *symmetric* in case it is mapped to itself by a large number of maps $\Omega(\pi)$, namely whenever π comes from a certain set in $\mathcal{F}$. Any particular *symmetric extension* is built out of such symmetric elements,[4] serving as *names* for the elements of the symmetric extension, together with a choice of filter on $\mathbb{P}$. We think of such a filter as a selection of conditions which have *correct* information about our symmetric extension, and we will refer to such conditions as being *correct*. The motivation for using a *filter* of conditions can be explained exactly as in [2]:

- If we consider the information that a condition q has to be correct, then any weaker condition p has less information than q, and this information should therefore also be correct. This corresponds to the upwards closure property of filters.
- If p and q are correct conditions, we consider the information that is jointly collected by p and q to be correct. We require that there is a condition that collects this joint information and that we consider to be correct. This corresponds to the property of a filter that any two of its elements are compatible, as witnessed by yet another element of the filter.

We require that for any condition $p \in P$, there exists a filter G of correct conditions of which p is an element, so that no condition is a priori incorrect. A number of natural axioms will make sure that we have *ground model control* over our symmetric extension, which we denote as $\mathcal{M}_{\mathbb{S}}[G]$, in a sufficiently simple way. We require the existence of a definable relation on our ground model, which, following [1], we call the $\mathbb{P}$-membership relation. It is supposed to relate to partial knowledge about the membership relation in symmetric extensions. If $a, b \in M$ and $p \in P$, we say that a is an element of b according to p, and write $a \in_p b$ in case the triple (p, a, b) stands in this relation.[5] We want to define a membership relation for $\mathcal{M}_{\mathbb{S}}[G]$, letting the object denoted by a be an element of the object denoted by b in case a is an element of b according to some correct condition (that is, $\exists p \in G\ a \in_p b$).

[4] In fact, we only use *hereditarily symmetric* elements later on.

[5] This relation corresponds to the relation that $(a, p) \in b$ in the standard setup, given that a, b are usual (hereditarily symmetric) forcing names. In this sense, the symmetric ground model objects that we will make use of as names can immediately be seen to be very similar to the usual (hereditarily) symmetric names for elements of symmetric extensions.

In order to be able to obtain a transitive model as our symmetric extension, we require the relation $\exists p \in P\ a \in_p b$ to be well-founded. The relation $\exists p \in G\ a \in_p b$ will usually not be extensional, but we nevertheless obtain a transitive $\in$-structure (which will serve as our generic extension $\mathcal{M}_{\mathbb{S}}[G]$) as the image of the homomorphism that is our *evaluation map* F_G, recursively defined by setting $F_G(b) = \{F_G(a) \mid a \in_G b\}$ for every symmetric b in M. In order to be able to show that $\mathcal{M}_{\mathbb{S}}[G]$ is well-defined and satisfies the axioms of ZF, we will need to require the following:

- Set-likeness of the $\mathbb{P}$-membership relation: for any symmetric $b \in M$,
  $\{a \mid \exists p \in P\ a \in_p b\}$ is a set in M.
- *High degrees of freedom* for the $\mathbb{P}$-membership relation: for any symmetric relation S on $M \times P$ in M, we find $b \in M$ for which $\{(a,p) \mid a \in_p b\} = S$.

Furthermore, we also require the existence of forcing predicates definably over $\mathcal{M}$, individually for each first order formula. We do not require any particular defining instances for these predicates, we only require them to be connected to truth in generic extensions by the following two axioms (these requirements correspond to what is usually known as the *forcing theorem* in any standard setup):

- Whatever holds in $\mathcal{M}_{\mathbb{S}}[G]$ is forced by some condition in G.
- Whatever is forced by some condition in G holds true in $\mathcal{M}_{\mathbb{S}}[G]$.

Let us mention two additional observations that this paper will help us make: First, our framework will help us to establish what the right notion of genericity with respect to symmetric systems is, namely that of symmetric rather than full genericity, a notion that was only recently introduced in [4]. Second, we will make the easy observation that one of the axioms in [1] and [2] was in fact unnecessary, as it easily folllows from the remaining axioms, namely the requirement that any condition in P forces at least as much as any weaker condition in P does.

2. The basic setup

Let $\mathcal{L}(\in)$ denote the collection of first order formulas in the language with the $\in$-predicate. We consider equality between sets to abbreviate the statement that they have the same elements. We start by providing the definition of a *symmetric framework*, which will be the basic formal concept in our approach.

Definition 2.1. *A* symmetric framework *is a tuple of the form*

$\left(\mathcal{M}, \mathbb{S}, \Omega, R, (\Vdash_\varphi)_{\varphi \in \mathcal{L}(\in)}, \mathfrak{G}\right)$ *with the following properties.*

- $\mathcal{M}$ *is a transitive set-size model of* ZF*.*
- $\mathbb{S} = (\mathbb{P}, \mathcal{G}, \mathcal{F}) \in M$*.*
  - $\mathbb{P} = \langle P, \leq \rangle$ *is a preorder with weakest element* 1.[6]
  - $\mathcal{G}$ *is a group of automorphism of* $\mathbb{P}$*.*
  - $\mathcal{F}$ *is a filter on the set of subgroups of* $\mathcal{G}$*.*
  - *We refer to such* $\mathbb{S}$ *as a* symmetric system*.*
- Ω *is a map with domain* $\mathcal{G}$*, and for* $\pi \in \mathcal{G}$*,* $\Omega(\pi) \colon M \to M$ *is such that* $\{(\pi, x) \mid x \in \Omega(\pi)\}$ *is definable (over* $\mathcal{M}$*).*
- *The* $\mathbb{P}$-membership relation R *is a definable (over* $\mathcal{M}$*) relation on* $P \times M \times M$*. We denote the property* $R(p, a, b)$ *as* $a \in_p b$.[7] *We also write* $b =_R \{(a, p) \mid a \in_p b\}$*.*
- $\mathfrak{G}$ *is a second order unary predicate on* P*, i.e. a unary predicate on* $\mathcal{P}(P)$*, and we require that* $\mathfrak{G}(G)$ *implies that* $G \subseteq P$ *is a filter. If* $\mathfrak{G}(G)$ *holds, we say that* G *is a* symmetrically generic filter*. Whenever we quantify over* G *in the following, we tacitly assume that we quantify over* G*'s such that* $\mathfrak{G}(G)$ *holds.*
- *For every* $\varphi \in \mathcal{L}(\in)$*,* $\Vdash_\varphi$ *is a definable (over* $\mathcal{M}$*) predicate (which we also call a* forcing relation for φ*) on* $P \times M^m$*, where* m *denotes the number of free first order variables of* φ*.*

  If $\langle q, a_0, \ldots, a_{m-1} \rangle \in \Vdash_\varphi$*, we also write* $q \Vdash \varphi(a_0, \ldots, a_{m-1})$*.*

3. The basic axioms

In this section, we present our basic axioms for symmetric frameworks.

(1) **Existence of generic filters:** $\forall p \in P\, \exists G\; p \in G$.[8]
(2) **Well-Foundedness:** The binary relation $\exists p \in P\; a \in_p b$ on M is well-founded.

Using axiom 2, we can define a notion of *name rank*, letting, for $a \in M$,

$$\operatorname{rank} a = \sup\{\operatorname{rank}(b) + 1 \mid \exists p \in P\; b \in_p a\}.$$

[6]We use preorders rather than (the perhaps more common restriction to) partial orders, dropping the requirement of antisymmetry (this more general context naturally appears for example in the case of (symmetric) iterations of forcing notions).

[7]In a standard forcing setup, this would correspond to the property that $(a, p) \in b$.

[8]Remember that by our above convention, we tacitly require here that $\mathfrak{G}(G)$ holds.

(3) **Extension:** For all $a, b \in M$, and $\pi \in \mathcal{G}$,

$$\Omega(\pi)(b) =_R \{(\Omega(\pi)(a), \pi(p)) \mid a \in_p b\}.$$

Let the *symmetry group* of $a \in M$ be $\mathrm{sym}(a) = \{\pi \in \mathcal{G} \mid \Omega(\pi)(a) = a\}$. We say that $a \in M$ is *symmetric* if $\mathrm{sym}(a) \in \mathcal{F}$, and we let N denote the collection of all symmetric objects (from M). We say (inductively, using the axiom of well-foundedness) that $b \in N$ is *hereditarily symmetric* if a is hereditarily symmetric whenever $\exists p \in P\ a \in_p b$. We let HS denote the collection of all hereditarily symmetric objects (from M). Elements of HS will serve as *names* for the elements of our symmetric extensions defined below.

Assume that G is such that $\mathfrak{G}(G)$ holds. Define a relation $\in_G$ on HS by letting $a \in_G b$ if $\exists p \in G\ a \in_p b$. Using axiom (2), this relation is well-founded, and since $\mathrm{HS} \in V$, we may thus recursively define our *evaluation function* F_G along the relation $\in_G$, letting $F_G(b) = \{F_G(a) \mid a \in_G b\}$ for each $b \in \mathrm{HS}$.[9] Let $\mathcal{M}_{\mathbb{S}}[G]$ denote the $\in$-structure on the transitive set $F_G[\mathrm{HS}]$:[10] That is, let $\mathcal{M}_{\mathbb{S}}[G] = \langle M_{\mathbb{S}}[G], \in\rangle$, where $M_{\mathbb{S}}[G] = F_G[\mathrm{HS}] = \{F_G(a) \mid a \in \mathrm{HS}\}$. We refer to $\mathcal{M}_{\mathbb{S}}[G]$ as a *symmetric extension* of M.

The next two axioms should be seen as the most crucial ones, and they state that a natural form of the forcing theorem holds, that is based on our forcing relations. Given a finite tuple $\vec{a} = \langle a_i \mid i < n\rangle$ of elements of HS (we simply say $\vec{a} \in \mathrm{HS}$ in this case), let $F_G(\vec{a}) = \langle F_G(a_i) \mid i < n\rangle$.

(4) **Truth Lemma:** For all $\varphi \in \mathcal{L}(\in)$, all $\vec{a} \in \mathrm{HS}$ and all G,

$$\mathcal{M}_{\mathbb{S}}[G] \models \varphi(F_G(\vec{a})) \text{ iff } \exists p \in G\ p \Vdash \varphi(\vec{a}).$$

(5) **Definability Lemma:** For all $\varphi \in \mathcal{L}(\in)$, all $\vec{a} \in \mathrm{HS}$ and $p \in P$,

$$p \Vdash \varphi(\vec{a}) \text{ iff } \forall G \ni p\ \ \mathcal{M}_{\mathbb{S}}[G] \models \varphi(F_G(\vec{a})).$$[11]

Our final two axioms make sure that our setup is reasonable, with the former assuming that names have set-like properties with respect

[9] It may seem like we are taking some sort of transitive collapse of the structure $\langle \mathcal{M}[G], \in_G\rangle$, however note that there is no reason to assume that $\in_G$ is extensional, or that $\in_G$ can be factorized in order to obtain an extensional relation.

[10] For the moment, this notation is somewhat ambiguous, for $\mathcal{M}_{\mathbb{S}}[G]$ may not only depend on $\mathcal{M}$, $\mathbb{S}$ and G, but also on the $\mathbb{P}$-membership relation. We will however show at the end of this section that under additional assumptions, $\mathcal{M}_{\mathbb{S}}[G]$ is uniquely determined.

[11] Note that we already required the forcing relations to be predicates of our model in our basic setup, however this axiom connects them with their intended meaning, and it thus seems justified to consider it to be our version of the *definability lemma*.

to the $\mathbb{P}$-membership relation, and the latter making sure that we have a sufficient amount of names available.[12]

(6) **Set-Likeness:** If $b \in \mathrm{HS}$, then $\{a \mid \exists p \in P\ a \in_p b\} \in M$.

(7) **Universality:** There is a map $\Gamma \colon M \to \mathrm{HS}$ that is definable over $\mathcal{M}$, such that if $S \in M$ is a *symmetric* subset of $\mathrm{HS} \times P$, that is,

$$\exists F \in \mathcal{F}\, \forall \pi \in F\, \forall (a,p) \in S\ (\Omega(\pi)(a), \pi(p)) \in S,$$

then $\Gamma(S) =_R S$, and $\Gamma(S)$ is the unique $T \in \mathrm{HS}$ for which $T =_R S$.

The statement of the following lemma was taken to be an axiom in [1] and [2], however it is easily provable (and would also have been easily provable in [1] or [2]) from axiom 5, which has been overlooked in earlier work on the subject.

Lemma 3.1. *For all $\varphi \in \mathcal{L}(\in)$, for all $\vec{a} \in \mathrm{HS}$, and $p, q \in P$, if $p \Vdash \varphi(\vec{a})$ and $q \leq p$, then $q \Vdash \varphi(\vec{a})$.*

Proof. Assume $p \Vdash \varphi(\vec{a})$ and $q \leq p$. Then, axiom 5 implies that

$$\forall G \ni p\ \ \mathcal{M}_{\mathbb{S}}[G] \models \varphi(F_G(\vec{a})).$$

But since any G is a filter, it contains p whenever it contains q, hence it clearly follows that $\forall G \ni q\ \ \mathcal{M}_{\mathbb{S}}[G] \models \varphi(F_G(\vec{a}))$, which again by axiom 5 is equivalent to $q \Vdash \varphi(\vec{a})$, as desired. □

We close this section by a lemma which in particular shows that $\mathcal{M}_{\mathbb{S}}[G]$ does in fact not depend on the choice of the $\mathbb{P}$-membership relation.

Lemma 3.2. *Assume that we have two symmetric frameworks which are based on the same model $\mathcal{M}$ and symmetric system $\mathbb{S} = (\mathbb{P}, \mathcal{G}, \mathcal{F})$:*

$$\left(\mathcal{M}, \mathbb{S}, \Omega, R, (\Vdash_\varphi)_{\varphi \in \mathcal{L}(\in)}, \mathfrak{G}\right)$$

and

$$\left(\mathcal{M}, \mathbb{S}, \Omega', R', (\Vdash'_\varphi)_{\varphi \in \mathcal{L}(\in)}, \mathfrak{G}'\right),$$

and that G is such that both $\mathfrak{G}(G)$ and $\mathfrak{G}'(G)$ hold. We will write $a \in_p b$ and $a \in'_p b$ in case $R(p,a,b)$ or $R'(p,a,b)$ hold. We will use HS' to denote the version of HS, we use F'_G to denote the version of F_G, and we use Γ' to denote the version of Γ provided by the latter symmetric framework.

If $a \in \mathrm{HS}$, then there is $b \in \mathrm{HS}'$ such that $F_G(a) = F'_G(b)$.

[12]The uniqueness requirement in axiom 7 below could be avoided, however it is very natural and easily available in any sort of setup for symmetric extensions.

Proof. Making use of the map Γ', we define a *translation function* $h\colon \mathrm{HS} \to \mathrm{HS}'$ by induction on name rank, and simultaneously show that for any $c \in \mathrm{HS}$ and $\pi \in \mathcal{G}$, $\Omega'(\pi)(h(c)) = h(\Omega(\pi)(c))$. For $c \in \mathrm{HS}$, consider the set $C = \{(d,p) \mid d \in_p c\}$, and let $F \in \mathcal{F}$ be such that $\forall \pi \in F\, \forall (d,p) \in C\ (\Omega(\pi)(d), \pi(p)) \in C$, using that c is symmetric. Let $C' = \{(h(d),p) \mid (d,p) \in C\} \subseteq \mathrm{HS}' \times P$. Let $\pi \in F$ and pick $(h(d),p) \in C'$. Then, $(\Omega'(\pi)(h(d)), \pi(p)) = (h(\Omega(\pi)(d)), \pi(p)) \in C'$, and thus we may invoke axiom 7, letting

$$h(c) = \Gamma'(C') \in \mathrm{HS}'.$$

Now if $\pi \in \mathcal{G}$, then

$$\Omega'(\pi)(h(c)) =_{R'} \{(\Omega'(\pi)(h(d)), \pi(p)) \mid d \in_p c\} = \{(h(\Omega(\pi)(d)), \pi(p)) \mid d \in_p c\}.$$

On the other hand, $h(\Omega(\pi)(c)) =_R \{(h(\Omega(\pi)(d)), \pi(p)) \mid d \in_p c\}$ as well, thus $\Omega'(\pi)(h(c)) = h(\Omega(\pi)(c))$ by the uniqueness requirement in axiom 7.

Concluding the proof of the lemma, we show by induction on name rank that for any $c \in \mathrm{HS}$, $F'_G(h(c)) = F_G(c)$. Let $c \in \mathrm{HS}$. Inductively,

$$F'_G(h(c)) = \{F'_G(h(d)) \mid \exists p \in G\ d \in_p c\} = \{F_G(d) \mid \exists p \in G\ d \in_p c\} = F_G(c),$$

as desired. □

4. Forcing predicates, density, and symmetry

We will use our axioms to verify some of the basic properties of forcing, and in particular to verify that the forcing predicates satisfy their usual defining clauses, by arguments that are partially similar to the arguments of [1, Section 4] or [2, Section 4]. However, we are making strong use of the universality axiom, the analogue of which was only introduced much later in both [1] and [2], already in the proof of Lemma 4.3. We start by observing that we obtain the usual defining clause for the forcing relation for negated formulae.

Lemma 4.1. *For all $\varphi \in \mathcal{L}(\in)$, $p \in P$ and $\vec{a} \in \mathrm{HS}$, we have that*

$$p \Vdash \neg\varphi(\vec{a}) \text{ iff } \forall q \le p\ q \nVdash \varphi(\vec{a}).$$

Proof. Let us assume that

(i) $p \Vdash \neg\varphi(\vec{a})$.

By axiom (5), equivalently

(ii) $\forall G \ni p\ \mathcal{M}[G] \models \neg\varphi(F_G(\vec{a}))$.

By axiom (4), this is equivalent to

(iii) $\forall G \ni p\, \forall q \in G\ q \nVdash \varphi(\vec{a})$.

We want to argue that this in turn is equivalent to our desired statement that

(iv) $\forall q \leq p\ q \not\Vdash \varphi(\vec{a})$.

Thus, assume first that (iii) holds, and let $q \leq p$. By axiom (1), we may pick a generic filter $G \ni q$, which will thus also contain p as an element. By (iii), we thus have that $q \not\Vdash \varphi(\vec{a})$, as desired.

Conversely, assume that (iv) holds. Let G be a generic filter that contains p as an element, and assume for a contradiction that there is $r \in G$ such that $r \Vdash \varphi(\vec{a})$. Since G is a filter, we may pick q below both p and r. By Lemma (3.1), it follows that $q \Vdash \varphi(\vec{a})$, contradicting (iv). □

The next lemma provides us with objects that represent ground model elements in our symmetric extensions.

Lemma 4.2 (Ground model elements)**.** *There is a definable map* $\check{}: M \to \mathrm{HS}$*,* $a \mapsto \check{a}$*, such that* $\forall a \in M\, \forall G$

$$F_G(\check{a}) = a \wedge \mathrm{sym}(\check{a}) = \mathcal{G}.$$

Proof. Using axiom (7), by recursion on von Neumann rank in M, for $b \in M$, let $\check{b} = \Gamma(\{(\check{a}, \mathbb{1}) \mid a \in b\})$. Now, for any b, $F_G(\check{b}) = b$ and $\mathrm{sym}(\check{b}) = \mathcal{G}$ is easily shown by induction on the rank of $\check{b}$, using that $\pi(\mathbb{1}) = \mathbb{1}$ for the latter. □

A subset D of P is *symmetrically dense* if it is dense (i.e., $\forall p \in \mathbb{P}\ \exists q \leq p\ q \in D$) and $\exists F \in \mathcal{F}\, \forall \pi \in F\ \pi[D] = D$. We will show that our axioms imply generic filters to intersect all symmetrically dense subsets of $\mathbb{P}$ in M.[13]

Lemma 4.3. *Let* $D \in M$ *be such that* $D \subseteq P$ *is symmetrically dense. If* G *is a generic filter, then* G *intersects* D*.*

Proof. Let $\dot{D} = \Gamma(\{(\check{\emptyset}, d) \mid d \in D\})$. Clearly, $\dot{D}$ is symmetric with $\mathrm{sym}(\dot{D}) = \mathcal{G}$. Let $p \in \mathbb{P}$ and assume $p \Vdash \dot{D} = \emptyset$. Then $\exists q \leq p\ q \in D$, hence $\check{\emptyset} \in_q \dot{D}$, so $F_G(\dot{D}) \neq \emptyset$ whenever $q \in G$, that is, by axiom 5, $q \Vdash \dot{D} \neq \emptyset$, contradicting Lemma 4.1. Thus, again by Lemma 4.1, $1 \Vdash \dot{D} \neq \emptyset$. It follows that for all G, $F_G(\dot{D}) \neq \emptyset$, and hence $D \cap G \neq \emptyset$. □

[13] It may be somewhat surprising that we do not obtain our generic filters to intersect *all* dense subsets of $\mathbb{P}$. However, it was already noted in [4] by Asaf Karagila and Jonathan Schilhan that this seems to be the right notion of genericity in the context of symmetric extension. This could be seen to further be supported by our lemma below, the proof of which does not extend to arbitrary dense subsets of P.

We next need another auxiliary result on symmetric open dense sets (which could easily be extended to arbitrary dense sets, but the current version is sufficient for our purposes). We say that a subset A of a preorder P is *open* if it is downward closed, that is if $p \in A$ and $q \leq p$, then also $q \in A$.

Lemma 4.4. *If $D \subseteq P$ is open and symmetric, $D \in M$, then D is dense below p if and only if*

$$(\dagger)\ \forall G \ni p\, D \cap G \neq \emptyset.$$

Proof. Assume first that ($\dagger$) holds. Let $r \leq p$, and using axiom (1), let G be a generic filter with $r \in G$. It follows that also $p \in G$, and thus using ($\dagger$), we obtain $s \in D \cap G$. Since D is open and G is a filter, we obtain q below both r and s that is an element of $D \cap G$, showing that D is dense below p.

On the other hand, assume that D is dense below p, and let G be a generic filter containing p as an element. Let $E = D \cup \{q \in P \mid \forall r \leq q\, r \notin D\}$. Then, E is clearly dense, as for any $q \in P$, either some $r \leq q$ is in D, or if not, then $q \in E$. But E is also symmetric. Let π be such that $\pi[D] = D$. If $q \in D$, then $\pi(q) \in D \subseteq E$. If $q \in E \setminus D$, this is because no $r \leq q$ is in D. But then, no $r \leq \pi(q)$ is in $\pi[D] = D$; that is, $\pi(q) \in E$. By Lemma 4.3, it follows that $G \cap E \neq \emptyset$. Since $p \in G$ and G is a filter, it thus follows that $G \cap D \neq \emptyset$, as desired. □

It is now possible to show that the usual defining clauses for the forcing relation can be recovered from our basic axioms. For $a, b \in \mathrm{HS}$ and $p \in P$, let $a \,\overline{\in}_p\, b$ abbreviate the statement that $\exists q \geq p\ a \in_q b$.

Lemma 4.5. *For any $p \in P$, $\varphi, \psi \in \mathcal{L}(\in)$, and $a, b \in \mathrm{HS}$,*

(1) $p \Vdash a \in b$ *iff* $\forall r \leq p\, \exists s \leq r\, \exists x \in \mathrm{HS}\ [x \,\overline{\in}_s\, b \wedge s \Vdash a = x]$.
(2) $p \Vdash a \subseteq b$ *iff* $\forall x \in \mathrm{HS}\, \forall r \in P\ [x \,\overline{\in}_r\, a \rightarrow \forall q \leq p, r\, \exists s \leq q\ s \Vdash x \in b]$.
(3) $p \Vdash a = b$ *iff* $[p \Vdash a \subseteq b \wedge p \Vdash b \subseteq a]$.
(4) $p \Vdash [\varphi \wedge \psi](\vec{a})$ *iff* $p \Vdash \varphi(\vec{a}) \wedge p \Vdash \psi(\vec{a})$.
(5) $p \Vdash [\varphi \vee \psi](\vec{a})$ *iff* $\forall r \leq p\, \exists q \leq r\ [q \Vdash \varphi(\vec{a}) \vee q \Vdash \psi(\vec{a})]$.
(6) $p \Vdash \exists x \varphi(x, \vec{a})$ *iff* $\forall r \leq p\, \exists q \leq r\, \exists x \in \mathrm{HS}\ q \Vdash \varphi(x, \vec{a})$.
(7) $p \Vdash \forall x \varphi(x, \vec{a})$ *iff* $\forall x \in \mathrm{HS}\ p \Vdash \varphi(x, \vec{a})$.

Proof. (1) Let us assume that (i) $p \Vdash a \in b$.
By axiom (5), this is equivalent to (ii) $\forall G \ni p\ F_G(a) \in F_G(b)$.
By the definition of F_G, this in turn is equivalent to

(iii) $\forall G \ni p\, \exists x \in \mathrm{HS}\ [F_G(a) = F_G(x) \wedge \exists q \in G\ x \in_q b]$.

Using axiom (4), we obtain the following equivalent form.

(iv) $\forall G \ni p\, \exists x \in \mathrm{HS}\ [\exists r \in G\ r \Vdash a = x \wedge \exists q \in G\ x \in_q b]$.

Now we make use of Lemma 3.1, equivalently obtaining that

(v) $\forall G \ni p\, \exists s \in G\, \exists x \in \mathrm{HS}\ [s \Vdash a = x \wedge x \mathrel{\overline{\in}_s} b]$.

Now note that $\{s \in P \mid \exists x \in M\ [s \Vdash a = x \wedge x \mathrel{\overline{\in}_s} b]\}$ is open with symmetry group $\mathrm{sym}(a) \cap \mathrm{sym}(b) \in \mathcal{F}$. Thus, as desired, Lemma 4.4 equivalently yields:

(vi) $\forall r \leq p\, \exists s \leq r\, \exists x \in \mathrm{HS}\ [x \mathrel{\overline{\in}_s} b \wedge s \Vdash a = x]$.

(2) Let us assume that (i) $p \Vdash a \subseteq b$.
By axiom (5), this is equivalent to (ii) $\forall G \ni p\ F_G(a) \subseteq F_G(b)$.
By the definition of F_G and axiom (4), this in turn is equivalent to

(iii) $\forall G \ni p\, \forall x \in \mathrm{HS}\, \forall r \in P\ [(x \mathrel{\overline{\in}_r} a \wedge r \in G) \to \exists s \in G\ s \Vdash x \in b]$.

Since all relevant r will be compatible with p, we may equivalently assume that $r \leq p$, and thus obtain the following equivalent form.

(iv) $\forall x \in \mathrm{HS}\, \forall r \leq p\, \forall G \ni r\ [x \mathrel{\overline{\in}_r} a \to \exists s \in G\ s \Vdash x \in b]$.

Now note that $\{s \in P \mid s \Vdash x \in b\}$ is open with symmetry group $\mathrm{sym}(x) \cap \mathrm{sym}(b) \in \mathcal{F}$. Thus, Lemma 4.4 equivalently yields:

(v) $\forall x \in \mathrm{HS}\, \forall r \leq p\ [x \mathrel{\overline{\in}_r} a \to \forall q \leq r\, \exists s \leq q\ s \Vdash x \in b]$.

Finally, it is easy to check that we equivalently obtain our desired statement below.

(vi) $\forall x \in \mathrm{HS}\, \forall r \in P\ [x \mathrel{\overline{\in}_r} a \to \forall q \leq r, p\, \exists s \leq q\ s \Vdash x \in b]$.

(3) is very easy. The remaining clauses are verified by induction on formula complexity, with (4) being very easy. Let us verify (5) and thus assume that

(i) $p \Vdash (\varphi \vee \psi)(\vec{a})$.

By axiom (5), this is equivalent to

(ii) $\forall G \ni p\ M_{\mathbb{S}}[G] \models (\varphi \vee \psi)(F_G(\vec{a}))$.

This in turn is equivalent to

(iii) $\forall G \ni p\ [M_{\mathbb{S}}[G] \models \varphi(\vec{a}) \vee M_{\mathbb{S}}[G] \models \psi(\vec{a})]$.

By axiom (4), we obtain the following equivalent form.

(iv) $\forall G \neq p\ [\exists q \in G\ q \Vdash \varphi(\vec{a}) \vee \exists q \in G\ q \Vdash \psi(\vec{a})]$.

Now note that $\{q \in P \mid q \Vdash \varphi(\vec{a}) \vee q \Vdash \psi(\vec{a})\}$ is open with symmetry group $\mathrm{sym}(a) \in \mathcal{F}$. Thus, Lemma 4.4 equivalently yields our desired equivalent form:

(v) $\forall q \leq p\, \exists r \leq q\ r \Vdash \varphi(\vec{a}) \vee r \Vdash \psi(\vec{a})$.

Let us verify (6) and thus assume that (i) $p \Vdash \exists x \varphi(x, \vec{a})$. By axiom (5), this is equivalent to (ii) $\forall G \ni p\ M_{\mathbb{S}}[G] \models \exists x\, \varphi(x, F_G(\vec{a}))$. This in turn is equivalent to

(iii) $\forall G \ni p\, \exists x \in \mathrm{HS}\, M_{\mathbb{S}}[G] \models \varphi(F_G(x), F_G(\vec{a}))$.

Now we use our induction hypothesis for φ, equivalently obtaining that

(iv) $\forall G \ni p\, \exists q \in G\, \exists x \in \mathrm{HS}\ q \Vdash \varphi(x, \vec{a})$.

Now note that $\{q \in P \mid \exists x \in \mathrm{HS}\ q \Vdash \varphi(x, \vec{a})\}$ is open and symmetric. Then, as desired, Lemma 4.4 equivalently yields:

(v) $\forall r \leq p\, \exists q \leq r\, \exists x \in \mathrm{HS}\ q \Vdash \varphi(x, \vec{a})$.

Finally, (7) is easy to verify, and we will leave this to the interested reader. □

The next lemma shows that, as one would perhaps hope, our forcing relations are symmetric.

Lemma 4.6. *For all $\varphi \in \mathcal{L}(\in)$, $\vec{a} \in \mathrm{HS}$, $p \in P$ and $\pi \in \mathcal{G}$,*

$$p \Vdash \varphi(a_0, \ldots, a_{m-1}) \text{ if and only if } \pi(p) \Vdash \varphi(\Omega(\pi)(a_0), \ldots, \Omega(\pi)(a_{m-1})).$$

Proof. By induction on formula complexity. For atomic formulas, we simultaneously argue for $\in$ and $=$ by induction on name rank (or, more precisely, by induction on pairs of name ranks, ordered lexicographically). By Lemma 4.5, $p \Vdash a \in b$ iff

$$\forall r \leq p\, \exists s \leq r\, \exists x \in M\ [x \mathrel{\overline{\in}_s} b \wedge s \Vdash a = x].$$

Note that $\mathrm{rank}(x) < \mathrm{rank}(b)$ in the above. Inductively, and using the symmetry axiom, we obtain that

$$\Omega(\pi)(x) \mathrel{\overline{\in}_{\pi(s)}} \Omega(\pi)(b) \wedge \pi(s) \Vdash \Omega(\pi)(a) = \Omega(\pi)(x).$$

Using Lemma 4.5, it follows that $\pi(p) \Vdash \Omega(\pi)(a) \in \Omega(\pi)(b)$. The reverse direction follows making use of $\pi^{-1} \in \mathcal{G}$. The argument for equality is analogous, using the respective statement in Lemma 4.5. For the case of negations, assume that $p \Vdash \neg\varphi(a_0, \ldots, a_{m-1})$. By Lemma 4.1, equivalently, $\forall q \leq p\ q \not\Vdash \varphi(a_0, \ldots, a_{m-1})$. Applying π, we obtain (inductively, using Lemma 4.1) that

$$\pi(p) \Vdash \neg\varphi(\Omega(\pi)(a_0), \ldots, \Omega(\pi)(a_{m-1})).$$

The reverse direction is again obtained by simply using π^{-1} in the same way. The remaining cases are essentially analogous to the case of negations, using the remaining clauses of Lemma 4.5. □

5. ZF IN SYMMETRIC EXTENSIONS

In this section, we consider the following statement.

(*) **Preservation of axioms:** $\forall G\ \mathcal{M}_{\mathbb{S}}[G] \models \mathrm{ZF}$.[14]

Let us start with the important remark that our axioms (1)–(7), as well as (*), hold in the standard setup for symmetric extensions, as described for example in [4]:[15] Given a countable transitive model $\mathcal{M}$ of ZF and a symmetric system $\mathbb{S} \in M$, interpreting $a \in_p b$ as $(a, p) \in b$, letting $\Omega(\pi)(b) = \{(\Omega(\pi)(a), \pi(p)) \mid (a, p) \in b\})$ for any $b \in M$, letting $\mathfrak{G}(G)$ hold if and only if G is a filter on $\mathbb{P}$ that intersects every symmetrically dense subset of P, and using the standard inductive definitions for the forcing predicates (which are exactly the ones we derived in Section 4), we arrive at a symmetric framework. The easy standard result known as the Rasiowa-Sikorski lemma implies that (1) for every $p \in P$, there is a (fully) $\mathbb{P}$-generic filter over $\mathcal{M}$ that contains p as an element. Axioms (2) and (3) are immediate from our above definitions. Verifying axioms (4) and (5) amounts to the proof of the forcing theorem in the standard setup (see for example [4]). Axiom (6) is immediate from our definitions, and axiom (7) follows taking, in the notation of that axiom, $T = S$, by a straightforward calculation using axiom (3) and the fact that any $F \in \mathcal{F}$ is closed under the taking of inverses (this is needed to check that $T \in \mathrm{HS}$). It is well-known [4] how to verify (*) with respect to $\mathcal{M}$ and $\mathbb{S}$ in this context.

We can however also derive axiom (*) from axioms (1)–(7). There are two different ways to do so. The first possibility is to make use of Lemma 3.2, showing that $\mathcal{M}_{\mathbb{S}}[G]$ is just the standard symmetric extension of $\mathcal{M}$ by the $\mathbb{S}$-generic filter G, and thus, again by the same standard arguments [4], $\mathcal{M}_{\mathbb{S}}[G] \models \mathrm{ZF}$. The second possibility is to actually verify the axioms of ZF in $\mathcal{M}_{\mathbb{S}}[G]$ using axioms (1)–(7). The advantage of this second option, which we choose in the below, is that the argument is self-contained.

Theorem 5.1. *Axioms (1)–(7) imply that (*)* $\mathcal{M}_{\mathbb{S}}[G] \models \mathrm{ZF}$.

Proof. Since $\mathcal{M}_{\mathbb{S}}[G]$ is a transitive $\in$-structure, it clearly satisfies Regularity and Extensionality. Using axiom (7), $\mathcal{M}_{\mathbb{S}}[G]$ satisfies Pairing:

[14]Due to axiom 5, this statement could equivalently be replaced by a scheme of axioms, consisting of statements of the form $1 \Vdash \varphi$ for every $\varphi \in \mathrm{ZF}$.

[15]Earlier references tend to make use of fully generic rather than just symmetrically generic filters, leading to a somewhat more restricted setting which simplifies the verification of (*), as it is possible to make use of the fact that symmetric extensions are submodels of fully generic extensions in this setting (as for example in [3]).

If $a, b \in \mathrm{HS}$, let $c =_R \{(a,1),(b,1)\}$, and let $F = \mathrm{sym}(a) \cap \mathrm{sym}(b) \in \mathcal{F}$, since $\mathcal{F}$ is a filter. Then, for every $\pi \in F$, clearly, $\pi(c) = c \in \mathrm{HS}$, and $F_G(c) = \{F_G(a), F_G(b)\}$. By Lemma 4.2, $\mathcal{M}_{\mathbb{S}}[G]$ satisfies Infinity.

Let us treat the union axiom: Let $a \in \mathrm{HS}$. We need to show that for some $b \in \mathrm{HS}$, $\bigcup F_G(a) \subseteq F_G(b)$. Let $X = \{c \mid \exists p\ c \in_p a\} \in M$ by axiom (6). Let $Y = \{d \mid \exists c \in X\, \exists q\ d \in_q c\} \in M$ by axiom (6). Using axiom (7), let $b =_R \{(d,1) \mid d \in Y\}$. It is straightforward to check that $F_G(b) \supseteq \bigcup F_G(a)$. It remains to show that $b \in \mathrm{HS}$. Let $\pi \in \mathrm{sym}(a)$. It follows that $\Omega(\pi)[X] = X$, which in turn implies that $\Omega(\pi)[Y] = Y$. It clearly follows that $\Omega(\pi)(b) = b$, and hence that $\mathrm{sym}(b) \supseteq \mathrm{sym}(a) \in \mathcal{F}$.

We now show that $\mathcal{M}_{\mathbb{S}}[G]$ satisfies collection: Let $a, t \in \mathrm{HS}$, let φ be a first order formula, and assume that $p \Vdash \forall x \in a\, \exists y\ \varphi(x,y,t)$. We will find $b \in \mathrm{HS}$ such that $p \Vdash \forall x \in a\, \exists y \in b\ \varphi(x,y,t)$. Let $X = \{c \mid \exists r\ c \in_r a\} \in M$. Using the axiom of collection in $\mathcal{M}$, let $Y \subseteq \mathrm{HS}$, $Y \in M$, be such that whenever $c \in X$ and $s \in P$ are such that $s \leq p$ and $s \Vdash c \in a$, if there is $y \in \mathrm{HS}$ such that $s \Vdash \varphi(c,y,t)$, then there is $y \in Y$ such that $s \Vdash \varphi(c,y,t)$. Let $Y^* = \{\Omega(\pi)(y) \mid y \in Y \wedge \pi \in \mathcal{G}\} \in M$. Let $b =_R \{(y,1) \mid y \in Y^*\} \in \mathrm{HS}$. Now if $c \in X$ and $s \leq p$ forces that $c \in a$, by Lemma 4.5, there is $u \leq s$ and $y \in \mathrm{HS}$ such that $u \Vdash \varphi(c,y,t)$. This shows that $p \Vdash \forall x \in a\, \exists y \in b\ \varphi(x,y,t)$, as desired.

Let us next show that $\mathcal{M}_{\mathbb{S}}[G]$ satisfies separation: Let $a, t \in \mathrm{HS}$ and let φ be a first order formula. Let $X = \{c \mid \exists p\ c \in_p a\} \in M$. Let

$$b =_R \{(c,p) \mid c \in X \ \wedge\ p \Vdash [c \in a \ \wedge\ \varphi(c,t)]\}.$$

Clearly, $b \in \mathrm{HS}$ for $\mathrm{sym}(b) \supseteq \mathrm{sym}(a) \cap \mathrm{sym}(t)$, since $\pi[X] = X$ for $\pi \in \mathrm{sym}(a)$, and by Lemma 4.6. But clearly also, $1 \Vdash b = \{x \in a \mid \varphi(x,t)\}$.

Finally, we argue that the powerset axiom is preserved to $\mathcal{M}_{\mathbb{S}}[G]$: Let $a \in \mathrm{HS}$, and let $X = \{c \mid \exists p\ c \in_p a\} \in M$. For $d \subseteq X \times P$ in M, let $x_d =_R d$. Let $b =_R \{(x_d, 1) \mid M \ni d \subseteq X \times P\}$. If $\pi \in \mathcal{G}$ and $d \subseteq X \times P$ in M, let $\pi^*[d] = \{(\Omega(\pi)(c), \pi(p)) \mid (c,p) \in d\}$. If $\pi \in \mathrm{sym}(a)$, then $\Omega(\pi)[X] = X$ and $\Omega(\pi)(x_d) = x_{\pi^*[d]}$, thus $\Omega(\pi)(b) = b$. Now assume that $e \in \mathrm{HS}$ is such that $F_G(e) \subseteq F_G(a)$ in $\mathcal{M}_{\mathbb{S}}[G]$. Let $p \in G$ force that $e \subseteq a$. Let $d = \{(c,r) \in X \times P \mid \exists q, f\ f \in_q e \ \wedge\ r \leq p, q \ \wedge\ r \Vdash f = c\}$ and let $x = x_d =_R d$. Since $F_G(x_d) \in F_G(b)$ by the definition of b, it only remains to check that $F_G(e) = F_G(x)$. Let $q \in G$ and $f \in \mathrm{HS}$ be such that $f \in_q e$. Then there is $r \leq p, q$ in G and $c \in X$ such that $r \Vdash f = c$, i.e., $(c,r) \in d$. This however implies that $F_G(f) \in F_G(x)$. On the other hand, if there is $r \in G$ and $c \in \mathrm{HS}$ such that $c \in_r x$, this means that $(c,r) \in d$. But then, there is $q \in G$ and $f \in_q e$ such that $r \Vdash f = c$. This implies that $F_G(c) \in F_G(e)$, as desired. □

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