THE ORDERING PRINCIPLE AND THE AXIOM OF DEPENDENT CHOICE

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Abstract. We introduce finite support iterations of symmetric systems, and use them to provide a strongly modernized proof of David Pincus' classical result that the axiom of dependent choice is independent over ZF with the ordering principle together with a failure of the axiom of choice.

1. Introduction

The ordering principle, OP, is the statement that every set can be linearly ordered. The axiom of choice, AC, in one of its equivalent forms, states that every set can be wellordered, and thus clearly implies OP. That this implication cannot be reversed was shown by Halpern and Lévy (see [1, Section 5.5]): The argument proceeds by showing that the basic Cohen model, which is well-known to satisfy ZF $+\neg$AC (see [1, Section 5.3]), satisfies OP. This model is obtained by first forcing to add countably many Cohen reals, and then passing to a symmetric submodel of this extension, in which we still have the set of those Cohen reals, but no wellordering of it. We may informally say that we *forget about* the wellordering in this submodel. On the other hand, it is easy to see that this submodel already fails to satisfy the axiom of dependent choice DC: The generic set of Cohen reals that were added is clearly infinite, however Dedekind-finite (see [1, Exercise 5.18]), i.e., ω does not inject into it. It is well-known (and an easy exercise) that DC implies the notions of being finite and of being Dedekind-finite to coincide.

The goal of our paper is to provide a new and strongly modernized proof of the following classical result of Pincus [5]:

Theorem 1 (Pincus)**.** [5] DC *is independent over* ZF $+$OP $+\neg$AC.

Given the properties of the basic Cohen model that we reviewed above, this amounts to verifying the relative consistency of ZF $+$OP $+$ DC $+\neg$AC, starting from ZF.

Pincus' paper makes use of the ramified forcing notation which developed directly out of Cohen's independence proof for CH. This old-fashioned way of presenting forcing already became outdated and essentially obsolete by the time [5] was published (see e.g. Shoenfield's [6]) and therefore, while he provides a very nice outline

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of his arguments, from a modern point of view, the details in his paper are difficult to grasp. For this reason, we think that providing a modern and essentially self-contained account of his result will be very interesting for and helpful to the set theoretic community. Furthermore, this paper provides an application of the technique of symmetric iterations that has been initiated by Asaf Karagila and has been further developed with the second author in [3]. Although finite support iterations have already appeared in some form in Karagila's [2], we provide a more compact and, at least in our view, simpler approach that follows more closely the familiar notation from usual forcing iterations. It is our hope that the presentation we give below can at some point become part of the general folklore, just as is the case for forcing. For this to happen, the method must be applicable in a broad sense, and our article is a proof-of-concept for this.

Our basic line of argument towards Pincus' result essentially follows the outline at the beginning of [5, Section 1]: The starting point is the basic Cohen model. It has a set of Cohen reals with no wellordering, and in fact, no countably infinite subset. In order to resurrect DC, one simply adds a surjection from ω onto this set, thus, however, even resurrecting AC. So what Pincus actually does is he adds not just one, but countably many such surjections, and then passes to a symmetric submodel of this second extension in which he forgets about the ordering of this set of surjections, thereby obtaining a new failure of DC. The failure of DC is thus shifted to a higher level of complexity. Now, the idea is to continue this process for ω_1-many stages, and that any possible failure of DC will somehow appear at an intermediate stage, and will actually be fixed by the very next stage, i.e., DC will hold in the final model. What we will actually show however, is something slightly stronger, namely that σ-covering holds between our symmetric extension and the corresponding full forcing extension, and that this in turn implies DC to hold in our symmetric extension. Finally, modifying the standard arguments for the basic Cohen model, it is still possible to show that OP holds in our symmetric extension, while AC fails in it.

Using this as a basic guideline, instead of following any of the further details in Pincus' paper, we came up with our own arguments, which we will provide below.

This paper is organised as follows: In Section 2, we will introduce some basic terminology regarding symmetric systems and extensions. In Section 3, we will briefly comment on how to deal with second order models in the context of symmetric extensions. In Section 4, we will introduce finite support iterations of symmetric systems. In Section 5, we will formally introduce the symmetric iteration that we will use to produce our desired model. In Section 6, we will verify that the ordering principle holds in our symmetric extension. In Section 7, we will show that DC holds in our symmetric extension, thus establishing Theorem 1.

2. Preliminaries

A symmetric system is a triple of the form $\mathcal{S} = (\mathbb{P}, \mathcal{G}, \mathcal{F})$, where $(\mathbb{P}, \leq)$ is a preorder, $\mathcal{G}$ is a group of automorphisms of $\mathbb{P}$ and $\mathcal{F}$ is a normal filter on the set of subgroups of $\mathcal{G}$. If $\dot{x}$ is a $\mathbb{P}$-name and $\pi \in \mathcal{G}$, we inductively let

$$\pi(\dot{x}) = \{(\pi(p), \pi(\dot{y})) \mid (p, \dot{y}) \in \dot{x}\},$$

and we let $\operatorname{sym}(\dot{x}) = \{\pi \in \mathcal{G} \mid \pi(\dot{x}) = \dot{x}\}$. The following fact is well-known and used throughout the paper:

Fact 2. *Let $\dot{x}_0, \ldots, \dot{x}_n$ be $\mathbb{P}$-names, π an automorphism of $\mathbb{P}$, $p \in \mathbb{P}$ and $\varphi(v_0, \ldots, v_n)$ some formula in the language of set theory. Then*

$$p \Vdash_{\mathbb{P}} \varphi(\dot{x}_0, \ldots, \dot{x}_n) \text{ if and only if } \pi(p) \Vdash_{\mathbb{P}} \varphi(\pi(\dot{x}_0), \ldots, \pi(\dot{x}_n)).$$

We say that a $\mathbb{P}$-name $\dot{x}$ is *symmetric* (according to $\mathcal{S}$) if $\operatorname{sym}(\dot{x}) \in \mathcal{F}$. We (inductively) say that $\dot{x}$ is *hereditarily symmetric* (according to $\mathcal{S}$) if $\dot{x}$ is symmetric and whenever $(p, \dot{y}) \in \dot{x}$, $\dot{y}$ is hereditarily symmetric. Given a symmetric system $\mathcal{S}$, we let $\mathrm{HS}_{\mathcal{S}}$ denote the collection of hereditarily symmetric $\mathbb{P}$-names, and we omit the subscript $\mathcal{S}$ when it is clear from context. We also refer to elements of HS as *$\mathcal{S}$-names*. The class HS is further stratified into sets $\mathrm{HS}_\alpha = \mathrm{HS}_{\mathcal{S},\alpha}$, α an ordinal, consisting of the $\mathcal{S}$-names of rank $< \alpha$.

An interesting fact is that we do not have to require our generic filters to be *fully generic* for $\mathbb{P}$: satisfying the weaker property of being *symmetrically generic* suffices. This will briefly be useful in Section 4, but isn't necessary for understanding the main result.

Definition 3. We say that a dense set $D \subseteq \mathbb{P}$ is *symmetric* if $\operatorname{sym}(D) = \{\pi \in \mathcal{G} : \pi[D] = D\} \in \mathcal{F}$. Let G be a filter on $\mathbb{P}$. Then G is *$\mathcal{S}$-generic*, or *symmetrically generic*, over M, if for every symmetric dense subset $D \subseteq \mathbb{P}$ in M, $G \cap D \neq \emptyset$.

If M is a transitive model of ZF, a symmetric extension of M via $\mathcal{S}$, or in other words, an $\mathcal{S}$-generic extension of M, is a model of the form $M[G]_{\mathcal{S}} = \{\dot{x}^G \mid \dot{x} \in \mathrm{HS}\}$ for an $\mathcal{S}$-generic filter G over M. We let $\Vdash_{\mathcal{S}}$ denote the (symmetric) forcing relation of the system $\mathcal{S}$, which is defined inductively just like the usual forcing relation, however restricting to hereditarily symmetric names (in particular also in the existential and universal quantification steps). The forcing theorem holds with respect to this relation and $\mathcal{S}$-generic extensions [3].

Fact 4 (see [3])**.** *Let G be $\mathcal{S}$-generic over M. Then we have the following:*

- The symmetric forcing theorem*: The forcing theorem holds with respect to $\Vdash_{\mathcal{S}}$ and $\mathcal{S}$-generic extensions of M.*
- *$M[G]_{\mathcal{S}} \models \mathrm{ZF}$.*
- *There is a $\mathbb{P}$-generic H over M so that $M[H]_{\mathcal{S}} = M[G]_{\mathcal{S}}$.*

The last item really says that there is no distinction between the models obtained via full versus symmetric generics.

If A is a set of $\mathbb{P}$-names, then $A^\bullet = \{\mathbb{1}\} \times A$ is the canonical $\mathbb{P}$-name for the set containing the elements of A (or more precisely, their evaluations by the generic filter). Similar notation is applied to sequences of $\mathbb{P}$-names, so that for instance $\langle \dot{a}_i : i < n \rangle^\bullet$ becomes the canonical name for the ordered tuple of $\dot{a}_i$, $i < n$.

We will sometimes use the following general fact, which says that we can uniformly find names for definable objects.

Fact 5. *Let $\mathcal{S} = (\mathbb{P}, \mathcal{G}, \mathcal{F})$ be a symmetric system, and let $\varphi(u, v_0, \ldots, v_n)$ be a formula in the language of set theory. Then, there is a definable class function F so that for any $\mathcal{S}$-names $\dot{x}_0, \ldots, \dot{x}_n$ and $p \in \mathbb{P}$ with*

$$p \Vdash_{\mathcal{S}} \exists! y \varphi(y, \dot{x}_0, \ldots, \dot{x}_n),$$

$\dot y = F(p, \dot x_0, \ldots, \dot x_n)$ is an $\mathcal{S}$-name with $\bigcap_{i\le n} \mathrm{sym}(\dot x_i) \le \mathrm{sym}(\dot y)$ so that

$$p \Vdash_{\mathcal{S}} \varphi(\dot y, \dot x_0, \ldots, \dot x_n).$$

Proof. Let γ be the least ordinal such that

$$p \Vdash_{\mathcal{S}} \exists y \in \mathrm{HS}^\bullet_\gamma\ \varphi(y, \dot x_0, \ldots, \dot x_n).$$

Let $\dot y$ be the set of all pairs $(q, \dot z) \in \mathbb{P} \times \mathrm{HS}_\gamma$ so that

$$q \Vdash \forall y(\varphi(y, \dot x_0, \ldots, \dot x_n) \to \dot z \in y)\}.$$

□

3. Symmetric extensions as second order models

Let $\mathcal{S} = (\mathbb{P}, \mathcal{G}, \mathcal{F})$ be a symmetric system in a model $\mathcal{M} = (M, \mathcal{C})$ of GB, that is Gödel-Bernays set theory, with M its domain of sets, and with $\mathcal{C}$ its domain of classes. In case we are starting with a model of ZF, it yields a model of GB when endowed with its definable classes. In $\mathcal{M}$, we say that a class $\dot X \subseteq \mathbb{P} \times \mathrm{HS}$ is a *class $\mathcal{S}$-name* if

$$\mathrm{sym}(\dot X) := \{\pi \in \mathcal{G} \mid \pi(\dot X) = \dot X\} \in \mathcal{F},$$

where $\pi(\dot X) = \{(\pi(p), \pi(\dot x)) : (p, \dot x) \in \dot X\}$. Let G be an $\mathcal{S}$-generic filter over $\mathcal{M}$, and let $\mathcal{N} = \mathcal{M}[G]_{\mathcal{S}} = (M[G]_{\mathcal{S}}, \mathcal{C}[G]_{\mathcal{S}})$, where $\mathcal{C}[G]$ is obtained by evaluating class names in $\mathcal{C}$ with G.[1] We will use uppercase letters to refer to classes and class names, while lowercase letters indicate sets or set names. When we allow for classes as parameters in first order formulas, we also mean to include additional atomic formulas of the form $x \in X$.

Proposition 6. *The symmetric forcing theorem can be extended to first order formulas using classes as parameters, and $\mathcal{N} \models \mathrm{GB}$.*

Proof. The verification of the extension of the symmetric forcing theorem is very much standard (proceeding exactly as for the usual forcing theorem) - see also [3]. Let us verify that the axiom of Collection holds in $\mathcal{N}$. So assume φ is a first order formula using class parameters, and $p \in \mathbb{P}$ is such that $p \Vdash_{\mathcal{S}} \forall x\, \exists y\, \varphi(x, y, \vec X)$, with $\vec X$ a finite sequence of class $\mathcal{S}$-names, and let $\dot a \in M$ be a $\mathbb{P}$-name. Since we may code any finite number of classes by a single class, it suffices to consider a single class $\mathcal{S}$-name $\dot X \in \mathcal{C}$, which we may also assume to code the set parameters appearing in φ. Let $\dot z = \{(q, \dot y) \mid \exists \dot x \in \mathrm{ran}(\dot a)\ \dot y$ is of minimal rank s.t. $q \Vdash_{\mathcal{S}} \varphi(x, \dot y, \dot X)\}$. Then, $\mathrm{sym}(\dot z) \ge \mathrm{sym}(\dot X) \cap \mathrm{sym}(\dot a) \in \mathcal{F}$, and $p \Vdash_{\mathcal{S}} \forall x \in \dot a\, \exists y \in \dot z\, \varphi(x, y, \dot X)$, thus witnessing Collection to hold in $\mathcal{N}$. Comprehension, and Class Comprehension, that is, the closure of $\mathcal{C}[G]$ under definability, are verified in similar ways (and somewhat more easily). We thus leave the details here to our readers. □

4. Finite support iterations of symmetric systems

Definition 7 (Two-step iteration, see e.g. [3])**.** Let $\mathcal{S} = (\mathbb{P}, \mathcal{G}, \mathcal{F})$ be a symmetric system and $\dot{\mathcal{T}} = (\dot{\mathbb{Q}}, \dot{\mathcal{H}}, \dot{\mathcal{E}})^\bullet$ be an $\mathcal{S}$-name for a symmetric system, where $\mathrm{sym}(\dot{\mathcal{T}}) = \mathcal{G}$. Then, we define the two-step iteration $\mathcal{S} * \dot{\mathcal{T}} = (\mathbb{P} *_{\mathcal{S}} \dot{\mathbb{Q}}, \mathcal{G} *_{\mathcal{S}} \dot{\mathcal{H}}, \mathcal{F} *_{\mathcal{S}} \dot{\mathcal{E}})$, where

[1] Note that the classes of $\mathcal{N}$ thus include all definable classes of $\mathcal{M}$ (for they have canonical symmetric names).

(1) $\mathbb{P} *_{\mathcal{S}} \dot{\mathbb{Q}}$ consists of all pairs $(p, \dot{q})$, where $p \in \mathbb{P}$ and $\dot{q}$ is an $\mathcal{S}$-name for an element of $\dot{\mathbb{Q}}$, together with the usual order on $\mathbb{P} * \dot{\mathbb{Q}}$,[2]
(2) $\mathcal{G} *_{\mathcal{S}} \dot{\mathcal{H}}$ consists of all pairs $(\pi, \dot{\sigma})$, where $\pi \in \mathcal{G}$ and $\dot{\sigma}$ is an $\mathcal{S}$-name for an element of $\dot{\mathcal{H}}$, and $(\pi, \dot{\sigma})$ is identified with the map
$$(p, \dot{q}) \mapsto (\pi(p), \dot{\sigma}(\pi(\dot{q}))),$$
(3) $\mathcal{F} *_{\mathcal{S}} \dot{\mathcal{E}}$ is generated by all groups of the form $(H_0, \dot{H}_1)$, where $H_0 \in \mathcal{F}$ and $\dot{H}_1$ is an $\mathcal{S}$-name for an element of $\dot{\mathcal{E}}$ with $H_0 \leq \operatorname{sym}(\dot{H}_1)$, and $(H_0, \dot{H}_1)$ is identified with $\{(\pi, \dot{\sigma}) : \pi \in H_0, \Vdash_{\mathcal{S}} \dot{\sigma} \in \dot{H}_1\}$.

A few remarks have to be made about this definition. The fact that we identify pairs with other types of objects should not lead to any confusion. When we write $\dot{\sigma}(\pi(\dot{q}))$, what we mean is a particular $\mathcal{S}$-name for the result of $\dot{\sigma}$ applied to $\pi(\dot{q})$.[3] While there is in fact a way to uniformly choose such a name (see [3]), the easiest way to make sense of this, and what we will actually do, is to simply identify a pair $(p, \dot{q})$ with the set of equivalent conditions $\{(p, \dot{r}) : \dot{r} \in \mathrm{HS}_\gamma, \Vdash_{\mathcal{S}} \dot{q} = \dot{r}\}$, where γ is least so that this set is nonempty. This has the added advantage that $\mathbb{P} *_{\mathcal{S}} \dot{\mathbb{Q}}$ really becomes a set, while technically, there are proper class many possible names for elements of $\dot{\mathbb{Q}}$. Again, this identification makes no difference in practice.

It is then relatively straightforward to check that $(\pi, \dot{\sigma})$ preserves the order on $\mathbb{P} *_{\mathcal{S}} \dot{\mathbb{Q}}$. One can compute ([3, proof of Lemma 3.2]) that
$$(\pi_0, \dot{\sigma}_0) \circ (\pi_1, \dot{\sigma}_1) = (\pi_0 \circ \pi_1, \dot{\sigma}_0 \circ \pi_0(\dot{\sigma}_1)),$$
and that
$$(\pi, \dot{\sigma})^{-1} = (\pi^{-1}, \pi^{-1}(\dot{\sigma}^{-1})),$$
where $\dot{\sigma}_0 \circ \pi_0(\dot{\sigma}_1)$ and $\dot{\sigma}^{-1}$ are $\mathcal{S}$-names for the respective objects. In particular, $(\pi, \dot{\sigma})$ is an automorphism, and $\mathcal{G} *_{\mathcal{S}} \dot{\mathcal{H}}$ forms a group.

Now, it is possible to make sense of (3), as it can be checked that the sets $(H_0, \dot{H}_1)$ are subgroups of $\mathcal{G} *_{\mathcal{S}} \dot{\mathcal{H}} = (\mathcal{G}, \dot{\mathcal{H}})$. It turns out that the filter $\mathcal{F} *_{\mathcal{S}} \dot{\mathcal{E}}$ generated by these subgroups is in fact a normal filter. This is a bit more tricky to prove and, letting $\bar{\pi} = (\pi_0, \dot{\pi}_1)$, it is generally not the case that $\bar{\pi}(H_0, \dot{H}_1)\bar{\pi}^{-1}$ is a group of the form $(K_0, \dot{K}_1)$ as in (3) again. On the other hand, if $H_0 \leq \operatorname{sym}(\pi_0^{-1}(\dot{\pi}_1^{-1}))$, which we may achieve by shrinking H_0, then $\bar{\pi}(H_0, \dot{H}_1)\bar{\pi}^{-1} = (\pi_0 H_0 \pi_0^{-1}, \dot{\pi}_1 \pi_0(\dot{H}_1)\dot{\pi}_1^{-1})$ – see [3, proof of Lemma 3.2].

Lemma 8 ([3, Lemma 3.2])**.** *$\mathcal{S} * \dot{\mathcal{T}}$ is a symmetric system.*

Moreover, one can prove a factorization theorem ([3, Theorem 3.3]) that expresses precisely that an extension via $\mathcal{S} * \dot{\mathcal{T}}$ is of the form $\mathcal{M}[G]_{\mathcal{S}}[H]_{\mathcal{T}}$ and vice-versa. There is some care to be taken though: It *does not follow* from G's $\mathbb{P}$-genericity over $\mathcal{M}$ and H's $\dot{\mathbb{Q}}^G$-genericity over $\mathcal{M}[G]_{\mathcal{S}}$, that $G * H = \{(p, \dot{q}) \in \mathbb{P} *_{\mathcal{S}} \dot{\mathbb{Q}} : p \in G \wedge \dot{q}^G \in H\}$ is itself $\mathbb{P} *_{\mathcal{S}} \dot{\mathbb{Q}}$-generic over $\mathcal{M}$. Rather, the factorization theorem states that if G is $\mathcal{S}$-generic (that is, symmetrically generic) over $\mathcal{M}$ and H is $\dot{\mathcal{T}}^G$-generic over $\mathcal{M}[G]_{\mathcal{S}}$, then $G * H$ is also $\mathcal{S} * \dot{\mathcal{T}}$-generic over $\mathcal{M}$. Conversely, if K is some $\mathcal{S} * \dot{\mathcal{T}}$-generic over $\mathcal{M}$, then $K = G * H$, where $G = \operatorname{dom} K$ is $\mathcal{S}$-generic

[2]Note that this is a dense subset of the usual forcing iteration $\mathbb{P} * \dot{\mathbb{Q}}$.

[3]$\pi(\dot{q})$ denotes the usual application of π to the $\mathcal{S}$-name $\dot{q}$. Note that since $\operatorname{sym}(\dot{\mathbb{Q}}) = \mathcal{G}$, $\pi(\dot{q})$ is again an $\mathcal{S}$-name for an element of $\dot{\mathbb{Q}}$.

over $\mathcal{M}$ and $H = \{\dot{q}^G : \dot{q} \in \operatorname{ran}(K)\}$ is $\dot{\mathcal{T}}^G$-generic over $\mathcal{M}[G]_{\mathcal{S}}$. In either case, $\mathcal{M}[G]_{\mathcal{S}}[H]_{\dot{\mathcal{T}}^G} = \mathcal{M}[G * H]_{\mathcal{S}*\dot{\mathcal{T}}}$.

On the level of the models alone, there is no difference between those obtained via full generics or those obtained via symmetric generics. Thus, it is nevertheless the case than an $\mathcal{S} * \dot{\mathcal{T}}$-extension obtained via a full generic is exactly the result of extending in succession using full generics of the respective systems, and vice-versa.

Definition 9 (Finite support iteration)**.** Let δ be an ordinal. Let

$$\langle \mathcal{S}_\alpha, \dot{\mathcal{T}}_\alpha : \alpha < \delta \rangle = \langle (\mathbb{P}_\alpha, \mathcal{G}_\alpha, \mathcal{F}_\alpha), (\dot{\mathbb{Q}}_\alpha, \dot{\mathcal{H}}_\alpha, \dot{\mathcal{E}}_\alpha)^\bullet : \alpha < \delta \rangle,$$

be such that each $\mathcal{S}_\alpha$ is a symmetric system, $\dot{\mathcal{T}}_\alpha$ is an $\mathcal{S}_\alpha$-name for a symmetric system and $\operatorname{sym}(\dot{\mathcal{T}}_\alpha) = \mathcal{G}_\alpha$. Then we call this sequence a *finite support iteration* of length δ if for each $\alpha < \delta$:

(1) (a) $\mathbb{P}_\alpha$ consists of sequences $\bar{p} = \langle \dot{p}(\beta) : \beta < \alpha \rangle$, where $\bar{p} \restriction \beta \in \mathbb{P}_\beta$ and $\dot{p}(\beta)$ is an $\mathcal{S}_\beta$ name for an element of $\dot{\mathbb{Q}}_\beta$, for all $\beta < \alpha$.
  (b) $\mathcal{G}_\alpha$ consists of automorphisms of $\mathbb{P}_\alpha$ represented, as detailed below, by sequences $\bar{\pi} = \langle \dot{\pi}(\beta) : \beta < \alpha \rangle$, where $\bar{\pi} \restriction \beta \in \mathcal{G}_\beta$ and $\dot{\pi}(\beta)$ is an $\mathcal{S}_\beta$ name for an element of $\dot{\mathcal{H}}_\beta$, for all $\beta < \alpha$.
  (c) $\mathcal{F}_\alpha$ is generated by subgroups of $\mathcal{G}_\alpha$ represented, as detailed below, by sequences $\bar{H} = \langle \dot{H}(\beta) : \beta < \alpha \rangle$, where $\bar{H} \restriction \beta \in \mathcal{F}_\beta$, $\dot{H}(\beta)$ is an $\mathcal{S}_\beta$ name for an element of $\dot{\mathcal{E}}_\beta$ and $\bar{H} \restriction \beta \leq \operatorname{sym}_{\mathcal{S}_\beta}(\dot{H}(\beta))$, for all $\beta < \alpha$.

(2) $\mathcal{S}_{\alpha+1} = \mathcal{S}_\alpha * \dot{\mathcal{T}}_\alpha$, where pairs $(\bar{p}, \dot{q})$, $(\bar{\pi}, \dot{\sigma})$, $(\bar{H}, \dot{K})$ as in Definition 7 are identified with sequences $\bar{p}^\frown \dot{q}$, $\bar{\pi}^\frown \dot{\sigma}$ and $\bar{H}^\frown \dot{K}$ respectively.

For $\alpha < \delta$ limit,

(3) (a) $\mathbb{P}_\alpha$ consists exactly of those $\bar{p}$ as above, so that $\Vdash_{\mathcal{S}_\beta} \dot{p}(\beta) = \mathbb{1}$ for all but finitely many $\beta < \alpha$, and $\bar{q} \leq \bar{p}$ iff $\bar{q} \restriction \beta \leq \bar{p} \restriction \beta$ for each $\beta < \alpha$,
  (b) $\mathcal{G}_\alpha$ consists exactly of those $\bar{\pi}$ as above, so that $\Vdash_{\mathcal{S}_\beta} \dot{\pi}(\beta) = \mathrm{id}$ for all but finitely many $\beta < \alpha$, and

$$\bar{\pi}(\bar{p}) = \bigcup_{\beta < \alpha} (\bar{\pi} \restriction \beta)(\bar{p} \restriction \beta),$$

  (c) $\mathcal{F}_\alpha$ is generated by the subgroups of the form $\bar{H} = \langle \dot{H}(\beta) : \beta < \alpha \rangle$ as above, where $\Vdash_{\mathcal{S}_\beta} \dot{H}(\beta) = \dot{\mathcal{H}}_\beta$ for all but finitely many $\beta < \alpha$, and $\bar{\pi} \in \bar{H}$ iff $\bar{\pi} \restriction \beta \in \bar{H} \restriction \beta$ for all $\beta < \alpha$.

As it stands, the above is just a definition. But of course, the way to read this in practice is as an instruction on how to recursively construct a finite support iteration. The definition says precisely what to do in each step of the construction. In the successor step, when constructing $\mathcal{S}_{\alpha+1} = \mathcal{S}_\alpha * \dot{\mathcal{T}}_\alpha$, we already know that this results in a symmetric system. On the other hand, to ensure that such a construction always makes sense we still need to check that if the limit step is defined as in (3), we really do obtain a symmetric system again.

Before doing so, let's make a few simple observations about finite-support iterations. Let $\operatorname{supp}(\bar{p}) := \{\alpha : \neg \Vdash_{\mathcal{S}_\alpha} \dot{p}(\alpha) = \mathbb{1}\}$, $\operatorname{supp}(\bar{\pi}) := \{\alpha : \neg \Vdash_{\mathcal{S}_\alpha} \dot{\pi}(\alpha) = \mathrm{id}\}$ and $\operatorname{supp}(\bar{H}) := \{\alpha : \neg \Vdash_{\mathcal{S}_\alpha} \dot{H}(\alpha) = \dot{\mathcal{H}}_\alpha\}$. We refer to the above as *support* in each case.

Lemma 10. *Let $\langle \mathcal{S}_\alpha, \dot{\mathcal{T}}_\alpha : \alpha < \delta \rangle$ be a finite support iteration as above, $\alpha < \delta$ arbitrary.*

(1) *All* $\bar{p} \in \mathbb{P}_\alpha$, $\bar{\pi} \in \mathcal{G}_\alpha$, $\bar{H} \in \mathcal{F}_\alpha$ *have finite support.*
(2) *For any* $\bar{p} \in \mathbb{P}_\alpha$, $\bar{\pi} \in \mathcal{G}_\alpha$, $\operatorname{supp}(\bar{p}) = \operatorname{supp}(\bar{\pi}(\bar{p}))$.
(3) *For any* $\bar{p} \in \mathbb{P}_\alpha$, $\bar{\pi} \in \mathcal{G}_\alpha$, $\beta \leq \alpha$, $\bar{\pi}(\bar{p}) \restriction \beta = (\bar{\pi} \restriction \beta)(\bar{p} \restriction \beta)$.
(4) *For any* $\beta \leq \alpha$, $\bar{p} \in \mathbb{P}_\beta$, $\bar{\pi} \in \mathcal{G}_\beta$ *and* $\bar{H} \in \mathcal{F}_\beta$, *we have that* $\bar{p}^\frown\langle \dot{\mathbb{1}}_\gamma : \gamma \in [\beta, \alpha)\rangle \in \mathbb{P}_\alpha$, $\bar{\pi}^\frown\langle \dot{\operatorname{id}}_\gamma : \gamma \in [\beta, \alpha)\rangle \in \mathcal{G}_\alpha$, $\bar{H}^\frown\langle \dot{\mathcal{H}}_\gamma : \gamma \in [\beta, \alpha)\rangle \in \mathcal{F}_\alpha$, *where* $\dot{\mathbb{1}}_\gamma$ *is a name for the trivial condition of* $\dot{\mathbb{Q}}_\gamma$ *and* $\dot{\operatorname{id}}_\gamma$ *for the identity in* $\dot{\mathcal{H}}_\gamma$. *In particular,* $\mathbb{P}_\beta = \{\bar{p} \restriction \beta : \bar{p} \in \mathbb{P}_\alpha\}$.
(5) *For any* $\bar{\pi}, \bar{\sigma} \in \mathcal{G}_\alpha$, $\bar{\pi} \circ \bar{\sigma} = \langle \dot{\pi}(\beta) \circ (\bar{\pi} \restriction \beta)(\dot{\sigma}(\beta)) : \beta < \alpha\rangle$ *and* $\operatorname{supp}(\bar{\pi} \circ \bar{\sigma}) \subseteq \operatorname{supp}(\bar{\pi}) \cup \operatorname{supp}(\bar{\sigma})$. *In particular,* $(\bar{\pi} \circ \bar{\sigma}) \restriction \beta = (\bar{\pi} \restriction \beta) \circ (\bar{\sigma} \restriction \beta)$, *for all* $\beta \leq \alpha$.
(6) *For any* $\bar{\pi} \in \mathcal{G}_\alpha$, $\bar{\pi}^{-1} = \langle (\bar{\pi} \restriction \beta)^{-1}(\dot{\pi}(\beta)^{-1}) : \beta < \alpha\rangle$ *and* $\operatorname{supp}(\bar{\pi}^{-1}) = \operatorname{supp}(\bar{\pi})$. *In particular,* $(\bar{\pi}^{-1}) \restriction \beta = (\bar{\pi} \restriction \beta)^{-1}$, *for all* $\beta \leq \alpha$.
(7) *For any of the generators* $\bar{H}, \bar{K} \in \mathcal{F}_\alpha$ *as in (1)(c) in the definition of a finite support iteration,* $\bar{E} = \langle \dot{E}(\beta) : \beta \leq \alpha\rangle \in \mathcal{F}_\alpha$, *where* $\dot{E}(\beta)$ *is a name for* $\dot{H}(\beta) \cap \dot{K}(\beta)$ *for each* $\beta < \alpha$, $\operatorname{supp}(\bar{E}) \subseteq \operatorname{supp}(\bar{H}) \cup \operatorname{supp}(\bar{K})$ *and* $\bar{E} \leq \bar{H} \cap \bar{K}$.
(8) *For each* $\bar{p} \in \mathbb{P}_\alpha$, $\bar{\pi} \in \mathcal{G}_\alpha$ *there are* $\bar{H}, \bar{K} \in \mathcal{F}_\alpha$ *so that* $\bar{H} \restriction \beta \leq \operatorname{sym}(\dot{p}(\beta))$ *and* $\bar{K} \restriction \beta \leq \operatorname{sym}((\bar{\pi} \restriction \beta)^{-1}(\dot{\pi}(\beta)^{-1}))$ *for each* $\beta < \alpha$.
(9) *For any* $\bar{\pi} \in \mathcal{G}_\alpha$ *and* $\bar{H} \in \mathcal{F}_\alpha$, *where for each* $\beta < \alpha$,
$$\bar{H} \restriction \beta \leq \operatorname{sym}((\bar{\pi} \restriction \beta)^{-1}(\dot{\pi}(\beta)^{-1})),$$
we have that
$$\bar{\pi}\bar{H}\bar{\pi}^{-1} = \langle \dot{\pi}(\beta)(\pi \restriction \beta)(\dot{H}(\beta))\dot{\pi}(\beta)^{-1} : \beta < \alpha\rangle$$
and $\operatorname{supp}(\bar{\pi}\bar{H}\bar{\pi}^{-1}) = \operatorname{supp}(\bar{H})$.
(10) *If* $\langle \mathbb{P}'_\beta, \dot{\mathbb{Q}}_\beta : \beta \leq \alpha\rangle$ *is the usual finite-support iteration of forcing notions, then* $\mathbb{P}_\alpha$ *is a dense subposet of* $\mathbb{P}'_\alpha$.

Proof. These are all straightforward inductions on α. For (5) and (6), use the inverse and composition formulas we have already given for the two-step iteration. For (9) use the analogous statement for two-step iterations we have mentioned above. Note that since our conditions have finite supports, limit stages in (10) are trivial. □

Lemma 11. *Let* $\langle \mathcal{S}_\beta, \dot{\mathcal{T}}_\beta : \beta < \alpha\rangle$ *be a finite-support iteration and let* $\mathcal{S}_\alpha = (\mathbb{P}_\alpha, \mathcal{G}_\alpha, \mathcal{F}_\alpha)$ *be defined as in (3) of Definition 9. Then,* $\mathcal{S}_\alpha$ *is a symmetric system.*

Proof. $\mathbb{P}_\alpha$ is clearly a forcing poset. Next, let $\bar{\pi} \in \mathcal{G}_\alpha$ and $\bar{p} \in \mathbb{P}_\alpha$ be given. According to Item (3) of Lemma 10, $(\bar{\pi} \restriction \beta)(\bar{p} \restriction \beta) \subseteq (\bar{\pi} \restriction \gamma)(\bar{p} \restriction \gamma)$ for every $\beta \leq \gamma < \alpha$, so $\bar{\pi}(\bar{p})$ is a sequence as in (1)(a) of Definition 9. Items (1) and (2) of Lemma 10 imply that $\operatorname{supp}(\bar{\pi}(\bar{p}))$ is still finite, so $\bar{\pi}(\bar{p}) \in \mathbb{P}_\alpha$. Clearly, $\bar{\pi}$ is also order-preserving and the inverse and composition formulas given in (5) and (6) above also work for the elements of $\mathcal{G}_\alpha$. Thus, $\mathcal{G}_\alpha$ is a group of automorphisms, and similarly, any $\bar{H}$ as in (3)(c) of Definition 9 is a subgroup of $\mathcal{G}_\alpha$. It remains to check that $\mathcal{F}_\alpha$ is normal. So let $\bar{H} \in \mathcal{F}_\alpha$, $\bar{\pi} \in \mathcal{G}_\alpha$ be arbitrary. $\operatorname{supp}(\bar{\pi}) \subseteq \beta$ for some $\beta \leq \alpha$ and using (7) and (8) above we can find $\bar{K} \in \mathcal{F}_\beta$, $\bar{K} \leq \bar{H} \restriction \beta$, so that $\bar{K} \restriction \gamma \leq \operatorname{sym}((\bar{\pi} \restriction \gamma)^{-1}(\dot{\pi}(\gamma)^{-1}))$ for each $\gamma < \beta$. Then $\bar{H}' := \bar{K}^\frown\bar{H} \restriction [\beta, \alpha) \leq \bar{H}$ and $\bar{H}' \in \mathcal{F}_\alpha$. From (9), also using (5), we can compute that
$$\bar{\pi}\bar{H}'\bar{\pi}^{-1} = \langle \dot{\pi}(\gamma)(\pi \restriction \gamma)(\dot{H}'(\gamma))\dot{\pi}(\gamma)^{-1} : \gamma < \alpha\rangle,$$

which has finite support and thus is in $\mathcal{F}_\alpha$. □

Lemma 12. *Let $\langle \mathcal{S}_\alpha, \dot{\mathcal{T}}_\alpha : \alpha < \delta\rangle$ be a finite support iteration as above. Fix some $\alpha < \delta$, let $\bar\pi, \bar\sigma \in \mathcal{G}_\alpha$, $\bar p \in \mathbb{P}_\alpha$, and assume that for all $\beta < \alpha$, $\bar p \restriction \beta \Vdash \dot\pi(\beta) = \dot\sigma(\beta)$. Then, for any $\mathbb{P}_\alpha$-name $\dot x$,*

$$\bar p \Vdash \bar\pi(\dot x) = \bar\sigma(\dot x).$$

Proof. This is essentially the same as [3, Lemma 5.5]. More precisely, work with a generic $G \ni \bar p$ and show by induction on $\beta \le \alpha$ that for any $\bar q$, $(\bar\pi \restriction \beta)(\bar q \restriction \beta) \in G$ iff $(\bar\sigma \restriction \beta)(\bar q \restriction \beta) \in G$. The rest then follows by induction on the rank of $\dot x$. □

A factorization theorem can also be proven for finite support iterations, similarly to the one for two-step iterations. We will not need this anywhere in our results, so the reader may immediately skip to the next section, but it is still important enough for the general theory so that we would like to include it.

Suppose that $\langle \mathcal{S}_\alpha, \dot{\mathcal{T}}_\alpha : \alpha \le \delta\rangle$ is a finite support iteration and $\alpha \le \delta$ is fixed. By recursion on the length δ, one defines an $\mathcal{S}_\alpha$-name $\langle \dot{\mathcal{S}}_{\alpha,\gamma}, \dot{\mathcal{T}}_{\alpha,\gamma} : \gamma \in [\alpha, \delta]\rangle^\bullet$ for a finite support iteration that naturally corresponds to the tail of the iteration. Simultaneously, one defines for each $\mathcal{S}_\delta$-name $\dot x$, an $\mathcal{S}_\alpha$-name $[\dot x]_{\alpha,\delta}$ for an $\dot{\mathcal{S}}_{\alpha,\delta}$-name, and similarly, for each $\mathcal{S}_\alpha$-name $\dot y$ for an $\dot{\mathcal{S}}_{\alpha,\delta}$-name, an $\mathcal{S}_\delta$-name $]\dot y[_{\alpha,\delta}$. Further, for any $\bar p \in \mathbb{P}_\delta$, $\bar\pi \in \mathcal{G}_\delta$ and $\bar H \in \mathcal{F}_\delta$, one defines $\mathcal{S}_\alpha$-names $[\bar p \restriction [\alpha, \delta)]$, $[\bar\pi \restriction [\alpha, \delta)]$ and $[\bar H \restriction [\alpha, \delta)]$ for respective objects in the system $\dot{\mathcal{S}}_{\alpha,\delta}$.

The recursive construction proceeds as follows: For $\delta = \alpha$, we let $\dot{\mathcal{S}}_{\alpha,\gamma}$ be a name for the trivial system $(\{\mathbb{1}\}, \{\mathrm{id}\}, \{\{\mathrm{id}\}\})$. $[\bar p \restriction [\alpha, \alpha)]$ is simply a name for $\mathbb{1}$. At each step δ, by recursion on the rank of names, we define

$$[\dot x]_{\alpha,\delta} = \{(\bar p \restriction \alpha, ([\bar p \restriction [\alpha, \delta)], [\dot z]_{\alpha,\delta})^\bullet) : (\bar p, \dot z) \in \dot x\},$$

and similarly,

$$]\dot y[_{\alpha,\delta} = \{(\bar p,]\dot z[_{\alpha,\delta}) : \bar p \restriction \alpha \Vdash_{\mathcal{S}_\alpha} ([\bar p \restriction [\alpha, \delta)], \dot z)^\bullet \in \dot y\}.$$

We let $\dot{\mathcal{T}}_{\alpha,\delta} = [\dot{\mathcal{T}}_\delta]_{\alpha,\delta}$. For $\delta = \gamma + 1$, we define

$$[\bar p \restriction [\alpha, \delta)] = ([\bar p \restriction [\alpha, \gamma)]^\frown [\dot p(\gamma)]_{\alpha,\gamma})^\bullet,$$

and for δ limit,

$$[\bar p \restriction [\alpha, \delta)] = \bigcup_{\gamma \in [\alpha,\delta)} [\bar p \restriction [\alpha, \gamma)].$$

Similarly for $[\bar\pi \restriction [\alpha, \delta)]$ and $[\bar H \restriction [\alpha, \delta)]$.

The properties claimed above are easily checked by induction on δ.

Theorem 13 (Factorization for finite support iterations)**.** *Whenever G is $\mathcal{S}_\alpha$-generic over $\mathcal{M}$ and H is $\dot{\mathcal{S}}^G_{\alpha,\delta}$-generic over $\mathcal{M}[G]_{\mathcal{S}_\alpha}$, then $G * H = \{\bar p : \bar p \restriction \alpha \in G \wedge [\bar p \restriction [\alpha, \delta)]^G \in H\}$ is $\mathcal{S}_\delta$-generic over $\mathcal{M}$. Similarly, whenever K is $\mathcal{S}_\delta$-generic over $\mathcal{M}$, then $K = G * H$, where $G = \{\bar p \restriction \alpha : \bar p \in K\}$ is $\mathcal{S}_\alpha$-generic over $\mathcal{M}$ and $H = \{[\bar p \restriction [\alpha, \delta)]^G : \bar p \in K\}$ is $\dot{\mathcal{S}}^G_{\alpha,\delta}$-generic over $\mathcal{M}[G]_{\mathcal{S}_\alpha}$.*

In either case, $([\dot x]^G)^H = \dot x^{G*H}$ for every $\mathcal{S}_\delta$-name $\dot x$ and $]\dot y[^{G*H} = (\dot y^G)^H$ for every $\mathcal{S}_\alpha$-name $\dot y$ for an $\dot{\mathcal{S}}_{\alpha,\delta}$-name. In particular, $\mathcal{M}[G * H]_{\mathcal{S}_\delta} = \mathcal{M}[G]_{\mathcal{S}_\alpha}[H]_{\dot{\mathcal{S}}^G_{\alpha,\delta}}$.

Proof. If $D \subseteq \mathbb{P}_\delta$ is open dense, show by induction on δ that the set

$$[D]_{\alpha,\delta} = \{(\bar p \restriction \alpha, [\bar p \restriction [\alpha, \delta)]) : \bar p \in D\}$$

is an $\mathcal{S}_\alpha$-name for an open dense subset of the forcing $\dot{\mathbb{P}}_{\alpha,\delta}$ corresponding to $\dot{\mathcal{S}}_{\alpha,\delta}$. Moreover, if $\bar{H} \leq \operatorname{sym}(D)$, then $\bar{H} \restriction \alpha \leq \operatorname{sym}([D]_{\alpha,\delta})$ and $\Vdash_{\mathcal{S}_\alpha} [\bar{H} \restriction [\alpha,\delta)] \leq \operatorname{sym}([D]_{\alpha,\delta})$. This is how we check that $G * H$ is $\mathcal{S}_\delta$-generic, given the genericity of G and H.

The other direction, starting from an $\mathcal{S}_\delta$-generic and obtaining the genericity of G and H, is completely analogous: From a name $\dot{D}$ for an open dense subset of $\dot{\mathbb{P}}_{\alpha,\delta}$, define

$$]\dot{D}[_{\alpha,\delta} = \{\bar{p} : \bar{p} \restriction \alpha \Vdash_{\mathcal{S}_\alpha} [\bar{p} \restriction [\alpha,\delta)] \in \dot{D}\}.$$

Everything else is just as straightforward and similar to [3, Theorem 3.3]. □

5. The symmetric extension

Definition 14. Given a forcing notion $\mathbb{R}$, we let $\mathcal{T}(\mathbb{R}) = (\mathbb{Q}, \mathcal{H}, \mathcal{E})$ denote the symmetric system where

- $\mathbb{Q}$ is the finite support product of ω-many copies of $\mathbb{R}$, i.e., $\mathbb{Q}$ consists of finite partial functions $p \colon \omega \to \mathbb{R}$ together with the extension relation given by $q \leq p$ iff $\operatorname{dom} p \subseteq \operatorname{dom} q$ and
$$\forall n \in \operatorname{dom} p \; q(n) \leq p(n),$$
- $\mathcal{H}$ is the group of finitary permutations of ω,[4] where $\pi \in \mathcal{H}$ acts on $\mathbb{Q}$ coordinate-wise, i.e., $\operatorname{dom} \pi(q) = \pi'' \operatorname{dom} q$ and
$$\pi(q)(n) = q(\pi^{-1}(n))$$
for every $n \in \omega$,
- $\mathcal{E}$ is generated by the subgroups of $\mathcal{H}$ of the form
$$\operatorname{fix}(e) = \{\pi \in \mathcal{H} : \forall n \in e \; \pi(n) = n\},$$
for $e \in [\omega]^{<\omega}$.

To see that $\mathcal{E}$ is normal, simply note that for any $\pi \in \mathcal{H}$, $\pi \operatorname{fix}(e) \pi^{-1} = \operatorname{fix}(\pi'' e)$. The system for the basic Cohen model (see [1, Section 5.3]) is exactly $\mathcal{T}(\mathbb{C})$, where $\mathbb{C}$ is Cohen forcing.

We will construct a finite support symmetric iteration of the form

$$\langle \mathcal{S}_\alpha, \dot{\mathcal{T}}_\alpha : 1 \leq \alpha \leq \omega_1 \rangle = \langle (\mathbb{P}_\alpha, \mathcal{G}_\alpha, \mathcal{F}_\alpha), (\dot{\mathbb{Q}}_\alpha, \dot{\mathcal{H}}_\alpha, \dot{\mathcal{E}}_\alpha)^\bullet : 1 \leq \alpha \leq \omega_1 \rangle,$$

where for each $\alpha < \omega_1$, $\dot{\mathcal{T}}_\alpha$ is an $\mathcal{S}_\alpha$-name for a symmetric system of the form $\mathcal{T}(\dot{\mathbb{R}}_\alpha)$, where $\dot{\mathbb{R}}_\alpha$ is an $\mathcal{S}_\alpha$-name for a forcing notion. We start by letting $\mathcal{S}_1$ be the basic Cohen system, i.e., $\mathcal{S}_1 = \mathcal{T}(\mathbb{C})$ where $\mathbb{C}$ is Cohen forcing, and we will inductively define the remainder of our symmetric iteration.[5]

Suppose we have constructed $\mathcal{S}_\delta$, for some $\delta \leq \omega_1$. Before defining $\dot{\mathbb{R}}_\delta$ (in case $\delta < \omega_1$), we first verify some general properties about the iteration up to δ.[6] To simplify notation, for the rest of this section, let us abbreviate $\mathcal{S} = \mathcal{S}_\delta$, $\mathbb{P} = \mathbb{P}_\delta$, $\mathcal{G} = \mathcal{G}_\delta$, and $\mathcal{F} = \mathcal{F}_\delta$.

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[4] A permutation $\pi \colon \omega \to \omega$ is *finitary* if $\pi(n) = n$ for all but finitely many $n \in \omega$.

[5] We start the iteration at index 1 rather than 0 for notational convenience related to the coherence of the indexing at steps below ω and after. One could also start with letting $\mathcal{S}_0$ be some trivial system and then ignore the first coordinate when writing $\bar{p}$, $\bar{\pi}$ or $\bar{H}$.

[6] We will only know precisely what $\mathcal{S}_\delta$ is once we have specified what happens in each step, but the description we have given so far is sufficient to make some general observations.

For $e \in [\delta \times \omega]^{<\omega}$, write e_α for the α^{th} section of e, i.e., $e_\alpha = \{n : (\alpha, n) \in e\}$. Let $\text{fix}(e) = \langle \text{fix}(e_\alpha)\check{} : \alpha < \delta\rangle \in \mathcal{F}$. While $\mathcal{F}$ contains more complicated groups, it usually suffices to only consider those of the form fix(e), as can be seen by the following:

Lemma 15. *Let $\dot{x} \in \text{HS}$, $\bar{p} \in \mathbb{P}$. Then there is $\bar{q} \leq \bar{p}$, $\dot{y} \in \text{HS}$ and $e \in [\delta \times \omega]^{<\omega}$ so that $\text{fix}(e) \leq \text{sym}(\dot{y})$ and $\bar{q} \Vdash \dot{x} = \dot{y}$.*

Proof. Let $\bar{H} \leq \text{sym}(\dot{x})$, $\bar{q} \leq \bar{p}$, and $e \in [\delta \times \omega]^{<\omega}$, so that

$$\bar{q} \restriction \alpha \Vdash_{\mathbb{P}_\alpha} \dot{H}(\alpha) = \text{fix}(e_\alpha)\check{}$$

for every $\alpha < \delta$. This is easy to achieve by extending $\bar{p}$ finitely often, deciding all $\dot{H}(\alpha)$ for $\alpha \in \text{supp}(\bar{H})$. Let γ be least so that $\dot{x} \in \text{HS}_\gamma$ and let $\dot{y}$ consist of all pairs $(\bar{r}, \dot{z}) \in \mathbb{P} \times \text{HS}_\gamma$ so that $\bar{r} \restriction \alpha \Vdash_{\mathbb{P}_\alpha} \dot{H}(\alpha) = \text{fix}(e_\alpha)\check{}$ for all $\alpha < \delta$, and $\bar{r} \Vdash \dot{z} \in \dot{x}$. Clearly $\bar{q} \Vdash \dot{x} = \dot{y}$.

Claim 16. $\text{fix}(e) \leq \text{sym}(\dot{y})$.

Proof. Let $\bar{\pi} \in \text{fix}(e)$, and let $(\bar{r}, \dot{z}) \in \dot{y}$ be arbitrary. Consider $\bar{\sigma} \in \mathcal{G}$ where each $\dot{\sigma}(\alpha)$ is so that the following is forced: "either $\dot{H}(\alpha) = \text{fix}(e_\alpha)\check{}$ and $\dot{\sigma}(\alpha) = \dot{\pi}(\alpha)$, or $\dot{H}(\alpha) \neq \text{fix}(e_\alpha)\check{}$ and $\dot{\sigma}(\alpha)$ is the identity". Such a name can be chosen uniformly by Fact 5. Then, $\bar{\sigma} \in \bar{H}$, and we note that also $\bar{\sigma}(\bar{r}) \restriction \alpha \Vdash \dot{H}(\alpha) = \text{fix}(e_\alpha)\check{}$, for each α, as $\bar{\sigma} \restriction \alpha \in \bar{H} \restriction \alpha \leq \text{sym}(\dot{H}(\alpha))$. Thus, also $\bar{\sigma}(\bar{r}) \restriction \alpha \Vdash \dot{\pi}(\alpha) = \dot{\sigma}(\alpha)$ for each α. By Lemma 12, we have that $\bar{\sigma}(\bar{r}) \Vdash \bar{\pi}(\dot{z}) = \bar{\sigma}(\dot{z})$. In particular, $\bar{\sigma}(\bar{r}) \Vdash \bar{\pi}(\dot{z}) \in \dot{x}$. On the other hand, we may also verify by induction that the conditions $\bar{\sigma}(\bar{r})$ and $\bar{\pi}(\bar{r})$ are equivalent. This implies that $(\bar{\pi}(\bar{r}), \bar{\pi}(\dot{z})) \in \dot{y}$, as desired. □

□

Something similar can be done on the level of the automorphisms themselves. Let f be a function from δ to finitary permutations of ω, so that for all but finitely many $\alpha < \delta$, $f(\alpha)$ is the identity. Then, we can consider $\tau_f = \langle \check{f}(\alpha) : \alpha < \delta\rangle \in \mathcal{G}$. f acts naturally on $\delta \times \omega$ via

$$f \cdot (\alpha, n) = (\alpha, f(\alpha)(n)).$$

For any $e \in [\delta \times \omega]^{<\omega}$,

$$\tau_f \, \text{fix}(e) \tau_f^{-1} = \text{fix}(f \cdot e),$$

where $f \cdot e = \{f \cdot (\alpha, n) : (\alpha, n) \in e\}$.

Lemma 17. *Let $\bar{\pi} \in \mathcal{G}$, $\bar{p} \in \mathbb{P}$, and let $\dot{x}$ be an arbitrary $\mathbb{P}$-name. Then, there is $\bar{q} \leq \bar{p}$ and f as above such that*

$$\bar{q} \Vdash \bar{\pi}(\dot{x}) = \tau_f(\dot{x}).$$

Moreover, whenever $\bar{\pi} \in \text{fix}(e)$, we can ensure that $\tau_f \in \text{fix}(e)$ as well.

Proof. Let $\bar{q}$ decide $\dot{\pi}(\alpha)$ for each α and then use Lemma 12. □

This shows that we can in fact consider the simpler system $\mathcal{S}' = (\mathbb{P}, \mathcal{G}', \mathcal{F}')$ where $\mathcal{G}'$ consists only of the automorphisms of the form τ_f and $\mathcal{F}'$ is generated by only the groups $\text{fix}(e) \cap \mathcal{G}'$. Observe for instance that Lemma 15 can be applied hereditarily to show that for any $\bar{p}$, and $\dot{x} \in \text{HS}_{\mathcal{S}}$, there is $\bar{q} \leq \bar{p}$ and $\dot{y} \in \text{HS}_{\mathcal{S}'}$ so that $\bar{q} \Vdash \dot{x} = \dot{y}$.

This is an instance of a much more general situation in which we want to consider only particular names for the conditions, automorphisms and generators of the filter at each iterand of our symmetric iteration. In the language of [3, Section 5], $\mathcal{S}'$ would be called a *reduced iteration.* It won't be necessary for us to actually pass to $\mathcal{S}'$, and we haven't even shown that $\mathcal{S}'$ is a symmetric system, albeit this is easy to check. Rather, it will be sufficient to use the previous lemmas directly in order to simplify our arguments.

For $(\alpha, n) \in \delta \times \omega$, let $\dot{g}_{\alpha,n}$ be a canonical name for the $\dot{\mathbb{R}}_\alpha$-generic added in the n^{th} coordinate of $\dot{\mathbb{Q}}_\alpha$. To be precise, let γ_α be least so that for any $\mathcal{S}_\alpha$-name $\dot{s}$ for an element of $\dot{\mathbb{R}}_\alpha$, there is $\dot{r} \in \mathrm{HS}_{\mathcal{S}_\alpha,\gamma_\alpha}$ with $\Vdash_{\mathcal{S}_\alpha} \dot{s} = \dot{r}$. Let

$$\dot{g}_{\alpha,n} = \{(\bar{p}, \dot{r}) : \bar{p} \in \mathbb{P}_{\alpha+1} \wedge \dot{r} \in \mathrm{HS}_{\mathcal{S}_\alpha,\gamma_\alpha} \wedge \bar{p} \restriction \alpha \Vdash_{\mathcal{S}_\alpha} \dot{r} = \dot{p}(\alpha)(n)\}.$$

Then, $\dot{g}_{\alpha,n} \in \mathrm{HS}$ and $\mathrm{fix}(\{\alpha, n\}) \leq \mathrm{sym}(\dot{g}_{\alpha,n})$. More generally, note that if $\bar{\pi} \in \mathcal{G}$ is such that $\Vdash \dot{\pi}(\alpha)(\check{n}) = \check{m}$, then we will have that $\bar{\pi}(\dot{g}_{\alpha,n}) = \dot{g}_{\alpha,m}$.

We define

$$\dot{A}_\delta = \{\bar{\pi}(\dot{g}_{\alpha,n}) : \bar{\pi} \in \mathcal{G}, (\alpha, n) \in \delta \times \omega\}^\bullet \in \mathrm{HS}.$$

Clearly, $\mathrm{sym}(\dot{A}_\delta) = \mathcal{G}$, and by Lemma 17,

$$\Vdash \dot{A}_\delta = \{\dot{g}_{\alpha,n} : (\alpha, n) \in \delta \times \omega\}^\bullet.$$

Concluding our definition, if $\delta < \omega_1$, we let $\dot{\mathcal{T}}_\delta = (\dot{\mathbb{Q}}_\delta, \dot{\mathcal{H}}_\delta, \dot{\mathcal{E}}_\delta)^\bullet$ be an $\mathcal{S}_\delta$-name for $\mathcal{T}(Coll(\omega, \dot{A}_\delta))$: recall that for a set A, $Coll(\omega, A)$ is the poset consisting of finite partial functions from ω to A ordered by extension. As $\mathrm{sym}(\dot{A}_\delta) = \mathcal{G} = \mathcal{G}_\delta$, Fact 5 shows that we can indeed require that $\mathrm{sym}(\dot{\mathcal{T}}_\delta) = \mathcal{G}_\delta$.

6. The ordering principle

In this section, we show that the ordering principle OP holds after performing the above-described symmetric iteration of length ω_1 over a ground model of ZFC with a definable wellorder of its universe. For example, we may work over Gödel's constructible universe. From now on, let $\mathcal{S} = \mathcal{S}_{\omega_1}$, $\mathbb{P} = \mathbb{P}_{\omega_1}$, $\mathcal{G} = \mathcal{G}_{\omega_1}$, and $\mathcal{F} = \mathcal{F}_{\omega_1}$.[7]

Lemma 18. *Let $\dot{x}_i \in \mathrm{HS}$, $\alpha < \omega_1$, $e_i \in [\alpha \times \omega]^{<\omega}$ and $\mathrm{fix}(e_i) \leq \mathrm{sym}(\dot{x}_i)$, for $i < n$. Let $\varphi(v_0, \ldots, v_{n-1})$ be a formula with all free variables shown, and let $\bar{p}, \bar{q} \in \mathbb{P}$. Then, whenever $\bar{p} \restriction \alpha = \bar{q} \restriction \alpha$, it holds that*

$$\bar{p} \Vdash_{\mathcal{S}} \varphi(\dot{x}_0, \ldots, \dot{x}_{n-1}) \textit{ iff } \bar{q} \Vdash_{\mathcal{S}} \varphi(\dot{x}_0, \ldots, \dot{x}_{n-1}).$$

Proof. Suppose $\bar{p} \Vdash_{\mathcal{S}} \varphi(\dot{x}_0, \ldots, \dot{x}_{n-1})$ but $\bar{r} \Vdash_{\mathcal{S}} \neg\varphi(\dot{x}_0, \ldots, \dot{x}_{n-1})$ for some $\bar{r} \leq \bar{q}$. It suffices to find $\bar{\pi} \in \bigcap_{i<n} \mathrm{fix}(e_i)$ so that $\bar{\pi}(\bar{r}) \parallel \bar{p}$, to yield a contradiction. Simply let $\dot{\pi}(\beta)$ be a name for the identity for all $\beta < \alpha$, thus already ensuring that $\bar{\pi} \in \bigcap_{i<n} \mathrm{fix}(e_i)$, and then define $\dot{\pi}(\beta)$, for $\beta \geq \alpha$ inductively as follows: When $\bar{\pi} \restriction \beta$ has been defined, simply let $\dot{\pi}(\beta)$ be a name for a finitary permutation of ω mapping the domain of $(\bar{\pi} \restriction \beta)(\dot{r}(\beta))$ away from the domain of $\dot{p}(\beta)$. If $a, b \subseteq \omega$ are finite, then a finitary permutation π such that $\pi[a] \cap b = \emptyset$ can of course be easily defined from a and b as parameters. So Fact 5 shows that such an $\mathcal{S}_\beta$-name $\dot{\pi}(\beta)$ exists.[8] □

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[7]In fact, while this is not needed for our main result, note that the results of this section would hold for any $\delta \leq \omega_1$ rather than just ω_1.

[8]Of course we are basically just showing that every tail of the iteration is homogeneous with respect to the group $\mathcal{G}$.

Define $\dot\Gamma = \{\bar\pi(\dot G) : \bar\pi \in \mathcal{G}\}^\bullet$, where $\dot G$ is the canonical name for the $\mathbb{P}$-generic filter. While $\dot\Gamma$ is not an $\mathcal{S}$-name in general, it is still a symmetric $\mathbb{P}$-name and it plays an important role in any symmetric system. The following is a quite general observation and shows that $M[G]_{\mathcal{S}} = \mathrm{HOD}^{M[G]}_{M(A)\cup\{\Gamma\}}$, where $\Gamma = \dot\Gamma^G$.[9]

Lemma 19. *Let $\dot x \in \mathrm{HS}$ and $e \in [\omega_1 \times \omega]^{<\omega}$ so that $\mathrm{fix}(e) \le \mathrm{sym}(\dot x)$. Whenever G is $\mathbb{P}$-generic over $\mathcal{M}$, $x = \dot x^G$ and $\Gamma = \dot\Gamma^G$, then x is definable in $\mathcal{M}[G]$ from elements of V, from Γ and from $\dot g^G_{\alpha,n}$ for $(\alpha,n) \in e$, as the only parameters.*

Proof. In $\mathcal{M}[G]$, define y to consist exactly of those z so that $z \in \dot x^H$ for some $H \in \Gamma$ with $\dot g^H_{\alpha,n} = \dot g^G_{\alpha,n}$ for all $(\alpha,n) \in e$. We claim that $x = y$. Clearly, $x \subseteq y$ as $G \in \Gamma$. Now suppose that $H \in \Gamma$ is arbitrary, so that $\dot g^H_{\alpha,n} = \dot g^G_{\alpha,n}$, for all $(\alpha,n) \in e$. Then $H = \bar\pi(\dot G)^G$, for some $\bar\pi \in \mathcal{G}$, and further, by Lemma 17, $H = \tau_f(\dot G)^G$ for some f. We obtain that $\dot g^H_{\alpha,n} = \tau_f(\dot g_{\alpha,n})^G = \dot g^G_{\alpha,n}$, for each $(\alpha,n) \in e$. But this is only possible if $\tau_f \in \mathrm{fix}(e)$. So also $\dot x^H = \tau_f(\dot x)^G = \dot x^G$ and we are done. □

The following is very specific to the way we chose $\dot{\mathcal{T}}_\alpha$:

Lemma 20. *Let $\bar p \in \mathbb{P}$, $\dot x \in \mathrm{HS}$ and $e \in [\omega_1 \times \omega]^{<\omega}$ be non-empty with $\mathrm{fix}(e) \le \mathrm{sym}(\dot x)$. Further, let $\alpha = \max \mathrm{dom}(e)$. Then there is $\bar q \le \bar p$ and $\dot y \in \mathrm{HS}$ with $\mathrm{fix}(\{\alpha\} \times e_\alpha) \le \mathrm{sym}(\dot y)$ so that $\bar q \Vdash \dot y = \dot x$.*

Proof. By the previous lemma, for any generic G, $\dot x^G$ is definable in $\mathcal{M}[G]$ from $\dot\Gamma^G$, $\langle \dot g^G_{\beta,n} : (\beta,n) \in e\rangle$ and parameters in V. But note that each $\dot g^G_{\beta,n}$, for $\beta < \alpha$, is itself definable from any $\dot g^G_{\alpha,m}$, as the latter enumerate $\dot A^G_\alpha$. Thus, $\dot x^G$ is already definable from $\dot\Gamma^G$, $\langle \dot g^G_{\alpha,n} : n \in e_\alpha\rangle$ and parameters in V. So we can find $\bar q \le \bar p$ and a formula φ so that

$$\bar q \Vdash_{\mathbb{P}} \dot x = \{z : \varphi(z, \dot\Gamma, \langle \dot g_{\alpha,n} : n \in e_\alpha\rangle^\bullet, \check v_0, \dots, \check v_k)\},$$

for some $v_0, \dots, v_k \in \mathcal{M}$. For some large enough γ, define

$$\dot y = \{(\bar r, \dot z) \in \mathbb{P} \times \mathrm{HS}_\gamma : \bar r \Vdash \varphi(\dot z, \dot\Gamma, \langle \dot g_{\alpha,n} : n \in e_\alpha\rangle^\bullet, \check v_0, \dots, \check v_k)\}.$$

We obtain that $\mathrm{fix}(\{\alpha\} \times e_\alpha) \le \mathrm{sym}(\dot y)$ and $\bar q \Vdash \dot y = \dot x$. □

Lemma 21. *Let $\alpha < \omega_1$, $\bar p \in \mathbb{P}$, $a_0, a_1 \in [\omega]^{<\omega}$ and $\dot x, \dot y \in \mathrm{HS}$ such that*

(1) $\bar p \Vdash \dot x = \dot y$,
(2) $\mathrm{fix}(\{\alpha\} \times a_0) \le \mathrm{sym}(\dot x)$,
(3) $\mathrm{fix}(\{\alpha\} \times a_1) \le \mathrm{sym}(\dot y)$.

Then, there is $\bar q \le \bar p$, $e \in [(\alpha+1) \times \omega]^{<\omega}$ and $\dot z \in \mathrm{HS}$ so that $e_\alpha = a_0 \cap a_1$, $\mathrm{fix}(e) \le \mathrm{sym}(\dot z)$ and $\bar q \Vdash \dot z = \dot x$.

Proof. First, applying Lemma 15, we find $\bar q \le \bar p$ such that $\mathrm{fix}(e') \le \mathrm{sym}(\dot q(\alpha))$ for some $e' \in [\alpha \times \omega]^{<\omega}$. We define $e = e' \cup \{\alpha\} \times (a_0 \cap a_1)$. Next, instead of $\dot y$, let us consider

$$\dot y' = \{(\bar r, \tau) : \exists (\bar s, \tau) \in \dot y\, (\bar r \le \bar q, \bar s)\},$$

and note that, $\bar q \Vdash \dot y' = \dot y = \dot x$. We let

$$\dot z = \bigcup_{\bar\pi \in \mathrm{fix}(e)} \bar\pi(\dot y').$$

[9]It is based on the fact that the names of the form $(\dot g_{\alpha_0,n_0}, \dots, \dot g_{\alpha_k,n_k})^\bullet$ for $(\alpha_i, n_i) \in \omega_1 \times \omega$ form a *respect-basis* for $\mathcal{S}$ (see more in [3, Section 6.4]).

Now clearly, $\text{fix}(e) \leq \text{sym}(\dot{z})$. We claim that $\bar{q} \Vdash \dot{z} = \dot{x}$. Towards this end, let G be an arbitrary generic containing $\bar{q}$. As $\dot{y}' = \text{id}(\dot{y}') \subseteq \dot{z}$, we have that $\dot{x}^G = \dot{y}^G = \dot{y}'^G \subseteq \dot{z}^G$. To see that $\dot{z}^G \subseteq \dot{x}^G$, we show that $\bar{\pi}(\dot{y}')^G \subseteq \dot{x}^G$, for every $\bar{\pi} \in \text{fix}(e)$. So fix $\bar{\pi} \in \mathcal{G}$ now. According to Lemma 17, there is f so that $\bar{\pi}(\dot{y}')^G = \tau_f(\dot{y}')^G$ and $\tau_f \in \text{fix}(e)$.

If $\tau_f(\bar{q}) \notin G$, clearly $\tau_f(\dot{y}')^G = \emptyset \subseteq \dot{x}^G$, as every condition appearing in a pair in $\tau_f(\dot{y}')$ is below $\tau_f(\bar{q})$.

So assume that $\tau_f(\bar{q}) \in G$. Consider for a moment the $\mathbb{Q}_\alpha$-generic H over $\mathcal{M}[G_\alpha]$ given by G, where $G_\alpha = \{\bar{r} \restriction \alpha : \bar{r} \in G\}$. More precisely,

$$H = \{\dot{s}^{G_\alpha} : \exists \bar{r} \in G(\dot{r}(\alpha) = \dot{s})\}.$$

We have that $s = \dot{q}(\alpha)^{G_\alpha} \in H$ and moreover, as $\tau_f \restriction \alpha \in \text{sym}(\dot{q}(\alpha))$,

$$\begin{aligned} f(\alpha)(s) &= f(\alpha)(\dot{q}(\alpha)^{G_\alpha}) \\ &= f(\alpha)\left((\tau_f \restriction \alpha)(\dot{q}(\alpha))^{G_\alpha}\right) \\ &= \tau_f(\bar{q})(\alpha)^{G_\alpha} \in H. \end{aligned}$$

Let $d = \{n \in \omega : f(\alpha)(n) \neq n\} \cup a_0 \cup \text{dom}(s)$, which is finite. By a density argument over $\mathcal{M}[G_\alpha]$, we can find a finitary permutation σ of ω that switches $a_1 \setminus a_0$ with a set disjoint from d, leaves everything else fixed, and is such that $\sigma(s) \in H$. In particular, σ fixes a_0. Moreover, note that $f(\alpha)(\sigma(s)) \in H$ as well: $\sigma(s) \restriction \sigma[a_1 \setminus a_0]$ is not moved by $f(\alpha)$, and $\sigma(s) \restriction (\omega \setminus \sigma[a_1 \setminus a_0]) \subseteq s$, where we know that $f(\alpha)(s) \in H$.

Back in $\mathcal{M}$, let $h(\beta) = \text{id}$ for every $\beta \in \omega_1 \setminus \{\alpha\}$ and let $h(\alpha) = \sigma$. Then, $\tau_h \in \text{fix}(\{\alpha\} \times a_0)$ and

$$\tau_h(\bar{q}) \restriction (\alpha+1), \tau_f(\tau_h(\bar{q})) \restriction (\alpha+1) \in G_{\alpha+1} = \{\bar{r} \restriction (\alpha+1) : \bar{r} \in G\}.$$

Then we obtain that $\tau_h(\bar{q}) \Vdash \dot{x} = \tau_h(\dot{y}') = \tau_h(\dot{y})$. By Lemma 18, this is already forced by $\tau_h(\bar{q}) \restriction (\alpha+1)^\frown\langle \dot{\mathbb{1}}_\beta : \beta \in [\alpha+1, \omega_1)\rangle \in G$. Thus, $\dot{x}^G = \tau_h(\dot{y}')^G = \tau_h(\dot{y})^G$.

Also, we note that $\tau_f \in \text{fix}(\{\alpha\} \times ((a_0 \cap a_1) \cup \sigma[a_1 \setminus a_0])) \leq \text{sym}(\tau_h(\dot{y}))$ (see the paragraph after Lemma 15). Hence, $\tau_f(\tau_h(\dot{y})) = \tau_h(\dot{y})$ and $\tau_f(\tau_h(\bar{q})) \Vdash \tau_f(\dot{x}) = \tau_h(\dot{y})$. Similarly to before, this implies that $\tau_f(\dot{x})^G = \tau_h(\dot{y})^G$. Since $\tau_f(\bar{q}) \Vdash \tau_f(\dot{x}) = \tau_f(\dot{y}')$ and $\tau_f(\bar{q}) \in G$, we have $\tau_f(\dot{x})^G = \tau_f(\dot{y}')^G$. So finally, we obtain that

$$\dot{x}^G = \tau_h(\dot{y})^G = \tau_f(\dot{x})^G = \tau_f(\dot{y}')^G,$$

which is what we wanted to show. □

Definition 22. Let $\bar{p} \in \mathbb{P}$, $\dot{x} \in \text{HS}$, $\alpha < \omega_1$ and $a \in [\omega]^{<\omega}$ be so that $\text{fix}(\{\alpha\} \times a) \leq \text{sym}(\dot{x})$. We say that α is a *minimal index for $\dot{x}$ below $\bar{p}$* if for any $\beta < \alpha$, $a' \in [\omega]^{<\omega}$ and $\dot{z} \in \text{HS}$ with $\text{fix}(\{\beta\} \times a') \leq \dot{z}$,

$$\bar{p} \Vdash \dot{x} \neq \dot{z}.$$

We say that $\{\alpha\} \times a$ *is a minimal support for $\dot{x}$ below $\bar{p}$* if α is a minimal index for $\dot{x}$ below $\bar{p}$ and for any $\bar{q} \leq \bar{p}$, $a' \in [\omega]^{<\omega}$ and $\dot{z} \in \text{HS}$ with $\text{fix}(\{\alpha\} \times a') \leq \text{sym}(\dot{z})$, if $\bar{q} \Vdash \dot{x} = \dot{z}$, then $a \subseteq a'$.

Corollary 23. *For any $\dot{x} \in \text{HS}$ and $\bar{p} \in \mathbb{P}$, there is $\bar{q} \leq \bar{p}$, $\dot{y} \in \text{HS}$, $\alpha < \omega_1$ and $a \in [\omega]^{<\omega}$ so that $\bar{q} \Vdash \dot{x} = \dot{y}$ and $\{\alpha\} \times a$ is a minimal support for $\dot{y}$ below $\bar{q}$.*

Proof. From Lemma 20, there is a minimal α where we can find $\bar{q} \leq \bar{p}$, $\dot{y}$ and a so that $\bar{q} \Vdash \dot{y} = \dot{x}$ and $\mathrm{fix}(\{\alpha\} \times a) \leq \mathrm{sym}(\dot{y})$. In that case, α is a minimal index for $\dot{y}$ below $\bar{q}$. Moreover then, fixing that minimal α, there is a $\subseteq$-minimal a for which we find $\bar{q}$, $\dot{y}$ as above. We claim that $\{\alpha\} \times a$ is a minimal support for $\dot{y}$ below $\bar{q}$. Otherwise, there are $\bar{q}' \leq \bar{q}$, a' and $\dot{z}$ with $\mathrm{fix}(\{\alpha\} \times a') \leq \mathrm{sym}(\dot{z})$ so that $\bar{q}' \Vdash \dot{y} = \dot{z}$ but $a \not\subseteq a'$. In particular $a \cap a'$ is a strict subset of a. According to Lemma 21, there is $\bar{q}'' \leq \bar{q}'$, $e \in [(\alpha+1) \times \omega]^{<\omega}$, $e_\alpha = a \cap a'$, and $\dot{z}'$ with $\mathrm{fix}(e) \leq \dot{z}'$, so that $\bar{q}'' \Vdash \dot{z}' = \dot{z} = \dot{y}$. If $\alpha = 0$, then a was not $\subseteq$-minimal. If $\alpha > 0$, then $a \cap a' \neq \emptyset$ since otherwise α was not minimal, by Lemma 20. But then again, according to Lemma 20, a was not $\subseteq$-minimal – contradiction. □

Note that in the above corollary, neither α nor the set a is necessarily unique. However, what is easily seen to be true using Lemma 21 is that if $\{\alpha\} \times a$ and $\{\beta\} \times b$ both are minimal supports for $\dot{y}$ below the same condition $\bar{q}$, then $\alpha = \beta$ and $a = b$.

Lemma 24. *There is an $\mathcal{S}$-name $\dot{<}$ for a linear order of $\dot{A}$, such that* $\mathrm{sym}(\dot{<}) = \mathcal{G}$.

Proof. In any model of ZF, we can consider the definable sequence of sets $\langle X_\alpha : \alpha \in \mathrm{Ord}\rangle$, obtained recursively by setting $X_0 = \omega$, $X_{\alpha+1} = (X_\alpha)^\omega$ and $X_\alpha = \bigcup_{\beta<\alpha} X_\beta$ for limit α. We can recursively define linear orders $<_\alpha$ on X_α, by letting $<_0$ be the natural order on ω, $<_{\alpha+1}$ be the lexicographic ordering on $X_{\alpha+1}$ obtained from $<_\alpha$ and for limit α, $x <_\alpha y$ iff, for β least such that $x \in X_\beta$, either $y \notin X_\beta$ or $x <_\beta y$. Then $<_{\omega_1}$ is a definable linear order of X_{ω_1}. Identifying the $\mathbb{R}_\alpha$-generics with the surjections they induce, note that $\dot{A}$ is forced to be contained in X_{ω_1}, and by Fact 5, there is an $\mathcal{S}$-name $\dot{<}$ as required. □

Proposition 25. *There is a class $\mathcal{S}$-name $\dot{F}$ for an injection of the symmetric extension into* $\mathrm{Ord} \times \dot{A}^{<\omega}$ *such that* $\mathrm{sym}(\dot{F}) = \mathcal{G}$. *In particular,* OP *holds in our symmetric extension.*

Proof. Fix a global well-order $\lhd$ of $\mathcal{M}$ and let G be $\mathbb{P}$-generic over $\mathcal{M}$. We will first provide a definition of an injection F in the full $\mathbb{P}$-generic extension $\mathcal{M}[G]$. Then, we will observe that all the parameters in this definition have symmetric names, which will let us directly build an $\mathcal{S}$-name for F.

For each $\alpha < \omega_1$, $a \in [\omega]^{<\omega}$ and each enumeration $h = \langle n_i : i < k\rangle$ of a, define $\dot{G}_{\alpha,a} = \{\dot{g}_{\alpha,n} : n \in a\}^\bullet$, and $\dot{t}_{\alpha,h} = \langle \dot{g}_{\alpha,n_i} : i < k\rangle^\bullet$. Let $\Gamma = \dot{\Gamma}^G$, $<= \dot{<}^G$ and $A = \dot{A}^G$. Given $x \in \mathcal{M}[G]_\mathcal{S}$, $F(x)$ will be found as follows:

First, let $(\bar{p}, \dot{z}, \alpha, a, h) \in \mathcal{M}$ be $\lhd$-minimal with the following properties:

1. (in $\mathcal{M}$) $\{\alpha\} \times a$ is a minimal support for $\dot{z}$ below $\bar{p}$,
2. (in $\mathcal{M}$) h is an enumeration of a so that $\bar{p}$ forces that $\dot{t}_{\alpha,h}$ enumerates $\dot{G}_{\alpha,a}$ in the order of $\dot{<}$,
3. there is $H \in \Gamma$, with $\bar{p} \in H$ and $\dot{z}^H = x$

Such a tuple certainly exists by Corollary 23 and since $G \in \Gamma$.

Claim 26. *For any $H, K \in \Gamma$ with $\bar{p} \in H, K$, the following are equivalent:*

(a) $(\dot{t}_{\alpha,h})^H = (\dot{t}_{\alpha,h})^K$,
(b) $\dot{z}^H = \dot{z}^K$.

Proof. Let $H, K \in \Gamma$, $\bar{p} \in H, K$. H is itself a $\mathbb{P}$-generic filter over $\mathcal{M}$ and $\dot{\Gamma}^H = \dot{\Gamma}^G = \Gamma$, as can be easily checked. Thus, there is $\bar{\pi} \in \mathcal{G}$ so that $K = \bar{\pi}(\dot{G})^H$. By

Lemma 17, there is f so that $K = \tau_f(\dot{G})^H$. Now note that $\tau_f(\dot{G})^H = \tau_f^{-1}[H]$ and $(\dot{t}_{\alpha,h})^K = (\dot{t}_{\alpha,h})^{\tau_f^{-1}[H]} = \tau_f(\dot{t}_{\alpha,h})^H$. Similarly, $\dot{z}^K = \tau_f(\dot{z})^H$.

Suppose that $(\dot{t}_{\alpha,h})^H = (\dot{t}_{\alpha,h})^K$. Then $(\dot{t}_{\alpha,h})^H = \tau_f(\dot{t}_{\alpha,h})^H$. The only way this is possible is if $f(\alpha)(n) = n$ for every $n \in a$. In other words, $\tau_f \in \operatorname{fix}(\{\alpha\} \times a)$. Thus $\dot{z}^H = \tau_f(\dot{z})^H = \dot{z}^K$.

Now suppose that $\dot{z}^H = \dot{z}^K = \tau_f(\dot{z})^H$. We have that $\operatorname{fix}(f \cdot (\{\alpha\} \times a)) = \tau_f \operatorname{fix}(\{\alpha\} \times a)\tau_f^{-1} \leq \operatorname{sym}(\tau_f(\dot{z}))$. Since $\{\alpha\} \times a$ is a minimal support of $\dot{z}$ below $\bar{p} \in H$, it follows that $a \subseteq f(\alpha)[a]$ and by a cardinality argument, $a = f(\alpha)[a]$. This also means that $\dot{G}_{\alpha,a} = \tau_f(\dot{G}_{\alpha,a})$. As $\bar{p}$ forces that $\dot{t}_{\alpha,h}$ is the $\dot{<}$-enumeration of $\dot{G}_{\alpha,a}$, we have that $\tau_f(\bar{p})$ forces that $\tau_f(\dot{t}_{\alpha,h})$ is the $\tau_f(\dot{<})$ enumeration of $\tau_f(\dot{G}_{\alpha,a})$. We have that $\bar{p} \in K = \tau_f^{-1}[H]$, so $\tau_f(\bar{p}) \in H$ and indeed $(\dot{t}_{\alpha,h})^K = \tau_f(\dot{t}_{\alpha,h})^H$ is the enumeration of $\tau_f(\dot{G}_{\alpha,a})^H = \dot{G}^H_{\alpha,a}$ according to $\tau_f(\dot{<})^H =<$, which is exactly what $(\dot{t}_{\alpha,h})^H$ is. □

By the claim, there is a unique $t \in A^{<\omega}$ so that $t = (\dot{t}_{\alpha,h})^H$, for some, or equivalently all, $H \in \Gamma$ with $\bar{p} \in H$ and $\dot{z}^H = x$. We let $F(x) = (\xi, t)$, where $(\bar{p}, \dot{z}, \alpha, a, h)$ is the ξ^{th} element of $\mathcal{M}$ according to $\lhd$. To see that this is an injection, assume that x and y both yield the same $(\bar{p}, \dot{z}, \alpha, a, h)$ and t. Let $H, K \in \Gamma$ with $\bar{p} \in H, K$, and with $\dot{z}^H = x$, $\dot{z}^K = y$. By definition $t = (\dot{t}_{\alpha,h})^H = (\dot{t}_{\alpha,h})^K$ and according to the claim $x = \dot{z}^H = \dot{z}^K = y$. This finishes the definition of F.

The definition we have just given can be rephrased as

$$F(x) = y \text{ iff } \varphi(x, y, \Gamma, <),$$

where φ is a first order formula using the class parameters Γ and $<$, and the only parameters that are not shown are parameters from $\mathcal{M}$, such as the class $\lhd$ or the class of $(\bar{p}, \dot{z}, \alpha, a, h)$ so that (1) and (2) hold. Simply let

$$\dot{F} = \{(\bar{p}, (\dot{x}, \dot{y})^\bullet) : \dot{x}, \dot{y} \in \mathrm{HS} \wedge \bar{p} \Vdash_{\mathbb{P}} \varphi(\dot{x}, \dot{y}, \dot{\Gamma}, \dot{<})\},$$

where the parameters from $\mathcal{M}$ in φ are replaced by their check-names. Then, $\dot{F} \subseteq \mathbb{P} \times \mathrm{HS}$, and $\operatorname{sym}(\dot{F}) = \mathcal{G}$, so $\dot{F}$ is a class $\mathcal{S}$-name, as desired. □

7. The axiom of dependent choice

In this section, we will show that the axiom of dependent choice DC holds in our above symmetric extension. We will use the well-known (and easy to verify) fact that DC holds if and only if any tree without maximal nodes contains an increasing chain of length ω.

Lemma 27. *For each $\alpha < \omega_1$, $\Vdash_{\mathbb{P}_\alpha} \dot{\mathbb{Q}}_\alpha$ is countable. In particular, $\mathbb{P}$ is ccc.*

Proof. If G is $\mathbb{P}_\alpha$-generic, $\dot{A}^G_\alpha = \{\dot{g}^G_{\beta,n} : (\beta, n) \in \alpha \times \omega\}$ is clearly countable in $\mathcal{M}[G]$. In particular, $Coll(\omega, \dot{A}^G_\alpha)$ and $\dot{\mathbb{Q}}^G_\alpha$ are countable forcing notions.

By Lemma 10, Item (10), $\mathbb{P}$ is just a dense subposet of the usual finite support iteration of the $\dot{\mathbb{Q}}_\alpha$, and must be ccc. □

Proposition 28. *Let G be $\mathbb{P}$-generic over $\mathcal{M}$. Then σ-covering holds between $\mathcal{M}[G]_\mathcal{S}$ and $\mathcal{M}[G]$. That is, whenever $x \in \mathcal{M}[G]$ is so that $\mathcal{M}[G] \models |x| = \omega$ and $x \subseteq \mathcal{M}[G]_\mathcal{S}$, there is $y \in \mathcal{M}[G]_\mathcal{S}$ so that $\mathcal{M}[G]_\mathcal{S} \models |y| = \omega$ and $x \subseteq y$.*

Proof. Let $x = \dot{x}^G$ for some $\mathbb{P}$-name $\dot{x} \in \mathcal{M}$. For some $p \in G$ and large γ, $p \Vdash \dot{x} \subseteq \mathrm{HS}_\gamma^\bullet \wedge \dot{x}$ is countable. Using the ccc of $\mathbb{P}$, we can find a countable set $c \subseteq \mathrm{HS}_\gamma$ so that $p \Vdash \dot{x} \subseteq c^\bullet$. Moreover, using Lemma 15, we can assume that for each $\dot{z} \in c$, there is $e \in [\omega_1 \times \omega]^{<\omega}$ so that $\mathrm{fix}(e) \leq \mathrm{sym}(\dot{z})$. Let $\alpha < \omega_1$ be large enough so that for each $\dot{z} \in c$ there is such e in $[\alpha \times \omega]^{<\omega}$.

Recall Lemma 19 and its proof: If $\mathrm{fix}(e) \leq \mathrm{sym}(\dot{z})$, and $\langle (\beta_i, n_i) : i < k\rangle$ enumerates e, then $y \in \dot{z}^G$ iff

$$\mathcal{M}[G] \models \varphi(y, \dot{z}, \langle \dot{g}_{\beta_i,n_i} : i < k\rangle, \dot{\Gamma}^G, \langle \dot{g}^G_{\beta_i,n_i} : i < k\rangle),$$

for the formula φ expressing that $y \in \dot{z}^H$ for some $H \in \dot{\Gamma}^G$ satisfying $\dot{g}^H_{\beta_i,n_i} = \dot{g}^G_{\beta_i,n_i}$ for all $i < k$. Since $\dot{g}^G_{\alpha,0}$ enumerates $\dot{A}^G_\alpha$, there is a sequence $\langle m_i : i < k\rangle$ so that $\langle \dot{g}^G_{\beta_i,n_i} : i < k\rangle = \langle \dot{g}^G_{\alpha,0}(m_i) : i < k\rangle$.

For any $\dot{z} \in c$, any $s \in (\alpha \times \omega)^{<\omega}$, and any $t \in \omega^{<\omega}$ of the same length, define $\dot{x}_{\dot{z},s,t}$ to consist of all $(\bar{p}, \dot{y}) \in \mathbb{P} \times \mathrm{HS}_\gamma$ so that

$$\bar{p} \Vdash \varphi(\dot{y}, (\dot{z})\check{}, \langle \dot{g}_{s(i)} : i < |s|\rangle\check{}, \dot{\Gamma}, \langle \dot{g}_{\alpha,0}(t(i)) : i < |t|\rangle^\bullet)$$

Then $\dot{x}_{\dot{z},s,t}$ is an $\mathcal{S}$-name with $\mathrm{fix}(\{(\alpha, 0)\}) \leq \mathrm{sym}(\dot{x}_{\dot{z},s,t})$. Letting $h \in \mathcal{M}$ be a surjection from ω to $\bigcup_{k\in\omega} c \times (\alpha \times \omega)^k \times \omega^k$, we find that

$$\dot{d} = \{(n, \dot{x}_{h(n)})^\bullet : n \in \omega\}$$

is an $\mathcal{S}$-name for a function with domain ω and with $x \subseteq \mathrm{ran}(\dot{d}^G)$, as desired □

Corollary 29. [4] $\Vdash_{\mathcal{S}}$ DC.

Proof. Consider a generic G and let $T \in \mathcal{M}[G]_{\mathcal{S}}$ be a tree without maximal nodes. Since $\mathcal{M}[G] \models$ AC, there is an increasing chain $\langle t_n : n \in \omega\rangle$ of T in $\mathcal{M}[G]$. By the previous proposition, there is a countable Y in $\mathcal{M}[G]_{\mathcal{S}}$ so that $\{t_n : n \in \omega\} \subseteq Y$. We may assume that $Y \subseteq T$. Recursively applying a pruning derivative to Y, whereby we remove all maximal elements in each step, we obtain a subtree $T' \subseteq Y$ without maximal elements. None of the t_n could ever have been removed, so T' is non-empty as e.g. $t_0 \in T'$. As T' is countable, and thus well-ordered, we do find an increasing chain $\langle s_n : n \in \omega\rangle$ of T', and thus also of T, in $\mathcal{M}[G]_{\mathcal{S}}$. □

Proposition 30. $\Vdash_{\mathcal{S}} \neg\mathrm{DC}_{\omega_1}$*, and thus in particular* $\Vdash_{\mathcal{S}} \neg$AC.

Proof. This is essentially the same argument that is used to show that AC fails in the basic Cohen model (see for example [1, Lemma 5.15]). Assume for a contradiction that there is $\dot{F} \in \mathrm{HS}$ such that $\bar{p} \in \mathbb{P}$ forces that $\dot{F}$ is an injection from ω_1 into $\{\dot{g}_{\alpha,n} \mid (\alpha, n) \in \omega_1 \times \omega\}^\bullet$ (clearly, one can construct such an injection under DC_{ω_1}). By Lemma 15 we can assume that $\mathrm{fix}(e) \leq \mathrm{sym}(\dot{F})$ for some $e \in [\omega_1 \times \omega]^{<\omega}$. Pick $\gamma < \omega_1$, $\bar{q} \leq \bar{p}$ and $\alpha > \max(\mathrm{dom}(e))$ such that $\bar{q} \Vdash \dot{F}(\check{\gamma}) = \dot{g}_{\alpha,0}$. By Lemma 18, this is already forced by $\bar{q}' = \bar{q} \restriction (\alpha + 1)^\frown\langle \dot{\mathbb{1}}_\beta : \beta \in [\alpha + 1, \omega_1)\rangle$. Let $\dot{\pi}(\beta)$ be a name for the identity for each $\beta \in \omega_1 \setminus \{\alpha\}$ and let $\dot{\pi}(\alpha)$ be a name for the permutation of ω switching 0 with the minimal $n > 0$ outside of the domain of $\dot{q}(\alpha)$. It suffices to note three things:

1. $\bar{\pi} \in \mathrm{fix}(e)$,
2. $\Vdash \bar{\pi}(\dot{g}_{\alpha,0}) \neq \dot{g}_{\alpha,0}$, and
3. $\bar{\pi}(\bar{q}') \parallel \bar{q}'$.

This clearly poses a contradiction, as $\bar{\pi}(\bar{q}') \Vdash \dot{F}(\check{\gamma}) \neq \dot{g}_{\alpha,0}$ while a compatible condition, $\bar{q}'$, forces the opposite. □

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