

DIPLOMARBEIT

A general Skolemisation Framework for Sequent Calculi and Expansion Tree Proofs

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1 Introduction

Skolemisation is a well-known method for handling quantifier dependencies in first-order sequent calculus. Here variables occurring in strong quantifier inferences are replaced by so-called skolemterms. Since its original introduction, several variants of skolemisation have been proposed, aiming to optimise proof complexity by modifying the dependency of skolemterms. Notable examples include Inner Skolemisation [1], Henkin Skolemisation [2] and Independence Friendly Logic [3]. A different method is variable-based conditions, such as those used in the original sequent calculus **LK** by Gentzen [4], or in more recent systems like $\mathbf{LK}^{++}$ introduced in [5].
Another modern approach to first-order proof theory involves expansion tree proofs. For these both skolem-based and variable-based conditions have been studied, as discussed in [6] and [7].
This thesis develops a general framework capable of modelling all of the above systems. It achieves this with the notion of skolem annotation, which record information in the superscripts of strong quantifiers. With these annotations, we are able to establish translations between different calculi and investigate their impact on proof complexity.
In chapter 2 we introduce our method of skolem annotations. For this we generalise the common definition of formulas and sequents to annotated and expanded annotated formulas and sequents.
Chapter 3 applies those annotations to sequent calculi, for both skolem and variable based approaches. We prove that for any proof in one of these systems, there exists a translation into the other that preserves proof size up to elementary equivalence. Additionally we show how proofs of annotated sequents can be reconstructed from proofs of their skolemised form and provide a linear bound on the complexity of this reconstruction. The last section includes an analysis of how different choices of annotation influence proof complexity. In chapter 4 we extend these translations to expansion tree proofs.
Chapter 5 establishes a connection between the sequent and expansion calculi of the previous chapters. We demonstrate that, for a given annotation, it is possible to switch freely between all of these systems.
In the final chapter 6 we explain whether existing skolemisation techniques like Inner or Atomic Skolemisation [8] and variable-based calculi like $\mathbf{LK}^{+}$ or $\mathbf{LK}^{++}$ can be represented within our framework by making appropriate choices of annotation. Additionally, we explore how two similar approaches to partial dependencies in the form of Henkin Quantifiers and Independence-Friendly Logic can be expressed as annotated formulas.

2 Preliminaries

This chapter introduces the notion of ordinary and annotated formulas, and provides a formal foundation for the annotated sequents that form the central object of study in this thesis. This is achieved by extending the traditional syntax with annotations in the form of superscripts of quantifiers.

2.1 Ordinary and annotated Formulas

In this thesis we consider first-order language over the following symbols:

1. Sets of *constants*:
   - Individual constants usually denoted as $c, d, e, f, ...$
   - Function constants $f, g, h, ...$ for each arity $0 \leq j$.
   - Predicate constants $P, G, R, ...$ for each arity $0 \leq j$.
2. Sets of *variables*:
   - Free variables $a, b, c, u, v, ...$
   - Bound variables $w, x, y, z, ...$
3. *logical* symbols
   - Logical constants $\top$ (true) and $\bot$ (false).
   - Connectives $\neg$ (not), $\wedge$ (and), $\vee$ (or) and $\supset$ (implies).
   - Quantifiers $\exists$ (exists) and $\forall$ (for all). We will sometimes use Q as a placeholder for both quantifiers.

We now recall the standard construction of terms, which serve as the building blocks for all formulas. These terms follow the usual rules from first-order logic.

Definition 2.1. The set of terms is defined inductively by the following 3 points:

1. All individual constants are terms.
2. All free variables are terms.
3. For a function constants f with n arguments and terms $t_1, ..., t_n$, $f(t_1, ..., t_n)$ is a term.

Terms will usually be denoted using the symbols $r, s, t, ...$

Having defined terms, we now turn to the construction of logical formulas.
Formulas as they are well known in literature will be called *ordinary formulas*. The definition coincides with the one in standard first-order logic, and will serve as the base structure for our extended notions later.

Definition 2.2. The set of ordinary formulas $\mathcal{F}$ is defined inductively by the following 3 points:

1. All *atomic formulas* are formulas. *Atomic formulas* have the form $R(t_1, ..., t_n)$, where R is a predicate symbol with n arguments and $t_1, ..., t_n$ are terms.

2. If A and B are formulas, then $\neg A$, $A \wedge B$, $A \vee B$ and $A \supset B$ are also formulas. We call $\neg, \wedge, \vee$ and $\supset$ the outermost logical symbols of the formula.

3. If A is a formula, a is a free variable and x is a bound variable which does not occur in A, then $\exists x\, A'$ $\forall x\, A'$ are formulas as well. Here A' refers to the formula A after replacing all occurrences of a in A by x. Note that in this definition, a must not appear in A.

   We call $\exists$ (or $\forall$) the outermost logical symbol of A and say that A' is in the *scope* of $\exists$ (or $\forall$).

We will use symbols $F, A, B, C, E, F, ...$ or $\Gamma, \Lambda, \Pi, \Delta, ...$ for formulas.

To allow the study of proof theory involving annotations we introduce a generalisation of ordinary formulas.
So called *annotated formulas* generalise the previous definition by adding superscripts of quantifiers in the form of skolemterms. These superscripts encode dependency information similarly to how skolemisation usually does. For this we postulate the set $\mathcal{SF}$ of skolem symbols for which we will use function constants $f, g, h, ...$

Definition 2.3. The set of annotated formulas $\mathcal{F}_{ann}$ is defined the same way as ordinary formulas with one additional case:

4. If A is a formula, a is a free variable and y is a bound variable which does not occur in A, then $\exists^{f(\overline{x})} y\, A'$ $(\forall^{f(\overline{x})} y\, A')$, where f is a skolemsymbol and $\overline{x}$ is a vector of bound variables not occurring in A', are formulas as well. Here A' refers to the formula A after replacing all occurrences of a in A by y. Note that in this definition, a must not appear in A.

   We call $\exists$ $(\forall)$ the outermost logical symbol of A and say that A' is in the *scope* of $\exists$ $(\forall)$.

Definition 2.4. We call the occurrence of an existential (universal) quantifier in a formula *positive* if it is under the scope of an odd (even) number of negations, and *negative* otherwise.

Definition 2.5. We define the sets $\mathcal{F}^+_{ann}, \mathcal{F}^-_{ann} \subseteq \mathcal{F}_{ann}$ of positive (negative) annotated formulas as follows:

- $F \in \mathcal{F}^+_{ann}$ if every *negative* quantifier in F has the form Qx and every *positive* quantifier in F has the form $Q^{f(\overline{x})}y$, with $\overline{x}$ being a subvector of $(x_1, ..., x_n)$, where $Q_1x_1, ..., Q_nx_n$ are the *negative* quantifiers such that $Q^{f(\overline{x})}y$ is in their scope and f is a unique skolemsymbol in F.
- $F \in \mathcal{F}^-_{ann}$ if every *positive* quantifier in F has the form Qx and every *negative* quantifier in F has the form $Q^{f(\overline{x})}y$, with $\overline{x}$ being a subvector of $(x_1, ..., x_n)$, where $Q_1x_1, ..., Q_nx_n$ are the *positive* quantifiers such that $Q^{f(\overline{x})}y$ is in their scope and f is a unique skolemsymbol in F.

Example 2.5.1. Consider the formulas

- $F_1 := \exists^{f(x)}y \forall^{g(y)}z\, P(y, z)$
- $F_2 := \exists x \forall^{g(z)}y\, P(w, y)$
- $F_3 := \exists x \forall^{g(x)}y\, P(x, y)$
- $F_4 := \forall x \exists^{g(x)}y\, P(x, y)$

All the above are annotated formulas, but only $F_3 \in \mathcal{F}^+_{ann}$ and $F_4 \in \mathcal{F}^-_{ann}$.

To account for substitutions within annotated formulas, especially those involving skolemterms, we define substitution behavior for strong quantifiers.

Definition 2.6. For a formula F containing a quantifier Q^sy with term s, we define $(Q^sy\, A)[x\backslash t] := Q^{s[x\backslash t]}y\, A[x\backslash t]$.

This ensures that both the skolemterm and the body of the quantifier reflect the substitution.
To integrate substitution into our calculi, we now have to define a set of mixed formulas which we will call *expanded annotated formulas*.

Definition 2.7. The set of expanded annotated formulas $\mathcal{F}_{exp}$ is defined the same way as annotated formulas with one additional case:

5. If A is a formula, a is a free variable and y is a bound variable which does not occur in A, then $\exists^{f(\tilde{t},\overline{x})}y\, A'$ ($\forall^{f(\tilde{t},\overline{x})}y\, A'$), where f is a skolemsymbol, $\overline{x}$ is a vector of bound variables not occurring in A' and $\tilde{t}$ are terms, is a formula as well. Here A' refers to the formula A after replacing all occurrences of a in A by y. Note that in this definition, a must not appear in A.

   We call $\exists$ (or $\forall$) the outermost logical symbol of A and say that A' is in the *scope* of $\exists$ (or $\forall$).

Polarity and the sets $\mathcal{F}^+_{exp}, \mathcal{F}^-_{exp} \subseteq \mathcal{F}_{exp}$ are defined the same way as for annotated formulas.

Expanded annotated formulas play a technical role in defining expansion proofs and in reasoning about quantifier instantiations. They appear in the intermediate stages of our proof transformations in later chapters.

2.2 Sequents and Inferences

Definition 2.8. An annotated sequent is an expression $\Gamma \vdash \Delta$, where $\Gamma \subseteq \mathcal{F}^{-}_{ann}$ and $\Delta \subseteq \mathcal{F}^{+}_{ann}$. A sequent is usually expressed as $\Gamma_1, ..., \Gamma_n \vdash \Delta_1, ..., \Delta_k$ or as

$$\left[\bigvee_{1\leq i\leq n} \neg\Gamma_i\right] \vee \left[\bigvee_{1\leq j\leq k} \Delta_j\right].$$

The negative (positive) quantifiers in Γ (Δ) are called *strong*, the positive (negative) quantifiers in Γ (Δ) are called *weak*.

The same definition is made for expanded annotated sequents.

Inferences (or rules) are expressions

$$\frac{\Gamma' \vdash \Delta'}{\Gamma \vdash \Delta}$$

where the upper sequent is called premise and the lower sequent is called conclusion. A rooted tree where the leaves are axioms and all non-leaf nodes are expanded annotated sequents such that each conclusion is obtained by its premise through an inference is called a derivation of the root sequent S. A subformula in the conclusion that is changed in an inference is called the principal formula.

Each rooted subtree in a derivation is a derivation of its root sequent.
A derivation whose root is an annotated sequent is called a proof.

Now that formulas and sequents have been defined, we will give a definition of a *skolem annotation*. This will be achieved by defining a mapping α from the set of ordinary formulas into the set of annotated formulas.

2.3 Skolem Annotations

Definition 2.9. A *skolem annotation* is a mapping $\alpha : \mathcal{F} \to \mathcal{F}_{ann}$ on formulas F such that for all strong quantifiers Qy in F

$$\alpha(Qy) := Q^{f(\overline{x})}y,$$

where $Q_1x_1, ..., Q_kx_k$ are the weak quantifiers such that Qy is in their scope, $\overline{x}$ is a subvector of $(x_1, ..., x_k)$ and f is a skolemsymbol that is chosen uniquely for all strong quantifiers.

For a sequent

$$S = \Gamma_1, ..., \Gamma_n \vdash \Delta_1, ..., \Delta_k$$

we define

$$\alpha(S) = \alpha(\Gamma_1), ..., \alpha(\Gamma_n) \vdash \alpha(\Delta_1), ..., \alpha(\Delta_k).$$

The quantifiers which receive the superscript are exactly the strong quantifiers in a sequent. We can therefore think of this mapping as adding a skolemterm with partial dependency of all weak quantifiers to the left for each strong quantifier. Note that for a strong quantifier Q, different to the standard notion of skolemisation, a skolemterm must not contain variables of all weak quantifiers such that Q is in their scope.

How the subvector of variables can be chosen arbitrarily and depends on the context. In order to compare different choices, we will define an ordering on the set of all skolem annotations.

Definition 2.10. We define an ordering on the set A of all skolem annotations of a formula F as $\alpha' \leq \alpha$ iff for all strong quantifiers Q_y in F, if $\alpha'(Q_y) = Q_y^{f'(\tilde{x})}$ and $\alpha(Q_y) = Q_y^{f(\overline{x})}$, then $\tilde{x}$ is a subvector of $\overline{x}$.

The smallest element in this ordering is the map which assigns a constant to each strong quantifier. The greatest element coincides with the standard definition of skolemisation.

3 Sequent Calculi

This chapter introduces two separate cut-free calculi for annotated sequents, $\mathbf{LK}^{\mathrm{sk}}$ and $\mathbf{LK}^{\mathrm{ac}}$. The first one revolves around skolemterms in strong quantifier inferences, while the latter uses so-called characteristic variables and puts an acyclicity condition on them. Furthermore, we introduce a notion of skolemisation that operates on annotated sequents. It will be shown that for a given annotation, both calculi yield proofs of equal or elementarily equivalent size.
In the final section, we investigate the influence of annotations on both the size of derivations and the validity of sequents.

3.1 Sequent Calculi

We begin by defining a base calculus that integrates both types of strong quantifier rules. We use a sequent calculus similar to **G3c** defined in [9], with the differences that we add axioms for $\top$ and $\bot$ and two types of strong quantifier inferences for annotated formulas. As common in literature and given by the famous Cut-Elimination Theorem by Gentzen [4], we will work only with cut-free proofs.

Definition 3.1. A derivation of an expanded annotated sequent is a tree built bottom up by the following axioms and rules:

$$\frac{}{A, \Gamma \vdash \Delta, A}\,\mathrm{ax} \qquad \frac{}{\bot, \Gamma \vdash \Delta}\,\mathrm{ax}\bot \qquad \frac{}{\Gamma \vdash \Delta, \top}\,\mathrm{ax}\top$$

where A is an atom. The logical rules are

$$\frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A}\,\neg_r \qquad \frac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta}\,\neg_l$$

$$\frac{A, \Gamma \vdash \Delta \qquad B, \Gamma \vdash \Delta}{A \vee B, \Gamma \vdash \Delta}\,\vee_l \qquad \frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \vee B}\,\vee_r$$

$$\frac{A, B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}\,\wedge_l \qquad \frac{\Gamma \vdash \Delta, A \qquad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}\,\wedge_r$$

$$\frac{\Gamma \vdash \Delta, A \qquad B, \Gamma \vdash \Delta}{A \supset B, \Gamma \vdash \Delta}\,\supset_l \qquad \frac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \supset B}\,\supset_r$$

The weak quantifier rules are

$$\frac{\Gamma \vdash \Delta, \exists x\, F, F[x\backslash t]}{\Gamma \vdash \Delta, \exists x\, F}\,\exists_r \qquad \frac{\forall x\, F, F[x\backslash s], \Gamma \vdash \Delta}{\forall x\, F, \Gamma \vdash \Delta}\,\forall_l$$

where t and s are arbitrary terms.

The strong quantifier rules are

$$\frac{F[x\backslash f(\bar{s})], \Gamma \vdash \Delta}{\exists^{f(\bar{s})} x\, F, \Gamma \vdash \Delta}\ \exists_l^{\mathrm{sk}} \qquad \frac{\Gamma \vdash \Delta, F[x\backslash f(\bar{t})]}{\Gamma \vdash \Delta, \forall^{f(\bar{t})} x\, F}\ \forall_r^{\mathrm{sk}}$$

where $f(\bar{t})$ and $f(\bar{s})$ are skolemterms or

$$\frac{F[x\backslash b], \Gamma \vdash \Delta}{\exists^{f(\bar{s})} x\, F, \Gamma \vdash \Delta}\ \exists_l^{\mathrm{ac}} \qquad \frac{\Gamma \vdash \Delta, F[x\backslash a]}{\Gamma \vdash \Delta, \forall^{f(\bar{t})} x\, F}\ \forall_r^{\mathrm{ac}}$$

where a, b are free variables, called the *characteristic variables* of the rules.

The rules $\exists_l^{\mathrm{sk}}$ and $\forall_r^{\mathrm{sk}}$ are called strong *skolem quantifier inferences*, and the rules $\exists_l^{\mathrm{ac}}$ and $\forall_r^{\mathrm{ac}}$ are called strong *acyclic quantifier inferences*.

This mixed calculus serves as the base for both $\mathbf{LK}^{\mathrm{sk}}$ and $\mathbf{LK}^{\mathrm{ac}}$ and will be the intermediate calculus when translating between them.

Remark 3.1.1. By using the logical equivalence $A \supset B \equiv \neg A \vee B$ for any two formulas A, B, we can replace the $\supset$-inferences by

$$\frac{\dfrac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta}\ \neg_l \qquad B, \Gamma \vdash \Delta}{\neg A \vee B, \Gamma \vdash \Delta}\ \vee_l$$

and

$$\frac{\dfrac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, \neg A, B}\ \neg_r}{\Gamma \vdash \Delta, \neg A \vee B}\ \vee_r$$

Therefore we will not specifically consider $\supset$-inferences in the following theorems.

We will use two notions of size to measure and compare the complexity of derivations within these systems. The length $l(\pi)$ simply counts the number of sequents occurring in a derivation, while the depth measures the maximal number of inference steps along any branch of the proof tree.

Definition 3.2. Let π be a derivation as in definition 3.1, then we define the length $l(\pi)$ as the number of sequents in π.

Definition 3.3. The $\mathrm{depth}(\pi)$ of the derivation is defined as the number of inferences in a maximal branch in π.

Remark 3.3.1. Let π be a derivation of an annotated sequent S. Since the rules in our calculi are at most binary, it follows that $l(\pi) \leq 2^{\mathrm{depth}(\pi)}$.

3.2 Skolem Calculus

Definition 3.4. An $\mathbf{LK}^{\mathrm{sk}}$-proof is a derivation π as in definition 3.1, starting with an annotated sequent such that all strong quantifier inferences in π are skolem quantifier inferences.

This calculus is very similar to the well-known **LK** introduced by Gentzen for ordinary formulas. The main difference in the strong quantifier inferences is that unlike the need for a separate *eigenvariable-condition*, all information is already stored in the annotation. Note that since the annotations explicitly determine the structure of skolemterms, we do not require a separate uniqueness constraint on skolemfunctions in the definition of $\mathbf{LK}^{\mathrm{sk}}$-proofs. Such regularity is already ensured by the annotation itself.

3.3 Acyclic Calculus

The second system we study, $\mathbf{LK}^{\mathrm{ac}}$, uses characteristic variables in place of skolemterms. To control their behaviour a dependency relation is introduced which we will assume to be acyclic.

It is very similar to the acyclicity condition in $\mathbf{LK}^{++}$ [5] or for expansion proofs found in [7], which will be discussed in a later chapter.

Definition 3.5. Let s be a term, then $\mathrm{FV}(s)$ is the set of all free variables occurring in s.

Definition 3.6. Let π be a derivation, then $\mathrm{FV}(\pi)$ is the set of all free variables which appear in acyclic quantifier inferences in π.

The following definitions formalise this notion of dependency and introduce the key acyclicity condition.

Definition 3.7. Let π be a derivation of an annotated sequent and $\mathrm{FV}(\pi)$ be the set of free variables in π. If $a \in \mathrm{FV}(\pi)$ is a characteristic variable with respective strong quantifier $Q^{f(\bar{t})}$ in π and b another free variable in π, then we define an ordering $<^0_\pi$ on $\mathrm{FV}(\pi)$ as

$$b <^0_\pi a \text{ if } b \in \mathrm{FV}(f(\bar{t})) \text{ and } b \text{ is a proper subterm of } f(\bar{t}).$$

The transitive closure $<_\pi$ of $<^0_\pi$ is called the *dependency relation* of π.

This dependency relation captures how characteristic variables are functionally dependent on one another through their occurrence in skolem annotations. We are now able to introduce the second calculus that we will analyse.

Definition 3.8. An $\mathbf{LK}^{\mathrm{ac}}$-proof is a derivation ψ as in definition 3.1, starting with an annotated sequent such that all strong quantifier inferences in π are acyclic quantifier inferences and $<_\psi$ is acyclic.

Additionally we will require the derivation to be weakly regular: the characteristic variables of any two acyclic quantifier inferences are the same if and only if the respective skolemterms in the superscripts of the strong quantifiers are the same.

Remark 3.8.1. Note that if a derivation π does not contain a strong quantifier rule, it contains neither skolemterms nor characteristic variables, which implies that $<_\pi$ is trivially acyclic. By definition 3.4 and 3.8 π is both an $\mathbf{LK}^{\mathrm{sk}}$ and $\mathbf{LK}^{\mathrm{ac}}$-proof at the same time.

3.4 Equivalence of the two Calculi

We now turn to the central theoretical result of this chapter: the equivalence of the calculi $\mathbf{LK}^{\mathrm{sk}}$ and $\mathbf{LK}^{\mathrm{ac}}$. More precisely, given an annotated sequent S, we will show that there exists a size preserving translation between proofs in $\mathbf{LK}^{\mathrm{sk}}$ and $\mathbf{LK}^{\mathrm{ac}}$.

The first direction establishes that skolem-style derivations can be transformed into acyclic ones. The main technical challenge is showing that replacing skolemterms with fresh characteristic variables leads to acyclicity.

Theorem 3.9. *Let S be an annotated sequent and π be an $\mathbf{LK}^{\mathrm{sk}}$-proof of S. Then there exists an $\mathbf{LK}^{\mathrm{ac}}$-proof ψ of the same sequent such that $l(\pi) = l(\psi)$ and $\mathrm{depth}(\pi) = \mathrm{depth}(\psi)$.*

Proof. We will prove the existence of a derivation ψ' of S such that the dependency relation $<_{\psi'}$ is acyclic by induction on the number of skolemterms in π.
If there are no skolemterms in π then there are no strong quantifier rules in π. As discussed in Remark 3.8.1, $\psi = \pi$ already is an $\mathbf{LK}^{\mathrm{ac}}$-proof of S.
In this case both equalities are trivially satisfied.

For the induction step assume that the set of skolemterms in π is not empty and let $f(\overline{t})$ be a skolemterm that is maximal with respect to the subterm relation. Let a be a free variable which doesn't occur in π and define ψ' as π where all occurrences of $f(\overline{t})$ are replaced by a. We want to show that $<_{\psi'}$ is acyclic.

If a is the only characteristic variable in ψ, then a cycle $a <_{\psi'} a$ would directly imply $a <^0_{\psi'} a$. This means $a \in \mathrm{FV}(f(\overline{t}))$, in other words a would have been inferred with a weak quantifier. Since a was chosen to not occur in π, this is impossible and therefore $<_{\psi'}$ is acyclic.

Assume there exists another characteristic variable $b \neq a$ in $\mathrm{FV}(\psi')$ with respective skolemterm $g(\overline{s})$ such that $a <^0_{\psi'} b$. By definition $a \in \mathrm{FV}(g(\overline{s}))$, which means that a is a proper subterm of $g(\overline{s})$. As a refers to the skolemterm $f(\overline{t})$ in π, this implies that in π, $f(\overline{t})$ is a proper subterm of $g(\overline{s})$.
Now assume that there exists a cycle $a <_{\psi'} a$. By transitivity of the above observation, this would imply that $f(\overline{t})$ is a proper subterm of itself. But this is not possible, which means that $<_{\psi'}$ has to be acyclic. As ψ' contains less skolemterms than π, we can apply the induction hypothesis and receive a derivation ψ where all skolemterms are replaced by characteristic variables and $<_\psi$ is acyclic. The derivation is weakly regular as all strong quantifiers have a unique skolemsymbol and therefore a unique free variable. By definition 3.8, ψ is an $\mathbf{LK}^{\mathrm{ac}}$-proof of S.

As no changes to the structure of π besides substitution have been done, the number of sequents and inferences stays the same. Therefore $l(\pi) = l(\psi)$ and $\mathrm{depth}(\pi) = \mathrm{depth}(\psi)$. □

The converse direction proceeds by replacing characteristic variables with their corresponding skolemterms, using the skolemterms from the annotation.

Theorem 3.10. *Let S be an annotated sequent and ψ be an $\mathbf{LK}^{\mathrm{ac}}$-proof of S. Then there exists an $\mathbf{LK}^{\mathrm{sk}}$-proof π of the same sequent such that $l(\psi) = l(\pi)$ and $\mathrm{depth}(\psi) = \mathrm{depth}(\pi)$.*

Proof. We will replace characteristic variables with skolemterms in ψ by induction on the number of characteristic variables.

If there are no such variables in ψ, then ψ does not contain any strong quantifier inferences. By definition 3.4 $\pi := \psi$ itself is an $\mathbf{LK}^{\mathrm{sk}}$-proof of S.
The two equalities are trivially satisfied.

For the induction step, assume that the set of characteristic variables in ψ is not empty. Let a be a characteristic variable that is minimal in $<_\psi$ and $Q^{f(\bar{t})}$ be the respective strong quantifier. This variable exists as $<_\psi$ is acyclic.
We define the derivation π' as ψ where all occurrences of a have been replaced by $f(\bar{t})$. In order to apply the induction hypothesis on π', we need to show that $<_{\pi'}$ is acyclic.
Let $b, c \in \mathrm{FV}(\pi')$, where b is a characteristic variable with skolemterm $g(\bar{s})$. Assume $c <^0_{\pi'} b$, then $c \in \mathrm{FV}(g(\bar{s})[a \backslash f(\bar{t})])$. If $c \in \mathrm{FV}(f(\bar{t}))$, then $b <_\psi a$ which is a contradiction to a being minimal. Therefore $c \in \mathrm{FV}(g(\bar{s}))$ which implies $b <^0_\psi c$.
A cycle in $<_{\pi'}$ would therefore imply a cycle in $<_\psi$. But since $<_\psi$ is acyclic by assumption, it follows that $<_{\pi'}$ is acyclic as well. We can therefore apply the induction hypothesis to π' and receive a derivation π of S where all characteristic variables have been replaced with skolemterms. This means that π is an $\mathbf{LK}^{\mathrm{sk}}$-proof of S.

As substitution does not affect the structure of the proof, it follows that $l(\psi) = l(\pi)$ and $\mathrm{depth}(\psi) = \mathrm{depth}(\pi)$.

□

The two theorems can be summarised in the following corollary.

Corollary 3.11. *Let S be an annotated sequent, then there exists an $\mathbf{LK}^{\mathrm{ac}}$-proof of S if and only if there exists an $\mathbf{LK}^{\mathrm{sk}}$-proof of it. The size and depth of minimal proofs of S in both calculi coincide.*

Proof. By theorem 3.9 and theorem 3.10 the existence of a proof in $\mathbf{LK}^{\mathrm{ac}}$ implies the existence of a proof in $\mathbf{LK}^{\mathrm{sk}}$ and vice-versa.

Let π and ψ be minimal proofs of S in $\mathbf{LK}^{\mathrm{sk}}$ and $\mathbf{LK}^{\mathrm{ac}}$ respectively. Let ψ' be the $\mathbf{LK}^{\mathrm{ac}}$-proof from applying theorem 3.9 to π and π' be the $\mathbf{LK}^{\mathrm{sk}}$-proof from applying theorem 3.10 to ψ.

By minimality of π and ψ respectively, and the equalities from both theorems we see that

$$\operatorname{depth}(\pi) \leq \operatorname{depth}(\pi') \overset{3.10}{=} \operatorname{depth}(\psi) \leq \operatorname{depth}(\psi') \overset{3.9}{=} \operatorname{depth}(\pi)$$

and

$$l(\pi) \leq l(\pi') \overset{3.10}{=} l(\psi) \leq l(\psi') \overset{3.9}{=} l(\pi)$$

This shows that both minimal proofs are equal in size.

□

This result confirms that we can freely switch between proofs between the two calculi.

3.5 Skolemisation of Annotated Sequents

We are now going to introduce skolemisation similar as in [6], but adjusted to work in our setting of annotated sequents or more general, expanded annotated sequents. This shortens the definition by a bit, as most information is already stored in the skolem annotation.

Definition 3.12. Let F be an expanded annotated formula. If F does not contain any strong quantifiers, for all $n \in \mathbb{N}$ we define the skolemisation as $\operatorname{sk}_n(F) := F$. If F contains a leftmost strong quantifier $Q^{f(\bar{t},\bar{x})}y$, we define

$$\operatorname{sk}_n(F) := \operatorname{sk}_{n+1}(F_{Q^{f(\bar{t},\bar{x})}y}[y \backslash f_n(\bar{t},\bar{x})]),$$

where $F_{Q^{f(\bar{t},\bar{x})}y}$ is F after omitting the quantifier occurrence $Q^{f(\bar{t},\bar{x})}y$.

Then the skolemisation of F is defined as $\operatorname{sk}(F) := \operatorname{sk}_0(F)$.

For an expanded annotated sequent

$$S = A_1, ..., A_n \vdash B_1, ..., B_k$$

we define the skolemisation of S as

$$\operatorname{sk}(S) = \operatorname{sk}(A_1), ..., \operatorname{sk}(A_n) \vdash \operatorname{sk}(B_1), ..., \operatorname{sk}(B_k)$$

The way that our version of skolemisation is split between the annotation α and the previously defined mapping sk gives it a very useful property:

Consider the annotated formula $\forall x \exists^{f(x)} y\, F$. Here

$$\operatorname{sk}(\forall x \exists^{f(x)} y\, F) = \forall x \operatorname{sk}(F)[y \backslash f(x)] = \forall x \left[\operatorname{sk}(\exists^{f(x)} y\, F)\right],$$

which is not true for the usual definition of skolemisation for ordinary formulas.

The following lemma will generalise this observation even further.

Definition 3.13. We say the skolemisation operator is compatible with the structure of a formula if

- $\mathrm{sk}(\neg A) = \neg\mathrm{sk}(A)$
- $\mathrm{sk}(A \vee B) = \mathrm{sk}(A) \vee \mathrm{sk}(B)$ and analogously for $\wedge$
- $\mathrm{sk}(\exists x\, A) = \exists x\, \mathrm{sk}(A)$ and analogously for weak universal quantifiers

The compatibility of the skolemisation operator with logical connectives allows us to lift properties from subformulas to formulas or even entire sequents.

Lemma 3.14. *Let F be an expanded annotated formula, then* sk *is compatible with the structure of F.*

Proof.

- If $F = \neg F'$, it is clear that $\mathrm{sk}(F) = \neg\mathrm{sk}(F')$.
- Let $F = A \vee B$ and $Q^{f(\tilde{t},\overline{x})} y\, F'$ be a strong quantifier in B, then some x_i may correspond to weak quantifiers in A. But as the superscript of such strong quantifiers in B is already defined in the annotation, both $\mathrm{sk}(A \vee B)$ and $\mathrm{sk}(A) \vee \mathrm{sk}(B)$ lead to the same substitution of $F'[y \backslash f(\overline{t}, \overline{x})]$.

  The same argument can be made for $\wedge$.
- If $F = \exists x\, F'$, then by the same argument about superscripts as above we have $\mathrm{sk}(F) = \exists x\, \mathrm{sk}(F')$.

  The same argument works for weak universal quantifiers.

□

Theorem 3.15. *Let S be an annotated sequent and π be an $\mathbf{LK}^{\mathrm{sk}}$-proof of S. Then there exists an $\mathbf{LK}^{\mathrm{sk}}$-proof ψ of $\mathrm{sk}(S)$ such that $l(\psi) \leq l(\pi)$ and $\mathrm{depth}(\psi) \leq \mathrm{depth}(\pi)$.*

Proof. This will be proven by induction on the number of inferences in π.
If S is an axiom, then $\mathrm{sk}(S)$ is an axiom too. In this case $\mathrm{sk}(S)$ is an $\mathbf{LK}^{\mathrm{sk}}$-proof ψ of itself with $l(\psi) = \mathrm{depth}(\psi) = 1$. This shows that the induction base holds.

If π has the form

$$\dfrac{\begin{array}{c}(\pi')\\ \Gamma \vdash \Delta, F'\end{array}}{\neg F', \Gamma \vdash \Delta}$$

then induction hypothesis gives a $\mathbf{LK}^{\mathrm{sk}}$-proof ψ of $\mathrm{sk}(\Gamma \vdash \Delta, F')$.
Using lemma 3.14, we define ψ as

$$\dfrac{\begin{array}{c}(\psi')\\ \mathrm{sk}(\Gamma \vdash \Delta, F')\end{array}}{\mathrm{sk}(\neg F', \Gamma \vdash \Delta)}$$

which is an $\mathbf{LK}^{\mathrm{sk}}$-proof of $\mathrm{sk}(S)$. The inequalities hold as

$$l(\psi) = l(\psi') + 1 \overset{\mathrm{IV}}{\leq} l(\pi') + 1 = l(\pi)$$

and

$$\mathrm{depth}(\psi) = \mathrm{depth}(\psi') + 1 \overset{\mathrm{IV}}{\leq} \mathrm{depth}(\pi') + 1 = \mathrm{depth}(\pi)$$

by induction hypothesis.

If π has the form

$$\frac{\begin{array}{cc}(\pi_1) & (\pi_2)\\ A, \Gamma \vdash \Delta & B, \Gamma \vdash \Delta\end{array}}{A \vee B, \Gamma \vdash \Delta}$$

then by induction hypothesis we get $\mathbf{LK}^{\mathrm{sk}}$-proofs ψ_1 of $\mathrm{sk}(A, \Gamma \vdash \Delta)$ and ψ_2 of $\mathrm{sk}(B, \Gamma \vdash \Delta)$. Using $\mathrm{sk}(A) \vee \mathrm{sk}(B) = \mathrm{sk}(A \vee B)$ from lemma 3.14, we can define ψ as

$$\frac{\begin{array}{cc}(\psi_1) & (\psi_2)\\ \mathrm{sk}(A, \Gamma \vdash \Delta) & \mathrm{sk}(B, \Gamma \vdash \Delta)\end{array}}{\mathrm{sk}(A \vee B, \Gamma \vdash \Delta)}$$

which is an $\mathbf{LK}^{\mathrm{sk}}$-proof of $\mathrm{sk}(S)$. The inequalities hold as

$$l(\psi) = l(\psi_1) + l(\psi_2) + 1 \overset{\mathrm{IV}}{\leq} l(\pi_1) + l(\pi_2) + 1 = l(\pi)$$

and

$$\mathrm{depth}(\psi) = \max\{\mathrm{depth}(\psi_1), \mathrm{depth}(\psi_2)\} + 1 \overset{\mathrm{IV}}{\leq} \max\{\mathrm{depth}(\pi_1), \mathrm{depth}(\pi_2)\} + 1 = \mathrm{depth}(\pi)$$

by induction hypothesis.

If π has the form

$$\frac{(\pi_0)}{\Gamma \vdash \Delta, \exists x\, F, F[x\backslash t]}$$
$$\Gamma \vdash \Delta, \exists x\, F$$

then by the induction hypothesis, there is an $\mathbf{LK}^{\mathrm{sk}}$-proof ψ_0 of $\mathrm{sk}(\Gamma \vdash \Delta, F[x\backslash t])$.
We define ψ as

$$\frac{(\psi_0)}{\mathrm{sk}(\Gamma \vdash \Delta, \exists x\, F, F[x\backslash t])}$$
$$\mathrm{sk}(\Gamma) \vdash \mathrm{sk}(\Delta), \exists x\, \mathrm{sk}(F)$$

.
Using lemma 3.14, we see that $\exists x\, \mathrm{sk}(F) = \mathrm{sk}(\exists x\, F)$, which then implies that ψ is an $\mathbf{LK}^{\mathrm{sk}}$-proof of $\mathrm{sk}(S)$ as well. The inequalities $l(\psi) \leq l(\pi)$ and $\mathrm{depth}(\psi) \leq \mathrm{depth}(\pi)$ follow directly from the induction hypothesis.

If π has the form

$$\dfrac{\begin{array}{c}(\pi_0)\\ \Gamma, F[x\backslash f(\bar{t})] \vdash \Delta\end{array}}{\Gamma, \exists^{f(\bar{t})} x\, F \vdash \Delta}$$

then the induction hypothesis gives an $\mathbf{LK}^{\mathrm{sk}}$-proof ψ of $\mathrm{sk}(\Gamma, F[x\backslash f(\bar{t})] \vdash \Delta)$.
Note that since $\mathrm{sk}(\exists^{f(\bar{t})} x\, F) = \mathrm{sk}(F[x\backslash f(\bar{t})])$, ψ is also an $\mathbf{LK}^{\mathrm{sk}}$-proof of $\mathrm{sk}(S)$, which is what we wanted to show.

As ψ contains one less inference than π, it follows that $l(\psi) \leq l(\pi)$ and $\mathrm{depth}(\psi) \leq \mathrm{depth}(\pi)$.

The dual cases are treated analogously.

□

This result shows that the known property of skolemising a sequent preserves provability in our skolem calculus as well, and that it does not increase proof complexity.

In the converse direction, we reconstruct a proof of the annotated sequent from its skolemised version. As we potentionally have to add strong quantifier inferences, the size of the proof may increase. But the complexity of this reversal only increases linearly on the number of strong quantifiers.

Definition 3.16. Let F be an expanded annotated formula, then $\mathrm{qocc}(F)$ is defined as the number of strong quantifiers in F.

The same definition is made for expanded annotated sequents.

Definition 3.17. Let F be an expanded annotated formula, then $\mathrm{extqocc}(F)$ is defined as the largest number k such that F can be written as $F = Q^{f_1(\bar{t}_1)} ... Q^{f_k(\bar{t}_k)}\, F'$, where $Q^{f_i(\bar{t}_i)}$ are strong quantifiers.
If no such k exists we define $\mathrm{extqocc}(F) := 0$.

For an expanded annotated sequent S, we define $\mathrm{extqocc}(S)$ as the sum over all expanded annotated formulas in S.

Example 3.17.1. Let $F := \exists^{f(a)} x \exists^{g(a,b)} y \forall z \exists^{h(a,z)} w\, P(a,x,y) \vee Q(b,z,w)$.

Then $\mathrm{qocc}(F) = 3$ and $\mathrm{extqocc}(F) = 2$ with $F' = \forall z \exists^{h(a,z)} w\, P(a,x,y) \vee Q(b,z,w)$.

Theorem 3.18. *Let S be an annotated sequent and ψ be an $\mathbf{LK}^{\mathrm{sk}}$-proof of $\mathrm{sk}(S)$. Then there exists an $\mathbf{LK}^{\mathrm{sk}}$-proof π of S such that $l(\pi) \leq l(\psi) * (\mathrm{qocc}(S) + 1)$ and $\mathrm{depth}(\pi) \leq \mathrm{depth}(\psi) + \mathrm{qocc}(S)$.*

Proof. We define $n := l(\psi)$ and $k := \mathrm{extqocc}(S)$ The theorem will be proven by double induction on k and n.

- Assume $n = 1$ and $k \geq 0$. Then $\mathrm{sk}(S)$ is an axiom and therefore S is an axiom too. In this case π is an $\mathbf{LK}^{\mathrm{sk}}$-proof of S with $l(\pi) = \mathrm{depth}(\pi) = 1$, which means that the induction base holds.

- Assume $n \geq 2$ and $k \geq 1$ and let ψ be an $\mathbf{LK}^{\mathrm{sk}}$-proof of $\mathrm{sk}(S)$.

  Then there exists an annotated expanded formula F in S such that $\mathrm{extqocc}(F) \geq 1$. Assume that $F = \exists^{f(\bar{t})} x\, F'$. By definition of skolemisation $\mathrm{sk}(F) = \mathrm{sk}(F')[x \backslash f(\bar{t})]$, which means that ψ is also an $\mathbf{LK}^{\mathrm{sk}}$-proof of

  $$\mathrm{sk}(S)' \coloneqq \mathrm{sk}(F')[x \backslash f(\bar{t})], \mathrm{sk}(\Gamma) \vdash \mathrm{sk}(\Delta).$$

  As $\mathrm{extqocc}(F') \leq \mathrm{extqocc}(F)$, we can apply the induction hypothesis to $\mathrm{sk}(S)'$ and receive an $\mathbf{LK}^{\mathrm{sk}}$-proof π' of $F'[x \backslash f(\bar{t})], \Gamma \vdash \Delta$.

  If we define π as

  $$\dfrac{\begin{array}{c}(\pi') \\ F'[x \backslash f(\bar{t})], \Gamma \vdash \Delta\end{array}}{\exists^{f(\bar{t})} y\, F', \Gamma \vdash \Delta}$$

  then π is an $\mathbf{LK}^{\mathrm{sk}}$-proof of S.

  The inequalities hold as

  $$\begin{aligned} l(\pi) &= l(\pi') + 1 \\ &\leq l(\psi') * (\mathrm{qocc}(S') + 1) + 1 && \text{IV} \\ &\leq l(\psi') * ((\mathrm{qocc}(S') + 1) + 1) \\ &= l(\psi') * (\mathrm{qocc}(S) + 1) \end{aligned}$$

  and

  $$\mathrm{depth}(\pi) = \mathrm{depth}(\pi') + 1 \overset{\text{IV}}{\leq} \mathrm{depth}(\psi') + \mathrm{qocc}(S') + 1 = \mathrm{depth}(\psi) + \mathrm{qocc}(S).$$

  The dual case $F = \forall^{f(\bar{t})} x\, F'$ can be treated analogously.

- For the last case, assume $n \geq 2$ and $k = 0$.

  - If ψ is

    $$\dfrac{\begin{array}{c}(\psi') \\ \mathrm{sk}(\Gamma \vdash \Delta, F')\end{array}}{\mathrm{sk}(\neg F', \Gamma \vdash \Delta)}$$

    then $F = \neg F'$. We can apply the induction hypothesis, as the number of inferences in ψ' is smaller than in ψ. This leads to an $\mathbf{LK}^{\mathrm{sk}}$-proof π' of $S' \coloneqq \Gamma \vdash \Delta, F'$.

    Define π as

    $$\dfrac{\begin{array}{c}(\pi') \\ \Gamma \vdash \Delta, F'\end{array}}{\neg F', \Gamma \vdash \Delta}$$

which is an $\mathbf{LK}^{\mathrm{sk}}$-proof of S.

By induction hypothesis $l(\pi') \leq l(\psi')*(\mathrm{qocc}(S')+1)$ and $\mathrm{depth}(\pi') \leq \mathrm{depth}(\psi')+\mathrm{qocc}(S')$, so

$$\begin{aligned} l(\pi) &= l(\pi') + 1 \\ &\leq l(\psi') * (\mathrm{qocc}(S') + 1) + 1 && \text{IV} \\ &\leq l(\psi') * ((\mathrm{qocc}(S') + 1) + 1) \\ &= l(\psi') * (\mathrm{qocc}(S) + 1) \end{aligned}$$

and

$$\mathrm{depth}(\pi) = \mathrm{depth}(\pi') + 1 \overset{\text{IV}}{\leq} \mathrm{depth}(\psi') + \mathrm{qocc}(S') + 1 \leq \mathrm{depth}(\psi) + \mathrm{qocc}(S).$$

– If ψ is

$$\frac{\begin{array}{cc}(\psi_1) & (\psi_2) \\ \mathrm{sk}(A, \Gamma \vdash \Delta) & \mathrm{sk}(B, \Gamma \vdash \Delta)\end{array}}{\mathrm{sk}(A \vee B, \Gamma \vdash \Delta)}$$

then $F = A \vee B$ and the induction hypothesis leads to $\mathbf{LK}^{\mathrm{sk}}$-proofs π_1 of $S_1 := A, \Gamma \vdash \Delta$ and π_2 of $S_2 := B, \Gamma \vdash \Delta$.

We define π as

$$\frac{\begin{array}{cc}(\pi_1) & (\psi_2) \\ A, \Gamma \vdash \Delta & B, \Gamma \vdash \Delta\end{array}}{A \vee B, \Gamma \vdash \Delta}$$

which is an $\mathbf{LK}^{\mathrm{sk}}$-proof of S.

By induction hypothesis $l(\pi_i) \leq l(\psi_i)*(\mathrm{qocc}(S_i)+1)$ and $\mathrm{depth}(\pi_i) \leq \mathrm{depth}(\psi_i)+\mathrm{qocc}(S_i)$; and since $\mathrm{qocc}(S_i) \leq \mathrm{qocc}(S)$ we get

$$\begin{aligned} l(\pi) &= l(\pi_1) + l(\pi_2) + 1 \\ &\leq l(\psi_1) * (\mathrm{qocc}(S_1) + 1) + l(\psi_2) * (\mathrm{qocc}(S_1) + 1) + 1 && \text{IV} \\ &\leq (l(\psi_1) + l(\psi_2) + 1) * (\mathrm{qocc}(S) + 1) \\ &= l(\psi) * (\mathrm{qocc}(S) + 1) \end{aligned}$$

and

$$\begin{aligned} \mathrm{depth}(\pi) &= \max\{\mathrm{depth}(\pi_1), \mathrm{depth}(\pi_2)\} + 1 \\ &\leq \max\{\mathrm{depth}(\psi_1) + \mathrm{qocc}(S_1), \mathrm{depth}(\psi_2) + \mathrm{qocc}(S_2)\} + 1 && \text{IV} \\ &\leq \max\{\mathrm{depth}(\psi_1) + \mathrm{qocc}(S), \mathrm{depth}(\psi_2) + \mathrm{qocc}(S)\} + 1 \\ &= \mathrm{depth}(\psi) + \mathrm{qocc}(S). \end{aligned}$$

– If ψ is

$$\frac{(\psi')}{\mathrm{sk}(\forall x\, F', F'[x\backslash t], \Gamma \vdash \Delta)}{\mathrm{sk}(\forall x\, F', \Gamma \vdash \Delta)}$$

then $F = \forall x\, F'$ and the induction hypothesis leads to an $\mathbf{LK}^{\mathrm{sk}}$-proof π' of $S' := F'[x\backslash t], \Gamma \vdash \Delta$.

We define π as

$$\frac{(\pi')}{\forall x\, F', F'[x\backslash t], \Gamma \vdash \Delta}{\forall x\, F', \Gamma \vdash \Delta}$$

which is an $\mathbf{LK}^{\mathrm{sk}}$-proof of S. The inequalities hold as

$$\begin{aligned} l(\pi) &= l(\pi') + 1 \\ &\leq l(\psi') * (\mathrm{qocc}(S') + 1) + 1 && \text{IV} \\ &\leq (l(\psi') + 1) * (\mathrm{qocc}(S) + 1) \\ &= l(\psi) * (\mathrm{qocc}(S) + 1) \end{aligned}$$

and

$$\begin{aligned} \mathrm{depth}(\pi) &= \mathrm{depth}(\pi') + 1 \\ &\leq \mathrm{depth}(\psi') + \mathrm{qocc}(S') + 1 && \text{IV} \\ &\leq \mathrm{depth}(\psi') + 1 + \mathrm{qocc}(S) \\ &= \mathrm{depth}(\psi) + \mathrm{qocc}(S) \end{aligned}$$

by induction hypothesis and since $\mathrm{qocc}(S') = \mathrm{qocc}(S)$.

The dual cases can be treated analogously.

□

Remark 3.18.1. As a skolemised sequent does not contain any strong quantifiers, $\mathbf{LK}^{\mathrm{ac}}$-proofs and $\mathbf{LK}^{\mathrm{sk}}$-proofs of it coincide. Therefore all results of this section work the same way in $\mathbf{LK}^{\mathrm{ac}}$.

3.6 Annotations and Proofs

We will now focus our attention on the skolem annotation itself. The first interesting observation is how the annotation of a sequent affects the size of the respective proofs. Let us illustrate this behaviour in the following example:

Example 3.18.1. Consider the formula $F = \exists x \forall y\, P(x) \supset P(y)$ and define two skolem annotations α, α' such that

$$\alpha(F) := \exists x \forall^{f(x)} y\, P(x) \supset P(y)$$
$$\alpha'(F) := \exists x \forall^{c} y\, P(x) \supset P(y)$$

This first annotation coincides with the standard skolemisation and leads to the following $\mathbf{LK}^{\mathrm{sk}}$-proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash P(c), P(f(c)) \supset P(f(c)), P(f(f(c))), \exists x \forall^{f(x)} y\, P(x) \supset P(y)}{\vdash P(c) \supset P(f(c)), P(f(c)) \supset P(f(f(c))), \exists x \forall^{f(x)} y\, P(x) \supset P(y)}\supset_r}{\vdash P(c) \supset P(f(c)), P(f(c)) \supset P(f(f(c))), \exists x \forall^{f(x)} y\, P(x) \supset P(y)}\supset_r}{\vdash P(c) \supset P(f(c)), \forall^{f(f(c))} y\, P(f(c)) \supset P(y), \exists x \forall^{f(x)} y\, P(x) \supset P(y)}\forall^{\mathrm{sk}}_r}{\vdash \forall^{f(c)} y\, P(c) \supset P(y), \forall^{f(f(c))} y\, P(f(c)) \supset P(y), \exists x \forall^{f(x)} y\, P(x) \supset P(y)}\forall^{\mathrm{sk}}_r}{\vdash \forall^{f(c)} y\, P(c) \supset P(y), \exists x \forall^{f(x)} y\, P(x) \supset P(y)}\exists_r}{\vdash \exists x \forall^{f(x)} y\, P(x) \supset P(y)}\exists_r$$

Now using the smaller annotation, in which the strong quantifier is chosen independent of the beforegoing weak quantifier, we can do the same with a much shorter proof.

$$\dfrac{\dfrac{\dfrac{\vdash P(c) \supset P(c), \exists x \forall^{f(x)} y\, P(x) \supset P(y)}{\vdash P(c) \supset P(c), \exists x \forall^{f(x)} y\, P(x) \supset P(y)}\supset_r}{\vdash \forall^{c} y\, P(c) \supset P(y), \exists x \forall^{f(x)} y\, P(x) \supset P(y)}\forall^{\mathrm{sk}}_r}{\vdash \exists x \forall^{c} y\, P(x) \supset P(y)}\exists_r$$

This example demonstrates how choosing a more concise annotation by reducing the dependency of the strong quantifier can shorten a proof.
This observation can be formalised in the following theorem.

Theorem 3.19. *Let S be a sequent, α a skolem annotation and π be an $\mathbf{LK}^{\mathrm{sk}}$-proof of $\alpha(S)$. For any $\alpha' \leq \alpha$, there exists an $\mathbf{LK}^{\mathrm{sk}}$-proof π' of $\alpha'(S)$ such that $l(\pi') = l(\pi)$.*

We will show the simple case where the annotations differ in a single variable only. The theorem then follows by simple induction. For this we define a mapping * that translates annotated sequents.

Definition 3.20. The mapping * is defined on terms as

$$s \to s^* := \begin{cases} a^* & := a \\ x^* & := x \\ t^* & := t \\ g(t_1, ..., t_k)^* & := g(t_1^*, ..., t_k^*) \\ f(t_1, ..., t_n)^* & := f'(t_2^*, ..., t_n^*) \end{cases}$$

where f, f' are skolem symbols and g an arbitrary function symbol. We define * on formulas and sequents by pointwise application.

Lemma 3.21. *Let A be an expanded annotated formula and t be a term. Then $(A[x\backslash t])^* = A^*[x\backslash t^*]$.*

Proof. This will be proven by induction on the structure of the formula A.

- If $A = R^n(t_1, ..., t_n)$ for an $n-ary$ predicate symbol, we have

  $$(R^n(t_1, ..., t_n)[x\backslash t])^* = (R^n(t_1[x\backslash t], ..., t_n[x\backslash t]))^* = R^n((t_1[x\backslash t])^*, ..., (t_n[x\backslash t])^*).$$

  So we need to show $(s[x\backslash t])^* = s^*[x\backslash t^*]$ for arbitrary terms s, t. In the case of s being a constant or free variable, this is clear. If $s = f(t_1, ..., t_n)$, then

  $$\begin{aligned}(f(t_1, ..., t_n)[x\backslash t])^* &= (f(t_1[x\backslash t], ..., t_n[x\backslash t]))^* \\ &= f'((t_2[x\backslash t])^*, ..., (t_n[x\backslash t])^*) \\ &= f'(t_2^*[x\backslash t^*], ..., t_n^*[x\backslash t^*]) \\ &= f'(t_2^*, ..., t_n^*)[x\backslash t^*] \\ &= (f(t_1, ..., t_n))^*[x\backslash t^*].\end{aligned}$$

  The case $s = g(t_1, ..., t_n)$ is treated the same way.

- If $A = B \vee C$, we have

  $$\begin{aligned}((B \vee C)[x\backslash t])^* &= ((B[x\backslash t] \vee C[x\backslash t]))^* \\ &= (B[x\backslash t])^* \vee (C[x\backslash t])^* && \text{IV} \\ &= B^*[x\backslash t^*] \vee C^*[x\backslash t^*] \\ &= (B^* \vee C^*)[x\backslash t^*].\end{aligned}$$

  All other non quantifier rules are treated analogously.

- If $A = Qy\,B$ where Q is a weak quantifier, we have

  $$(Qy\,B[x\backslash t])^* = Qy\,(B[x\backslash t])^* \overset{\text{IV}}{=} Qy\,B^*[x\backslash t^*]$$

  by definition and induction hypothesis.

- If $A = Q^{h(\bar{t})}y\,B$ with skolemterm $h(\bar{t})$, we have

  $$(Q^{h(\bar{t})}y\,B[x\backslash t])^* = Q^{h(\bar{t})^*}y\,(B[x\backslash t])^* \overset{\text{IV}}{=} Q^{h(\bar{t})^*}y\,B^*[x\backslash t^*]$$

  by definition and induction hypothesis.

□

Lemma 3.22. *Let S be a sequent, α a skolem annotation and π be an $\mathbf{LK}^{\mathrm{sk}}$-proof of $\alpha(S)$. Then π^* is an $\mathbf{LK}^{\mathrm{sk}}$-proof of $\alpha(S)^*$ such that $l(\pi') = l(\pi)$ and $\mathrm{depth}(\pi') = \mathrm{depth}(\pi)$.*

Proof. We show the theorem by induction on the number of inferences in π. If π is a single axiom $A, \Gamma \vdash \Delta, A$, then by definition of * we have $\pi^* := A^*, \Gamma^* \vdash \Delta^*, A^*$.

If π has the form

$$\frac{\begin{array}{c}(\psi)\\ A, \Gamma \vdash \Delta\end{array}}{\Gamma \vdash \Delta, \neg A}$$

then by induction hypothesis and definition of * we get

$$\frac{\begin{array}{c}(\psi^*)\\ A^*, \Gamma^* \vdash \Delta^*\end{array}}{\Gamma^* \vdash \Delta^*, \neg A^*}$$

. All other logical inferences follow analogously.

If π has the form

$$\frac{\begin{array}{c}(\psi)\\ \Gamma \vdash \Delta, \exists x\, A, A[x\backslash t]\end{array}}{\Gamma \vdash \Delta, \exists x\, A}$$

,
as $(\exists\, xA)^* = \exists x\, A^*$ by definition, it suffices to show $(A[x\backslash t])^* = A^*[x\backslash t^*]$. But this has already been proven in lemma 3.21.

As the translation does not change the number of inferences, it is clear that $l(\pi') = l(\pi)$ and $\mathrm{depth}(\pi') = \mathrm{depth}(\pi)$.

□

Corollary 3.23. *Let S be a sequent and α, α' annotations such that $\alpha' \leq \alpha$. Then $\alpha'(S)$ admits shorter proofs than $\alpha(S)$*

Proof. By theorem 3.19 any proof of $\alpha(S)$ leads to a proof of $\alpha'(S)$ with the same size. Shorter proofs of $\alpha'(S)$ may exist. □

This corollary serves as motivation for finding a proper annotation for each individual sequent to shorten proofs. But one has to be careful to find the proper annotation, as there are invalid formulas which can be proven by choosing an annotation. This can be seen in the following example.

Example 3.23.1. Consider the sequent $S = \forall x \exists y\ x \leq y \supset \exists z \forall x\ x \leq z$. This obviously does not hold in the natural numbers with the ordering $(\mathbb{N}, \leq)$.
But by using the skolem annotation α such that

$$\alpha(S) = \forall x \exists^c y\ x \leq y \supset \exists z \forall^{f(z)} x\ x \leq z$$

we can prove the sequent:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash f(c) \leq c \supset f(c) \leq c}{\vdash f(c) \leq c \supset \forall^{f(c)} x\ x \leq c}\,\forall_r^{\mathrm{sk}}}{\vdash \exists^c y\ f(c) \leq y \supset \forall^{f(c)} x\ x \leq c}\,\exists_l^{\mathrm{sk}}}{\vdash \exists^c y\ f(c) \leq y \supset \exists z \forall^{f(z)} x\ x \leq z}\,\exists_r}{\vdash \forall x \exists^c y\ x \leq y \supset \exists z \forall^{f(z)} x\ x \leq z}\,\forall_l$$

This counterexample shows that an overly small annotation may result in a proof of a normally invalid sequent. Smaller annotations reduce the size of proofs, but it has to be sufficiently large such that the validity or invalidity of the initial sequent is preserved.

4 Expansion Proofs

This chapter is centered around expansion proofs. Analogously to the sequent calculi in the previous chapter, we will consider two types of expansion proofs, which differ primarily in how they treat strong quantifiers. The first type corresponds to skolemisation, similar to the definition in [6], while the second introduces an acyclicity condition on selected variables as the one proposed in [7]. In the final section, we examine the interaction between expansion proofs and skolemised formulas.

4.1 Expansions

We begin by formalising the notion of expansions for annotated formulas. The following definition generalises the one introduced in [6] to arbitrary annotations, resulting in a base for both skolem and variable-based quantification.

Definition 4.1. Expansions and dual expansions of expanded annotated formulas are defined inductively as

1. $\bot$ is an expansion of any formula.
2. Any atom A is an expansion of itself.
3. If E is an expansion of A, then $\neg E$ is a dual expansion of $\neg A$.
4. If E_1 is an expansion of A and E_2 is an expansion of B, then $E_1 \vee E_2$ is an expansion of $A \vee B$.
5. If $E_1, ..., E_n$ are expansions of $A[x\backslash t_i]$ with terms $t_1, ..., t_n$, then $\exists x\, A +^{t_1} E_1 ... +^{t_n} E_n$ is an expansion of $\exists x\, A$.
6. If E is an expansion of $A[x\backslash f(\overline{x}, \overline{t})]$, then $\forall^{f(\overline{x},\overline{t})} x\, A +^{f(\overline{x},\overline{t})} E$ is an expansion of $\forall^{f(\overline{x},\overline{t})} x\, A$. We call this a skolem quantifier node.
7. If E is an expansion of $A[x\backslash a]$ and a is a free variable that has not been used in a strong quantifier inference in E, then $\forall^{f(\overline{t})} x\, A +^{a} E$ is an expansion of $\forall^{f(\overline{t})} x\, A$. We call this a variable quantifier node with *selected variable* a.

We call 5. *weak quantifier nodes* and 6. and 7. *strong quantifier nodes*. Dual expansions are defined the same way by replacing *expansion* with *dual expansion*, $\bot$ with $\top$ and $\forall$ with $\exists$.

Expansions of expanded annotated sequents are defined in the following way: If $S := \Gamma_1, ..., \Gamma_n \vdash \Delta_1, ..., \Delta_k$ is an annotated sequent, then $\Pi_1, ..., \Pi_n \vdash \Lambda_1, ..., \Lambda_k$ is an expansion tree of S if $\Pi_1, ..., \Pi_n$ are dual expansions of $\Gamma_1, ..., \Gamma_n$ and $\Lambda_1, ..., \Lambda_k$ being dual expansions of $\Delta_1, ..., \Delta_k$.
As in the case of sequent calculi, we are interested in measuring the size and complexity of proofs. For this, we introduce a notion of size based on the structure of expansions.

Definition 4.2. Let E be an expansion or dual expansion, then we define the *logical complexity* $|E|$ as:

$$
\begin{aligned}
|E| &= 0 \text{ if } E \text{ is an atom} \\
|\neg E| &= |E| + 1 \\
|E_1 \vee E_2| &= |E_1| + |E_2| + 1 \\
|Qx A +^{t_1} E_1 ... +^{t_n} E_n| &= \sum_{i=1}^{n} (|E_i| + 1) \text{ if } Qx \text{ is a weak quantifier} \\
|Q^{f(\bar{t})} x\, A +^{f(\bar{t})} E| &= |E| + 1 \text{ if } Qx \text{ is a strong quantifier}
\end{aligned}
$$

For an expansion $\Pi \vdash \Lambda$ of a sequent we define $|\Pi \vdash \Lambda| := \sum_{E \in \Pi \cup \Lambda} |E|$.

Let S be an annotated sequent and E be an expansion of it. We can think of E as a rooted tree where the root is the sequent S, the edges refer to quantifier inferences (denoted as $+^t$) and the nodes are subexpansions of E. We say that an edge $+^t$ is *above* an edge $+^s$, if both edges are on the same branch and $+^s$ is closer to the root S than $+^t$.
In terms of the annotated sequent S, if Q^t and Q^s are the quantifiers corresponding to the edges, then Q^s is in the scope of Q^t.

4.2 Skolem Expansions

The first class of expansion proofs we consider are those in which all strong quantifiers are introduced via skolemterms. For this we first need to define two associated formulas: the shallow and deep formulas.

Definition 4.3. For a dual expansion or expansion E, we define the shallow formula $\mathrm{Sh}(E)$ and the quantifier-free deep formula $\mathrm{Dp}(E)$ inductively:

$$
\begin{aligned}
\mathrm{Sh}(E) &= E \text{ if } E \text{ is an atom} \\
\mathrm{Sh}(\neg E) &= \neg \mathrm{Sh}(E) \\
\mathrm{Sh}(E_1 \vee E_2) &= \mathrm{Sh}(E_1) \vee \mathrm{Sh}(E_2) \\
\mathrm{Sh}(E_1 \wedge E_2) &= \mathrm{Sh}(E_1) \wedge \mathrm{Sh}(E_2) \\
\mathrm{Sh}(Qx\, A +^{t_1} E_1 \ldots +^{t_n} E_n) &= Qx\, A \\
\mathrm{Sh}(Q^{f(\bar{t})} x\, A +^{f(\bar{t})} E) &= Q^{f(\bar{t})} x\, A \\
\mathrm{Sh}(Q^{f(\bar{t})} x\, A +^{a} E) &= Q^{f(\bar{t})} x\, A
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Dp}(E) &= E \text{ if } E \text{ is an atom} \\
\mathrm{Dp}(\neg E) &= \neg \mathrm{Dp}(E) \\
\mathrm{Dp}(E_1 \vee E_2) &= \mathrm{Dp}(E_1) \vee \mathrm{Dp}(E_2) \\
\mathrm{Dp}(E_1 \wedge E_2) &= \mathrm{Dp}(E_1) \wedge \mathrm{Dp}(E_2) \\
\mathrm{Dp}(\exists x\, A +^{t_1} E_1 \ldots +^{t_n} E_n) &= \bigvee_{1 \leq i \leq n} \mathrm{Dp}(E_i) \\
\mathrm{Dp}(\forall x\, A +^{t_1} E_1 \ldots +^{t_n} E_n) &= \bigwedge_{1 \leq i \leq n} \mathrm{Dp}(E_i) \\
\mathrm{Dp}(Q^{f(\bar{t})} x\, A +^{f(\bar{t})} E) &= \mathrm{Dp}(E) \\
\mathrm{Dp}(Q^{f(\bar{t})} x\, A +^{a} E) &= \mathrm{Dp}(E)
\end{aligned}
$$

For an expansion tree $E = \Pi \vdash \Lambda$ of a sequent S we define the shallow formula as

$$
\mathrm{Sh}(E) = \left[\bigvee_{1 \leq i \leq n} \neg \mathrm{Sh}(\Pi_i) \right] \vee \left[\bigvee_{1 \leq j \leq k} \mathrm{Sh}(\Lambda_j) \right].
$$

The same definition is done for the deep formula $\mathrm{Dp}(E)$.
It is worth noting that the shallow formula $\mathrm{Sh}(E)$ of an expansion E does not necessarily coincide with the original annotated formula as it may contain any number of $\top$ and $\bot$ in place of atoms or subformulas. But by substituting these constants with appropriate subformulas of the original sequent, we can always reconstruct the initial input.

Definition 4.4. An expansion A of an annotated formula F such that all strong quantifier nodes in A are skolem quantifier nodes and its deep formula $\mathrm{Dp}(A)$ is a tautology is called an $\mathbf{EP}^{\mathrm{sk}}$-proof.

$\mathbf{EP}^{\mathrm{sk}}$-proofs of annotated sequent are defined in the canonical way.

Remark 4.4.1. The same way as with $\mathbf{LK}^{\mathrm{sk}}$-proofs we do not need the additional condition of the skolemsymbols being unique for each strong quantifier, as we are only considering expansions of annotated sequents.

The next example illustrates how skolem expansions behave in practice. In particular, it demonstrates how different annotations yield different expansions and how the choice of skolemterms impacts the size of the resulting proof.

Example 4.4.1. Consider the formula $F \coloneqq \exists x \forall y\, P(x) \supset P(y)$ from Example 3.18.1. For each of the two annotations we can find the following expansions

$$E_1 \coloneqq \exists x \forall^{f(x)} y\, P(x) \supset P(y) +^{c} \left[\left(\forall^{f(c)} y\, P(c) \supset P(y) \right) +^{f(c)} \left(P(c) \supset P(f(c)) \right) \right] \\ +^{f(c)} \left[\left(\forall^{f(f(c))} y\, P(f(c)) \supset P(y) \right) +^{f(f(c))} \left(P(f(c)) \supset P(f(f(c))) \right) \right]$$

and

$$E_2 \coloneqq \exists x \forall^{c} y\, P(x) \supset P(y) +^{c} \left[\left(\forall^{c} y\, P(c) \supset P(y) \right) +^{c} \left(P(c) \supset P(c) \right) \right].$$

As both deep formulas are tautologies, these are both $\mathbf{EP}^{\mathrm{sk}}$-proofs of the given annotated formula. Similar to Example 3.18.1 the smaller annotation leads to a smaller expansion.

4.3 Acyclic Expansions

We now turn to an alternative formulation of expansion proofs that uses selected variables instead of skolemterms. Here we require a condition on the use of selected variables to prevent circular dependencies. This leads to the notion of acyclic expansion proofs of annotated sequents, which generalise the system proposed in [7].
To define this formally, we first introduce a dependency relation on the edges in an expansion tree. This relation describes how a selected variable depends on earlier inferences, with respect to the annotation of the sequent.

Definition 4.5. Let A be an expansion tree and $\mathbf{Q}(A)$ the set of edges in A. If a is a selected variable of a quantifier occurrence $Q^{f(\bar{t})}$, we define an ordering $<^0_A$ on $\mathbf{Q}(A)$ by

- $+^t <^0_A +^a$ if t is an element of $\bar{t}$ and appears in a weak quantifier inference above a in A,
- $+^a <^0_A +^t$ if a is a proper subterm of t.

The transitive closure $<_A$ of $<^0_A$ is called the *dependency relation* of A.

Using this relation, we can define the class of acyclic expansion proofs. These are expansions in which no circular dependencies occur and only variable quantifier nodes are permitted.

Definition 4.6. An expansion A of an annotated formula F such that its deep formula $\mathrm{Dp}(A)$ is a tautology, all strong quantifier nodes in A are variable quantifier nodes and the *dependency relation* $<_A$ is acyclic is called an $\mathbf{EP}^{\mathrm{ac}}$-proof.
As with $\mathbf{LK}^{\mathrm{ac}}$-proofs we require weak regularity: the selected variables of any two acyclic quantifier nodes are the same if and only if the respective skolemterms in the superscripts of the strong quantifiers are the same.
$\mathbf{EP}^{\mathrm{ac}}$-proofs of annotated sequent are defined in the canonical way. Note that in this case the weak regularity has to be fulfilled with respect to all annotated formulas in the sequent.

Example 4.6.1. Similar to example 4.4.1 we can find two expansions

$$A_1 \coloneqq \exists x \forall^{f(x)} y\, P(x) \supset P(y) +^u \left[\left(\forall^{f(u)} y\, P(u) \supset P(y)\right) +^v (P(u) \supset P(v))\right] \\ +^v \left[\left(\forall^{f(v)} y\, P(v) \supset P(y)\right) +^w (P(v) \supset P(w))\right]$$

and

$$A_2 \coloneqq \exists x \forall^v y\, P(x) \supset P(y) +^v [(\forall^v y\, P(v) \supset P(y)) +^v (P(v) \supset P(v))]\,.$$

Here $\mathbf{Q}(A_1) = \{u, v, v, w\}$ where the relation is $u <^0_{A_1} v$, $v <^0_{A_1} v$, $v <^0_{A_1} w$ and $\mathbf{Q}(A_2) = \{v, v\}$ where $<^0_{A_2} = \varnothing$. This shows that both dependency relations are acyclic and therefore both A_1 and A_2 are $\mathbf{EP}^{\mathrm{ac}}$-proofs of the respective annotated formula.
On the other hand choosing u instead of w in A_1 would mean that the dependency relation contains a cycle.

4.4 Equivalence between the Expansion Calculi

Going back to the annotated formula $\exists x \forall^{f(x)} y\, P(x) \supset P(y)$ we can see that the $\mathbf{EP}^{\mathrm{sk}}$-proof E_1 from Example 4.4.1 and the $\mathbf{EP}^{\mathrm{ac}}$-proof A_1 from Example 4.6.1 are seemingly identical, the only difference being the substitution of skolemterms $\{c, f(c), f(f(c))\}$ by variables $\{u, v, w\}$ in an appropriate way.
As an analogy to section 3.4 we will now show that this holds in general, by proving that this substitution is a size preserving translation between $\mathbf{EP}^{\mathrm{sk}}$-proofs and $\mathbf{EP}^{\mathrm{ac}}$-proofs.

Theorem 4.7. *Let A be an $\mathbf{EP}^{\mathrm{ac}}$-proof of an annotated sequent S. Then there exists an $\mathbf{EP}^{\mathrm{sk}}$-proof E of the same sequent such that $|A| = |E|$.*

Proof. We will prove by induction on the number of selected variables in A that there exists an expansion E where all such variables are replaced by skolemterms.

If A does not contain any selected variables, then it does not contain any acyclic quantifier nodes. In this case $E \coloneqq A$ itself is an $\mathbf{EP}^{\mathrm{sk}}$-proof and clearly $|A| = |E|$.

Now assume that the set of selected variables is not empty and let a be one that is minimal in $<_A$. This exists as the relation is acyclic. If $Q^{f(\bar{t})}$ is the respective strong quantifier, we define A' to be the expansion where all occurrences of a are replaced by $f(\bar{t})$. Note that superscripts $Q^{f(\bar{s})}$ may also be affected by this substitution.
In order to apply the induction hypothesis on A', we need to show that A' is an expansion proof and $<_{A'}$ is acyclic. Replacing terms does not affect $\mathrm{Dp}(A')$ being a tautology, so it only remains to show the acyclicity condition.
Let b be a selected variable and s' be a term in A'. By definition of A', there exists a term s in A such that $s' = s[a \backslash f(\bar{t})]$. There are two possibilities:

- If $s' <^0_{A'} b$, then s' appears in a weak quantifier inference above b. As no changes have been done to the structure of A during the substitution, this implies that s appears above b in A, meaning $s <^0_A b$.
- If $b <^0_{A'} s'$, this means that b is a subterm of s'. If b was a subterm of $f(\overline{t})$, then it would be a subterm of at least one $t_k \in \overline{t}$, which by definition means $b <^0_A t_k$. Note that $t_k <^0_A a$ always holds due to how the annotation was chosen. By transitivity $b <_A a$, which is a contradiction to the minimality of a. Therefore b is not a subterm of $f(\overline{t})$. As $s' = s[a \backslash f(\overline{t})]$, b is already a subterm of s, meaning $b <^0_A s$.

Therefore $<_{A'}$ containing a cycle would imply the existence of a cycle in $<_A$. But since $<_A$ is by assumption acyclic, $<_{A'}$ is acyclic as well and we can apply the induction hypothesis to A' to get an expansion proof E where all selected variables are replaced by skolemterms. As all strong quantifier nodes in E are skolem type, by definition E is an $\mathbf{EP}^{\mathrm{sk}}$-proof of S.

As no changes other than substitution of variables have been done to A, it follows directly that $|A| = |E|$.

□

Theorem 4.8. *Let E be an $\mathbf{EP}^{\mathrm{sk}}$-proof of an annotated sequent S. Then there exists an $\mathbf{EP}^{\mathrm{ac}}$-proof A of the same sequent such that $|E| = |A|$.*

Proof. This will be proven by induction on the number of skolemterms in E by replacing all such terms by selected variables and showing that the resulting expansion proof A satisfies the acyclicity condition.

If there are no skolemterms in E, then E does not contain any strong quantifier nodes. Therefore $A := E$ does not contain any selected variables either and $<_A$ is trivially acyclic. Therefore A an $\mathbf{EP}^{\mathrm{ac}}$-proof of S and $|E| = |A|$.

Now assume that the set of skolemsymbols in E is non empty. If $f(\overline{t})$ is a skolemterm that is maximal with respect to the subterm relation, then let a be a free variable which does not occur in E. We define A' as E where all occurrences of $f(\overline{t})$ are replaced by a.

As substitution does not affect $\mathrm{Dp}(E)$ being a tautology, A' is an expansion proof with less skolemterms than E. We can therefore apply the induction hypothesis and get an expansion proof A of S. It remains to show that the dependency relation $<_A$ is acyclic.

Assume that $<_A$ contains a cycle, then it contains a cycle $b <_A b$ where b is a selected variable with strong quantifier $Q^{g(\overline{s})}$. Let t', s' be terms in A, we have the following two cases:

- If $+^{t'} <^0_A +^b$, this implies that t' is a subterm of $g(\overline{s})$.
- If $+^b <^0_A +^{s'}$, then by definition $g(\overline{s})$ is a subterm of s'.

The above observations imply that $g(\overline{s})$ is a proper subterm of itself, which is impossible. Therefore $<_A$ has to be acyclic.
As we have replaced each skolemterm with a unique selected variable, A is weakly regular and therefore an $\mathbf{EP}^{\mathrm{ac}}$-proof of S.

As no changes other than substitution of variables have been applied to E, it follows directly that $|E| = |A|$.

□

Corollary 4.9. *Let S be an annotated sequent, then there exists an $\mathbf{EP}^{\mathrm{ac}}$-proof of S if and only if there exists an $\mathbf{EP}^{\mathrm{sk}}$-proof of it. The size of minimal proofs of S in both calculi coincide.*

Proof. Both results follow immediately from theorem 4.8 and theorem 4.7 with the same arguments as in corollary 3.11. □

4.5 Expansion Proofs and Skolemisation

We now return to the topic of skolemisation as introduced in definition 2.10.
The same way as examined in the relationship between sequent proofs of a formula and its skolemised version, we now study how expansion proofs interact with skolemisation. We first show that any expansion of an annotated formula can be converted into an expansion of its skolemised form. From this we can prove the statement for annotated sequents.

Theorem 4.10. *Let F be an annotated formula and E be an (dual) expansion of F. Then there exists an (dual) expansion E' of $\mathrm{sk}(F)$ such that $\mathrm{Dp}(E') = \mathrm{Dp}(E)$ and $|E'| \leq |E|$.*

Proof. We will prove the existence of E' by induction on the logical complexity $|E|$. If $|E| = 0$, then F is an atom. In this case $\mathrm{sk}(F)$ is an atom too and as atoms are expansions of itself, $E' := \mathrm{sk}(F)$ is an expansion of $\mathrm{sk}(F)$. Here $\mathrm{Dp}(E') = \mathrm{sk}(F) = F = \mathrm{Dp}(E)$ and even equality holds in $|E'| = |E|$.

Now let $|E| > 0$.
If $E = E_1 \wedge E_2$ and the formula is $F = A \wedge B$, then E_1 is an expansion of A and E_2 is an expansion of B.
Applying the induction hypothesis gives two expansions E_1' of $\mathrm{sk}(A)$ and E_2' of $\mathrm{sk}(B)$. By lemma 3.14

$$\mathrm{sk}(A) \wedge \mathrm{sk}(B) = \mathrm{sk}(A \wedge B),$$

so $E' := E_1' \wedge E_2'$ is an expansion of $\mathrm{sk}(F)$.
Both $\mathrm{Dp}(E_1') = \mathrm{Dp}(E_1)$ and $\mathrm{Dp}(E_2') = \mathrm{Dp}(E_2)$ by induction hypothesis, which implies that

$$\mathrm{Dp}(E') = \mathrm{Dp}(E_1' \wedge E_2') \overset{\mathrm{IV}}{=} \mathrm{Dp}(E_1 \wedge E_2) = \mathrm{Dp}(E)$$

As for the inequality, applying the induction hypothesis to E_i implies that

$$|E'| = |E_1'| + |E_2'| + 1 \overset{\mathrm{IV}}{\leq} |E_1| + |E_2| + 1 = |E|.$$

A similar argument can be made for expansions of $F = \neg F'$ and $F = A \vee B$.

If $E = \exists x\, A +^{t_1} E_1 ... +^{t_n} E_n$ where E_i are expansions of $A[x\backslash t_i]$, then the induction hypothesis gives expansions E_i' of $\mathrm{sk}(A[x\backslash t_i])$ for all $1 \leq i \leq n$. If we define $E' := \exists x\, \mathrm{sk}(A) +^{t_1} E_1' ... +^{t_n} E_n'$, then E' is an expansion of $\exists x\, \mathrm{sk}(A)$. Using $\exists x\, \mathrm{sk}(A) = \mathrm{sk}(\exists x\, A)$ from lemma 3.14 we see that E' is also an expansion of $\mathrm{sk}(F)$.
By induction hypothesis $\mathrm{Dp}(E_i') = \mathrm{Dp}(E_i)$ for all $1 \leq i \leq n$ from which

$$\begin{aligned} \mathrm{Dp}(E') &= \bigvee_{1 \leq i \leq n} \mathrm{Dp}(E_i') \\ &= \bigvee_{1 \leq i \leq n} \mathrm{Dp}(E_i) && \text{IV} \\ &= \mathrm{Dp}(E) \end{aligned}$$

follows.
The induction hypothesis gives $\mathrm{depth}(E_i)' \leq \mathrm{depth}(E_i)$, and therefore

$$|E'| = \sum_{i=1}^{n} (|E_i'| + 1) \overset{\text{IV}}{\leq} \sum_{i=1}^{n} (|E_i| + 1) = |E|.$$

If $E = \forall^{f(\bar{t})} x\, A +^{f(\bar{t})} E$, where E is an expansion of $A[x\backslash f(\bar{t})]$, then the induction hypothesis gives an expansion E' of $\mathrm{sk}(A[x\backslash f(\bar{t})])$. By definition of skolemisation it follows that $\mathrm{sk}(\forall^{f(\bar{t})} x\, A) = \mathrm{sk}(A[x\backslash f(\bar{t})])$. This implies that E' is already an expansion of $\mathrm{sk}(\forall^{f(\bar{t})} x\, A)$ and therefore of $\mathrm{sk}(F)$.
In this case both $\mathrm{Dp}(E) = \mathrm{Dp}(E')$ and the inequality $|E'| \leq |E|$ follow directly from the induction hypothesis.
The same argument can be done for dual expansions. □

Corollary 4.11. *Let S be an annotated sequent and E be an $\mathbf{EP}^{\mathrm{sk}}$-proof of S. Then there exists an $\mathbf{EP}^{\mathrm{sk}}$-proof E' of $\mathrm{sk}(S)$ such that $|E'| \leq |E|$.*

Proof. Let $E = \Pi \vdash \Lambda$ and $S = \Gamma \vdash \Delta$. Applying theorem 4.10 to all formulas Π_i and Λ_j gives dual expansions Π_i' of $\mathrm{sk}(\Gamma_i)$ and expansions Λ_j' of $\mathrm{sk}(\Delta_j)$ with $\mathrm{Dp}(\Pi_i) = \mathrm{Dp}(\Pi_i')$ and $\mathrm{Dp}(\Lambda_j) = \mathrm{Dp}(\Lambda_j')$. Now $E' = \Pi_1', ..., \Pi_n' \vdash \Lambda_1', ..., \Lambda_k'$ is an expansion of $\mathrm{sk}(S)$.
As

$$\begin{aligned} \mathrm{Dp}(E) &= \left[\bigvee_{1 \leq i \leq n} \neg \mathrm{Dp}(\Pi_i) \right] \vee \left[\bigvee_{1 \leq j \leq k} \mathrm{Dp}(\Lambda_j) \right] \\ &= \left[\bigvee_{1 \leq i \leq n} \neg \mathrm{Dp}(\Pi_i') \right] \vee \left[\bigvee_{1 \leq j \leq k} \mathrm{Dp}(\Lambda_j') \right] \\ &= \mathrm{Dp}(E') \end{aligned}$$

it is an $\mathbf{EP}^{\mathrm{sk}}$-proof of $\mathrm{sk}(S)$.

The inequality is a direct consequence of theorem 4.10 as

$$\begin{aligned} |E'| &= \sum_{A' \in \Pi \cup \Lambda} |A'| \\ &\leq \sum_{A \in \Pi \cup \Lambda} |A| \\ &= |E|. \end{aligned}$$

□

We also want to show that this process can be reversed. That is, any expansion of a skolemised sequent can be converted into one of the original sequent. Here the size of the resulting expansion may increase by a linear factor depending on the number of quantifiers in the original formula.

Theorem 4.12. *Let F be an annotated formula and E be an (dual) expansion of* $\mathrm{sk}(F)$*. Then there exists an (dual) expansion E' of F such that* $\mathrm{Dp}(E) = \mathrm{Dp}(E')$ *and* $|E'| \leq |E|(\mathrm{qocc}(F) + 1)$*.*

Proof. We will prove this theorem for expansions. The proof for dual expansions can be done analogously.

Let $n := |E|$ and $k := \mathrm{extqocc}(F)$.
We will prove the existence of an expansion E' of F by double induction on k and n.

- Assume $n = 0$ and $k \geq 0$. Then $\mathrm{sk}(F)$ is an atom, which implies that F is an atom too. In this case $E' := F$ is an expansion of itself. It follows that $\mathrm{Dp}(E') = \mathrm{sk}(F) = F = \mathrm{Dp}(E)$ and $|E'| = |E|$.

- Assume $n \geq 1$ and $k \geq 1$.

  Then F can be written as $F := \forall^f(\bar{t})x\, F_0$. By definition of skolemisation $\mathrm{sk}(F) = \mathrm{sk}(F_0)[x \backslash f(\bar{t})]$, so E is also an expansion of $\mathrm{sk}(F_0)[x \backslash f(\bar{t})]$.

  As $\mathrm{extqocc}(F_0) < k$, we can apply the induction hypothesis and receive an expansion E_0 of $F_0[x \backslash f(\bar{t})]$ such that $\mathrm{Dp}(E) = \mathrm{Dp}(E_0)$.

  We now define $E' := F +^{f(\bar{t})} E_0$ which is an expansion of F. By definition $\mathrm{Dp}(E') = \mathrm{Dp}(E_0)$ and therefore $\mathrm{Dp}(E) = \mathrm{Dp}(E')$.

  The inequality holds as

  $$|E'| = |E_0| + 1 \overset{\text{IV}}{\leq} |E|(\mathrm{qocc}(F_0) + 1) + 1 \leq |E|((\mathrm{qocc}(F_0) + 1) + 1) = |E|(\mathrm{qocc}(F) + 1).$$

- Assume $n \geq 1$ and $k = 0$.

– If $E = \neg E_0$ is an expansion of $\mathrm{sk}(F) = \mathrm{sk}(\neg F_0)$, then by lemma 3.14 E_0 is an expansion of $\mathrm{sk}(F_0)$. We apply the induction hypothesis to E_0 and receive an expansion E_0' of F_0. Then $E' \coloneqq \neg E_0'$ is an expansion of F.

We have

$$\mathrm{Dp}(E') = \neg\mathrm{Dp}(E_0') \overset{\mathrm{IV}}{=} \neg\mathrm{Dp}(E_0) = \mathrm{Dp}(E)$$

and the inequality holds as

$$|E'| = |E_0'| + 1 \overset{\mathrm{IV}}{\leq} |E_0|(\mathrm{qocc}(F_0) + 1) + 1 \leq |E|(\mathrm{qocc}(F) + 1).$$

– If $E = E_1 \vee E_2$ is an expansion of $\mathrm{sk}(F) = \mathrm{sk}(A \vee B)$, then by lemma 3.14 E_1 is an expansion of $\mathrm{sk}(A)$ and E_2 is an expansion of $\mathrm{sk}(B)$. Applying the induction hypothesis to $\mathrm{sk}(A)$ and $\mathrm{sk}(B)$ gives expansions E_1' of A and E_2' of B. If we define $E' \coloneqq E_1' \vee E_2'$, then E is an expansion of F.

By induction hypothesis $\mathrm{Dp}(E_1') = \mathrm{Dp}(E_1)$ and $\mathrm{Dp}(E_2') = \mathrm{Dp}(E_2)$, which gives

$$\mathrm{Dp}(E') = \mathrm{Dp}(E_1' \wedge E_2') \overset{\mathrm{IV}}{=} \mathrm{Dp}(E_1 \wedge E_2) = \mathrm{Dp}(E).$$

The inequality follows as

$$\begin{aligned}
|E'| &= |E_1'| + |E_2'| + 1 \\
&\leq |E_1|(\mathrm{qocc}(A) + 1) + |E_2|(\mathrm{qocc}(B) + 1) + 1 && \text{IV} \\
&\leq (|E_1| + |E_2| + 1)(\mathrm{qocc}(F) + 1) \\
&\leq (|E|)(\mathrm{qocc}(F) + 1).
\end{aligned}$$

The same argument can be made for $E = E_1 \wedge E_2$.

– If $E = \mathrm{sk}(\exists x\, A) +^{t_1} E_1 ... +^{t_n} E_n$ where E_i are expansions of $\mathrm{sk}(A)[x\backslash t_i]$, then the induction hypothesis leads to expansions E_i' of $A[x\backslash t_i]$. Let $E' \coloneqq \exists x\, A +^{t_1} E_1' ... +^{t_n} E_n'$ which is an expansion of $\exists x\, A$.

By induction hypothesis $\mathrm{Dp}(E_i') = \mathrm{Dp}(E_i)$ for all $1 \leq i \leq n$. This implies

$$\begin{aligned}
\mathrm{Dp}(E') &= \bigvee_{1 \leq i \leq n} \mathrm{Dp}(E_i') \\
&= \bigvee_{1 \leq i \leq n} \mathrm{Dp}(E_i) && \text{IV} \\
&= \mathrm{Dp}(E).
\end{aligned}$$

The inequality follows from

$$|E'| = \sum_{i=1}^{n} (|E_i'| + 1) \overset{\mathrm{IV}}{\leq} \sum_{i=1}^{n} (|E_i|(\mathrm{qocc}(A) + 1) + 1) = |E|(\mathrm{qocc}(F) + 1).$$

□

Corollary 4.13. *Let S be an annotated sequent and E be an $\mathbf{EP}^{\mathrm{sk}}$-proof of $\mathrm{sk}(S)$. Then there exists an $\mathbf{EP}^{\mathrm{sk}}$-proof E' of S such that $|E'| \leq |E|(\mathrm{qocc}(S)+1)$.*

Proof. Let $E = \Pi \vdash \Lambda$ and $S = \Gamma \vdash \Delta$. Applying theorem 4.12 to all formulas Π_i and Λ_j gives expansions Π'_i of Γ_i and Λ'_j of Δ_j. Now $E' = \Pi'_1, ..., \Pi'_n \vdash \Lambda'_1, ..., \Lambda'_k$ is an expansion of S.
As

$$\begin{aligned}\mathrm{Dp}(E) &= \left[\bigvee_{1\leq i\leq n} \neg\mathrm{Dp}(\Pi_i)\right] \vee \left[\bigvee_{1\leq j\leq k} \mathrm{Dp}(\Lambda_j)\right] \\ &= \left[\bigvee_{1\leq i\leq n} \neg\mathrm{Dp}(\Pi'_i)\right] \vee \left[\bigvee_{1\leq j\leq k} \mathrm{Dp}(\Lambda'_j)\right] \\ &= \mathrm{Dp}(E')\end{aligned}$$

it is an $\mathbf{EP}^{\mathrm{sk}}$-proof of S.
The inequality follows from theorem 4.12 as

$$\begin{aligned}|E'| &= \sum_{A'\in\Pi\cup\Lambda} |A'| \\ &\leq \sum_{A\in\Pi\cup\Lambda} |A|(\mathrm{qocc}(S_j)+1) \\ &= \sum_{i=1}^{n} |\Pi_i|(\mathrm{qocc}(\Gamma_i)+1) + \sum_{j=1}^{k} |\Lambda_j|(\mathrm{qocc}(\Delta_j)+1) \\ &\leq \sum_{i=1}^{n} |\Pi_i|(\mathrm{qocc}(S)+1) + \sum_{j=1}^{k} |\Lambda_j|(\mathrm{qocc}(S)+1) \\ &= |E|(\mathrm{qocc}(S)+1).\end{aligned}$$

□

5 Connection between Sequent and Expansion Calculi

Building on the results from earlier sections, we now bring the four notions of proof, acyclic and skolem systems in both sequent calculus and expansion calculus frameworks together. The central theorem of this section connects sequent and expansion calculi as a last step to show equivalence between all systems that were introduced in this thesis.

As the first step we translate acyclic $\mathbf{LK}^{\mathrm{ac}}$-proofs into $\mathbf{EP}^{\mathrm{ac}}$-proofs.
In order to prove this, we need a way to merge two (dual) expansions of the same formula as can be seen in [6], which works the same way in our framework.

Definition 5.1. Let A and B be expansions of an expanded annotated formula F. Then $A \cup B$ is inductively defined by the following cases:

- If $A = \bot$, then $A \cup B := B$ and analogously for $B = \bot$
- If $A = A_1 \vee A_2$ and $B = B_1 \vee B_2$, then $A \cup B := (A_1 \cup B_1) \vee (A_2 \cup B_2)$
- If $A = A_1 \wedge A_2$ and $B = B_1 \wedge B_2$, then $A \cup B := (A_1 \cup B_1) \wedge (A_2 \cup B_2)$
- If $A = \neg A'$ and $B = \neg B'$, then $A \cup B := \neg(A' \cup B')$
- If

$$A = \exists x\, F' +^{t_1} A_1 ... +^{t_n} A_n +^{s_1} A_{n+1} ... +^{s_l} A_{n+l}$$
$$B = \exists x\, F' +^{t_1} B_1 ... +^{t_n} B_n +^{r_1} B_{n+1} ... +^{r_m} B_{n+m}$$

  where $\{s_1, ..., s_l\} \cap \{r_1, ..., r_m\} = \varnothing$, then

$$\begin{aligned} A \cup B := \exists x\, F' +^{t_1} (A_1 \cup B_1) ... +^{t_n} (A_n \cup B_n) \\ +^{s_1} A_{n+1} ... +^{s_l} A_{n+l} \\ +^{r_1} B_{n+1} ... +^{r_m} B_{n+m}. \end{aligned}$$

- If $A = \exists^{f(\bar{t})} x\, F' +^{f(\bar{t})} A'$ and $B = \exists^{f(\bar{t})} x\, F' +^{f(\bar{t})} B'$ then

$$A \cup B := \forall x\, F' +^{f(\bar{t})} (A' \cup B').$$

For dual expansions, we switch all $\bot$ for $\top$ and $\forall$ for $\exists$.

If S is a sequent and $S_1 = \Pi \vdash \Lambda$ and $S_2 = \Gamma \vdash \Delta$ are expansions of S, then we define the merge as

$$S_1 \cup S_2 := (\Pi_1 \cup \Gamma_1), ..., (\Pi_n \cup \Gamma_n) \vdash (\Lambda_1 \cup \Delta_1), ..., (\Lambda_k \cup \Delta_k)$$

Lemma 5.2. *If A and B are expansion of an expanded annotated formula F, then* $\mathrm{Dp}(A) \vee \mathrm{Dp}(B) \Rightarrow \mathrm{Dp}(A \cup B)$. *If A and B are dual expansions, then* $\mathrm{Dp}(A) \vee \mathrm{Dp}(B) \Leftarrow \mathrm{Dp}(A \cup B)$. *If S_1 and S_2 are expansions of a sequent S, then* $\mathrm{Dp}(S_1) \vee \mathrm{Dp}(S_2) \Rightarrow \mathrm{Dp}(S_1 \cup S_2)$

Proof. The statement will be proven by induction on the structure of expansions A and B.

- If $A = \bot$, then $\mathrm{Dp}(A) \vee \mathrm{Dp}(B) = \mathrm{Dp}(B) = \mathrm{Dp}(A \cup B)$ and analogously for $B = \bot$.
- If $A = \neg A'$ and $B = \neg B'$ are dual expansions for A', B' expansions, then

$$\begin{aligned} \mathrm{Dp}(A \cup B) &= \neg \mathrm{Dp}(A' \cup B') \\ &\Rightarrow \neg \mathrm{Dp}(A' \vee B') && \text{IV} \\ &= \neg(\mathrm{Dp}(A') \vee \mathrm{Dp}(B')) \\ &\Rightarrow \neg \mathrm{Dp}(A') \vee \neg \mathrm{Dp}(B') \\ &= \mathrm{Dp}(A) \vee \mathrm{Dp}(B). \end{aligned}$$

- If $A = A_1 \vee A_2$ and $B = B_1 \vee B_2$ then

$$\begin{aligned} \mathrm{Dp}(A) \vee \mathrm{Dp}(B) &= (\mathrm{Dp}(A_1) \vee \mathrm{Dp}(B_1)) \vee (\mathrm{Dp}(A_2) \vee \mathrm{Dp}(B_2)) \\ &= (\mathrm{Dp}(A_1) \vee \mathrm{Dp}(A_2)) \vee (\mathrm{Dp}(B_1) \vee \mathrm{Dp}(B_2)) \\ &\Rightarrow (\mathrm{Dp}(A_1 \cup A_2)) \vee (\mathrm{Dp}(B_1 \cup B_2)) && \text{IV} \\ &= \mathrm{Dp}(A \cup B). \end{aligned}$$

- If $A = A_1 \wedge A_2$ and $B = B_1 \wedge B_2$, then

$$\begin{aligned} \mathrm{Dp}(A) \vee \mathrm{Dp}(B) &= (\mathrm{Dp}(A_1) \wedge \mathrm{Dp}(B_1)) \vee (\mathrm{Dp}(A_2) \wedge \mathrm{Dp}(B_2)) \\ &\Rightarrow (\mathrm{Dp}(A_1) \vee \mathrm{Dp}(A_2)) \wedge (\mathrm{Dp}(B_1) \vee \mathrm{Dp}(B_2)) \\ &\Rightarrow (\mathrm{Dp}(A_1 \cup A_2)) \wedge (\mathrm{Dp}(B_1 \cup B_2)) && \text{IV} \\ &= \mathrm{Dp}(A \cup B). \end{aligned}$$

  Note that unlike in the $\vee$-case, the second line is an implication only.

- If

$$\begin{aligned} A &= \exists x\, F' +^{t_1} A_1 \ldots +^{t_n} A_n +^{s_1} A_{n+1} \ldots +^{s_l} A_{n+l} \\ B &= \exists x\, F' +^{t_1} B_1 \ldots +^{t_n} B_n +^{r_1} B_{n+1} \ldots +^{r_m} B_{n+m} \end{aligned}$$

  then

$$\begin{aligned} \mathrm{Dp}(A) \vee \mathrm{Dp}(B) &= \left[\bigvee_{1 \le j \le n+l} \mathrm{Dp}(A_j)\right] \vee \left[\bigvee_{1 \le j \le n+m} \mathrm{Dp}(B_j)\right] \\ &= \left[\bigvee_{1 \le j \le n} (\mathrm{Dp}(A_j) \vee \mathrm{Dp}(B_j))\right] \vee \left[\bigvee_{1 \le j \le l} \mathrm{Dp}(A_{n+j})\right] \vee \left[\bigvee_{1 \le j \le m} \mathrm{Dp}(B_{n+j})\right] \\ &\Rightarrow \left[\bigvee_{1 \le j \le n} (\mathrm{Dp}(A_j) \cup \mathrm{Dp}(B_j))\right] \vee \left[\bigvee_{1 \le j \le l} \mathrm{Dp}(A_{n+j})\right] \vee \left[\bigvee_{1 \le j \le m} \mathrm{Dp}(B_{n+j})\right] \quad \text{IV} \\ &= \mathrm{Dp}(A \cup B). \end{aligned}$$

If $A = \forall^{f(\bar{t})} x\, F' +^{f(\bar{t})} A'$ and $B = \forall^{f(\bar{t})} x\, F' +^{f(\bar{t})} B'$ then

$$\begin{aligned} \mathrm{Dp}(A) \vee \mathrm{Dp}(B) &= \mathrm{Dp}(A') \vee \mathrm{Dp}(B') \\ &\Rightarrow \mathrm{Dp}(A') \cup \mathrm{Dp}(B') && \text{IV} \\ &= \mathrm{Dp}(A \cup B). \end{aligned}$$

The statement for dual expansions can be proven the same way.

If S_1 and S_2 are expansions of the same sequent S, then using what we have already proven we get

$$\begin{aligned} \mathrm{Dp}(S_1) \vee \mathrm{Dp}(S_2) &= \left[\bigvee_{1\le j\le n} (\Pi_j \vee \Gamma_j)\right] \vee \left[\bigvee_{1\le j\le k} (\Lambda_j \vee \Delta_j)\right] \\ &\Rightarrow \left[\bigvee_{1\le j\le n} (\Pi_j \cup \Gamma_j)\right] \vee \left[\bigvee_{1\le j\le k} (\Lambda_j \cup \Delta_j)\right] && \text{IV} \\ &= \mathrm{Dp}(S_1 \cup S_2). \end{aligned}$$

□

By a similarly quick induction we can show that for two expansions A and B, $|A \cup B| \le |A| + |B|$.
This together with the above lemma enables us to prove the first implication.

Theorem 5.3. *Let π be an $\mathbf{LK}^{\mathrm{ac}}$-proof of an annotated sequent S. Then there exists an $\mathbf{EP}^{\mathrm{ac}}$-proof of the same sequent such that $|E| \le l(\pi)$.*

Proof. We will first prove the existence of a tautological expansion A of S by induction on the number of inferences in π.
Any axiom $\Gamma, F \vdash F, \Delta$ in $\mathbf{LK}^{\mathrm{ac}}$ has an expansion $A \coloneqq \top, F \vdash F, \bot$. Its deep formula is trivially tautological. This shows that the induction base holds.

If π has the form

$$\dfrac{\begin{matrix}(\pi_0)\\ A, \Gamma \vdash \Delta\end{matrix}}{\Gamma \vdash \Delta, \neg A}$$

then by induction hypothesis, there exists a tautological expansion $E, \Pi \vdash \Lambda$ of $A, \Gamma \vdash \Delta$. Then $\Pi \vdash \Lambda, \neg E$ is an expansion of the bottom sequent. It is tautological as

$$\mathrm{Dp}(\Pi \vdash \Lambda, \neg E) = \neg \mathrm{Dp}(E) \vee \mathrm{Dp}(\Pi \vdash \Lambda) = \mathrm{Dp}(E, \Pi \vdash \Lambda)$$

where the latter is tautological by induction hypothesis. The inequality holds because

$$|\Pi \vdash \Lambda, \neg E| = |\Pi| + |\Lambda| + |\neg E| = |\Pi| + |\Lambda| + |E| + 1 \overset{\text{IV}}{\le} |\pi_0| + 1 = l(\pi).$$

If π has the form

$$\frac{\begin{matrix}(\pi_1) & (\pi_2)\\ F_1, \Gamma \vdash \Delta & F_2, \Gamma \vdash \Delta\end{matrix}}{F_1 \vee F_2, \Gamma \vdash \Delta} \vee_l$$

then by induction hypothesis, we get tautological expansions $A_1 := E_1, \Pi_1 \vdash \Lambda_1$ and $A_2 := E_2, \Pi_2 \vdash \Lambda_2$ of $F_1, \Gamma \vdash \Delta$ and $F_2, \Gamma \vdash \Delta$. Define $A := E_1 \vee E_2, (\Pi_1 \cup \Pi_2) \vdash (\Lambda_1 \cup \Lambda_2)$. To see that A is tautological, note that by lemma 5.2

$$\begin{aligned}\mathrm{Dp}(E_1, \Pi_1 \vdash \Lambda_1) \wedge \mathrm{Dp}(E_2, \Pi_2 \vdash \Lambda_2) &= \neg\mathrm{Dp}(E_1) \vee \mathrm{Dp}(\Pi_1 \vdash \Lambda_1) \wedge \neg\mathrm{Dp}(E_2) \vee \mathrm{Dp}(\Pi_2 \vdash \Lambda_2)\\ &= \neg\mathrm{Dp}(E_1 \wedge E_2) \vee \mathrm{Dp}(\Pi_1 \vdash \Lambda_1) \vee \mathrm{Dp}(\Pi_2 \vdash \Lambda_2)\\ &\Rightarrow \mathrm{Dp}(E_1 \vee E_2, (\Pi_1 \cup \Pi_2) \vdash (\Lambda_1 \cup \Lambda_2)).\end{aligned}$$

Since A_1 and A_2 are tautological by induction hypothesis, it follows that A is as well. The inequality follows as

$$|A| \leq |E_1, \Pi_1 \vdash \Lambda_1| + |E_2, \Pi_2 \vdash \Lambda_2| + 1 \overset{\text{IV}}{\leq} l(\pi_1) + l(\pi_2) + 1 = l(\pi).$$

where the first inequality is an application of $|B_1 \cup B_2| \leq |B_1| + |B_2|$.

The dual $\wedge$-case can be treated analogously.
If π has the form

$$\frac{(\pi_0)}{\Gamma \vdash \Delta, \exists x\, F, F[x\backslash t]}$$

then by induction hypothesis we get a tautological expansion $A' = \Pi \vdash \Lambda, \exists x\, F +^{t_1} E_1 ... +^{t_n} E_n, E_{n+1}$ of $\Gamma \vdash \Delta, \exists x\, F, F[x\backslash t]$. Let $E' := (\exists x\, F +^{t_1} E_1 ... +^{t_n} E_n) \cup (\exists x\, F +^{t_{n+1}} E_{n+1})$, then $A := \Pi \vdash \Lambda, E'$ is an expansion of $\Gamma \vdash \Delta, \exists x\, F$. Again with lemma 5.2 we can see that

$$\begin{aligned}\mathrm{Dp}(A') &= \mathrm{Dp}(\Pi \vdash \Lambda) \vee \mathrm{Dp}(\exists x\, F +^{t_1} E_1 ... +^{t_n} E_n) \vee \mathrm{Dp}(E_{n+1})\\ &\Rightarrow \mathrm{Dp}(\Pi \vdash \Lambda, (\exists x\, F +^{t_1} E_1 ... +^{t_n} E_n) \cup (\exists x\, F +^{t_{n+1}} E_{n+1}))\\ &= \mathrm{Dp}(A).\end{aligned}$$

By induction hypothesis the expansion A is tautological. For the inequality, note that

$$\begin{aligned}|A| &\leq |\Pi \vdash \Lambda, \exists x\, F +^{t_1} E_1 ... +^{t_n} E_n| + |E_{n+1}| + 1\\ &= |\Pi \vdash \Lambda, \exists x\, F +^{t_1} E_1 ... +^{t_n} E_n, E_{n+1}| + 1\\ &\leq l(\pi_0) + 1 \qquad \text{IV}\\ &= l(\pi).\end{aligned}$$

If π has the form

$$\dfrac{\begin{matrix}(\pi_0)\\ F[x\backslash a], \Gamma \vdash \Delta\end{matrix}}{\exists^{f(\bar{t})} x\, F, \Gamma \vdash \Delta}$$

then by induction hypothesis we get an expansion $E, \Pi \vdash \Lambda$ of $F[x\backslash a], \Gamma \vdash \Delta$.

Let $E' \coloneqq \exists^{f(\bar{t})} x\, F + !E$, then $E', \Pi \vdash \Lambda$ is an expansion of $\exists^{f(\bar{t})} x\, F, \Gamma \vdash \Delta$.

As $\mathrm{Dp}(\exists^{f(\bar{t})} x\, F +^a E, \Pi \vdash \Lambda) = \mathrm{Dp}(E', \Pi \vdash \Lambda)$, by induction hypothesis, the expansion is tautological. For the inequality, note that

$$|E', \Pi \vdash \Lambda| = |\Pi \vdash \Lambda| + |E'| = |\Pi \vdash \Lambda| + |E| + 1 \overset{\text{IV}}{\leq} l(\pi_0) + 1 = l(\pi).$$

The $\forall$-cases are proven the same way.

What remains is to show that the dependency relation $<_A$ is acyclic.
Let $a <_E^0 t <_E^0 b$ where a, b are selected variables and t is a term. This means $a \in \mathrm{FV}(t)$ and $t \in \bar{t}$, which implies $a <_\pi b$ by definition.
Assume that $<_A$ contains a cycle. By the above observation this would imply a cycle in $<_\pi$, a contradiction.
Therefore $<_A$ is acyclic, which shows that A is an $\mathbf{EP}^{\mathrm{ac}}$-proof. □

5.1 Expansion Calculus

Having completed the direction from sequent calculus to expansions, we now want to transform an $\mathbf{EP}^{\mathrm{sk}}$-proof in an $\mathbf{LK}^{\mathrm{sk}}$-proof. For this we define a sequent calculus $\mathbf{LK}^{\mathrm{sk}}_E$ which works on expansions. This serves as an intermediate step which will be useful for proving the first part of this direction. The definition of this calculus is analogous to the one in [6], but for annotated sequents.

Definition 5.4. A $\mathbf{LK}^{\mathrm{sk}}_E$-proof of an expansion E is a tree built bottom up by the following axioms and rules

$$\dfrac{}{A, \Pi \vdash \Lambda, A}\,\mathrm{ax} \qquad \dfrac{}{\bot, \Pi \vdash \Lambda}\,\mathrm{ax}\bot \qquad \dfrac{}{\Pi \vdash \Lambda, \top}\,\mathrm{ax}\top$$

where A is an axiom. The logical rules are

$$\dfrac{E, \Pi \vdash \Lambda}{\Pi \vdash \Lambda, \neg E}\,\neg_r \qquad \dfrac{\Pi \vdash \Lambda, E}{\neg E, \Pi \vdash \Lambda}\,\neg_l$$

$$\dfrac{E_1, \Pi \vdash \Lambda \qquad E_2, \Pi \vdash \Lambda}{E_1 \vee E_1, \Pi \vdash \Lambda}\,\vee_l \qquad \dfrac{\Pi \vdash \Lambda, E_1, E_2}{\Pi \vdash \Lambda, E_1 \vee E_2}\,\vee_r$$

$$\dfrac{E_1, E_2, \Pi \vdash \Lambda}{E_1 \wedge E_2, \Pi \vdash \Lambda}\,\wedge_l \qquad \dfrac{\Pi \vdash \Lambda, E_1 \qquad \Pi \vdash \Lambda, E_2}{\Pi \vdash \Lambda, E_1 \wedge E_1}\,\wedge_r$$

The quantifier rules are

$$\frac{\Pi \vdash \Lambda, \exists x\, A +^{t_1} E_1 ... +^{t_{k-1}} E_{k-1} +^{t_{k+1}} E_{k+1} ... +^{t_n} E_n, E_k}{\Pi \vdash \Lambda, \exists\, x A +^{t_1} E_1 ... +^{t_n} E_n}\ \exists_r$$

where E_k is an expansion of $A[x\backslash t_k]$ and

$$\frac{E, \Pi \vdash \Lambda}{\exists^{f(\bar{t})} y\, A +^{f(\bar{t})} E, \Pi \vdash \Lambda}\ \exists_l^{\mathrm{sk}}$$

where E is an expansion of $A[y\backslash f(\bar{t})]$.
The dual definitions are made for $\forall_l$ and $\forall_r$.

The calculus is defined in such a way that its derivations correspond to the structure of $\mathbf{LK}^{\mathrm{sk}}$-proofs.

We now proceed to show that each expansion proof in the skolem expansion calculus corresponds to a derivation in this new calculus of comparable size.

Lemma 5.5. *Let E be an $\mathbf{EP}^{\mathrm{sk}}$-proof of an annotated sequent S, then there exists an $\mathbf{LK}^{\mathrm{sk}}_E$-proof π of the same sequent such that* $\mathrm{depth}(\pi) \leq |E|$.

Proof. The proof π will be constructed by induction on the structure of the expansion. The case where E is an atom or $\top$, $\pi := E$ is an axiom in $\mathbf{LK}^{\mathrm{sk}}_E$.

If $E = \Pi \vdash \Lambda, \neg E'$. then we now need to show that $\Pi, E' \vdash \Lambda$ is a tautological expansion in order to apply the induction hypothesis and receive an $\mathbf{LK}^{\mathrm{sk}}_E$-proof π' of $\Pi, E' \vdash \Lambda$.
But this follows immediately from E being tautological because

$$\mathrm{Dp}\left(\Pi \vdash \Lambda, \neg E'\right) = \neg\mathrm{Dp}\left(\Pi\right) \vee \mathrm{Dp}\left(\neg E'\right) \vee \mathrm{Dp}\left(\Lambda\right) = \neg\mathrm{Dp}\left(\Pi\right) \vee \neg\mathrm{Dp}\left(E'\right) \vee \mathrm{Dp}\left(\Lambda\right) = \mathrm{Dp}\left(\Pi, E' \vdash \Lambda\right).$$

It follows that

$$\pi := \frac{\begin{array}{c}(\pi')\\ \Pi, E' \vdash \Lambda\end{array}}{\Pi \vdash \Lambda, \neg E'}\ .$$

is an $\mathbf{LK}^{\mathrm{sk}}_E$-proof of E.
The inequality holds since

$$\mathrm{depth}(\pi) = \mathrm{depth}(\pi') + 1 \overset{\mathrm{IV}}{\leq} |\Pi \vdash \Lambda, E'| + 1 = |\Pi \vdash \Lambda, \neg E'|.$$

Let $E = E_1 \vee E_2, \Pi \vdash \Lambda$.
As $\neg\mathrm{Dp}\left(E_1 \vee E_2\right) = \neg\mathrm{Dp}\left(E_1\right) \wedge \neg\mathrm{Dp}\left(E_2\right)$, we have $\mathrm{Dp}\left(E_1 \vee E_2, \Pi \vdash \Lambda\right) \Rightarrow \mathrm{Dp}\left(E_1, \Pi \vdash \Lambda\right) \wedge \mathrm{Dp}\left(E_1, \Pi \vdash \Lambda\right)$. This implies that both subexpansions are tautological as well and we can apply the induction hypothesis get to π_1 and π_2 such that

$$\pi := \frac{\begin{array}{cc}(\pi_1) & (\pi_2)\\ E_1, \Pi \vdash \Lambda & E_2, \Pi \vdash \Lambda\end{array}}{E_1 \vee E_2, \Pi \vdash \Lambda}\ .$$

is an $\mathbf{LK}^{\mathrm{sk}}_E$-proof of E.
For the inequality note that

$$\mathrm{depth}(\pi) = \max\{\mathrm{depth}(\pi_1), (\pi_2)\} + 1 \overset{\mathrm{IV}}{\leq} \max\{|E_1, \Pi \vdash \Lambda|, |E_2, \Pi \vdash \Lambda|\} + 1 \leq |E_1 \vee E_2, \Pi \vdash \Lambda|.$$

In the case of $E = \Pi \vdash \Lambda, \exists x\, F +^{t_1} E_1 \ldots +^{t_n} E_n$ we have

$$\mathrm{Dp}\left(\exists x\, F +^{t_1} E_1 \ldots +^{t_{k-1}} E_{k-1} +^{t_{k+1}} E_{k+1} \ldots +^{t_n} E_n, E_k\right) = \left(\bigvee_{1 \leq i \leq n, i \neq k} E_i\right) \vee E_k$$

and

$$\mathrm{Dp}\left(\exists x\, F +^{t_1} E_1 \ldots +^{t_n} E_n\right) = \bigvee_{1 \leq i \leq n} E_i.$$

This shows the two deep formulas being equal, which allows us to apply the induction hypothesis and get a proof π'. We now define π as

$$\dfrac{\begin{array}{c}(\pi') \\ \Pi \vdash \Lambda, \exists x\, F +^{t_1} E_1 \ldots +^{t_{k-1}} E_{k-1} +^{t_{k+1}} E_{k+1} \ldots +^{t_n} E_n, E_k\end{array}}{\Pi \vdash \Lambda, \exists x\, F +^{t_1} E_1 \ldots +^{t_n} E_n}$$

. For the inequality, note that

$$\begin{aligned}\mathrm{depth}(\pi) &= \mathrm{depth}(\pi') + 1 \\ &\leq |\Pi \vdash \Lambda, \exists x\, F +^{t_1} E_1 \ldots +^{t_{k-1}} E_{k-1} +^{t_{k+1}} E_{k+1} \ldots +^{t_n} E_n, E_k| + 1 \qquad \mathrm{IV} \\ &= |\Pi \vdash \Lambda, \exists x\, F +^{t_1} E_1 \ldots +^{t_n} E_n|.\end{aligned}$$

Let $E = \exists^{f(\bar{t})} x\, F +^{f(\bar{t})} E', \Pi \vdash \Lambda$. Since $\mathrm{Dp}(\exists^{f(\bar{t})} x\, F +^{f(\bar{t})} E') = \mathrm{Dp}\,(E')$ we can apply the induction hypothesis to get a proof π' of $E', \Pi \vdash \Lambda$.
We define π as

$$\dfrac{\begin{array}{c}(\pi') \\ E', \Pi \vdash \Lambda\end{array}}{\exists^{f(\bar{t})} x\, F +^{f(\bar{t})} E', \Pi \vdash \Lambda}$$

.
The inequality follows by induction as

$$\mathrm{depth}(\pi) = \mathrm{depth}(\pi') + 1 \overset{\mathrm{IV}}{\leq} |E', \Pi \vdash \Lambda| + 1 = |\exists^{f(\bar{t})} x\, F +^{f(\bar{t})} E', \Pi \vdash \Lambda|.$$

All the dual cases are proven analogously. □

The following example illustrates this construction.

Example 5.5.1. Consider the annotated sequent $S := \exists^c x\, \neg P(x) \vee Q \vdash \exists y\, (\neg P(y) \vee Q)$. Defining the expansions

$$E_1 := [\exists^c x\, \neg P(x) +^c \neg P(c)] \vee Q,$$

and

$$E_2 := \exists^c x\, \neg P(x) \vee Q +^c [\neg P(c) \vee Q],$$

which satisfy $\mathrm{Dp}(E_1) = \mathrm{Dp}(E_2)$ and $|E_1| = |E_2| = 3$.
Then $E := E_1 \vdash E_2$ is a $\mathbf{EP}^{\mathrm{sk}}$-proof of S since its deep formula

$$\mathrm{Dp}(E) = \neg \mathrm{Dp}(E_1) \vee \mathrm{Dp}(E_2)$$

is tautological.

Following the steps in the proof of theorem 5.5 we receive a $\mathbf{LK}^{\mathrm{sk}}_E$-proof π of E in the form of

$$\dfrac{\dfrac{\dfrac{\dfrac{}{Q \vdash \exists^c x\, (\neg P(x) \vee Q), \neg P(c), Q}\,\mathrm{ax} \quad \dfrac{\dfrac{}{\neg P(c) \vdash \exists^c x\, (\neg P(x) \vee Q), \neg P(c), Q}\,\mathrm{ax}}{\exists^c x\, \neg P(x) +^c \neg P(c) \vdash \exists^c x\, (\neg P(x) \vee Q), \neg P(c), Q}\,\exists_l}{[\exists^c x\, \neg P(x) +^c \neg P(c)] \vee Q \vdash \exists^c x\, (\neg P(x) \vee Q), \neg P(c), Q}\,\vee_l}{[\exists^c x\, \neg P(x) +^c \neg P(c)] \vee Q \vdash \exists^c x\, (\neg P(x) \vee Q), \neg P(c) \vee Q}\,\vee_r}{[\exists^c x\, \neg P(x) +^c \neg P(c)] \vee Q \vdash \exists^c x\, (\neg P(x) \vee Q) +^c [\neg P(c) \vee Q]}\,\exists_r$$

Here $\mathrm{depth}(\pi) = 5 \leq 6 = |E|$.

In this example we see that replacing the expansions in each step by their shallow formulas leads to a $\mathbf{LK}^{\mathrm{sk}}$-proof of S. This is not true in general though, as the shallow formula of an expansion may contain $\top$ and $\bot$. But as remarked in a previous chapter, those cases can be addressed by substitutions that restore the original structure of the formula.

We therefore summarise that derivations in the expansion calculus $\mathbf{LK}^{\mathrm{sk}}_E$ can be translated back to standard sequent calculus proofs while preserving depth.

Lemma 5.6. *Let S be an annotated sequent and E be an $\mathbf{EP}^{\mathrm{sk}}$-proof of S. If π is an $\mathbf{LK}^{\mathrm{sk}}_E$-proof of E, then there exists an $\mathbf{LK}^{\mathrm{sk}}$-proof ψ of the sequent S such that* $\mathrm{depth}(\pi) = \mathrm{depth}(\psi)$

Proof. The translation of the inferences itself is straightforward. This leads to a proof ψ' of $\mathrm{Sh}(E)$. From this the $\mathbf{LK}^{\mathrm{sk}}$-proof ψ of S can then be obtained by replacing $\top$ and $\bot$ by subformulas of S.
Both the translation of inferences and replacing subformulas doesn't affect the depth of the proof, so we have

$$\mathrm{depth}(\pi) = \mathrm{depth}(\psi') = \mathrm{depth}(\psi).$$

□

Combining the last two results, we obtain a translation from expansion proofs to sequent proofs. The depth of the resulting LK-proof is at most exponential in the size of the expansion proof.

Corollary 5.7. *Let E be an $\mathbf{EP}^{\mathrm{sk}}$-proof of an annotated sequent S, then there exists an $\mathbf{LK}^{\mathrm{sk}}$-proof ψ of the sequent S such that $l(\psi) \leq 2^{|E|}$.*

Proof. By lemma 5.5 there exists an $\mathbf{LK}^{\mathrm{sk}}_{E}$-proof π of E and by lemma 5.6 there exists an $\mathbf{LK}^{\mathrm{sk}}$-proof ψ of S.
Using the inequalities from both lemmata we get

$$\mathrm{depth}(\psi) = \mathrm{depth}(\pi) \leq |E|.$$

Since the rules in our calculi are at most binary, it follows that

$$l(\psi) \leq 2^{\mathrm{depth}(\psi)} \leq 2^{|E|}.$$

□

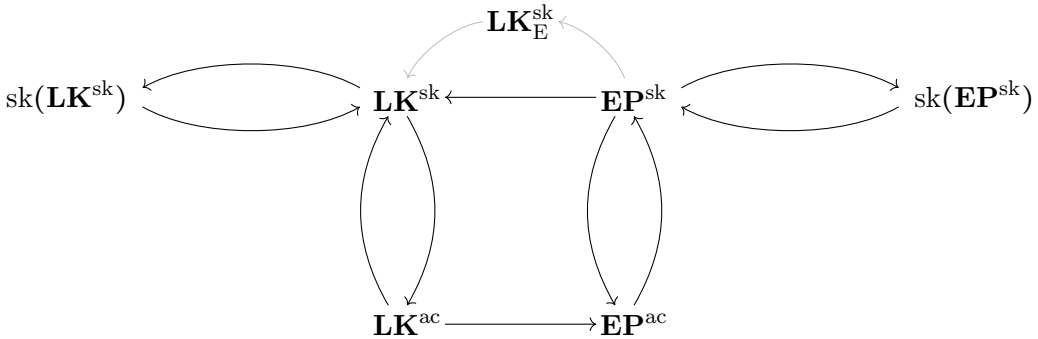
5.2 The Main Theorem

This completes the diagram of translations between the four proof systems. We can now summarise the results from all previous chapters in the main theorem of this thesis.

Theorem 5.8. *Let S be a sequent and α be a skolem annotation. Then the following are equivalent:*

- *There exists an $\mathbf{LK}^{\mathrm{ac}}$-proof of $\alpha(S)$.*
- *There exists an $\mathbf{EP}^{\mathrm{ac}}$-proof of $\alpha(S)$.*
- *There exists an $\mathbf{EP}^{\mathrm{sk}}$-proof of $\alpha(S)$.*
- *There exists an $\mathbf{LK}^{\mathrm{sk}}$-proof of $\alpha(S)$.*
- *There exists an $\mathbf{EP}^{\mathrm{sk}}$-proof of $\mathrm{sk}(\alpha(S))$.*
- *There exists an $\mathbf{LK}^{\mathrm{sk}}$-proof of $\mathrm{sk}(\alpha(S))$.*

This will be proven by showing the following implications between the respective calculi:



where $\mathrm{sk}(\mathbf{LK}^{\mathrm{sk}})$ *is short for* $\mathbf{LK}^{\mathrm{sk}}$*-proofs of* $\mathrm{sk}(\alpha(S))$ *and the same for* $(\mathrm{sk}(\mathbf{EP}^{\mathrm{sk}}))$.

Additionally, there exist the following size estimations when switching between the different calculi:

- *Given an* $\mathbf{LK}^{\mathrm{sk}}$*-proof* π *and an* $\mathbf{LK}^{\mathrm{ac}}$*-proof* ψ *of* $\alpha(S)$ *and an* $\mathbf{LK}^{\mathrm{sk}}$*-proof* π' *of* $\mathrm{sk}(\alpha(S))$ *the following equalities or inequalities hold:*

$$l(\psi) = l(\pi) \leq l(\pi') \leq l(\pi) * (\mathrm{qocc}(S) + 1)$$

$$\mathrm{depth}(\psi) = \mathrm{depth}(\pi) \leq \mathrm{depth}(\pi') \leq \mathrm{depth}(\pi) * (\mathrm{qocc}(S) + 1)$$

- *Given an* $\mathbf{EP}^{\mathrm{sk}}$*-proof* E *and an* $\mathbf{EP}^{\mathrm{ac}}$*-proof* A *of* $\alpha(S)$ *and an* $\mathbf{EP}^{\mathrm{sk}}$*-proof* E' *of* $\mathrm{sk}(\alpha(S))$ *the following equalities or inequalities hold:*

$$|A| = |E| \leq |E'| \leq |E| * (\mathrm{qocc}(S) + 1)$$

- *For an* $\mathbf{EP}^{\mathrm{sk}}$*-proof* E *and an* $\mathbf{LK}^{\mathrm{sk}}$*-proof* π *of* $\alpha(S)$ *we conclude*

$$|E| \leq l(\pi) \leq 2^{|E|}.$$

  The same holds for acyclic sequent and expansion proofs.

In particular, we have shown that every derivation in one of the calculi can be efficiently translated into each of the others, preserving complexity either linearly or at most exponentially.

6 Embeddings

In this chapter, we explore how various well-known calculi found in the literature can be embedded into our framework through the use of suitable annotations. This demonstrates the flexibility of our system and its ability to represent different logical systems under a unified formalism. We begin by recalling various forms of skolemisation known in literature and give equivalent skolem annotations. We then show that the classical Gentzen-style sequent calculus **LK** and more recent approaches such as $\mathbf{LK}^+$ can be embedded into our system by appropriately selecting annotations. We do this by showing that the conditions on strong quantifier inferences of the calculi are equivalent to our definition of acyclicity of the dependency relation.
In the last section we address similar approaches to partial dependencies from literature, namely Henkin quantifiers [2] and IF-Logic [3], and define annotations that transform them into annotated formulas.

6.1 Formula Based Dependencies

In this section we examine various annotation strategies based on well-established skolemisation methods from the literature [1] and [8]. Each strategy corresponds to a different way of determining the dependencies of strong quantifiers on preceeding weak quantifiers. As the names suggest, each of the annotations refer to forms of skolemisation that are known in literature.

We begin with the standard skolemisation, which introduces maximal dependencies.

Definition 6.1. Let F be a formula, then we define the annotation α_{std} such that for all strong quantifiers Qy in F,

$$\alpha_{\text{std}}(Qy) \coloneqq Q^{f(x_1,\ldots,x_k)}y,$$

where $Q_1x_1, \ldots, Q_kx_k$ are the weak quantifiers such that Qy is in their scope and f is a skolemsymbol that is chosen uniquely for all strong quantifiers.
We call α_{std} the *Standard Annotation*.

This standard form captures all weak quantifier dependencies in the scope of each strong quantifier. It ensures correctness but can lead to unnecessary complexity in the resulting dependency graph.
Next, we look at a less strict annotation that corresponds more closely to the internal structure of the formula.

Definition 6.2. Let F be a formula, then we define the annotation α_{inn} such that for all strong quantifiers Qy where $Qy\,F'$ is a subformula of F,

$$\alpha_{\text{inn}}(Qy) \coloneqq Q^{f(\overline{x})}y,$$

where $Q_1x_1, ..., Q_kx_k$ are the weak quantifiers such that Qy is in their scope, $\overline{x}$ is the subvector of $(x_1, .., x_k)$ of all bound variables x_i which occur in F' and f is a skolemsymbol that is chosen uniquely for all strong quantifiers.
We call α_{inn} the *Inner Annotation*.

This variant, referred to as the Inner Annotation, reduces dependencies by considering only those variables that appear in the scope of the quantified formula. It typically yields simpler proofs compared to the standard annotation.
An even less strict annotation focuses solely on atomic formulas.

Definition 6.3. Let F be a formula, then we define the annotation α_{atom} such that for all strong quantifiers Qy where $Qy\,F'$ is a subformula of F,

$$\alpha_{\text{atom}}(Qy) \coloneqq Q^{f(\overline{x})}y,$$

where $Q_1x_1, ..., Q_kx_k$ are the weak quantifiers such that Qy is in their scope, $\overline{x}$ is the subvector of $(x_1, .., x_k)$ of all bound variables x_i which occur in atomic formulas together with y and f is a skolemsymbol that is chosen uniquely for all strong quantifiers.
We call α_{atom} the *Atomic Annotation*.

This strategy, which we call the Atomic Annotation, introduces the minimal necessary dependencies by restricting attention to bound variables inside atomic subformulas.
The following example illustrates the effect of these different annotations on a concrete sequent.

Example 6.3.1. Consider the sequent

$$S \coloneqq \ \vdash \forall w \forall x \exists y\,[(P(x,y) \vee Q(w)) \wedge (\exists z(\neg R(z) \vee S(x)))]$$

Applying all three annotations gives the following three annotated formulas:

$$\alpha_{\text{std}}(S) = \ \vdash \forall w \forall x \exists^{f(w,x)} y\,[(P(x,y) \vee Q(w)) \wedge (\exists^{g(w,x)} z\,(\neg R(z) \vee S(x)))]$$

$$\alpha_{\text{inn}}(S) = \ \vdash \forall w \forall x \exists^{f(w,x)} y\,[(P(x,y) \vee Q(w)) \wedge (\exists^{g(x)} z\,(\neg R(z) \vee S(x)))]$$

$$\alpha_{\text{atom}}(S) = \ \vdash \forall w \forall x \exists^{f(x)} y\,[(P(x,y) \vee Q(w)) \wedge (\exists^{c} z\,(\neg R(z) \vee S(x)))]$$

As seen in this example, we have $\alpha_{\text{atom}} \leq \alpha_{\text{inn}} \leq \alpha_{\text{std}}$. As a consequence of Corollary 3.23, this hierarchy confirms that the smaller annotations allow for shorter proofs.

6.2 The Classic LK

We now address the classical sequent calculus **LK** introduced by Gentzen, the system that forms the foundational framework for many modern proof-theoretic investigations, as in [4]. For this we recite the version **G3c** similar to our version from Chapter. As always we will only consider cut-free proofs.

Definition 6.4. An **LK**-proof of a sequent S is a tree built bottom up by the following rules ending with axioms. The axioms are

$$\frac{}{A, \Gamma \vdash \Delta, A}\,\mathrm{ax} \qquad \frac{}{\bot, \Gamma \vdash \Delta}\,\mathrm{ax}\bot \qquad \frac{}{\Gamma \vdash \Delta, \top}\,\mathrm{ax}\top$$

where A is an atom. The logical rules are

$$\frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A}\,\neg_r \qquad \frac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta}\,\neg_l$$

$$\frac{A, \Gamma \vdash \Delta \qquad B, \Gamma \vdash \Delta}{A \vee B, \Gamma \vdash \Delta}\,\vee_l \qquad \frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \vee B}\,\vee_r$$

$$\frac{A, B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}\,\wedge_l \qquad \frac{\Gamma \vdash \Delta, A \qquad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}\,\wedge_r$$

$$\frac{\Gamma \vdash \Delta, A \qquad B, \Gamma \vdash \Delta}{A \supset B, \Gamma \vdash \Delta}\,\supset_l \qquad \frac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \supset B}\,\supset_r$$

The weak quantifier rules are

$$\frac{\Gamma \vdash \Delta, \exists x\, F, F[x\backslash t]}{\Gamma \vdash \Delta, \exists x\, F}\,\exists_r \qquad \frac{\forall x\, F, F[x\backslash s], \Gamma \vdash \Delta}{\forall x\, F, \Gamma \vdash \Delta}\,\forall_l$$

where t and s are arbitrary terms.

The strong quantifier rules are

$$\frac{F[x\backslash b], \Gamma \vdash \Delta}{\exists x\, F, \Gamma \vdash \Delta}\,\exists \qquad \frac{\Gamma \vdash \Delta, F[x\backslash a]}{\Gamma \vdash \Delta, \forall x\, F}\,\forall$$

where a, b are free variables that does not appear in the lower sequent. We call a (respective b) the *eigenvariable* of the inference. This condition on a is called the *eigenvariable-condition*.

We say an **LK**-proof π satisfies the eigenvariable-condition if all strong quantifier inferences in π satisfy it.

To relate this classical system to our annotated calculus, we need to define an annotation that relates **LK**-proofs to $\mathbf{LK}^{\mathrm{ac}}$. This will be achieved by finding a skolem annotation such that the eigenvariable-condition encapsulates the acyclicity condition on the dependency relation. The annotation that achieves this is the standard annotation α_{std}.

Definition 6.5. Let π be an **LK**-proof of asequent S. We say π is regular if all strong quantifier inferences infer a unique characteristic variable.
Note that this is stricter than the usual condition that two characteristic variables are the same if the skolemterm in the annotated strong quantifiers is the same.

Theorem 6.6. *Let S be a sequent. Then there exists a size preserving translation of regular* **LK***-proof π of S into* $\mathbf{LK}^{\mathrm{ac}}$*-proof π' of $\alpha_{\mathrm{std}}(S)$.*

Proof. We obtain our candidate π' by a direct translation of inferences in π into $\mathbf{LK}^{\mathrm{ac}}$. What remains to show is that the eigenvariable condition of π implies that the dependency relation $<_{\pi'}$ is acyclic.

Let $a' <^0_{\pi'} b'$, where b' is a characteristic variable in

$$\frac{\Gamma \vdash \Delta, F[x\backslash b']}{\Gamma \vdash \Delta, \forall^{f(\bar{t})} x\, F}\ \forall^{\mathrm{ac}}_r$$

and $a' \in \mathrm{FV}(f(\bar{t}))$. By definition of α_{std}, a' has been inferred in a weak quantifier inference before and the respective variable a in π would appear in the conclusion.
Now assume that there exists a cycle in $b <_{\pi'} b$. By the above considerations the characteristic variable would appear in the lower sequent, which is impossible by the eigenvariable-condition.

□

The theorem asserts that any cut-free regular **LK**-proof can be seen as an $\mathbf{LK}^{\mathrm{ac}}$-proof by using the skolem annotation α_{std}.

6.3 $\mathbf{LK}^{+}$ and $\mathbf{LK}^{++}$

We now turn our attention to the calculi $\mathbf{LK}^{+}$ and $\mathbf{LK}^{++}$ as defined by Aguilera and Baaz in [5], which can be seen as a generalization of **LK**. In these systems, the eigenvariable condition is replaced by a set of three conditions that control how variables are introduced. Similar to our $\mathbf{LK}^{\mathrm{ac}}$, these calculi introduce an acyclicity condition. It is based on *side-variables*.

Definition 6.7. Consider a strong $\exists$-inference in a proof π

$$\frac{F(a, b, \overline{s}), \Gamma \vdash \Delta}{\exists x\, F(x, b, \overline{s}), \Gamma \vdash \Delta}\ \exists_l\ .$$

We call b a *side variable* of a and write $a \sqsubset_\pi b$.

The same definition is made for any strong universal inferences in π.

This allows us to define the two calculi. While they both share the acyclicity condition, they require a different form of regularity. The first one is the one commonly used in sequential calculi, while the other one coincides with the one we defined in $\mathbf{LK}^{\mathrm{ac}}$.

Definition 6.8. The calculus $\mathbf{LK}^{+}$ is defined the same way as **LK**, but the eigenvariable-condition for strong quantifiers in a proof is dropped and instead the strong quantifier inferences have to meet the following 3 criteria:

- (substitutability) the characteristic variables do not appear in the conclusion of π.
- (side-variable condition) $\sqsubset_\pi$ is acyclic.

- (weak-regularity) The characteristic variable is not the characteristic variable of another strong quantifier inference in π.

Definition 6.9. The calculus $\mathbf{LK}^{++}$ is defined the same way as $\mathbf{LK}^{+}$, but instead of the weak-regularity, all strong quantifier inferences in a proof π have to meet the following less strict criteria:

- (very weak-regularity) if a characteristic variable is the characteristic variable of another strong quantifier inference, it has to have the same principal formula.

Consider a $\mathbf{LK}^{\mathrm{ac}}$-proof π of an annotated sequent S.

- The weak-regularity is the same as the regularity that we required in the previous **LK** section.
- The substitutability is always fulfilled as we require the bottom sequent in a proof to be an annotated sequent, which by definition only contains bound variables.

The only difference lies in the side-variable condition, but by choosing the right annotation we can show that the definition of $\mathbf{LK}^{+}$-proofs coincides with $\mathbf{LK}^{\mathrm{ac}}$-proofs.
The annotation that accomplishes this turns out to be the Inner Annotation from the previous section.

Theorem 6.10. *Let S be a sequent and $\alpha_{\mathrm{inn}}(S)$ be its inner annotated sequent. Then there exists a size preserving translation between $\mathbf{LK}^{+}$-proofs π of S and $\mathbf{LK}^{\mathrm{ac}}$-proofs π' of $\alpha_{\mathrm{inn}}(S)$.*

Proof. The translation of inferences is straightforward in both directions. The interesting part is showing that the side-variable condition of π is equivalent to the acyclicity of the dependency relation in π'.
Assume that the side-variable condition holds. We have to show that $<_{\pi'}$ is acyclic.
If $a' <^0_{\pi'} b'$, where b' is inferred with a strong quantifier $Q^{f(\bar{t})}x$, then by definition $a' \in \mathrm{FV}(f(\bar{t}))$. By choice of the annotation α_{inn}, a appears in F, which implies that a is a side variable of b and therefore $b \sqsubset_\pi a$.
Therefore acyclicity of $\sqsubset_\pi$ implies acyclicity of $<_{\pi'}$.

As for the other direction, assume that the dependency relation is acyclic. Let $b \sqsubset_\pi a$, where a is a characteristic variable of a strong quantifier in π and b is a side-variable of a. Let $Q^{f(\bar{t})}x$ be the respective strong quantifier in π'. By definition of the annotation this means that $b' \in \mathrm{FV}(f(\bar{t}))$ in π', which implies $a' <^0_{\pi'} b'$. The same way as in the previous case, acyclicity of $<_{\pi'}$ implies acyclicity of $\sqsubset_\pi$ which shows that both conditions are equivalent.
□

This lemma shows that, under the annotation α_{inn}, we can fully embed $\mathbf{LK}^{+}$-proofs into our framework.

$\mathbf{LK}^{++}$-proofs on the other hand can not be captured using our definition of skolem annotations, as the very weak-regularity is too general. It stands in conflict with our requirement of choosing one unique skolemsymbol for each strong quantifier inside a formula.
Two strong quantifier inferences of subformulas $Q^{f(\overline{x})}y\,F$ and $Q^{f(\tilde{x})}z\,F$ of an annotated sequent S can share the same characteristic variable as their principal formula is the same. But since their skolemterms are different, this can not be properly translated into our types of calculi.

6.4 Henkin Quantifiers and Independence Friendly Logic

Contrary to previous section where dependencies were based around occurrences within subformulas, we now examine forms of skolemisation that allow for arbitrary dependencies. We consider two different forms well known in literature, namely Henkin quantifiers [2] and Independence Friendly Logic [3]. While Henkin quantifiers directly define which weak quantifiers are affecting a strong quantifier, IF Logic excludes variables.

Let us first look at an example of a Henkin Quantifier:

Example 6.10.1. Let

$$F = \begin{pmatrix} \forall v & \forall w & \exists x \\ & \forall y & \exists z \end{pmatrix} (P(w,x,y) \wedge Q(w,y,z)) \vee P(x,v).$$

In this example x depends on v, w while z only depends on y. It can be rewritten as an annotated formula F' using two skolemterms:

$$F' = \forall v \forall w \exists^{f(v,w)} x \forall y \exists^{g(y)} z\, (P(w,x,y) \wedge Q(w,y,z)) \vee P(x,v)$$

As seen in the example, Henkin quantifiers are matrices of quantifiers where weak quantifiers only affect strong quantifiers within the same row.

Definition 6.11. Let F be a formula in prenex form with strong quantifiers $Q_1x_1, ...Q_nx_n$ and weak quantifiers $Q_1y_1, ..., Q_ky_k$. If F' is the principal formula of F, a Henkin Quantifier Q_H is a matrix

$$Q_H := \begin{pmatrix} Q_{11}y_{11} & \dots & Q_{1n_1}y_{1n_1} & Q_{11}x_{11}, ..., Q_{1k_1}x_{1k_1} \\ & \vdots & & \vdots \\ Q_{l1}y_{l1} & \dots & Q_{ln_l}y_{ln_l} & Q_{l1}x_{l1}, ..., Q_{lk_l}x_{lk_l} \end{pmatrix}$$

where all $x_{ij} \in x_1, ..., x_n$ and $y_{ij} \in y_1, ..., y_k$.
We call Q_HF' a Henkin formula and define the set of all Henkin formulas as $\mathcal{F}_{\text{Hen}}$.

Definition 6.12. Let $F = Q_HF'$ be a Henkin formula . Then we define the annotation $\alpha_{\text{Hen}} : \mathcal{F}_{\text{Hen}} \to \mathcal{F}_{ann}$ such that for all strong quantifiers Qy_{ij} in Q_H

$$\alpha_{\text{Hen}}(Qy_{ij}) := Q^{f(x_{i1},\dots,x_{ik_i})}y_{ij},$$

where the weak quantifiers $Q_{i1}x_{i1}, ..., Q_{ik_i}x_{ik_i}$ are in the same row as Qy_{ij} in Q_H and f is a skolemsymbol that is chosen uniquely for all strong quantifiers.
We call α_{Hen} the *Henkin Annotation*.

The second version we will examine in this section, namely Independence Friendly Logic, defines partial dependencies by excluding them using a /-notation.
For example the IF-formula

$$F = \forall x(\exists y/\forall x)P(x,y)$$

would imply that y does not depend on x. For now we will only consider IF-formulas where /-expressions only appear together with strong quantifiers.

Definition 6.13. If F is a formula and Qy a strong quantifier that occurs in F, then the expression F' that is the result of replacing Qy by

$$(Qy/Q_1y_1...Q_ky_k)$$

where $Q_1y_1...Q_ky_k$ are weak quantifiers such that Qy is in their scope is called an **IF**-formula.
The set of all **IF**-formulas is called $\mathcal{F}_{\mathrm{IF}}$.

Example 6.13.1. Let

$$F = \forall v\forall w(\exists x/\forall w)\forall y(\exists z/\forall v\forall w)(P(w,x,y) \wedge Q(w,y,z)) \vee P(x,v).$$

In this example x depends on v while z only depends on y. It can be rewritten as an annotated formula F' using two skolemterms:

$$F' = \forall v\forall w\exists^{f(v)}x\forall y\exists^{g(y)}z\,(P(w,x,y) \wedge Q(w,y,z)) \vee P(x,v)$$

We can see in the above example that the definition of the skolem annotation that translates IF-formulas into annotated formulas is very straightforward.

Definition 6.14. Let F be an **IF**-formula. Then we define the annotation $\alpha_{\mathrm{IF}} : \mathcal{F}_{\mathrm{IF}} \to \mathcal{F}_{ann}$ such that for all strong quantifiers $(Qy/Q_1x_1...Q_kx_k)$

$$\alpha_{\mathrm{IF}}(Qy) := Q^{f(x_1,\ldots,x_k)}y,$$

where f is a skolemsymbol that is chosen uniquely for all strong quantifiers.
We call α_{IF} the *IF Annotation*.

As mentioned at the beginning we only considered IF-formulas where the /-expressions occur in direct combination with with strong quantifiers. But IF-logic allows for even more generalisation, for example expressions like

$$F = \forall x\exists y\,[A(y)(\vee/\forall x)B(y)]$$

are also allowed. In the subformula $A(y)$ the bound variable y depends on x, while in $B(y)$ it is independent of x. A full definition of all possible IF-formulas can be found in [3].
A skolemised form of this formula could be

$$F_{\mathrm{sk}} = \forall x\,[A(f(x)) \vee B(c)].$$

As our definition of annotations connects each strong quantifier with one skolemsymbol, the full scope of IF-formulas can not be captured in our skolem annotations.

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