Why does induction require cut?

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Abstract

This short note aims at putting the above question into perspective and to provide a brief but precise answer for the case of Peano arithmetic.

We work with classical first-order logic. A *sentence* is a formula without free variables, a *theory* is a set of sentences (the axioms of the theory). An example for a theory is minimal arithmetic Q in the language $L = \{0, s, +, \times, =\}$ which is defined by a finite number of Π_1-axioms, see e.g. [2]. The theory *Peano arithmetic (PA)* is defined as Q plus all first-order induction axioms, i.e. all formulas of the form

$$(\psi(0) \land \forall x\, (\psi(x) \to \psi(s(x)))) \to \forall x\, \psi(x).$$

for ψ being an arbitrary formula. The theory $\mathrm{I}\Sigma_k$ is defined as Q plus all induction axioms of the above form where ψ is a Σ_k-formula.

Since we want to speak about cut-elimination we will work with the sequent calculus. Which variant of the sequent calculus we use is not of importance for the points discussed here; for the sake of precision let us fix it to be the calculus **LK** of [1]. A sequent is denoted as $\Gamma \longrightarrow \Delta$. For a theory T and a formula φ we write $T \vdash \varphi$ if there is a finite set $T_0 \subseteq T$ and an **LK**-proof of the sequent $T_0 \longrightarrow \varphi$. By the completeness theorem this is equivalent to φ being true in all models of T.

Theorem 1 (cut-elimination)**.** If there is an **LK**-proof of a sequent $\Gamma \longrightarrow \Delta$, then there is a cut-free **LK**-proof of $\Gamma \longrightarrow \Delta$.

An important feature of cut-free proofs is that they have the subformula property. In the context of first-order logic this means that every formula that occurs in a cut-free proof of the sequent $\Gamma \longrightarrow \Delta$ is an instance of a subformula of a formula that occurs in $\Gamma \longrightarrow \Delta$. A proof that has the subformula property is also called *analytic*.

Since the cut-elimination theorem considers arbitrary first-order sequents, it can also be applied to theories containing induction axioms:

Corollary 1. If $\mathrm{PA} \vdash \varphi$ then there is a finite $A_0 \subseteq \mathrm{PA}$ and a cut-free **LK**-proof of the sequent $A_0 \longrightarrow \varphi$.

So we see that *in the sense of the above corollary*, inductive theories do *not* require cut; we can obtain **LK**-proofs of sequents of the form $A_0 \longrightarrow \varphi$ with $A_0 \subseteq \mathrm{PA}$ which have the subformula property, i.e., every formula occurring in such a proof is an instance of a subformula of $A_0 \longrightarrow \varphi$. *However*, A_0 may contain induction axioms on induction formulas which are not instances of subformulas of φ, i.e. non-analytic induction formulas. Therefore the answer to the question posed in the title is rooted in the necessity of non-analytic induction formulas.

The necessity of non-analytic induction formulas follows for example from Gödel's second incompleteness theorem. Recall that, by arithmetising the syntax of formulas and proofs, one can formulate the consistency of an arithmetical theory as an arithmetical sentence. More specifically, for all $k \geq 1$ there is a Π_1-sentence $\mathrm{Con}(\mathrm{I}\Sigma_k)$ expressing the consistency of $\mathrm{I}\Sigma_k$, see for example [2]. We then have:

Theorem 2. For all $k \geq 1$: $\mathrm{PA} \vdash \mathrm{Con}(\mathrm{I}\Sigma_k)$ but $\mathrm{I}\Sigma_k \nvdash \mathrm{Con}(\mathrm{I}\Sigma_k)$.

Note that this result embodies a very strong non-analyticity requirement: given any $k \geq 1$, in order to prove $\mathrm{Con}(\mathrm{I}\Sigma_k)$ not only do we need a non-analytic induction formula, but we need one with at least k quantifier alternations even though $\mathrm{Con}(\mathrm{I}\Sigma_k)$ is only a Π_1-sentence.

Coming back to the question posed in the title, this theorem entails the necessity of cut in the following sense. First, formulate induction as the inference rule

$$\frac{\Gamma \longrightarrow \Delta, \psi(0) \quad \Gamma, \psi(x) \longrightarrow \Delta, \psi(s(x))}{\Gamma \longrightarrow \Delta, \forall x\, \psi(x)}\ \mathrm{Ind}$$

with the usual side condition and ψ being an arbitrary formula. Observe that $\mathrm{PA} \vdash \varphi$ iff there is an $\mathbf{LK}+\mathrm{Ind}$-proof of $Q \longrightarrow \varphi$. Now, in contrast to $\mathbf{LK}$, the calculus $\mathbf{LK}+\mathrm{Ind}$ does not have cut-elimination:

Corollary 2. There is a formula φ s.t. $Q \longrightarrow \varphi$ has an $\mathbf{LK} + \mathrm{Ind}$-proof but no cut-free $\mathbf{LK} + \mathrm{Ind}$-proof.

Proof. Let $\varphi = \mathrm{Con}(\mathrm{I}\Sigma_k)$ for *any* $k \geq 2$. Then, by Theorem 2, $\mathrm{PA} \vdash \mathrm{Con}(\mathrm{I}\Sigma_k)$ and consequently there is an $\mathbf{LK} + \mathrm{Ind}$-proof of $Q \longrightarrow \mathrm{Con}(\mathrm{I}\Sigma_k)$. On the other hand, suppose there would be a cut-free $\mathbf{LK} + \mathrm{Ind}$-proof of $Q \longrightarrow \mathrm{Con}(\mathrm{I}\Sigma_k)$. Then, due to the subformula property, all formulas, and in particular: all induction formulas, in this proof would be Σ_k thus contradicting Theorem 2. $\square$

The reason for the failure of cut-elimination in $\mathbf{LK} + \mathrm{Ind}$ can thus be seen to be the fact that the elimination of cuts would require the elimination of non-analytic induction formulas which is impossible.

References

[1] Sam Buss. An Introduction to Proof Theory. In Sam Buss, editor, *The Handbook of Proof Theory*, pages 2–78. North-Holland, 1999.

[2] Sam Buss. First-Order Proof Theory of Arithmetic. In Sam Buss, editor, *The Handbook of Proof Theory*, pages 79–147. North-Holland, 1999.