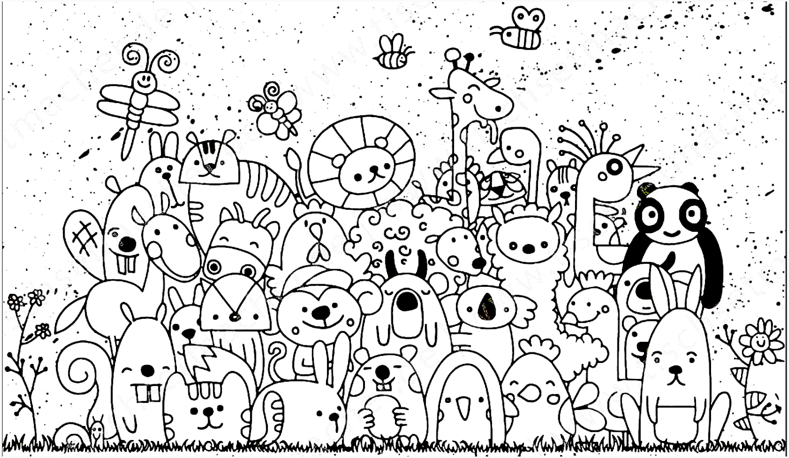
Hunting bears in random planar maps

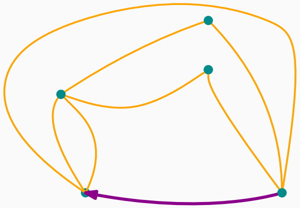
Eva-Maria Hainzl (TU Wien), SSAAEC, August 2023



Planar maps

Rooted planar maps with n edges

= connected graph with multiple edges and loops allowed embedded on a sphere (genus = 0) with one oriented (root) edge



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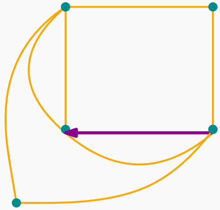
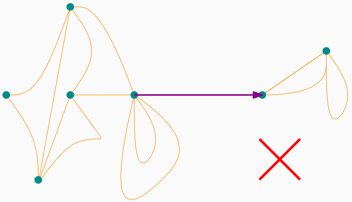
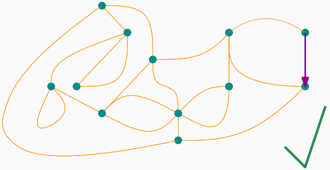


Planar maps

Maps with simple boundaries

= maps with a boundary consisting of as many edges as vertices

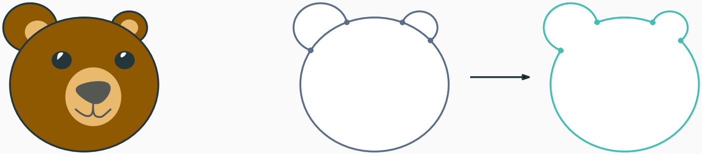




Bears in planar maps

Bears

A map consisting of one *simple* 4-gon and 2 *simple* 2-gons



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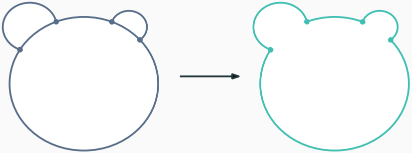
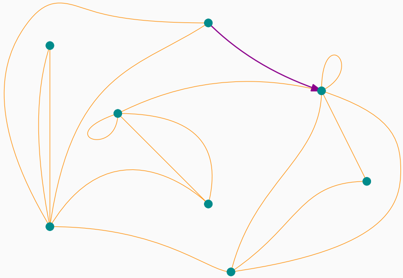


Bears have also *simple* boundaries.

Some basic questions

How many bears do we expect in a uniformly at random chosen map with *n* edges?

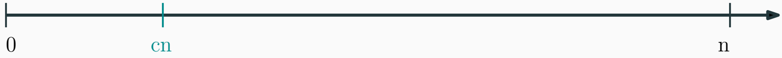
Can we say something about the distribution of bear counts?



Pattern occurrences in planar maps

Bender, Gao, Richmond, 1992

A random rooted map[1] with n edges almost surely contains at least cn copies of a pattern $\mathcal{P}$[2].



[1] on a surface with genus g
[2] if $\mathcal{P}$ is planar and almost surely none if it is not
[3] adjusted with a (regular critical) Boltzmann distribution

Pattern occurrences in planar maps

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A random rooted map[1] with n edges almost surely contains at least cn copies of a pattern $\mathcal{P}$[2].

Drmota, Stufler, 2017

Let X_n be the number of occurrences of $\mathcal{P}$ in a random rooted planar map[3] with n edges. Then

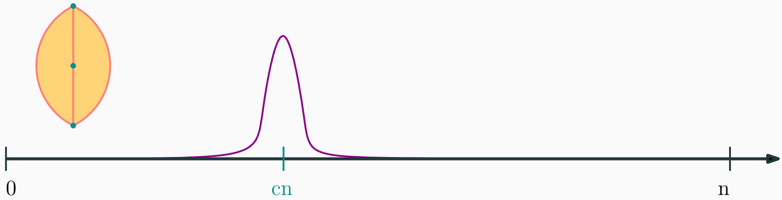
$$\mathbb{E}[X_n] = \Theta(n).$$

[1] on a surface with genus g
[2] if $\mathcal{P}$ is planar and almost surely none if it is not
[3] adjusted with a (regular critical) Boltzmann distribution

Gao, Wormald, 2004

Let $\mathcal{P}$ be a triangulation which cannot self-intersect and X_n be the number occurrences of $\mathcal{P}$ in a random triangulation with n edges. Then

$$\frac{X_n - \mu_n}{\sigma_n} \to \mathcal{N}(0,1).$$



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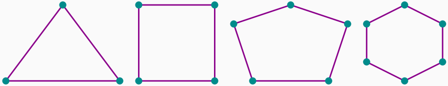
Drmota, Noy, Yu, 2020

The number of simple k-gons in a random planar map satisfies a CLT (as above).



Gao, Wormald, 2000

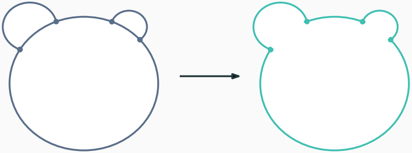
Asymptotic normality determined by high moments, and submap counts of random maps, Probability Theory and Related Fields volume 130, pages 368–376, 2004

Drmota, Noy, Yu, 2020

Universal singular exponents in catalytic variable equations, Journal of Combinatorial Theory, Series A Volume 185, January 2022

Theorem (Drmota, H., Wormald, 2023+)

Let $\mathcal{P}$ be a map with simple boundary. Then the number of occurrences of $\mathcal{P}$ in a random planar map satisfies a central limit theorem.



Proof ideas: Main tool

Theorem (Gao, Wormald, 2004)

Suppose that $\mu_n \to \infty, \sigma_n \log^2 \sigma_n = o(\mu_n), \mu_n = o(\sigma_n^3)$ and $(X_n)_{n\geq 1} \geq 0$ satisfies

$$\mathbb{E}\left[(X_n)_k\right] \sim \mu_n^k \exp\left(\frac{k^2}{2}\frac{\sigma_n^2 - \mu_n}{\mu_n^2}\right)$$

uniformly for all k in the range $c\mu_n/\sigma_n \leq k \leq c'\mu_n/\sigma_n$ for some constants $c' > c > 0$. Then

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$$\frac{X_n - \mu_n}{\sigma_n} \to \mathcal{N}(0, 1).$$

Note: $\mu_n/\sigma_n = \Theta(\sqrt{n})$

Proof ideas: Main tool

X_n = (random) number of patterns in map with n edges

What is $\mathbb{E}\left[(X_n)_k\right]$?

$$\mathbb{E}\left[(X_n)_k\right] = \sum_{\ell \geq k} \ell(\ell-1)\cdots(\ell-k+1)\frac{m_{n,\ell}}{m_n}$$

$m_n =$ the number of planar maps with n edges
$m_{n,\ell} =$ the number of planar maps with n edges and ℓ bear occurrences

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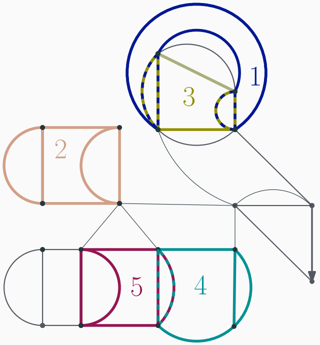
Proof ideas: Main tool

What is $\ell(\ell-1)\cdots(\ell-k+1)m_{n,\ell}$?

the number of maps on n edges with k labelled patterns (among arbitrary many).

$\Rightarrow$ count maps
with $\geq k$ bears
and label k of them!



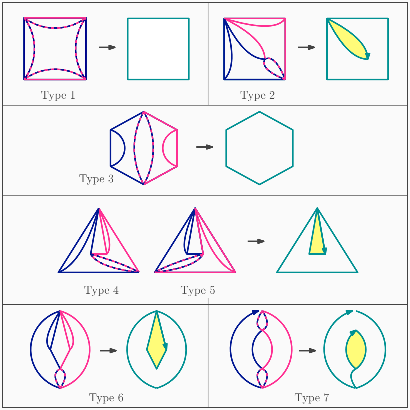
Proof ideas: Two steps

First important observation

For large n,
labelled bears are monogamous or confident singles!

First important observation

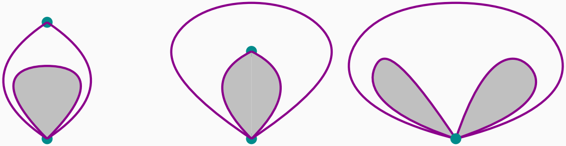
Only maps with single bears and pairs of bears contribute



Proof ideas: Two steps

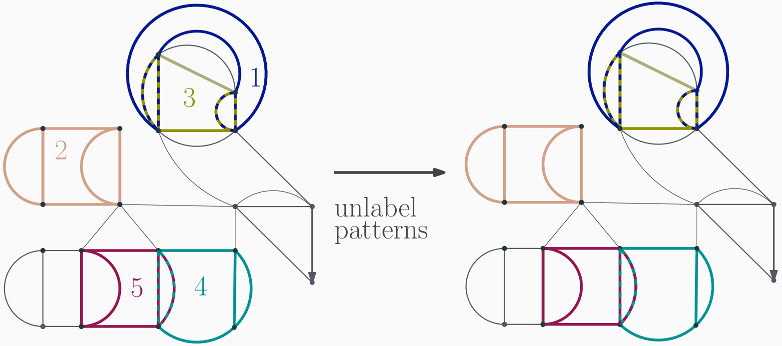
A consequence of Drmota, Noy, Yu 2020

Let $\mathcal{F}$ be a face of *specific shape*. Then the number of occurrences of $\mathcal{F}$ in a random planar map satisfies a CLT.



Second step

Counting maps with n edges and k labelled bears, where each bear intersects at most one other



Proof ideas: Two steps

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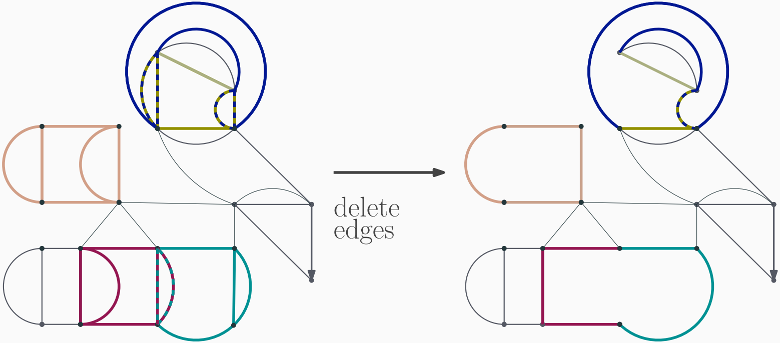

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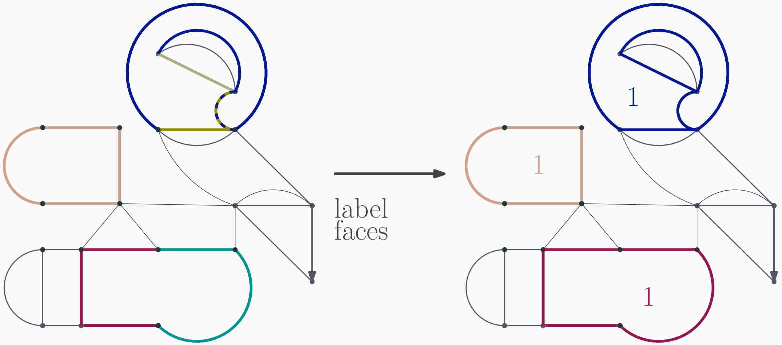

Proof ideas

The rest is just...

Again, we use (2.4), (2.5) and (3.3) such that we obtain

$$\begin{aligned}\mathbb{E}[(X_n)_2] &= r_0^2\Big(f_{x_{t(0)}}(\mathbf{1})^2(n-2d_0)^2 + \big(f_{x_{t(0)}x_{t(0)}}(\mathbf{1}) + 2f_{x_{t(0)}}(\mathbf{1})g_{x_{t(0)}}(\mathbf{1})\big)n + O(1)\Big)\frac{m_{n-2d_0}}{m_n}\\ &\quad + 2\left(\sum_{i=1}^T r_i f_{x_{t(i)}}(\mathbf{1})n + O(1)\right)\frac{m_{n-d_i}}{m_n}\\ &= \frac{r_0^2 f_{x_{t(0)}}(\mathbf{1})^2}{12^{2d_0}}\left((n-2d_0)^2 + \frac{f_{x_{t(0)}x_{t(0)}}(\mathbf{1}) + 2f_{x_{t(0)}}(\mathbf{1})g_{x_{t(0)}}(\mathbf{1})}{f_{x_{t(0)}}(\mathbf{1})^2}n\right)\left(1+\frac{5d_0}{n}+O\left(\frac{1}{n^2}\right)\right)\\ &\quad + 2\left(\sum_{i=1}^T \frac{r_i}{12^{d_i}} f_{x_{t(i)}}(\mathbf{1})n\right)\left(1+O\left(\frac{1}{n}\right)\right)\\ &= \frac{r_0^2 f_{x_{t(0)}}(\mathbf{1})^2}{12^{2d_0}}n^2\\ &\quad + \left(\frac{r_0^2\big(f_{x_{t(0)}x_{t(0)}}(\mathbf{1}) + 2f_{x_{t(0)}}(\mathbf{1})g_{x_{t(0)}}(\mathbf{1}) + d_0 f_{x_{t(0)}}(\mathbf{1})^2\big)}{12^{2d_0}} + \sum_{i=1}^T \frac{2r_i f_{x_{t(i)}}(\mathbf{1})}{12^{d_i}}\right)n + O(1)\end{aligned}$$

Now, it is straight forward to compute the variance by

$$\begin{aligned}\mathbb{V}(X_n) &= \mathbb{E}(X_n(X_n-1)) + \mathbb{E}(X_n) - \mathbb{E}(X_n)^2\\ &= \frac{r_0^2 f_{x_{t(0)}}(\mathbf{1})^2}{12^{2d_0}}n^2 + \left(\frac{r_0^2\big(f_{x_{t(0)}x_{t(0)}}(\mathbf{1}) + 2f_{x_{t(0)}}(\mathbf{1})g_{x_{t(0)}}(\mathbf{1}) + d_0 f_{x_{t(0)}}(\mathbf{1})^2\big)}{12^{2d_0}} + \sum_{i=1}^T \frac{2r_i f_{x_{t(i)}}(\mathbf{1})}{12^{d_i}}\right)n\\ &\quad + \frac{r_0 f_{x_{t(0)}}(\mathbf{1})}{12^{d_0}}n - \frac{r_0^2 f_{x_{t(0)}}(\mathbf{1})^2}{12^{2d_0}}\left(n + \frac{3d_0}{2} + \frac{g_{x_{t(0)}}(\mathbf{1})}{f_{x_{t(0)}}(\mathbf{1})} + O\left(\frac{1}{n}\right)\right)^2\\ &= \frac{1}{12^{2d_0}}\left(r_0^2 f_{x_{t(0)}x_{t(0)}}(\mathbf{1}) - 2d_0 r_0^2 f_{x_{t(0)}}(\mathbf{1})^2 + 12^{d_0} r_0 f_{x_{t(0)}}(\mathbf{1}) + \sum_{i=1}^T \frac{2r_i f_{x_{t(i)}}(\mathbf{1})}{12^{d_i-2d_0}}\right)n + O(1)\end{aligned}$$

5.5. **Proof of Theorem 2.** Analogously to Section 3 we compute the k-th factorial moment for which we hope to compute asymptotics of the same form as in Theorem 1 such that we can derive our main result as a direct consequence.
Remember that the k-th factorial moment is the fraction of the number of maps on n edges with k labelled pattern occurrences of the total number of maps on n edges. Further, we hope to be able to restrict ourselves to count maps on n edges with k labelled pattern occurrences where each labelled pattern intersects with at most one other labelled pattern occurrence. As we have already seen for bear patterns,

$$\mathbb{E}[(X_n)_k] = \frac{m^{\circ}_{n,k}}{m_n} \sim \frac{m^{\circ,\times}_{n,k}}{m_n}$$

where $m^{\circ}_{n,k}$ is the number of maps with n edges and k labelled pattern occurrence (among arbitrary many occurrences) and $m^{\circ,\times}_{n,k}$ is the subset of all such maps where each labelled pattern occurrence intersects at most one other labelled pattern occurrence.
For a map in $m^{\circ,\times}_{n,k}$, let p_i be the the number of pairs of intersecting labelled pattern occurrences with intersection type i and let s be the number of labelled pattern occurrences which do not intersect with any other. Of course it has to hold $s + \sum 2p_i = k$. Now instead of counting maps with n edges with s single labelled patterns and p_i intersecting pairs of labelled patterns of type i, $1 \le i \le I$, we unlabel the patterns and delete the d_i edges according to the rules we set for intersection type i. Thus, we end up with a map on $n - sd_0 - \sum p_i d_i$ edges and $s + \sum p_i$ faces which are marked by the intersection type which they contained. We already have counting variables for these faces in our functional equation in 4.1 such that we can easily compute that the number of

such maps is

$$\frac{m_{n-sd_0-\sum p_i d_i, s, \mathbf{p}}}{s!p_1!p_2!\cdots p_T!} = \left[z^{n-sd_0-\sum p_i d_i}\right]\frac{1}{s!p_1!p_2!\cdots p_T!}\partial^s_{x_{t(0)}}\partial^{p_1}_{x_{t(1)}}\partial^{p_2}_{x_{t(2)}}\cdots\partial^{p_T}_{x_{t(T)}}M(z,\mathbf{x})\big|_{\mathbf{x}=\mathbf{1}}$$

where $\mathbf{p} = (p_1, p_2, \ldots p_T)$. Note that in case that $t(i_1) = t(i_2) = \cdots = t(i_\ell)$, the derivative of the generating function will give the falling factorial factor $f(f-1)\cdots(f - \sum_{j=1}^{\ell} p_{i_j} + 1)$ before the coefficient counting maps with f faces and the factor $\prod_{j=1}^{\ell} 1/(p_{i_j}!)$ can be interpreted as unlabeling the faces with p_{i_1} smallest labels and assigning them intersection type i_1, then unlabeling the next p_{i_2} faces and assigning them intersection type i_2 and so on. Or we could think about unlabelling all $\sum_{j=1}^{\ell} p_{i_j}$ faces and then choosing from the marked faces the ones that end up in the respective intersection types. This would give of course the same factor of

$$\frac{1}{(\sum_{j=1}^{\ell} p_{i_j})!}\binom{\sum_{j=1}^{\ell} p_{i_j}}{p_{i_1}, p_{i_2}, \ldots p_{i_\ell}} = \frac{1}{p_{i_1}!p_{i_2}!\cdots p_{i_T}!}.$$

When we reinsert the deleted edges, we have to mulitply of course by the number of ways to do so and label the patterns in the end in $k!$ ways. Thus, we will get an extra factor of

$$k! r_0^s r_1^{p_1} r_2^{p_2} \cdot r_T^{p_T}$$

which in total sums up to

$$\frac{m^{\circ,\times}_{n,k}}{m_n} = k! \sum_{s+2\sum p_i = k} \frac{r_0^s \prod r_i^{p_i}}{s! \prod p_i!} \frac{m_{n-D,s,\mathbf{p}}}{m_{n-D}} \cdot \frac{m_{n-D}}{m_n}$$

where $D = d_0 s + \sum_{i=0}^m p_i d_i$. Consequently,

$$\begin{aligned}\frac{m^{\circ,\times}_{n,k}}{m_n} &= k! \sum_{P=0}^{\lfloor k/2\rfloor} \sum_{\sum p_i = P} \frac{r_0^{k-2P}\prod r_i^{p_i}}{(k-2P)!\prod p_i!} \frac{m_{n-D,k-2P,\mathbf{p}}}{m_{n-D}} \cdot \frac{m_{n-D}}{m_n}\\ &= \frac{k!}{12^D} \sum_{P=0}^{\lfloor k/2\rfloor} \sum_{\sum p_i = P} \frac{r_0^{k-2P}\prod r_i^{p_i}}{(k-2P)!\prod p_i!} f_{x_{t(0)}}(\mathbf{1})^{k-2P}\left(\prod_{i=1}^m f_{x_{t(i)}}(\mathbf{1})^{p_i}\right)\\ &\qquad \cdot (n-D)^{k-P} e^{\left(\frac{1}{2(n-D)}\langle\bar{\mathbf{p}},\Sigma\bar{\mathbf{p}}\rangle\right)}\left(1 + O\left(\frac{D}{n}\right)\right)\end{aligned}$$

where $\bar{\mathbf{p}} = (k-2P, p_1, p_2, \ldots p_T)^T$ and $(\Sigma)_{i+1,j+1} = \frac{f_{x_{t(i)}x_{t(j)}}(\mathbf{1})}{f_{x_{t(i)}}(\mathbf{1})f_{x_{t(j)}}(\mathbf{1})}$. Note that Lemma 3 actually gives an expression with $\langle \mathbf{v}, \bar{\Sigma}\mathbf{v}\rangle$ in the exponential function, where $v = (v_1, v_2, \ldots v_\ell)$ and v_i is the sum of all p_j with $t(j) = i$ while $(\bar{\Sigma})_{i,j} = \frac{f_{x_i x_j}(\mathbf{1})}{f_{x_i}(\mathbf{1})f_{x_j}(\mathbf{1})}$. However it is easy to see by elementary computations that we can expand this expression to $\langle\bar{\mathbf{p}},\Sigma\bar{\mathbf{p}}\rangle$.

Next, we factor out the main asymptotic term of μ_n^k which has already appeared in the sum.

$$\begin{aligned}\frac{m^{\circ,\times}_{n,k}}{m_n} &= \left(\frac{r_0}{12^{d_0}} f_{x_{t(0)}}(\mathbf{1})n\right)^k \sum_{P=0}^{\lfloor k/2\rfloor} \sum_{\sum p_i = P} \frac{k!}{12^{D-kd_0}(k-2R)!}\left(1 - \frac{D}{n}\right)^k\\ &\qquad \cdot \prod_{i=1}^T \frac{1}{p_i!}\left(\frac{r_i f_{x_{t(i)}}(\mathbf{1})}{(n-D) r_0^2 f_{x_{t(0)}}(\mathbf{1})^2}\right)^{p_i} e^{\left(\frac{1}{2(n-D)}\langle\bar{\mathbf{p}},\Sigma\bar{\mathbf{p}}\rangle\right)}\left(1 + O\left(\frac{D}{n}\right)\right)\end{aligned}$$

In the next step we rewrite

$$\langle\bar{\mathbf{p}},\Sigma\bar{\mathbf{p}}\rangle = k^2 \frac{f_{x_{t(0)}x_{t(0)}}(\mathbf{1})}{f_{x_{t(0)}}(\mathbf{1})f_{x_{t(0)}}(\mathbf{1})} + \sum_{i=1}^T q_i p_i$$

where q_i is according to the calculation linear in P and k (without mixed terms). Hence,

$$\exp\left(\frac{1}{2(n-D)}\sum_{i=1}^T q_i p_i\right) = \prod_{i=1}^T\left(1 + O\left(\frac{k}{n}\right)\right)^{p_i}.$$

Further we factor out more terms independent of any p_i's and since $D = O(k)$, the above simplifies to

$$\begin{aligned}\frac{m^{\circ,\times}_{n,k}}{m_n} &= \left(\frac{r_0}{12^{d_0}} f_{x_{t(0)}}(\mathbf{1})n\right)^k \exp\left(\frac{k^2}{2n}\frac{f_{x_{t(0)}x_{t(0)}}(\mathbf{1})}{f_{x_{t(0)}}(\mathbf{1})f_{x_{t(0)}}(\mathbf{1})}\right)\\ &\quad \cdot \sum_{P=0}^{\lfloor k/2\rfloor} \sum_{\sum p_i = P} \frac{k!}{12^{D-kd_0}(k-2P)!}\left(1 - \frac{D}{n}\right)^k \prod_{i=1}^T \frac{1}{p_i!}\left(\frac{r_i f_{x_{t(i)}}(\mathbf{1})}{n r_0^2 f_{x_{t(0)}}(\mathbf{1})^2}\left(1 + O\left(\frac{k}{n}\right)\right)\right)^{p_i}\end{aligned}$$

By now, we can see a pattern emerge reminiscent of the product of several exponential sums. In fact, we notice that only the terms with $P = o(k)$ are asymptotically relevant because of the factorials in the denominator and we may therefore use the asymptotics $\frac{k!}{(k-2P)!} = k^{2P}\left(1 + O\left(\frac{1}{k}\right)\right)$ and

$$\left(1 - \frac{d_0(k-2P) + \sum_{i=1}^m p_i d_i}{n}\right)^k = \exp\left(-\frac{d_0 k^2}{n}\right)\left(1 + O\left(\frac{1}{k}\right)\right)$$

such that we finally obtain the desired form

$$\begin{aligned}\frac{m^{\circ,\times}_{n,k}}{m_n} &= \left(\frac{r_0}{12^{d_0}} f_{x_{t(0)}}(\mathbf{1})n\right)^k \exp\left(\frac{k^2}{2n}\left(\frac{f_{x_{t(0)}x_{t(0)}}(\mathbf{1})}{f_{x_{t(0)}}(\mathbf{1})f_{x_{t(0)}}(\mathbf{1})} - 2d_0\right)\right)\left(1 + O\left(\frac{1}{k}\right)\right)\\ &\quad \cdot \sum_{P=0}^{o(k)} \sum_{\sum p_i = P} \prod_{i=1}^T \frac{1}{p_i!}\left(\frac{k^2}{n}\frac{12^{d_i-2d_0} r_i f_{x_{t(i)}}(\mathbf{1})}{r_0^2 f_{x_{t(0)}}(\mathbf{1})^2}\left(1 + O\left(\frac{1}{k}\right)\right)\right)^{p_i}\\ &= \left(\frac{r_0}{12^{d_0}} f_{x_{t(0)}}(\mathbf{1})n\right)^k \exp\left(\frac{k^2}{2n}\left(\frac{f_{x_{t(0)}x_{t(0)}}(\mathbf{1})}{f_{x_{t(0)}}(\mathbf{1})^2} - 2d_0 + 2\sum_{i=1}^T \frac{12^{d_i-2d_0} r_i f_{x_{t(i)}}(\mathbf{1})}{r_0^2 f_{x_{t(0)}}(\mathbf{1})^2}\right)\right)\left(1 + O\left(\frac{1}{k}\right)\right)\\ &= (\mu_n)^k \exp\left(\frac{k^2}{2n}\left(\frac{f_{x_{t(0)}x_{t(0)}}(\mathbf{1})}{f_{x_{t(0)}}(\mathbf{1})^2} - 2d_0 + 2\sum_{i=1}^T \frac{12^{d_i-2d_0} r_i f_{x_{t(i)}}(\mathbf{1})}{r_0^2 f_{x_{t(0)}}(\mathbf{1})^2}\right)\right)\left(1 + O\left(\frac{1}{k}\right)\right)\end{aligned}$$

Now, by Lemma 4 we can conclude that

$$\mathbb{E}[(X_n)_k] \sim \frac{m^{\circ,\times}_{n,k}}{m_n}$$

and indeed, by Lemma 5, the expression in the exponential function equals exactly the desired

$$\frac{k^2}{2}\frac{\sigma_n^2 - \mu_n}{\mu_n^2} = \frac{k^2}{2n}\frac{r_0^2 f_{x_{t(0)}x_{t(0)}}(\mathbf{1}) - 2d_0 r_0^2 f_{x_{t(0)}}(\mathbf{1})^2 + 2\sum_{i=1}^T 12^{d_i-2d_0} r_i f_{x_{t(i)}}(\mathbf{1})}{r_0^2 f_{x_{t(0)}}(\mathbf{1})^2}(1 + o(1))$$

such that

$$\mathbb{E}[(X_n)_k] \sim \mu_n^k \exp\left(\frac{k^2}{2}\frac{\sigma_n^2 - \mu_n}{\mu_n^2}\right)$$

and Theorem 1 finally yields Theorem 2.

6. References

References

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[2] Michael Drmota, Marc Noy, and Guan-Ru Yu. Universal singular exponents in catalytic variable equations. *Journal of Combinatorial Theory, Series A*, 185:105522, 2022.
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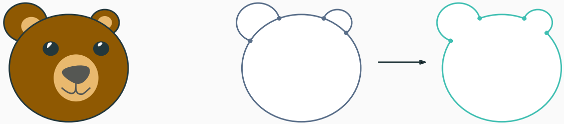
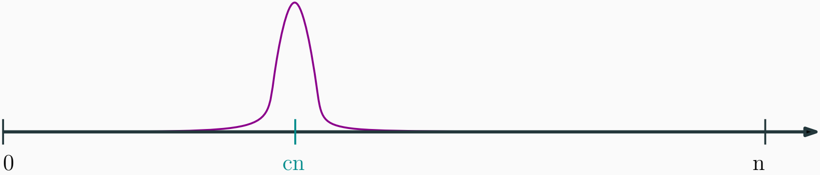
Pattern occurrences in planar maps

Theorem (Drmota, H., Wormald, 2023+)

Let $\mathcal{P}$ be a map with simple boundary. Then the number of occurrences of $\mathcal{P}$ in a random planar map satisfies a central limit theorem.





Thank you for the attention :)