Applications of Logic in Algebra: clones

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Outline

1. Background
2. Descriptive set theory
3. Infinite Combinatorics
4. Forcing
5. Structure or nonstructure?

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- **general problem:** Analyse the relationships between different algebras on the same set;
  by how much is $(\mathbb{Q}, +, \cdot)$ "richer" than $(\mathbb{Q}, +)$?
- **specific problem:** Which algebras are *complete*? (i.e., all functions are term functions)?

Definition

Fix a set X. We write $\mathcal{O}^{(n)}$ for the set of n-ary operations:
$\mathcal{O}^{(n)} = X^{X^n}$*, and we let* $\mathcal{O} = \mathcal{O}_X = \bigcup_{n=1,2,\ldots} \mathcal{O}^{(n)}$.
A clone on X *is a set $C \subseteq \mathcal{O}$ which contains all the projection functions and is closed under composition.*

Equivalently, a clone is the set of term functions of some universal algebra on X.

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The set of clones on X forms a complete lattice: $\mathbf{CLONE}(X)$.

Definition: For any $C \subseteq \mathcal{O}$ let $\langle C \rangle$ be the clone generated by C.

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If X is infinite, then

- $|\mathcal{O}_X| = 2^{|X|}$,
- $|\mathbf{CLONE}(X)| = 2^{2^{|X|}}$,
- and only little is known about the structure of **CLONE**(X).

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Now let X be any set.

Example

Assume that $\leq$ *is a nontrivial partial order on* X*, and that all functions in* $C \subseteq \mathcal{O}$ *are monotone with respect to* $\leq$*.*
Then $\langle C \rangle \neq \mathcal{O}$*.*

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We write $\mathrm{Pol}(\leq)$, $\mathrm{Pol}(\theta)$, $\mathrm{Pol}(A)$, … for the clone of all functions respecting $\leq$, θ, A, …

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Theorem

Let X be a finite set, and let $C \subseteq D$ be clones on X. Then the interval $[C, D]$ in $\mathbf{CLONE}(X)$ is

- *either finite,*
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Note 2 Not true for clones on infinite sets; all cardinalities are possible.

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The clone T_2

The following “canonisation theorem” follows from Ramsey’s theorem:

For every function $f : \omega \times \omega \to \omega$ we can find infinite sets A, B such that $f{\restriction}(A \times B) \cap \nabla$ (with $\nabla := \{(x, y) : x < y\}$)

- is injective,
- or depends injectively only on x: $f(x, y) = h_1(x)$, h_1 1-1
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This theorem motivates the definition of a clone; namely, the clone of all functions for which the first case ("injective") never happens.

An open question concerning T_2

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Let $\hat{T}_2$ be the set of all function $f : \omega^k \to \omega$ such that no term

$$u_0(f(u_1(?), u_2(?), ...u_n(?))$$

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Open Question

Does T_2 generate $\hat{T}_2$?

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For unary functions f, g we define $f \leq_D g$ iff $f \in \langle C_D \cup \{g\}\rangle$ (iff f is bounded by a finite composition of functions from $\{g, h_A\}$ for some $A \in D$.)

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Theorem

Assuming CH, we can construct an ultrafilter D such that $\leq_D$ is linear without last element.

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In ZFC, the existence of such a clone is still open.

Growth clones on uncountable cardinals

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Assume $2^\kappa = \kappa^+$. Then there is a clone C on κ such that the interval $[C, \mathcal{O})$ contains a cofinal linear order of type κ^+.

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Clones above the idempotent clone

Let C_{ip} be the clone of all idempotent operations:

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Clones above the idempotent clone

Let C_{ip} be the clone of all idempotent operations:

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For every filter D (including the trivial filter $\mathcal{P}(X)$) on X let C_D be clone of D-idempotent functions.

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Theorem

Every clone in the interval $[C_{\mathrm{ip}}, \mathcal{O}]$ is of the form C_D for some D. Hence, the interval $[C_{\mathrm{ip}}, \mathcal{O}]$ is (as a lattice) isomorphic to the family of open subsets of βX.

(This translates a problem from algebra to topology.)