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Structure and nonstructure in the clone lattice

"Applications of Mathematical Logic in Universal Algebra"

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THE GOAL of general algebra:

Classify all algebras!

What is an algebra?

$(M, *, +, ', \ldots)$

(many operations allowed, even uncountable many.)

Subgoals:

- Describe all groups
  - ... all abelian groups
  - ... all finite groups
  - ... all finite abelian groups
  - ... all simple groups
  - ... all finite simple groups
  - ...
- all fields
  - all finite fields
  - all pairs $K \leq L$ of fields, together with their Galois group
  - ...
- ...

Comparing algebras

E.g., $(\mathbb{Z}, +)$ is "smaller" than $(\mathbb{Q}, +)$

E.g. $(\mathbb{Q}, +, \cdot)$ ist "smaller" than $(\mathbb{C}, +, \cdot)$.

"Completion"

Algebras with the same base set

- semigroup $(\mathbb{Q}, +)$
- group $(\mathbb{Q}, +, -)$
- ring $(\mathbb{Q}, +, -, \cdot)$
- field $(\mathbb{Q}, +, -, \cdot, /)$

semigroup $<$ group $<$ ring $<$ field

ONE goal of general algebra:

Fix a set X.
Describe all algebras on X!

We identify algebras with the same term functions.

E.g., group $(G, +, -) = (G, -)$.
[because $x + y = x - ((x - x) - y)$.]

I.e., we describe algebras not by their "basic operations", but rather by their set of term functions; we compare two algebras by $\subseteq$-comparing their sets of term functions.

Example (term functions)

Let $(G, +, -)$ be an abelian group.

- unary term functions: x, $-x$, $x + x$, $x - x$, $x + x + x$, $\ldots$, $n \cdot x$ $(n \in \mathbb{Z})$
- binary term functions: $n_1 \cdot x_1 + n_2 \cdot x_2$ $(n_1, n_2 \in \mathbb{Z})$
- k-ary term functions: $\sum_{i=1}^{k} n_i \cdot x_i$

Clones

The set of all term functions (of any arity) of an abelian group

- is a subset of the set
$$\mathscr{O}_X = \bigcup_{k=1}^{\infty} \{f \mid \ f : X^k \to X\}$$
of *all* finitary operations on X.
- contains all projections π_k^n
(where $\pi_k^n(x_1, \ldots, x_n) = x_k$).
- is closed under substitution

A subset $\mathscr{C} \subseteq \mathscr{O}_X$ is called "Clone on X", if $\mathscr{C}$ contains all projections π_k^n and is closed under substitution.

Substitution: If $f, g_1, \ldots, g_k \in \mathscr{C}$, then $f(g_1, \ldots, g_k) \in \mathscr{C}$:

$$\begin{array}{rrcl} f: & X^k & \rightarrow & X \\ g_1, \ldots, g_k: & X^n & \rightarrow & X \\ \hline f(g_1, \ldots, g_k): & X^n & \rightarrow & X \end{array}$$

Example: Clone of all linear functions (on a fixed vector space). Clone of all polynomials. Clone of all continuous functions. Clone of measure preserving functions.

Clone $\mathscr{O}^{(1)}$ of all essentially unary functions.

ONE goal of general algebra:

Let X be a fixed set.
Describe all clones on X!
(= all algebras on X, identifying algebras with the same term functions.)

Equivalently: (?)
Describe the set $Cl(X)$ of all clones on X!

Structure of $Cl(X)$

Let $C \subseteq \mathscr{O}_X$ be a set of operations. We write $\langle C \rangle$ for the smallest clone containing C.

If $\mathscr{C}_1$ and $\mathscr{C}_2$ are clones, then so are

$\mathscr{C}_1 \cap \mathscr{C}_2$

$\langle \mathscr{C}_1 \cup \mathscr{C}_2 \rangle$

Thus the set $Cl(X)$, ordered by $\subseteq$, is a lattice.

Additional structure: e.g., conjugation.

In fact $Cl(X)$ is a *complete* lattice, and it is algebraic. (Each element is sup of compact elements; equivalently: isomorphic to the subalgebra lattice of some universal algebra.)

Notation: $\mathscr{O}_X = \sup Cl(X)$.
$\mathscr{O}^{(1)}$ = clone of unary functions.

A few natural questions

Completeness: For which sets $F \subseteq \mathscr{O}$ do we have $\langle F \rangle = \mathscr{O}$?

(remember nand and nor)

Relative completeness: Given $\mathscr{C}$: For which sets $F \subseteq \mathscr{C}$ do we have $\langle F \rangle = \mathscr{C}$?

Distance: How far are $\mathscr{C} \subseteq \mathscr{D}$ apart?

- How large/complicated is the interval $[\mathscr{C}, \mathscr{D}]$?
- Find small F with $\langle \mathscr{C} \cup F \rangle = \mathscr{D}$.

Structure: Classify clones, e.g. partition them according to their unary (or binary, etc) part. For each monoid $M \subseteq X^X$, describe the set of all clones $\mathscr{C}$ whose unary part is M.

This is an interval.

A few answers

On finite base sets X,

- There are $2^{\aleph_0}$ many clones
- Each interval is either $\leq \aleph_0$ or of size $2^{\aleph_0}$.
- There are finitely many coatoms in the clone lattice, all explicitly known
- $\langle F \rangle = \mathscr{O}$ iff F is not contained in any coatom.

Example: If $\leq$ is a bounded partial order, X finite, then the set of monotone functions is a clone, and a coatom in the clone lattice.

Conversely, all coatoms in the clone lattice look "similar" to this example.

Not true on infinite sets!

1. coatoms not all known (and never will be).

   example

2. Is every clone $\neq \mathscr{O}$ below a coatom? – maybe no. (for uncountable sets: **open**)

   proof: 207

Example for $X = \mathbb{N}$

For the moment, we treat unary functions as trivial. So we are interested in the interval $[\mathscr{O}^{(1)}, \mathscr{O}]$ of all clones containing all unary functions.

- This interval is small:
  1. Exactly 2 coatoms
  2. $\mathscr{O} = \langle \mathscr{O}^{(1)} \cup \{p\} \rangle$ for any 1-1 $p : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$.
     (Hence: Every element is below a coatom.)
- This interval is large: Of cardinality $2^{\mathfrak{c}}$.

  Proof sketch

An open question about binary clones

Definition:

T_1 = The set of all "almost unary" binary functions:

$\exists f \, \forall n \, \forall k \; F(n,k) \leq f(n)$

or $\exists f \, \forall n \, \forall k \; F(n,k) \leq f(k)$.

T_2 = The set of all "never 1-1" functions:

Whenever $F \restriction \{(n,k) \in A \times A : n \leq k\}$ is 1-1, then A must be finite
(and also the dual holds: $\cdots n \geq k \cdots$)

These clones play an important role in the interval $[\mathscr{O}^{(1)}, \mathscr{O}]$.

Theorem [Pinsker]: The clones $\mathscr{C}$ whose binary part is T_1 form a chain of type $1+\omega^*$.

Question:
Describe all clones whose binary part is T_2.
(How many are there? More than 1?)