More clones from ideals

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Outline

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Clones

We consider **A**lgebras $(X, f, g, \ldots)$ on a **fixed set** X,
and rank them according to their richness of term functions.

Note: In general, our algebras will have *many* operations.

Example

$(\mathbb{Q}, +) < (\mathbb{Q}, +, \cdot) < (\mathbb{Q}, +, -, \cdot) = (\mathbb{Q}, -, \cdot)$.

- **general problem:** Analyse the relationships between different algebras on the same set;
  by how much is $(\mathbb{Q}, +, \cdot)$ "richer" than $(\mathbb{Q}, +)$?
- **specific problem:** Which algebras are *complete*?
  (i.e., all functions are term functions)?
- Which are *precomplete*? (i.e., will become complete when adding any new function)

Definition

Fix a set X. We write $\mathcal{O}^{(n)}$ for the set of n-ary operations: $\mathcal{O}^{(n)} = X^{X^n}$, and we let $\mathcal{O} = \mathcal{O}_X = \bigcup_{n=1,2,\dots} \mathcal{O}^{(n)}$.
A clone on X is a set $C \subseteq \mathcal{O}$ which contains all the projection functions and is closed under composition.

Equivalently, a clone is the set of term functions of some universal algebra on X.

Fact

The set of clones on X forms a complete Lattice: **CLONE**(X).

Definition: For any $C \subseteq \mathcal{O}$ let $\langle C \rangle$ be the clone generated by C.
We write $C(f)$ for $\langle C \cup \{f\} \rangle$.

Size of **CLONE**(X)

If X is finite, then $\mathcal{O}_X$ is countable.

- If $|X| = 1$, then $\mathcal{O}_X$ is trivial.
- If $|X| = 2$, then **CLONE**(X) is countable, and completely understood. ("Post's Lattice")
- If $3 \leq |X| < \aleph_0$, then $|\textbf{CLONE}(X)| = 2^{\aleph_0}$, and not well understood.

If X is infinite, then

- $|\mathcal{O}_X| = 2^{|X|}$,
- $|\textbf{CLONE}(X)| = 2^{2^{|X|}}$, (we will see many proofs of this fact)
- and only little is known about the structure of **CLONE**(X).

Completeness

Example

The functions $\wedge, \vee, \mathtt{true}, \mathtt{false}$ *do not generate all operations on* $\{\mathtt{true}, \mathtt{false}\}$.

Proof: All these functions are monotone, and $\neg$ is not.

Now let X be any set.

Example

Assume that $\leq$ *is a nontrivial partial order on* X*, and that all functions in* $C \subseteq \mathcal{O}$ *are monotone with respect to* $\leq$.
Then $\langle C \rangle \neq \mathcal{O}$.

Polymorphisms

Let X be a set, $C \subseteq \mathcal{O}_X$.

- If all functions in C respect some order $\leq$ on X,
- or: if all functions in C respect some nontrivial equivalence relation θ
- or: if all functions in C respect some nontrivial fixed set $A \subset X$
  (i.e., $f[A^k] \subseteq A$)
- or ...

then $\langle C \rangle \neq \mathcal{O}$.

We write $\mathrm{Pol}(\leq)$, $\mathrm{Pol}(\theta)$, $\mathrm{Pol}(A)$, ... for the clone of all functions respecting $\leq$, θ, A, ...
Instead of unary (A) or binary ($\leq$, θ) relations, we may also consider n-ary or even infinitary relations.

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Pol( ) and precomplete clones

- Every set of the form $\mathrm{Pol}(A_1) \cap \mathrm{Pol}(A_2) \cap \mathrm{Pol}(\theta_3) \cap \cdots$ is a clone.
- Conversely, every clone is the intersection of (at most countably many) sets of the form $\mathrm{Pol}(R)$.

The "maximal" or "precomplete" clones are the coatoms in the clone lattice.
$C \neq \mathcal{O}$ is precomplete iff $C(f) = \mathcal{O}$ for all $f \in \mathcal{O} \setminus C$.

- Every precomplete clone is of the form $\mathrm{Pol}(R)$ for some relation R.

Question

Which relations R give rise to precomplete clones?

This is nontrivial, already for binary relations.

Precomplete clones on finite sets

Example

Let $\emptyset \neq A \neq X$.
Then $\mathrm{Pol}(A)$ is precomplete.

Example

Let X be finite. Let θ be a nontrivial equivalence relation.
Then $\mathrm{Pol}(\theta)$ is precomplete.

Theorem (Rosenberg, 1970)

There is an explicit list of all (finitely many, depending on the cardinality of X) relations R such that $\mathrm{Pol}(R)$ is precomplete.

Moreover, every clone $C \neq \mathcal{O}$ is below some precomplete clone. This gives an effective (but not efficient) method for checking $\langle D \rangle = \mathcal{O}$, for all $D \subseteq \mathcal{O}$. (Check $f \in D \Rightarrow f \in \mathrm{Pol}(R)$, for all relevant R.)

Precomplete clones on infinite sets

Example

Let $\emptyset \neq A \neq X$.
Then $\mathrm{Pol}(A)$ *is precomplete.*

Example

Let θ be a nontrivial equivalence relation with finitely many classes.
Then $\mathrm{Pol}(\theta)$ *is precomplete.*

For which R is $\mathrm{Pol}(R)$ precomplete?
Is every $C \neq \mathcal{O}$ below some precomplete clone?

Outline

Fixpoint clones

Definition

Let $A \subseteq X$. $\operatorname{fix}(A)$ is the set of all functions f satisfying

$\forall x \in A: \ f(x, \ldots, x) = x.$

This is a clone.

Definition

Let F be a filter on X. $\operatorname{fix}((F))$ is defined as $\bigcup_{A \in F} \operatorname{fix}(A)$, i.e.,

$$\operatorname{fix}((F)) = \{g : \exists A \in F \, \forall x \in A : \ g(x, \ldots, x) = x\}$$

- $\operatorname{fix}((F))$ is a clone.
- If F is the principal filter generated by the set A, then $\operatorname{fix}((F)) = \operatorname{fix}(A)$.
- larger filter $\Rightarrow$ larger clone.
- maximal filter $\Rightarrow$ maximal clone.

Fixpoint clones, application

Let $C_0 := \mathrm{fix}(X)$, i.e. the clone of all functions f satisfying $f(x, \ldots, x) = x$ for *all* $x \in X$.
Let $C_1 := \mathrm{fix}(\emptyset) = \mathcal{O}$, the clone of all functions. Then the interval $[C_0, C_1]$ in the clone lattice is rather complicated, and yet we can "explicitly" describe it.

Theorem (Goldstern-Shelah, 2004)

The clones in the interval $[C_0, C_1]$ are exactly the clones $\mathrm{fix}((F))$, for all possible filters (including the trivial filter $\mathfrak{P}(X)$). (Maximal=precomplete clones correspond to ultrafilters.)

So this interval is order isomorphic to the lattice of closed subsets of βX (with reverse inclusion).
(Leave it to **T**opologists . . .
. . . who work on **B**oolean spaces.)

Outline

Clones from ideals

Definition

Let I be a nontrivial ideal on the set X containing all small sets. $f : X^k \to X$ preserves I if $\forall A \in I : f[A^k] \in I$. We write $\mathrm{Pol}((I))$ for the set of all functions preserving I.

- $\mathrm{Pol}((I))$ is a clone.
- If I is the principal ideal generated by the set A, then $\mathrm{Pol}((I)) = \mathrm{Pol}(A)$.
- larger ideal $\not\Rightarrow$ larger clone.
- maximal ideal $\Rightarrow$ maximal clone.
- However, many other ideals also yield maximal clones.

  $I^{-\circ} := \{A \subseteq X : \forall B \in [A]^\omega : [B]^\omega \cap I \neq \emptyset\}$.

  If $I = I^{-\circ}$, then $\mathrm{Pol}((I))$ is maximal.

Ideal clones, application

Let $X := 2^{<\omega}$, the full binary tree. Every $\eta \in 2^\omega$ defines a branch $b_\eta = \{\eta{\restriction}n : n \in \omega\}$ through this tree.
For every subset $A \subseteq 2^\omega$ we define an ideal I_A:

$$I_A = \{E \subseteq 2^{<\omega} : \forall \eta \in A \, |b_\eta \cap E| < \aleph_0\}$$

Easy to check that $I_A = I_A^{-\circ}$, and that the ideals I_A are all different.

Theorem (Beiglböck-Goldstern-Heindorf-Pinsker, 2007)

While the ideals I_A are not maximal, the clones $\mathrm{Pol}((I_A))$ *are (for nontrivial A).*

This gives an explicit example of $2^{\mathfrak{c}}$ many precomplete clones on a countable set. (Even without AC.)

Question

Find such examples on uncountable sets.

Equivalence relations

Example

Let θ be a nontrivial equivalence relation on a finite set. Then $\mathrm{Pol}(\theta)$ is a precomplete clone.

Example

Let θ be a nontrivial equivalence relation on any set, with finitely many classes. Then $\mathrm{Pol}(\theta)$ is a precomplete clone.

Definition

Let $\mathcal{E}$ be a directed family of equivalence relations (coarser and coarser).
Define $\mathrm{Pol}((\mathcal{E}))$ *as the set of all functions* $f : X^k \to X$ *with:*
for all $E \in \mathcal{E}$ *there is* $E' \in \mathcal{E}$ *such that: whenever* $\vec{x}\, E\, \vec{y}$*, then* $f(\vec{x}) E' f(\vec{y})$.

When is $\mathrm{Pol}((\mathcal{E}))$ precomplete? Difficult. Because…

Fact

For every ideal I *there is a family* $\mathcal{E}$ *as above such that* $\mathrm{Pol}((I)) = \mathrm{Pol}((\mathcal{E}))$.

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Growth clones

Definition

Let $X = \omega = \{0, 1, 2, \ldots\}$ for simplicity. For every infinite $A = \{a_0 < a_1 < \cdots\} \subseteq X$ we define $\mathrm{bound}(A)$ as the set of functions which do not jump to far in A:

$$\mathrm{bound}(A) := \{f : \exists k \forall i : \vec{x} < a_i \Rightarrow f(\vec{x}) < a_{i+k}\}$$

(This is a clone.)

A similar construction is possible for uncountable **S**ets.

Growth clones, continued

Definition

Let $X = \omega$ again. For every filter F of subsets of X we define $\mathrm{bound}((F)) := \bigcup_{A \in F} \mathrm{bound}(A)$.

$$\mathrm{bound}((F)) := \{f : \exists A \in F\, \exists k\, \forall i : \vec{x} < a_i^A \Rightarrow f(\vec{x}) < a_{i+k}^A\}$$

(where $a_0^A < a_1^A < \cdots$ is the increasing enumeration of A).

- $\mathrm{bound}((F))$ is a clone.
- If F is the principal filter generated by the set A, then $\mathrm{bound}((F)) = \mathrm{bound}(A)$.
- larger filter $\Rightarrow$ larger clone.
- maximal filter $\not\Rightarrow$ maximal clone.
- (In fact, $\mathrm{bound}((F))$ is never a maximal clone.)

Growth clones, application

Theorem

G-Shelah, 2006*
Assume GCH. Then on every infinite set X there is a filter F such that, letting $C := \mathrm{bound}((F))$, we know the interval $[C, \mathcal{O})$ quite well,
and it is (more or less) a quite saturated linear order with no last element.

In particular: not every clone is below a precomplete clone.

References: See `http://arXiv/abs/math/0701030`, a survey paper by Goldstern-Pinsker.