Is the clone lattice on infinite sets dually atomic?

Martin Goldstern

Institute of Discrete Mathematics and Geometry
Vienna University of Technology

June 2007

Outline

Outline

Clones

Definition

Fix a set X. We write $\mathcal{O}^{(n)}$ for the set of n-ary operations: $\mathcal{O}^{(n)} = X^{X^n}$, and we let $\mathcal{O} = \mathcal{O}_X = \bigcup_{n=1,2,\ldots} \mathcal{O}^{(n)}$.
A clone on X *is a set $C \subseteq \mathcal{O}$ which contains all the projection functions and is closed under composition.*

Equivalently, a clone is the set of term functions of some universal algebra on X.

Fact

The set of clones on X forms a complete lattice: $\mathbf{CLONE}(X)$.

Clones

Definition

Fix a set X. We write $\mathcal{O}^{(n)}$ for the set of n-ary operations: $\mathcal{O}^{(n)} = X^{X^n}$, *and we let* $\mathcal{O} = \mathcal{O}_X = \bigcup_{n=1,2,\dots} \mathcal{O}^{(n)}$.
A clone on X *is a set $C \subseteq \mathcal{O}$ which contains all the projection functions and is closed under composition.*

Equivalently, a clone is the set of term functions of some universal algebra on X.

Fact

The set of clones on X forms a complete lattice: $\mathbf{CLONE}(X)$.

Clones

Definition

Fix a set X. We write $\mathcal{O}^{(n)}$ for the set of n-ary operations:
$\mathcal{O}^{(n)} = X^{X^n}$*, and we let* $\mathcal{O} = \mathcal{O}_X = \bigcup_{n=1,2,\dots} \mathcal{O}^{(n)}$.
A clone on X *is a set $C \subseteq \mathcal{O}$ which contains all the projection functions and is closed under composition.*

Equivalently, a clone is the set of term functions of some universal algebra on X.

Fact

The set of clones on X forms a complete lattice: $\mathbf{CLONE}(X)$.

Clones

Definition

Fix a set X. We write $\mathcal{O}^{(n)}$ for the set of n-ary operations:
$\mathcal{O}^{(n)} = X^{X^n}$*, and we let* $\mathcal{O} = \mathcal{O}_X = \bigcup_{n=1,2,\dots} \mathcal{O}^{(n)}$.
A clone on X *is a set* $C \subseteq \mathcal{O}$ *which contains all the projection functions and is closed under composition.*

Equivalently, a clone is the set of term functions of some universal algebra on X.

Fact

The set of clones on X forms a complete lattice: $\mathbf{CLONE}(X)$.

Clones

Definition

Fix a set X. We write $\mathcal{O}^{(n)}$ for the set of n-ary operations: $\mathcal{O}^{(n)} = X^{X^n}$*, and we let* $\mathcal{O} = \mathcal{O}_X = \bigcup_{n=1,2,\dots} \mathcal{O}^{(n)}$.
A clone on X *is a set $C \subseteq \mathcal{O}$ which contains all the projection functions and is closed under composition.*

Equivalently, a clone is the set of term functions of some universal algebra on X.

Fact

The set of clones on X forms a complete lattice: $\mathbf{CLONE}(X)$.

Coatoms

$\mathcal{O}$ is the full clone (the set of all finitary operations on X).

A *coatom* or *precomplete clone* or *maximal clone* is a clone $C \neq \mathcal{O}$ such that there is no clone strictly between C and $\mathcal{O}$.

Example

Let $\emptyset \subsetneq A \subsetneq X$. Then the set

$$\mathrm{Pol}(A) := \{f : f[A^k] \subseteq A\}$$

is a coatom in the clone lattice.

Coatoms

$\mathcal{O}$ is the full clone (the set of all finitary operations on X).
A *coatom* or *precomplete clone* or *maximal clone* is a clone $C \neq \mathcal{O}$ such that there is no clone strictly between C and $\mathcal{O}$.

Example

Let $\emptyset \subsetneq A \subsetneq X$. Then the set

$$\mathrm{Pol}(A) := \{f : f[A^k] \subseteq A\}$$

is a coatom in the clone lattice.

Coatoms

$\mathcal{O}$ is the full clone (the set of all finitary operations on X).
A *coatom* or *precomplete clone* or *maximal clone* is a clone $C \neq \mathcal{O}$ such that there is no clone strictly between C and $\mathcal{O}$.

Example

Let $\emptyset \subsetneq A \subsetneq X$. Then the set

$$\mathrm{Pol}(A) := \{f : f[A^k] \subseteq A\}$$

is a coatom in the clone lattice.

Rosenberg's theorem

Let X be finite. Then

- **CLONE**(X) has finitely many coatoms ("precomplete clones").
- All of these coatoms are explicitly known,
- they have the form $\mathrm{Pol}(\leq)$ for some order $\leq$, or ...
- Every clone other than $\mathcal{O}$ is contained in a coatom.

This gives a decision procedure for the question

Is $\langle C \rangle = \mathcal{O}$*?*

Rosenberg's theorem

Let X be finite. Then

- **CLONE**(X) has finitely many coatoms ("precomplete clones").
- All of these coatoms are explicitly known,
- they have the form $\mathrm{Pol}(\leq)$ for some order $\leq$, or ...
- Every clone other than $\mathcal{O}$ is contained in a coatom.

This gives a decision procedure for the question

Is $\langle C \rangle = \mathcal{O}$*?*

Rosenberg’s theorem

Let X be finite. Then

- **CLONE**(X) has finitely many coatoms (“precomplete clones”).
- All of these coatoms are explicitly known,
- they have the form Pol$(\leq)$ for some order $\leq$, or ...
- Every clone other than $\mathcal{O}$ is contained in a coatom.

This gives a decision procedure for the question

Is $\langle C \rangle = \mathcal{O}$*?*

Rosenberg's theorem

Let X be finite. Then

- **CLONE**(X) has finitely many coatoms ("precomplete clones").
- All of these coatoms are explicitly known,
- they have the form Pol$(\leq)$ for some order $\leq$, or . . .
- Every clone other than $\mathcal{O}$ is contained in a coatom.

This gives a decision procedure for the question

Is $\langle C \rangle = \mathcal{O}$*?*

Rosenberg's theorem

Let X be finite. Then

- **CLONE**(X) has finitely many coatoms ("precomplete clones").
- All of these coatoms are explicitly known,
- they have the form Pol$(\leq)$ for some order $\leq$, or . . .
- Every clone other than $\mathcal{O}$ is contained in a coatom.

This gives a decision procedure for the question

Is $\langle C \rangle = \mathcal{O}$*?*

Rosenberg's theorem

Let X be finite. Then

- **CLONE**(X) has finitely many coatoms ("precomplete clones").
- All of these coatoms are explicitly known,
- they have the form Pol$(\leq)$ for some order $\leq$, or ...
- Every clone other than $\mathcal{O}$ is contained in a coatom.

This gives a decision procedure for the question

Is $\langle C \rangle = \mathcal{O}$*?*

Rosenberg's theorem

Let X be finite. Then

- **CLONE**(X) has finitely many coatoms ("precomplete clones").
- All of these coatoms are explicitly known,
- they have the form Pol$(\leq)$ for some order $\leq$, or ...
- Every clone other than $\mathcal{O}$ is contained in a coatom.

This gives a decision procedure for the question

Is $\langle C\rangle = \mathcal{O}$*?*

Question

Is the clone lattice on infinite sets also dually atomic?

- If YES, describe all coatoms. (Hopeless)
- If NO, find some other cofinal set.

Question

Is the clone lattice on infinite sets also dually atomic?

- If YES, describe all coatoms. (Hopeless)
- If NO, find some other cofinal set.

Question

Is the clone lattice on infinite sets also dually atomic?

- If YES, describe all coatoms. (Hopeless)
- If NO, find some other cofinal set.

Question

Is the clone lattice on infinite sets also dually atomic?

- If YES, describe all coatoms. (Hopeless)
- If NO, find some other cofinal set.

Question

Is the clone lattice on infinite sets also dually atomic?

- If YES, describe all coatoms. (Hopeless)
- If NO, find some other cofinal set.

Question

Is the clone lattice on infinite sets also dually atomic?

- If YES, describe all coatoms. (Hopeless)
- If NO, find some other cofinal set.

Question

Is the clone lattice on infinite sets also dually atomic?

- If YES, describe all coatoms. (Hopeless)
- If NO, find some other cofinal set.

Question

Is the clone lattice on infinite sets also dually atomic?

- If YES, describe all coatoms. (Hopeless)
- If NO, find some other cofinal set.

Generators

For any set F of finitary operations on X, let $\langle F\rangle$ be the smallest clone containing F.
Often we fix a "base clone" B, and write $\langle F\rangle_B$ instead of $\langle B\cup F\rangle$

Fact

Let $B \subsetneq \mathcal{O}$ be a clone such that $\mathcal{O}$ is finitely generated over B:

$$\mathcal{O} = \langle f_1, \ldots, f_k\rangle_B$$

Then the interval $[B, \mathcal{O}]$ is dually atomic.

Generators

For any set F of finitary operations on X, let $\langle F\rangle$ be the smallest clone containing F.

Often we fix a “base clone” B, and write $\langle F\rangle_B$ instead of $\langle B\cup F\rangle$

Fact

Let $B \subsetneq \mathcal{O}$ be a clone such that $\mathcal{O}$ is finitely generated over B:

$$\mathcal{O} = \langle f_1,\ldots,f_k\rangle_B$$

Then the interval $[B,\mathcal{O}]$ is dually atomic.

Generators

For any set F of finitary operations on X, let $\langle F \rangle$ be the smallest clone containing F.
Often we fix a “base clone” B, and write $\langle F \rangle_B$ instead of $\langle B \cup F \rangle$

Fact

Let $B \subsetneq \mathcal{O}$ be a clone such that $\mathcal{O}$ is finitely generated over B:

$$\mathcal{O} = \langle f_1, \ldots, f_k \rangle_B$$

Then the interval $[B, \mathcal{O}]$ is dually atomic.

Generators

For any set F of finitary operations on X, let $\langle F \rangle$ be the smallest clone containing F.
Often we fix a "base clone" B, and write $\langle F \rangle_B$ instead of $\langle B \cup F \rangle$

Fact

Let $B \subsetneq \mathcal{O}$ be a clone such that $\mathcal{O}$ is finitely generated over B:

$$\mathcal{O} = \langle f_1, \ldots, f_k \rangle_B$$

Then the interval $[B, \mathcal{O}]$ is dually atomic.

Outline

Consistently: no.

Theorem (G-Shelah 2003)

Assume CH. Then the clone lattice on countable sets is not dually atomic.

(GoSh:808, Transactions of the AMS)

Theorem (G-Shelah 2006)

Let κ be a regular cardinal. Assume $2^\kappa = \kappa^+$. Then the clone lattice on a set of size κ is not dually atomic.

(GoSh:884)

Consistently: no.

Theorem (G-Shelah 2003)

Assume CH. Then the clone lattice on countable sets is not dually atomic.

(GoSh:808, Transactions of the AMS)

Theorem (G-Shelah 2006)

Let κ be a regular cardinal. Assume $2^\kappa = \kappa^+$. Then the clone lattice on a set of size κ is not dually atomic.

(GoSh:884)

Consistently: no.

Theorem (G-Shelah 2003)

Assume CH. Then the clone lattice on countable sets is not dually atomic.

(GoSh:808, Transactions of the AMS)

Theorem (G-Shelah 2006)

Let κ be a regular cardinal. Assume $2^\kappa = \kappa^+$. Then the clone lattice on a set of size κ is not dually atomic.

(GoSh:884)

Consistently: no.

Theorem (G-Shelah 2003)

Assume CH. Then the clone lattice on countable sets is not dually atomic.

(GoSh:808, Transactions of the AMS)

Theorem (G-Shelah 2006)

Let κ be a regular cardinal. Assume $2^\kappa = \kappa^+$. Then the clone lattice on a set of size κ is not dually atomic.

(GoSh:884)

Consistently: no.

Theorem (G-Shelah 2003)

Assume CH. Then the clone lattice on countable sets is not dually atomic.

(GoSh:808, Transactions of the AMS)

Theorem (G-Shelah 2006)

Let κ be a regular cardinal. Assume $2^\kappa = \kappa^+$. Then the clone lattice on a set of size κ is not dually atomic.

(GoSh:884)

Consistently: no.

Theorem (G-Shelah 2003)

Assume CH. Then the clone lattice on countable sets is not dually atomic.

(GoSh:808, Transactions of the AMS)

Theorem (G-Shelah 2006)

Let κ be a regular cardinal. Assume $2^\kappa = \kappa^+$. Then the clone lattice on a set of size κ is not dually atomic.

(GoSh:884)

Consistently: no.

Theorem (G-Shelah 2003)

Assume CH. Then the clone lattice on countable sets is not dually atomic.

(GoSh:808, Transactions of the AMS)

Theorem (G-Shelah 2006)

Let κ be a regular cardinal. Assume $2^\kappa = \kappa^+$. Then the clone lattice on a set of size κ is not dually atomic.

(GoSh:884)

Outline

Growth clones

We fix a linear order on the base set X (or better: identify X with its cardinal number).
Let $C_{\max} := \{f : \forall \vec{x}\, f(\vec{x}) \leq \max(\vec{x})\}$.

Above the clone $C_{\max}$:

- Clones are downward closed.
- Clones are (more or less) determined by their unary functions
- Typical clones are described by *growth* conditions, e.g. the clone of all functions of subexponential growth.
- compact = principal:

$$\langle f_1, \ldots, f_k \rangle_{C_{\max}} = \langle \max(f_1, \ldots, f_k) \rangle_{C_{\max}}$$

Growth clones

We fix a linear order on the base set X (or better: identify X with its cardinal number).

Let $C_{\max} := \{f : \forall \vec{x}\, f(\vec{x}) \leq \max(\vec{x})\}$.

Above the clone $C_{\max}$:

- Clones are downward closed.
- Clones are (more or less) determined by their unary functions
- Typical clones are described by *growth* conditions, e.g. the clone of all functions of subexponential growth.
- compact = principal:

$$\langle f_1, \ldots, f_k \rangle_{C_{\max}} = \langle \max(f_1, \ldots, f_k) \rangle_{C_{\max}}$$

Growth clones

We fix a linear order on the base set X (or better: identify X with its cardinal number).
Let $C_{\max} := \{f : \forall \vec{x}\, f(\vec{x}) \le \max(\vec{x})\}$.

Above the clone $C_{\max}$:

- Clones are downward closed.
- Clones are (more or less) determined by their unary functions
- Typical clones are described by *growth* conditions, e.g. the clone of all functions of subexponential growth.
- compact = principal:

$$\langle f_1, \dots, f_k \rangle_{C_{\max}} = \langle \max(f_1, \dots, f_k) \rangle_{C_{\max}}$$

Growth clones

We fix a linear order on the base set X (or better: identify X with its cardinal number).
Let $C_{\max} := \{f : \forall \vec{x}\, f(\vec{x}) \leq \max(\vec{x})\}$.

Above the clone $C_{\max}$:

- Clones are downward closed.
- Clones are (more or less) determined by their unary functions
- Typical clones are described by *growth* conditions, e.g. the clone of all functions of subexponential growth.
- compact = principal:

$$\langle f_1, \ldots, f_k \rangle_{C_{\max}} = \langle \max(f_1, \ldots, f_k) \rangle_{C_{\max}}$$

Growth clones

We fix a linear order on the base set X (or better: identify X with its cardinal number).
Let $C_{\max} := \{f : \forall \vec{x}\, f(\vec{x}) \leq \max(\vec{x})\}$.

Above the clone $C_{\max}$:

- Clones are downward closed.
- Clones are (more or less) determined by their unary functions
- Typical clones are described by *growth* conditions, e.g. the clone of all functions of subexponential growth.
- compact = principal:

$$\langle f_1, \ldots, f_k \rangle_{C_{\max}} = \langle \max(f_1, \ldots, f_k) \rangle_{C_{\max}}$$

Growth clones

We fix a linear order on the base set X (or better: identify X with its cardinal number).
Let $C_{\max} := \{f : \forall \vec{x}\, f(\vec{x}) \leq \max(\vec{x})\}$.

Above the clone $C_{\max}$:

- Clones are downward closed.
- Clones are (more or less) determined by their unary functions
- Typical clones are described by *growth* conditions, e.g. the clone of all functions of subexponential growth.
- compact = principal:

$$\langle f_1, \dots, f_k \rangle_{C_{\max}} = \langle \max(f_1, \dots, f_k) \rangle_{C_{\max}}$$

Growth clones

We fix a linear order on the base set X (or better: identify X with its cardinal number).
Let $C_{\max} := \{f : \forall \vec{x}\, f(\vec{x}) \leq \max(\vec{x})\}$.

Above the clone $C_{\max}$:

- Clones are downward closed.
- Clones are (more or less) determined by their unary functions
- Typical clones are described by *growth* conditions, e.g. the clone of all functions of subexponential growth.
- compact = principal:

$$\langle f_1, \ldots, f_k \rangle_{C_{\max}} = \langle \max(f_1, \ldots, f_k) \rangle_{C_{\max}}$$

Main idea

Reduce the problem from comparing clones to comparing functions:

Program

Find a "base clone" B such that the relation

$$f \in \langle g \rangle_B$$

is a linear *quasiorder on $\mathcal{O}$.*
Clones above B will then correspond to Dedekind cuts in this order.

Main idea

Reduce the problem from comparing clones to comparing functions:

Program

Find a "base clone" B such that the relation

$$f \in \langle g \rangle_B$$

is a linear *quasiorder on $\mathcal{O}$.*
Clones above B will then correspond to Dedekind cuts in this order.

Main idea

Reduce the problem from comparing clones to comparing functions:

Program

Find a "base clone" B such that the relation

$$f \in \langle g \rangle_B$$

is a linear *quasiorder on* $\mathcal{O}$.

Clones above B will then correspond to Dedekind cuts in this order.

Main idea

Reduce the problem from comparing clones to comparing functions:

Program

Find a "base clone" B such that the relation

$$f \in \langle g \rangle_B$$

is a linear *quasiorder on* $\mathcal{O}$.
Clones above B will then correspond to Dedekind cuts in this order.

Main idea, continued

Slightly weaker:

Program

Find a "base clone" B and a partial order $<$ on $\mathcal{O}$ such that:

- *The relation $<$ is a preference relation, i.e., the relation $x \sim y :\Leftrightarrow x \not< y \wedge y \not< x$ is an equivalence relation*
- $\forall f, g : f < g \Rightarrow f \in \langle g \rangle_B \Rightarrow f \lesssim g$

Clones above B will then be contained in Dedekind cuts of the linear order $\mathcal{O}/\sim$.

IMPORTANT POINT: Make sure that the quotient order has no last element.

Main idea, continued

Slightly weaker:

Program

Find a "base clone" B and a partial order $<$ on $\mathcal{O}$ such that:

- *The relation $<$ is a preference relation, i.e., the relation $x \sim y :\Leftrightarrow x \not< y \wedge y \not< x$ is an equivalence relation*
- $\forall f, g : f < g \Rightarrow f \in \langle g \rangle_B \Rightarrow f \lesssim g$

Clones above B will then be contained in Dedekind cuts of the linear order $\mathcal{O}/\sim$.

IMPORTANT POINT: Make sure that the quotient order has no last element.

Main idea, continued

Slightly weaker:

Program

Find a "base clone" B and a partial order $<$ on $\mathcal{O}$ such that:

- *The relation $<$ is a preference relation, i.e., the relation $x \sim y :\Leftrightarrow x \not< y \wedge y \not< x$ is an equivalence relation*
- $\forall f, g : f < g \Rightarrow f \in \langle g \rangle_B \Rightarrow f \lesssim g$

Clones above B will then be contained in Dedekind cuts of the linear order $\mathcal{O}/\sim$.

IMPORTANT POINT: Make sure that the quotient order has no last element.

Main idea, continued

Slightly weaker:

Program

Find a "base clone" B and a partial order $<$ on $\mathcal{O}$ such that:

- *The relation $<$ is a preference relation, i.e., the relation $x \sim y :\Leftrightarrow x \not< y \wedge y \not< x$ is an equivalence relation*
- $\forall f, g : f < g \Rightarrow f \in \langle g \rangle_B \Rightarrow f \lesssim g$

Clones above B will then be contained in Dedekind cuts of the linear order $\mathcal{O}/\sim$.

IMPORTANT POINT: Make sure that the quotient order has no last element.

Main idea, continued

Slightly weaker:

Program

Find a "base clone" B and a partial order $<$ on $\mathcal{O}$ such that:

- *The relation $<$ is a preference relation,*
  i.e., the relation $x \sim y :\Leftrightarrow x \not< y \wedge y \not< x$ is an equivalence relation
- $\forall f, g : f < g \Rightarrow f \in \langle g \rangle_B \Rightarrow f \lesssim g$

Clones above B will then be contained in Dedekind cuts of the linear order $\mathcal{O}/\sim$.

IMPORTANT POINT: Make sure that the quotient order has no last element.

Main idea, continued

Slightly weaker:

Program

Find a "base clone" B and a partial order $<$ on $\mathcal{O}$ such that:

- *The relation $<$ is a preference relation,*
  i.e., the relation $x \sim y :\Leftrightarrow x \not< y \wedge y \not< x$ is an equivalence relation
- $\forall f, g : f < g \Rightarrow f \in \langle g \rangle_B \Rightarrow f \lesssim g$

Clones above B will then be contained in Dedekind cuts of the linear order $\mathcal{O}/\sim$.

IMPORTANT POINT: Make sure that the quotient order has no last element.

Main idea, continued

Slightly weaker:

Program

Find a "base clone" B and a partial order $<$ on $\mathcal{O}$ such that:

- *The relation $<$ is a preference relation,*
  i.e., the relation $x \sim y :\Leftrightarrow x \not< y \ \wedge y \not< x$ is an equivalence relation
- $\forall f, g : f < g \ \Rightarrow \ f \in \langle g \rangle_B \ \Rightarrow \ f \lesssim g$

Clones above B will then be contained in Dedekind cuts of the linear order $\mathcal{O}/\sim$.

IMPORTANT POINT: Make sure that the quotient order has no last element.

Main idea, continued

Slightly weaker:

Program

Find a "base clone" B and a partial order $<$ on $\mathcal{O}$ such that:

- *The relation $<$ is a preference relation, i.e., the relation $x \sim y :\Leftrightarrow x \not< y \wedge y \not< x$ is an equivalence relation*
- $\forall f, g : f < g \Rightarrow f \in \langle g \rangle_B \Rightarrow f \lesssim g$

Clones above B will then be contained in Dedekind cuts of the linear order $\mathcal{O}/\sim$.

IMPORTANT POINT: Make sure that the quotient order has no last element.

Random facts from the proof

Fact

Let $(L, \leq)$ be a well-order without last element.
For each unbounded set $A \subseteq L$ define

$$h_A(x) := \min\{y \in L : x < y\}$$

Let U be a filter of unbounded set on L.
Then $f \in \langle h_A : A \in U\}\rangle_{\max}$ iff there is some $A \in U$ and some $k \in \{1, 2, \ldots\}$ such that $f(\vec{x}) \leq h_A^{(k)}(\max(\vec{x}))$ for all $\vec{x}$.

(Most of the clones that we construct will be such "filter clones" $\langle h_A : A \in U\}\rangle_{\max}$.)

Random facts from the proof

Fact

Let $(L, \leq)$ be a well-order without last element.
For each unbounded set $A \subseteq L$ define

$$h_A(x) := \min\{y \in L : x < y\}$$

Let U be a filter of unbounded set on L.
Then $f \in \langle h_A : A \in U\}\rangle_{\max}$ iff there is some $A \in U$ and some $k \in \{1, 2, \ldots\}$ such that $f(\vec{x}) \leq h_A^{(k)}(\max(\vec{x}))$ for all $\vec{x}$.

(Most of the clones that we construct will be such "filter clones" $\langle h_A : A \in U\}\rangle_{\max}$.)

Random facts from the proof

Fact

Let $(L, \leq)$ *be a well-order without last element.*
For each unbounded set $A \subseteq L$ *define*

$$h_A(x) := \min\{y \in L : x < y\}$$

Let U *be a filter of unbounded set on* L.
Then $f \in \langle h_A : A \in U\}\rangle_{\max}$ *iff there is some* $A \in U$ *and some* $k \in \{1, 2, \ldots\}$ *such that* $f(\vec{x}) \leq h_A^{(k)}(\max(\vec{x}))$ *for all* $\vec{x}$.

(Most of the clones that we construct will be such "filter clones" $\langle h_A : A \in U\}\rangle_{\max}$.)

Random facts from the proof

Fact

Let $(L, \leq)$ be a well-order without last element.
For each unbounded set $A \subseteq L$ define

$$h_A(x) := \min\{y \in L : x < y\}$$

Let U be a filter of unbounded set on L.
Then $f \in \langle h_A : A \in U\}\rangle_{\max}$ iff there is some $A \in U$ and some $k \in \{1, 2, \ldots\}$ such that $f(\vec{x}) \leq h_A^{(k)}(\max(\vec{x}))$ for all $\vec{x}$.

(Most of the clones that we construct will be such "filter clones" $\langle h_A : A \in U\}\rangle_{\max}$.)

Random facts from the proof

Fact

Let $(L, \leq)$ *be a well-order without last element.*
For each unbounded set $A \subseteq L$ *define*

$$h_A(x) := \min\{y \in L : x < y\}$$

Let U *be a filter of unbounded set on* L*.*
Then $f \in \langle h_A : A \in U\}\rangle_{\max}$ *iff there is some* $A \in U$ *and some* $k \in \{1, 2, \ldots\}$ *such that* $f(\vec{x}) \leq h_A^{(k)}(\max(\vec{x}))$ *for all* $\vec{x}$*.*

(Most of the clones that we construct will be such "filter clones" $\langle h_A : A \in U\}\rangle_{\max}$.)

Random facts from the proof

Fact

Let $(L, \leq)$ be a well-order without last element.
For each unbounded set $A \subseteq L$ define

$$h_A(x) := \min\{y \in L : x < y\}$$

Let U be a filter of unbounded set on L.
Then $f \in \langle h_A : A \in U\}\rangle_{\max}$ iff there is some $A \in U$ and some $k \in \{1, 2, \ldots\}$ such that $f(\vec{x}) \leq h_A^{(k)}(\max(\vec{x}))$ for all $\vec{x}$.

(Most of the clones that we construct will be such "filter clones" $\langle h_A : A \in U\}\rangle_{\max}$.)

Random facts from the proof

Fact

Let $(L, \leq)$ *be a well-order without last element. For each unbounded set* $A \subseteq L$ *define*

$$h_A(x) := \min\{y \in L : x < y\}$$

Let U *be a filter of unbounded set on* L*. Then* $f \in \langle h_A : A \in U\}\rangle_{\max}$ *iff there is some* $A \in U$ *and some* $k \in \{1, 2, \ldots\}$ *such that* $f(\vec{x}) \leq h_A^{(k)}(\max(\vec{x}))$ *for all* $\vec{x}$*.*

(Most of the clones that we construct will be such “filter clones” $\langle h_A : A \in U\}\rangle_{\max}$.)

More random facts from the proof

Fact

Let $(U_i : i \in I)$ be ultrafilters on disjoint sets $(A_i : i \in I)$. Let V be an ultrafilter on I, $A := \bigcup_i A_i$. Then

$$\int U_i \, d_V i := \{B \subseteq A : \{i : B \cap A_i \in U_i\} \in V\}$$

is an ultrafilter on A.

This fact is a special case (trees with exactly 3 levels) of the folowing more general (but still very easy) fact:

Fact

Let $(T, \leq)$ be a tree with a least element. Assume that T does not contain any infinite chains.

For each internal node $\eta \in T$ let D_η be an ultrafilter on $\mathrm{Succ}_T(\eta)$. Then this family naturally induces an ultrafilter on the set of external nodes (or: branches) of T.

More random facts from the proof

Fact

Let $(U_i : i \in I)$ be ultrafilters on disjoint sets $(A_i : i \in I)$. Let V be an ultrafilter on I, $A := \bigcup_i A_i$. Then

$$\int U_i \, d_V i := \{B \subseteq A : \{i : B \cap A_i \in U_i\} \in V\}$$

is an ultrafilter on A.

This fact is a special case (trees with exactly 3 levels) of the folowing more general (but still very easy) fact:

Fact

Let $(T, \leq)$ be a tree with a least element. Assume that T does not contain any infinite chains.
For each internal node $\eta \in T$ let D_η be an ultrafilter on $\mathrm{Succ}_T(\eta)$.
Then this family naturally induces an ultrafilter on the set of external nodes (or: branches) of T.

More random facts from the proof

Fact

Let $(U_i : i \in I)$ be ultrafilters on disjoint sets $(A_i : i \in I)$. Let V be an ultrafilter on I, $A := \bigcup_i A_i$. Then

$$\int U_i \, d_V i := \{B \subseteq A : \{i : B \cap A_i \in U_i\} \in V\}$$

is an ultrafilter on A.

This fact is a special case (trees with exactly 3 levels) of the folowing more general (but still very easy) fact:

Fact

Let $(T, \leq)$ be a tree with a least element. Assume that T does not contain any infinite chains.
For each internal node $\eta \in T$ let D_η be an ultrafilter on $\mathrm{Succ}_T(\eta)$.
Then this family naturally induces an ultrafilter on the set of external nodes (or: branches) of T.

More random facts from the proof

Fact

Let $(U_i : i \in I)$ be ultrafilters on disjoint sets $(A_i : i \in I)$. Let V be an ultrafilter on I, $A := \bigcup_i A_i$. Then

$$\int U_i \, d_V i := \{B \subseteq A : \{i : B \cap A_i \in U_i\} \in V\}$$

is an ultrafilter on A.

This fact is a special case (trees with exactly 3 levels) of the folowing more general (but still very easy) fact:

Fact

Let $(T, \leq)$ be a tree with a least element. Assume that T does not contain any infinite chains.
For each internal node $\eta \in T$ let D_η be an ultrafilter on $\mathrm{Succ}_T(\eta)$.
Then this family naturally induces an ultrafilter on the set of external nodes (or: branches) of T.

More random facts from the proof

Fact

Let $(U_i : i \in I)$ be ultrafilters on disjoint sets $(A_i : i \in I)$. Let V be an ultrafilter on I, $A := \bigcup_i A_i$. Then

$$\int U_i \, d_V i := \{B \subseteq A : \{i : B \cap A_i \in U_i\} \in V\}$$

is an ultrafilter on A.

This fact is a special case (trees with exactly 3 levels) of the folowing more general (but still very easy) fact:

Fact

Let $(T, \leq)$ be a tree with a least element. Assume that T does not contain any infinite chains.

For each internal node $\eta \in T$ let D_η be an ultrafilter on $\mathrm{Succ}_T(\eta)$.

Then this family naturally induces an ultrafilter on the set of external nodes (or: branches) of T.

More random facts from the proof

Fact

Let $(U_i : i \in I)$ be ultrafilters on disjoint sets $(A_i : i \in I)$. Let V be an ultrafilter on I, $A := \bigcup_i A_i$. Then

$$\int U_i \, d_V i := \{B \subseteq A : \{i : B \cap A_i \in U_i\} \in V\}$$

is an ultrafilter on A.

This fact is a special case (trees with exactly 3 levels) of the folowing more general (but still very easy) fact:

Fact

Let $(T, \leq)$ be a tree with a least element. Assume that T does not contain any infinite chains.

For each internal node $\eta \in T$ let D_η be an ultrafilter on $\mathrm{Succ}_T(\eta)$.

Then this family naturally induces an ultrafilter on the set of external nodes (or: branches) of T.

More random facts from the proof

Fact

Let $(U_i : i \in I)$ be ultrafilters on disjoint sets $(A_i : i \in I)$. Let V be an ultrafilter on I, $A := \bigcup_i A_i$. Then

$$\int U_i \, d_V i := \{B \subseteq A : \{i : B \cap A_i \in U_i\} \in V\}$$

is an ultrafilter on A.

This fact is a special case (trees with exactly 3 levels) of the folowing more general (but still very easy) fact:

Fact

Let $(T, \leq)$ be a tree with a least element. Assume that T does not contain any infinite chains.

For each internal node $\eta \in T$ let D_η be an ultrafilter on $\mathrm{Succ}_T(\eta)$. Then this family naturally induces an ultrafilter on the set of external nodes (or: branches) of T.

More random facts from the proof

Fact

Let $(U_i : i \in I)$ be ultrafilters on disjoint sets $(A_i : i \in I)$. Let V be an ultrafilter on I, $A := \bigcup_i A_i$. Then

$$\int U_i \, d_V i := \{B \subseteq A : \{i : B \cap A_i \in U_i\} \in V\}$$

is an ultrafilter on A.

This fact is a special case (trees with exactly 3 levels) of the folowing more general (but still very easy) fact:

Fact

Let $(T, \leq)$ be a tree with a least element. Assume that T does not contain any infinite chains.

For each internal node $\eta \in T$ let D_η be an ultrafilter on $\mathrm{Succ}_T(\eta)$.

Then this family naturally induces an ultrafilter on the set of external nodes (or: branches) of T.

More random facts from the proof

Fact

Let $(U_i : i \in I)$ be ultrafilters on disjoint sets $(A_i : i \in I)$. Let V be an ultrafilter on I, $A := \bigcup_i A_i$. Then

$$\int U_i \, d_V i := \{B \subseteq A : \{i : B \cap A_i \in U_i\} \in V\}$$

is an ultrafilter on A.

This fact is a special case (trees with exactly 3 levels) of the folowing more general (but still very easy) fact:

Fact

Let $(T, \leq)$ be a tree with a least element. Assume that T does not contain any infinite chains.
For each internal node $\eta \in T$ let D_η be an ultrafilter on $\mathrm{Succ}_T(\eta)$.
Then this family naturally induces an ultrafilter on the set of external nodes (or: branches) of T.

More random facts from the proof

Fact

Let $(U_i : i \in I)$ be ultrafilters on disjoint sets $(A_i : i \in I)$. Let V be an ultrafilter on I, $A := \bigcup_i A_i$. Then

$$\int U_i \, d_V i := \{B \subseteq A : \{i : B \cap A_i \in U_i\} \in V\}$$

is an ultrafilter on A.

This fact is a special case (trees with exactly 3 levels) of the folowing more general (but still very easy) fact:

Fact

Let $(T, \leq)$ be a tree with a least element. Assume that T does not contain any infinite chains.

For each internal node $\eta \in T$ let D_η be an ultrafilter on $\mathrm{Succ}_T(\eta)$.

Then this family naturally induces an ultrafilter on the set of external nodes (or: branches) of T.