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TOOLS FOR YOUR FORCING CONSTRUCTION

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Abstract. A preservation theorem is a theorem of the form: "If $\langle P_\alpha, Q_\alpha : \alpha < \delta\rangle$ is an iteration of forcing notions, and every Q_α satisfies φ in V^{P_α}, then P_δ satisfies φ."
We give a simplified version of a general preservation theorem for countable support iteration due to Shelah. This version is particularly useful for problems dealing with sets of reals. We give several examples of applications, among them "countable support iteration of proper ω^ω-bounding forcing notions is ω^ω-bounding." We also review the basic facts about countable support iteration and proper forcing, as well as Souslin proper forcing notions.

0 Introduction

The main objective of this paper is to present a simplified version of Shelah's preservation theorems. We hope that the reader will find here powerful tool for his/her forcing constructions, as well as good open problems in this abstract area of mathematical research: The study of (iterated) forcing.

Iterated forcing is a powerful tool for proving independence results. In an iterated forcing argument the "ground model" V_0 is first extended to a model V_1 using some forcing notion Q_0. Then V_1 is extended to some universe V_2, using some forcing notion Q_1 in V_1, etc. After ω many steps, a model V_ω containing $\bigcup_n V_n$ is constructed, and the iteration can continue—usually up to ω_1, ω_2 or some large cardinal.

In an iterated forcing argument there two main points we have to take care of:

(1) In each extension by a single forcing notion Q_i we have to add certain generic objects that we want to have present in the final model.

(2) At no stage are we allowed to add certain objects that we do not want to have in the final model.

The argument given to deal with (1) usually depends heavily on the special properties of the forcing notions Q_i. Similarly, dealing with (2) at a successor step $i+1$ is done by arguments that are characteristic for the forcing notion Q_i.

A "preservation theorem" is a theorem that deals with problem (2) at limit stages, i.e. a theorem ensuring that no "unwanted" objects are introduced at limit stages whenever all the forcing notions Q_i that are used satisfy certain "niceness" conditions.

For example, the problem of preserving certain cardinals and cofinalities is an instance of (2). The earliest preservation theorem we are aware of is the following ([from 13])

> The finite support limit of an iteration in which all iterands satisfy the countable chain condition also satisfies the countable chain condition (and thus preserves cofinalities and cardinals)

The corresponding theorem for countable support iteration is the following ([10]):

> The countable support limit of an iteration in which all iterands are proper is itself proper (and thus does not collapse $\aleph_1$). Also, starting from CH, if the length of the iteration is $\leq \aleph_2$ and all iterands have size $\leq \aleph_1$, then the limit satisfies the $\aleph_2$-cc, so all cardinals and cofinalities are preserved.

The following example shows that the question of what reals are introduced in limit stages of an iteration is usually nontrivial:

Example 1 0.1: *Assume that V_0 satisfies CH, Q_0 adds no new reals to V_0, Q_1 adds $\aleph_1^{V_0}$ many new reals to V_1, Q_2 adds $\aleph_2^{V_0}$ many new reals to V_2, etc. Let $P_n := Q_0 * Q_1 * \cdots * Q_{n-1}$. So V_n is the generic extension of V_0 by P_n.*

Then in V_ω we will have at least $\aleph_\omega$ many new reals. But as the cofinality of the continuum must be uncountable, this means that in V_ω will have at least $\aleph_{\omega+1}^{V_0}$ many new reals. Since only $\aleph_\omega$ many of them appear in intermediate stages, "most" of these reals appear only in the limit stage. This shows that it is not trivial to keep control over which reals are added in a limit stages by controlling what reals are added in intermediate stages.

Example 2 0.2: *Consider an iteration $\langle P_n, Q_n : n < \omega\rangle$, where for all n, $\Vdash_{P_n}$ "Q_n is nontrivial, i.e., contains incompatible conditions."*

If we define V_ω as V^{P_ω}, where P_ω is the finite support limit of the iteration sequence $\langle P_n, Q_n : n < \omega\rangle$, then (no matter what the forcing notions Q_n are), P_ω will add Cohen reals over the ground model. (Let $\Vdash_n$ "$q_n^0 \perp_{Q_n} q_n^1$," then the function $f : \omega \to 2$, defined by $f(n) = 0 \Leftrightarrow q_n^0 \in G(n)$ will be a Cohen real, where $G(n)$ is the generic filter on Q_n.)

This example shows an **inherent limitation of the method of finite support iteration**: In limit stages of cofinality ω, Cohen reals are always added.

The general problem of whether Cohen reals can be added in a limit stage of a **countable support iteration** is **open**:

Assume P_ω is the countable support iteration of $\langle P_n, Q_n : n < \omega\rangle$, where for all n we have:

$$\Vdash_{P_n} \text{"There are no Cohen reals over } V\text{"}$$

Does this imply $\Vdash_{P_\omega}$ "There are no Cohen reals over V"?

At this moment it seems impossible to apply the preservation theorem we prove below to this problem directly. There are, however, stronger properties (of forcing notions) than "not adding Cohen reals", that can be shown to be preserved under countable support iteration. The best known of them are "${}^\omega\omega$-bounding" and "Laver property." (see 6.3 and 6.25, below)

Remark 0.3: *The same proof that shows that finite support iteration always adds Cohen reals in stage ω also shows an* **inherent limitation of countable support iteration***: In stage ω_1 (or indeed at any stage of cofinality ω_1) we add an "ω_1-Cohen set," i.e., a generic filter for the forcing notion $Fn(\omega_1, 2, \omega_1)$ of all countable partial functions from ω_1 to 2. At first generic filters for this forcing notion seem innocent for problems concerning sets of reals, as $Fn(\omega_1, 2, \omega_1)$ does not add any reals. However, a simple density argument shows that the continuum of the ground model is collapsed to $\aleph_1$. This makes it impossible to have $\mathfrak{c} > \aleph_2$ at any limit stage of uncountable cofinality during a countable support iteration.*

For a specific example of where a preservation theorem can be used, consider the problem of proving

$$\mathrm{Con}(\mathbf{Cov}(\mathcal{N}) + \mathfrak{b} = \mathfrak{c} + \neg\mathbf{Unif}(\mathcal{N}))$$

using iterated forcing.

Explanation:

$\mathbf{Cov}(\mathcal{N})$ means: the real line cannot be covered by less than $\mathfrak{c}$ (=continuum) many sets from $\mathcal{N}$, the ideal of null (=measure zero) sets.

$\mathfrak{b} = \mathfrak{c}$ means: Whenever $F \subseteq {}^\omega\omega$ is a family of less than $\mathfrak{c}$ many functions, there exists a function g bounding every element of F, where "g bounds f" means $\exists k\, \forall n \geq k\; g(n) > f(n)$.

$\neg\mathbf{Unif}(\mathcal{N})$ means that there is a set of reals of cardinality $< \mathfrak{c}$ that does not have measure zero.

There are two approaches to get a model satisfying the condition above:

(1) Start with a model where (some big fragment of) Martin's axiom holds and $\mathfrak{c}$ is big. Then construct a short iteration of length $\kappa < \mathfrak{c}$. In each iteration stage $i < \kappa$, add a real r_i in such a way that at the end the set $\{r_i : i < \kappa\}$ is nonmeasurable. Preserve enough of Martin's axiom to ensure that in the final model, $\mathbf{Cov}(\mathcal{N}) + \mathfrak{b} = \mathfrak{c}$ holds.

(2) Start with the constructible universe L (or some model satisfying enough of GCH) and construct a long iteration of length $\kappa > \aleph_1$. In each stage $i < \kappa$, add a function f_i and a real r_i such that f_i is not bounded by any function constructed so far, and r_i is not in any measure zero set constructed so far. Assuming that no new reals appear in stage κ, in the final model we will have $\mathfrak{c} = \kappa$, the set $\{r_i : i < \kappa\}$ will be a set of reals that cannot be covered by $< \kappa$ many measure zero sets, and $\{f_i : i < \kappa\}$ will witness $\mathfrak{b} = \mathfrak{c}$. We have to take care not to cover the set of constructible reals by a measure zero set. This will ensure that there is a non-null set of size $\aleph_1 < \mathfrak{c}$.

At the end of section 6, we will consider approach (2). Since a Cohen real makes the set of all old reals a measure zero set, we should not add Cohen reals during the iteration. So we cannot use finite support iteration.

r_i will be a random real. It is easy to see that any set that is not of measure zero in V will also not be of measure zero after adding a random real.

f_i will be a Laver real. [7] showed that Laver reals also do not make non null sets into null sets.

This two observations show that if in stage i, the set of constructible reals was not null, the also in stage $i+1$ this property will hold. So it remains to show that in limit stages the constructible reals are not covered by a measure zero set. For this purpose we will use the iteration theorem proved below (5.14), with parameters given in application 2 (6.8, 6.12)

Notation for forcing 0.4: A forcing notion $P = \langle P, \leq_P, \mathbb{1}_P \rangle$ is a set P equipped with a transitive reflexive relation $\leq_P$ and a greatest element $\mathbb{1}_P$. $\Vdash_P$ denotes the forcing relation of P.

We interpret $p \geq q$ (or $p \geq_P q$) as "q extends p," "q is stronger than p," or "q has more information then p." p is "incompatible" with q ($p \perp q$ or $p \perp_P q$) means $\neg \exists r \in P : p \geq r \,\&\, q \geq r$.

Note that $p \perp q \Leftrightarrow p \Vdash q \notin G_P$.

A set $D \subseteq P$ is called dense if $\forall p \in P \, \exists q \in D : q \leq p$. D is open iff $\forall q \in D \, \forall r \leq q : r \in D$. D is an antichain if $\forall p, q \in D : p \neq q \Rightarrow p \perp q$. D is predense if $\forall p \in P \, \exists d \in D \; d \not\perp p$. D is predense below p iff $\forall q \leq p \, \exists d \in D \; d \not\perp p$.

D is a "filter" if any two elements of D are compatible with a witness in D, i.e. $\forall p, q \in D \, \exists t \in D \; p \geq r \,\&\, q \geq r$.

If V is a model of set theory, $G \subseteq P$ is called generic (over V) (for P), if $G \cap D \neq 0$ for all dense $D \in V$. We often omit mentioning P and/or V, if the context makes it clear what P and V should be.

The class of P-names is defined by $\in$-recursion: $\underset{\sim}{x}$ is a P-name iff every element of $\underset{\sim}{x}$ is a pair of the form $\langle \underset{\sim}{y}, p \rangle$ where $\underset{\sim}{y}$ is a P-name and $p \in P$.

To avoid having to work with a proper class, we introduce an equivalence relation $\equiv$ on P-names by $\underset{\sim}{x} \equiv \underset{\sim}{y}$ iff $\mathbb{1}_P \Vdash_P \underset{\sim}{x} = \underset{\sim}{y}$, and choose a set of representatives from each class, say, those of least possible rank.

So for example, the **set** of P-names for functions from ω to ω is the set of all representatives $\underset{\sim}{x}$ such that $\mathbb{1} \Vdash \underset{\sim}{x} : \omega \to \omega$. (It can easily be seen that this is in fact a set.)

Usually we write (variables for) P-names with a tilde, as in $\underset{\sim}{x}$. For a P-name $\underset{\sim}{x}$ and a generic $G \subseteq P$, we let $\underset{\sim}{x}[G]$ be the evaluation of $\underset{\sim}{x}$ by G:

$$\underset{\sim}{x}[G] := \{\underset{\sim}{y}[G] : \langle \underset{\sim}{y}, p\rangle \in \underset{\sim}{x}, p \in G\}$$

But if M is a model of a large fragment of ZFC, then we let $M[G] := \{\underset{\sim}{x}[G] : M \models \underset{\sim}{x} \text{ is a } P\text{-name}\}$. Sometimes (if G is clear from the context) we use for the evaluation of a name by G the same variable as the one used for the name, but leaving out the tilde, i.e., if G is given, x abbreviates $\underset{\sim}{x}[G]$, p abbreviates $\underset{\sim}{p}[G]$, etc. Similarly p^1_n is $\underset{\sim}{p}^1_n[G]$, etc.)

For an element x of V, $\widehat{x}$ or $(x)\widehat{\ }$ is the standard P-name for x, $\widehat{x} := \{\langle \widehat{y}, \mathbb{1}_P\rangle : y \in x\}$ but we usually write x for $\widehat{x}$.

For a forcing P, G_P is (depending on the context) either $= \{\langle \widehat{p}, p\rangle : p \in P\}$, the canonical name for the generic object (also called the generic filter) added by P, or a variable ranging over all V-generic filters $G \subseteq P$.

We will freely use the following "existential completeness lemma", sometimes without explicitly mentioning it:

Lemma 0.5: *For any forcing P, any formula $\varphi(x)$ there is a name $\underset{\sim}{\tau}$ such that*

$$\mathbb{1} \Vdash \exists x\, \varphi(x) \;\Leftrightarrow\; \varphi(\underset{\sim}{\tau})$$

(See [8] for a proof.)

We will also need the following well-known definitions and facts about forcing:

Definition 0.6: *For p, q elements of a forcing notion P, we define $p \geq^* q$ iff for all $r \leq q$, r is compatible with p.*

Fact 0.7: *$p \geq^* q$ iff $q \Vdash_P p \in G_P$. In particular, $p \geq q$ implies $p \geq^* q$.*

Most properties of $\leq$ are also shared by $\leq^*$. E.g., $p \perp q$ iff there is no condition r satisfying $p \geq^* r$ and $q \geq^* r$. Also, if $p \Vdash \varphi$, and $p \geq^* q$, then $q \Vdash \varphi$.

Definition 0.8: *If P, Q are forcing notions, we say that $i : P \to Q$ is a "dense* embedding", iff*

(1) $\forall p_1, p_2 \in P$: $p_1 \leq^*_P p_2$ iff $i(p_1) \leq^*_Q i(p_2)$. ("i is an embedding with respect to $\leq^*$")

(2) $\forall q \in Q\, \exists p \in P : q \geq^* i(p)$. ("$i$ is dense")*

Note that i is not necessarily 1-1. If (1) and (2) hold with $\leq$ instead of $\leq^*$, then we say that i is a dense embedding.

Definition 0.9: *If there exists a dense* embedding $i : P \to Q$, we say that P and Q are equivalent, and we write $P \approx Q$.*

This is justified by the following fact:

Fact 0.10: *Assume $i : P \to Q$ is a dense* embedding. Then whenever $G \subseteq P$ is generic, then the set H defined by*

$$H := \{q \in Q : \exists p \in G, i(p) \leq q\}$$

is generic for Q. Conversely, if a set $H \subseteq Q$ is generic, then the set G, defined by

$$G := \{p \in P : i(p) \in H\}$$

is generic for P.

Moreover, in both cases $V[G] = V[H]$, and there is a canonical translation function that maps P-names $\underset{\sim}{x}$ to corresponding Q-names $\underset{\sim}{x}'$, and conversely. We will often identify names with their image under this translation function.

We have that $p \Vdash_P \varphi(\underset{\sim}{x})$ iff $i(p) \Vdash_Q \varphi(\underset{\sim}{x}')$.

The proof is a routine computation.

Finally, we define the concept of an "interpretation":

Definition 0.11: *Assume Q is a forcing notion, $\underset{\sim}{f}$, is a Q-name of a functions in ${}^\omega\omega$, f^* is a function in ${}^\omega\omega$, $\langle p_n : n < \omega \rangle$ an increasing sequence of conditions.*

We say that $\langle p_n : n < \omega \rangle$ interprets $\underset{\sim}{f}$ as f^*, if for all n, $p_n \Vdash \underset{\sim}{f} \restriction n = f^* \restriction n$.

We say that f^* is an interpretation of $\underset{\sim}{f}$ if there exists an increasing sequence as above.

The main theorem (5.14) of this paper is a simplified version of a theorem of Shelah ([11, XVIII]). See also [10, V, VI], [12] and [7] for precursors. The proof presented here is a joint work of Judah and the author.

Theorem 8.4 has been proved for various instances by several people.

Souslin forcing and Souslin Proper forcing were introduced in [5].

Contents of this paper:

In section 1, we give a review of composition and iteration of forcing notions in a general context.

In section 2 we introduce finite support iteration and show that the countable chain condition is preserved in finite support limits.

In section 3 we explain the concept of properness, and give a simple proof of Shelah's theorem "properness is preserved under countable support iteration." This proof will serve as a basis for the proof of the preservation theorem in section 6.

In section 4 we continue the review of countable and finite support iteration, by considering the relationship between an intermediate model and the final model.

In section 5 we formulate and prove a general preservation for countable support iteration.

In section 6 we show how to apply the preservation theorem for countable support iteration by giving several examples of properties that can be preserved in this framework.

In section 7 we review the concept of "Souslin Proper forcing", and we prove the corresponding preservation theorem.

In section 8 we formulate and prove a general preservation theorem for finite support iteration, and we give examples.

Those readers who are interested only in finite support iteration can skip section 3 and all references to countable support iteration in section 4 without loss of continuity. A dual remark applies to readers only interested in countable support iteration.

We conclude this introduction by mentioning some open problems concerning countable support iteration and proper forcing:

Problem (Judah) 0.12: *Is "MA(Axiom A) + projective measurability" equiconsistent with the existence of 2 weakly compact cardinals?*

From [1] it essentially follows that MA(Axiom A) is equiconsistent with one weakly compact cardinal.

Problem (Judah) 0.13: *Find a sequence $\langle \sqsubset_n : n \in \omega \rangle$, such that "$Q$ preserves $\sqsubset$" iff Q does not add random reals.*

The most outstanding problem concerning countable support iteration is the following:

Problem (Judah-Shelah) 0.14: *If $\langle P_n, Q_n : n < \omega \rangle$ is a countable support iteration with countable support limit P_ω. Does*

$$\forall n : \ P_n \textit{ does not add Cohen reals over } V$$

imply that P_ω does not add Cohen reals over V?

(For a partial solution see [6].)

I want to thank Saharon Shelah for fruitful discussions about iterated forcing.

1 Composition and iteration of forcing

We briefly review a few facts about composition of forcing and iterated forcing.

If P is a forcing notion, and $\underset{\sim}{Q}$ is a P-name of a forcing notion, then we can force in V^P with $\underset{\sim}{Q}$ to obtain a new extension $(V^P)^Q$. There is a single forcing notion $P * \underset{\sim}{Q}$ (the "composition" of P and $\underset{\sim}{Q}$) such that the extension $V^{P*\underset{\sim}{Q}}$ is canonically isomorphic to $(V^P)^{\underset{\sim}{Q}}$.

Definition 1.1: *Assume that P is a forcing notion, and $\underset{\sim}{Q}$ is a P-name for a forcing notion. Then we let $P * \underset{\sim}{Q}$ be the set of all pairs $\langle p, \underset{\sim}{q}\rangle$ such that $p \in P$ and $p \Vdash_Q$ "$\underset{\sim}{q} \in \underset{\sim}{Q}$".*

We let $\langle p, q\rangle \geq_{P*\underset{\sim}{Q}} \langle p', q'\rangle$ iff $p \geq p'$ and $p' \Vdash_P$ "$q \geq_{\underset{\sim}{Q}} q'$".

Remark 1.2: *As defined, $P * \underset{\sim}{Q}$ is a proper class. However, as in [4], we choose for each class of equivalent P-names a representative. Then the official definition of $P * \underset{\sim}{Q}$ is: The set of all pairs $\langle p, \underset{\sim}{q}\rangle$, where $p \in P$, $\underset{\sim}{q}$ a representative P-name, and $p \Vdash \underset{\sim}{q} \in \underset{\sim}{Q}$. Then $P * \underset{\sim}{Q}$ is a set.*

A similar remark will apply to iteration of forcing.

([8] chooses a different way to avoid proper classes. However, this solutions causes anomalies that we want to avoid. See [8, Exercise VIII E2–E4] and [9].)

Fact 1.3: *Assume that P, $\underset{\sim}{Q}$ are as above. Then (see [8])*

(1) If $G \subseteq P$ is generic over V, $H \subseteq \underset{\sim}{Q}[G]$ generic over $V[G]$, then

$$G * H := \{\langle p, \underset{\sim}{q}\rangle \in P * \underset{\sim}{Q} : p \in G, \underset{\sim}{q}[G] \in H\}$$

is generic for $P * \underset{\sim}{Q}$ over V,and $V[G * H] = V[G][H]$.

(2) Conversely, if $J \subseteq P * \underset{\sim}{Q}$ is generic over V, then $G := \{p : \exists \underset{\sim}{q}\ \langle p, \underset{\sim}{q}\rangle \in J\}$ is generic for P over V, $H := \{\underset{\sim}{q}[G] : \exists p \in G, \langle p, \underset{\sim}{q}\rangle \in J\}$ is generic for $\underset{\sim}{Q}$ over $V[G]$, and $J = G * H$.

(3) Moreover, $P * \underset{\sim}{Q}$-names can be translated to P-names for Q-names, i.e., for every $P * \underset{\sim}{Q}$-name $\underset{\sim}{x}$ there is $\underset{\sim}{x}'$, a P-name for a $\underset{\sim}{Q}$-name such that whenever G, H are as above, then

$$\underset{\sim}{x}[G * H] = (\underset{\sim}{x}'[G])[H]$$

Conversely, if $\underset{\sim}{x}'$ is P-name for a Q-name, we can find a corresponding $P * \underset{\sim}{Q}$-name $\underset{\sim}{x}$.

We often identify $\underset{\sim}{x}$ and $\underset{\sim}{x}'$.

Definition 1.4: *Let κ be ordinal, and let I be an ideal on κ containing all finite sets. (I is not necessarily a proper ideal.) By induction on $\varepsilon < \kappa$ we will define what an I-supported iteration of length ε is, and what the I-supported limit of such an iteration is.*

"finite support" means that we let I be the ideal of finite subsets of κ, and "countable support" means that we let I be the (possibly improper) ideal of all countable subsets of κ. In this context, "countable" is understood to include "finite".

(1) $\bar{Q}^\varepsilon := \langle P_\alpha, \underset{\sim}{Q}_\alpha : \alpha < \varepsilon \rangle$ is an I-supported iteration iff for all $\alpha < \varepsilon$,

$P_\alpha = \lim_I \langle P_\beta, \underset{\sim}{Q}_\beta : \beta < \alpha \rangle$, and

$\Vdash_{P_\alpha}$ "$Q_\alpha = \langle \underset{\sim}{Q}_\alpha, \leq_{\underset{\sim}{Q}_\alpha}, \mathbb{1}_{\underset{\sim}{Q}_\alpha} \rangle$ is a forcing notion".

(2)

(a) The underlying set of $\lim_I \bar{Q}^\varepsilon$ is the set of all partial functions p with $\mathrm{dom}(p) \in I$ satisfying

$$\forall \beta \in \mathrm{dom}(p) : p{\restriction}\beta \Vdash_\beta p(\beta) \in \underset{\sim}{Q}_\beta$$

(b) For $p, q \in \lim_I \langle P_\alpha, \underset{\sim}{Q}_\alpha : \alpha < \varepsilon \rangle$ we define $p \geq_{\lim_I \bar{Q}^\varepsilon} q$ iff

$$\forall \beta \in \mathrm{dom}(p) \cup \mathrm{dom}(q) : \quad q{\restriction}\beta \Vdash p(\beta) \geq_{\underset{\sim}{Q}_\beta} q(\beta)$$

(where we agree to let $p(\beta) = \mathbb{1}_{\underset{\sim}{Q}_\beta}$ for $\beta \notin \mathrm{dom}(p)$.)

(c) $\mathbb{1}_{\lim_I \bar{Q}^\varepsilon} = \emptyset$.

Whenever we consider a I-supported iteration $\langle P_\alpha, \underset{\sim}{Q}_\alpha : \alpha < \varepsilon \rangle$, we automatically define P_ε to be the I-supported limit of this iteration. This allows us to avoid the more awkward notation $\langle P_\alpha, \underset{\sim}{Q}_\beta : \alpha \leq \varepsilon, \beta < \varepsilon \rangle$.

Example 1.5: *$P_0 = \{\emptyset\}$. $\underset{\sim}{Q}_0$ is a P_0-name for a forcing notion. Since P_0 is the trivial forcing notion, $G_0 := P_0$ is a "generic filter", and $V = V[G_0]$. We identify $\underset{\sim}{Q}_0$ with $Q_0 := \underset{\sim}{Q}_0[G_0]$.*

P_1 is the set of all partial functions from $\{\emptyset\}$ to Q_0, i.e., P_1 is isomorphic to Q_0, similarly, P_2 is isomorphic to $Q_0 * \underset{\sim}{Q}_1$ by the map that sends $p \in P_2$ to $\langle p(0), p(1) \rangle \in Q_0 * \underset{\sim}{Q}_1$.

Remark 1.6: *By expanding definitions it is easy to see that in general the forcing notion $P_{\alpha+1} := \lim_I \langle P_\beta, \underset{\sim}{Q}_\beta : \beta < \alpha + 1 \rangle$ is isomorphic to $P_\alpha * \underset{\sim}{Q}_\alpha$, via the map that sends $p \in P_{\alpha+1}$ to $\langle p{\restriction}\alpha, p(\alpha) \rangle$.*

The following two facts follow easily from the definitions.

Fact 1.7: *If ε is a limit ordinal, then*

(1) $p \in P_\varepsilon$ iff $\mathrm{dom}(p) \subseteq \varepsilon$ is in I, and $\forall \alpha < \varepsilon$, $p{\restriction}\alpha \in P_\alpha$.

(2) For $p, q \in P_\varepsilon$, $p \leq_\varepsilon q$ iff $\forall \alpha < \varepsilon$, $p{\restriction}\alpha \leq_\alpha q{\restriction}\alpha$.

See also 1.19.

Fact 1.8: *If $\varepsilon = \alpha + 1$, then*

(1) $p \in P_\varepsilon$ iff $p{\restriction}\alpha \in P_\alpha$, and $p{\restriction}\alpha \Vdash_\alpha p(\alpha) \in Q_\alpha$. (Again, this includes the case that $\alpha \notin \mathrm{dom}(p)$, where we declare $p(\alpha) = \mathbb{1}_{Q_\alpha}$.)

(2) For $p, q \in P_\varepsilon$, $p \geq_\varepsilon q$ iff $p{\restriction}\alpha \geq_\alpha q{\restriction}\alpha$ and $q{\restriction}\alpha \Vdash_\alpha p(\alpha) \geq_{Q_\alpha} q(\alpha)$.

(3) For $p, q \in P_\varepsilon$, $p \geq^*_\varepsilon q$ iff $p{\restriction}\alpha \geq^*_\alpha q{\restriction}\alpha$ and $q{\restriction}\alpha \Vdash_\alpha p(\alpha) \geq^*_{Q_\alpha} q(\alpha)$.

Fact 1.9: *For all $\alpha \leq \varepsilon$:*

(1) $P_\alpha \subseteq P_\varepsilon$. In fact, $P_\alpha = \{p \in P_\varepsilon : \mathrm{dom}(p) \subseteq \alpha\}$.

(2) If $p, q \in P_\alpha$, then $p \geq_\alpha q$ iff $p \geq_\varepsilon q$.

(3) If $p \in P_\alpha$, $q \in P_\varepsilon$, then $p \geq_\varepsilon q$ iff $p \geq_\alpha q{\restriction}\alpha$.

Proof: (1) and (2) follow immediately from the definition. For (3), note that for $\beta \in \mathrm{dom}(q) - \alpha$, $p(\beta) = \mathbb{1}_{\underset{\sim}{Q}_\beta}$, so $q{\restriction}\beta \Vdash p(\beta) \geq q(\beta)$.

Fact and Notation 1.10: *Let $\langle P_\alpha, Q_\alpha : \alpha < \varepsilon \rangle$ be an iteration. If $G_\alpha \subseteq P_\alpha$ is generic, and $H \subseteq \underset{\sim}{Q}_\alpha[G_\alpha]$ is any set, we let*

$$G_\alpha * H = \{r \in P_{\alpha+1} : r{\restriction}\alpha \in G_\alpha,\ r(\alpha)[G_\alpha] \in H\}$$

Then: $G_\alpha * H$ is generic iff G_α is generic and $H \subseteq \underset{\sim}{Q}_\alpha[G_\alpha]$ is generic over $V[G_\alpha]$.

Conversely, writing $G(\alpha)$ for $\{q(\alpha)[G_\alpha] : q \in G_{\alpha+1}\}$, we know that is a generic filter on $\underset{\sim}{Q}_\alpha[G_\alpha]$ over the model $V[G_\alpha]$, and $G_{\alpha+1} = G_\alpha * G(\alpha)$.

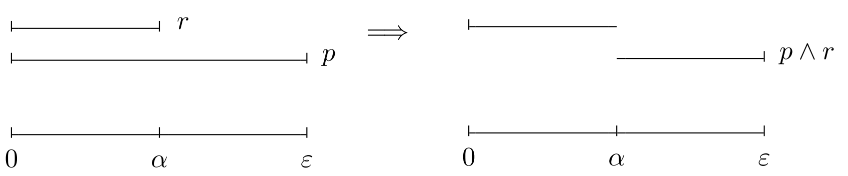
Proof: This is just a restatement of 1.3, using 1.6.

Remark 1.11: *The recursion theorem tells us that if F is a function (possibly a class) then for all ε there exists an I-supported iteration $\bar{Q} = \langle P_\alpha, \underset{\sim}{Q}_\alpha : \alpha < \varepsilon \rangle$ such that:*

If for all $\alpha < \varepsilon$, $F(\bar{Q}{\restriction}\alpha)$ is (defined and) a P_α-name for a forcing notion, then for all $\alpha < \varepsilon$ $\Vdash_\alpha \underset{\sim}{Q}_\alpha = F(\bar{Q}{\restriction}\alpha)$

For the following, let $\langle P_\alpha, \underset{\sim}{Q}_\alpha : \alpha < \varepsilon \rangle$ be an I-supported iteration forcing iteration, and let P_ε be the I-supported limit of this iteration, and let α range over $\varepsilon \cup \{\varepsilon\}$.

Definition 1.12: *Assume $\alpha \leq \varepsilon$. If $p \in P_\varepsilon$, $r \in P_\alpha$, and $p{\restriction}\alpha \geq r$, then we let $p \wedge r := r \cup p{\restriction}[\alpha, \varepsilon)$.*



Fact 1.13: *Assume p, r are as above. Then:*

(1) $p \wedge r \in P_\varepsilon$

(2) $p \wedge r \leq p$.

(3) *If $p_1 \geq p_2$, then $p_1 \wedge r \geq p_2 \wedge r$.*

Proof: By induction on β we can show (simultaneously) $(p \wedge r)\restriction\beta \in P_\beta$ and $(p \wedge r)\restriction\beta \leq p\restriction\beta$:

If $\beta \leq \alpha$, we have $(p \wedge r)\restriction\beta = r\restriction\beta \leq_\beta p\restriction\beta$.

If $\beta > \alpha$, β limit, then by induction hypothesis for all $\gamma < \beta$ we have $(p\wedge r)\restriction(\gamma + 1) \in P_{\gamma+1}$, so $(p \wedge r)\restriction\gamma\Vdash(p \wedge r)(\gamma) \in \underset{\sim}{Q}_\gamma$. So by 1.7, $(p \wedge r)\restriction\beta) \in P_\beta$. Also by induction hypothesis for all $\gamma < \beta$ we have $(p\wedge r)\restriction\gamma \leq_\gamma p\restriction\gamma$, so by 1.7, $(p\wedge r) \leq_\beta p$.

If $\beta > \alpha$ is a successor ordinal, say $\beta = \gamma + 1$, then $(p \wedge r)\restriction\gamma \in P_\gamma$ by induction hypothesis, and $(p \wedge r)\restriction\gamma\Vdash(p \wedge r)(\gamma) = p(\gamma) \in \underset{\sim}{Q}_\gamma$, as $(p \wedge r)\restriction\gamma \leq p\restriction\gamma$. So by 1.8, $(p \wedge r)\restriction\beta \in P_\beta$. Also by induction hypothesis have $(p \wedge r)\restriction\gamma \leq_\gamma p\restriction\gamma$, so by 1.8, $(p \wedge r) \leq_\beta p$.

(3): By induction, $(p_1 \wedge r)\restriction\beta \geq (p_2 \wedge r)\restriction\beta$.

Fact 1.14: *If $\alpha \leq \varepsilon$, $p \in P_\alpha$, $q \in P_\varepsilon$, then $p \perp_\varepsilon q$ iff $p \perp_\alpha q\restriction\alpha$.*

Proof: By 1.9(3), if $r \in P_\varepsilon$, $r \leq_\varepsilon p, q$, then $r\restriction\alpha \leq_\alpha p, q\restriction\alpha$. Conversely, if $r \in P_\alpha$, $r \leq_\alpha p, q\restriction\alpha$, then $r \wedge q \leq_\varepsilon p, q$.

Fact 1.15:

(1) *If $A \subseteq P_\alpha$ is a maximal antichain, $\alpha \leq \varepsilon$, then A is also a maximal antichain in P_ε.*

(2) *If G_ε is generic for P_ε, then $G_\alpha := G_\varepsilon \cap P_\alpha$ is generic for P_α.*

Proof of (1): A is an antichain in P_ε, by 1.14. Assume that $r \in P_\varepsilon$, $r \perp_\varepsilon a$ for all $a \in A$. Then again by 1.14, $r\restriction\alpha \perp_\alpha a$ for all $a \in A$, so A is not maximal in P_α.

(2) follows immediately from (1).

Notation 1.16: *We write $\leq_\alpha$ for $\leq_{P_\alpha}$, similarly $\Vdash_\alpha$, etc. We may write G_α for G_{P_α}, and $G(\alpha)$ for G_{Q_α}.*

When we talk about a I-supported iteration $\langle P_\alpha, Q_\alpha : \alpha < \varepsilon\rangle$, it is understood that P_ε is defined as the I-supported limit of this iteration.

If $\beta \leq \alpha$, and $\langle P_\alpha, Q_\alpha : \alpha < \varepsilon\rangle$ is an iteration, then G_β always denotes $G_\alpha \cap P_\beta$.

When we fix a ground model $V = V_0$, and consider an iteration $\langle P_\alpha, Q_\alpha : \alpha < \varepsilon\rangle \in V_0$, we write V_α for $V[G_\alpha]$. **Note that this conflicts with the notation that some authors use for the sets of rank $< \alpha$.**

Fact 1.17: *Assume that λ is a limit ordinal. Then for a generic $G_\lambda \subseteq P_\lambda$, for all $p \in P_\lambda$,*

$$p \in G_\lambda \Leftrightarrow \forall\alpha < \lambda\ p\restriction\alpha \in G_\alpha$$

Proof: Assume not, then there exists a condition q forcing this.

So $q\Vdash_\alpha \forall\beta < \lambda\, p{\restriction}\beta \in G_\beta \,\&\, p \notin G_\lambda$. For $\alpha \in \mathrm{dom}(p) \cup \mathrm{dom}(q)$ we let $q'(\alpha)$ be a name such that

$$\Vdash_\alpha q'(\alpha) \le q(\alpha) \,\&\, (q'(\alpha) \le p(\alpha) \vee q'(\alpha) \perp p(\alpha))$$

(We can get such $q'(\alpha)$ using 0.5) Now we claim that for all α, $q'{\restriction}\alpha \Vdash q'(\alpha) \le p(\alpha)$. Assume not, then there exists a generic filter G_α containing $q'{\restriction}\alpha$ such that $p(\alpha)[G_\alpha] \perp q'(\alpha)[G_\alpha]$. So we can extend G_α to a generic $G_{\alpha+1} := G_\alpha * G(\alpha)$ containing $q'{\restriction}\alpha+1$ but not $p{\restriction}\alpha+1$. Now we can extend $G_{\alpha+1}$ to G_ε, containing q' (hence q) but not $p{\restriction}\alpha+1$, a contradiction.

Corollary 1.18: *For $p \in P_\lambda$, we have $p \in G_\lambda$ iff for all $\beta < \lambda$, $p(\beta)[G_\beta] \in G(\beta)$.*

Proof of the corollary: By induction on $\beta < \lambda$ we can show $p{\restriction}\beta \in G_\beta$. Limit steps are handled by 1.17, and successor steps by 1.10.

Corollary 1.19: *If ε is a limit ordinal, then*

For $p, q \in P_\varepsilon$, $p \ge^*_\varepsilon q$ iff $\forall\alpha < \varepsilon$, $p{\restriction}\alpha \ge^*_\alpha q{\restriction}\alpha$.

Proof: If for all $\alpha < \varepsilon$ $q{\restriction}\alpha \Vdash_\alpha p{\restriction}\alpha \in G_\alpha$, then $q\Vdash_\varepsilon p \in G_\varepsilon$, by 1.17.

Conversely, if for some $\alpha < \varepsilon$ we have $p{\restriction}\alpha \not\ge^* q{\restriction}\alpha$, then there is $r \in P_\alpha$, $r \le q{\restriction}\alpha$, $r \perp p{\restriction}\alpha$. Letting $q' := q \wedge r$ we get $q' \Vdash p \notin G_\varepsilon$. Since $q \ge q'$, we do NOT have $q \Vdash p \in G_\varepsilon$, so $q \not\le^* p$.

The following fact shows that in finite support iteration of ccc forcing notions and in countable support iteration of proper forcing notions no new reals are added in limit steps of cofinality $> \omega$.

Lemma 1.20: *Assume $\langle P_\alpha, Q_\alpha : \alpha < \varepsilon\rangle$ is an iteration, and δ is a limit ordinal of cofinality $> \omega$, and all there are no conditions in P_δ whose domain is unbounded in δ. (In particular, this will be true for finite or countable support iteration.) Then:*

$$\Vdash_\delta \textit{If } cf(\delta) > \omega, \textit{ then } {}^\omega\omega \cap V[G_\delta] = \bigcup_{\alpha<\delta} {}^\omega\omega \cap V[G_\alpha]$$

Proof: For any $\alpha < \delta$ we define a P_α-name $\underset{\sim}{f}_\alpha$ satisfying the following:

$$\forall n \Vdash_\alpha \text{“If there is } p \in G_\alpha,\ j \in \omega \text{ such that } V \models p \Vdash_\delta \underset{\sim}{f}(\check n) = \check\jmath, \\ \text{then } \underset{\sim}{f}_\alpha(n) = j$$

(Note that it is possible to define such a function, since any two $p, p' \in G_\alpha$ must be compatible (in P_δ) hence cannot force two different values for $\underset{\sim}{f}(n)$.)

We will show that $\Vdash_\delta \exists\alpha \underset{\sim}{f} = \underset{\sim}{f}_\alpha$.

Work in $V[G_\delta]$. For any n, let $j_n := \underset{\sim}{f}[G_\delta](n)$, and find a condition $p_n \in G_\delta$ such that $V \models p_n \Vdash_\delta \underset{\sim}{f}(n) = \check\jmath_n$.

Since (in $V[G_\delta]$) we have $cf(\delta) > \omega$, and the supports of the p_n are bounded in δ, we can find $\alpha < \delta$ such that for all n, $p_n \in P_\alpha$, and hence also $p_n \in G_\alpha$. Clearly $\underset{\sim}{f}[G_\delta] = \underset{\sim}{f}_\alpha[G_\alpha]$.

2 Finite support iteration and ccc

A forcing notion P is said to satisfy the countable chain condition, if there is no uncountable set of pairwise incompatible conditions.

Fact 2.1: *"P has the* ccc*" is equivalent to the following statement:*

> *Whenever $\underset{\sim}{\alpha}$ is a P-name, and $\Vdash$ "$\underset{\sim}{\alpha}$ is an ordinal", then there exists a countable set A of ordinals such that $\Vdash \underset{\sim}{\alpha} \in A$.*

(Equivalently, if $\Vdash \underset{\sim}{x} \in V$ for some P-name $\underset{\sim}{x}$, then there exists a countable set A in V such that $\Vdash \underset{\sim}{x} \in A$.)

Proof: Assume that P has the ccc. Let $C \subseteq P$ be a maximal set of conditions satisfying

(1) $p \in C \Rightarrow \exists \beta\, p \Vdash \underset{\sim}{\alpha} = \widehat{\beta}$.

(2) $p_1, p_2 \in C \Rightarrow p_1 \perp_P p_2$.

It is easy to see that C must be a maximal antichain in P (because the set of conditions satisfying (1) is dense).

Let $A := \{\beta : \exists p \in C : p \Vdash \underset{\sim}{\alpha} = \widehat{\beta}\}$. Then A is countable because C is countable, and clearly $\Vdash \underset{\sim}{\alpha} \in A$. [Proof: Every condition $p \in C$ forces $\underset{\sim}{\alpha} \in A$. A condition forcing $\underset{\sim}{\alpha} \notin A$ cannot be compatible with any condition in C, which contradicts the above observation that C is a maximal antichain.]

Conversely, assume that $\langle p_i : i < \omega_1 \rangle$ is an antichain. Let $\underset{\sim}{\alpha}$ be a name of an ordinal such that

$$\Vdash \exists i\, p_i \in G_P \Rightarrow p_{\underset{\sim}{\alpha}} \in G_P$$

Assume, towards a contradiction, that there is a countable set A of ordinals such that $\Vdash \underset{\sim}{\alpha} \in A$. Let $i \in \omega_1 - A$. Then $p_i \Vdash p_i \in G_P$, but for all $j \neq i$, $p_i \Vdash p_j \notin G$. So $p_i \Vdash \underset{\sim}{\alpha} = i \notin A$, a contradiction.

The same proof also shows: P has the ccc, iff

> Whenever $\underset{\sim}{x}$ is a P-name, and $\Vdash$ "$\underset{\sim}{x} \in V$", then there exists a countable set A such that $\Vdash \underset{\sim}{x} \in A$.

Corollary 2.2: *If P has the* ccc*, and $\underset{\sim}{X}$ is a name of a countable set $\subseteq V$, then there exists a countable set A in V such that $\Vdash \underset{\sim}{X} \subseteq A$.*

The next theorem can be phrased "ccc is preserved under composition of forcing notions."

Corollary 2.3: *Assume P has the* ccc*, and $\Vdash_P$ "$\underset{\sim}{Q}$ has the* ccc*." Then $P * \underset{\sim}{Q}$ has the* ccc*.*

Proof: Let $\underset{\sim}{\alpha}$ be a $P * \underset{\sim}{Q}$-name of an ordinal. We consider $\underset{\sim}{\alpha}$ as a P-name for a $\underset{\sim}{Q}$-name of an ordinal. So since $\Vdash_P$ "$\underset{\sim}{Q}$ has the ccc," (and using the existential completeness lemma) we can find a P-name $\underset{\sim}{A}$ such that

(1) $\Vdash_P$ "$\underset{\sim}{A}$ is countable"
(2) $\Vdash_P$ "$\Vdash_Q \underset{\sim}{\alpha} \in \underset{\sim}{A}$".

By 2.2 we can find a countable set A' in V such that

$$\Vdash_P A \subseteq A'.$$

So $\Vdash_P \Vdash_{\underset{\sim}{Q}} \underset{\sim}{\alpha} \in A'$, hence $\Vdash_{P * \underset{\sim}{Q}} \underset{\sim}{\alpha} \in A'$.

The next theorem shows that the property of satisfying the countable chain condition is satisfied under finite support iteration.

Theorem 2.4: *Assume that $\langle P_\alpha, Q_\alpha : \alpha < \varepsilon\rangle$ is a finite support iteration of forcing notions, and*

$$\forall \alpha < \varepsilon : \ \Vdash_\alpha \text{“}Q_\alpha \textit{ has the } \text{ccc”}$$

Then P_ε has the ccc.

The proof of this theorem proceeds by induction on ε. Successor steps are handled by 2.3 and 1.6.

As for limit steps, from the induction hypothesis we can conclude that

$$(*) \qquad\qquad \forall \alpha < \varepsilon \ P_\alpha \models \text{ccc}$$

We will show that $(*)$ implies that P_ε has the ccc. We need to use the following combinatorial fact (called the Δ-system lemma):

Lemma 2.5: *Assume that $\langle F_i : i < \omega_1\rangle$ is a family of finite sets. Then there is a set $S \subseteq \omega_1$ of size $\aleph_1$ such that the family $\langle F_i : i \in S\rangle$ is a Δ-system, i.e.*

$$\exists F \textit{ finite } \forall i, j \in S : i \neq j \Rightarrow F_i \cap F_j \subseteq F$$

F is called a "kernel", "root" or "heart" of the Δ-system $\langle F_i : i \in S\rangle$.

See [8, II.1.5] for a proof.

Proof of 2.4, limit step:

Assume (toward a contradiction) that $\langle p_i : i < \omega_1\rangle$ is an antichain of conditions in P_ε. Since $\langle \mathrm{dom}(p_i) : i < \omega_1\rangle$ is a family of finite sets, the Δ-lemma applies.

So we can find a set $S \subseteq \omega_1$ of size $\aleph_1$ and a finite set F such that for all distinct i and j in S, $\mathrm{dom}(p_i) \cap \mathrm{dom}(p_j) \subseteq F$. Let $F \subseteq \alpha < \varepsilon$.

The family $\langle p_i \restriction \alpha : i \in S\rangle$ is an uncountable familiy of conditions in P_α. By our assumption $(*)$, it cannot be an antichain. So there are $i \neq j$ in S and a condition $r \in P_\alpha$ such that $p_i \geq r$ and $p_j \geq r$.

Now define $p \in P_\varepsilon$ as follows:

$p \restriction \alpha = r$.
$\forall \gamma \in \mathrm{dom}(p_i) - \alpha$: $p(\gamma) = p_i(\gamma)$.
$\forall \gamma \in \mathrm{dom}(p_j) - \alpha$: $p(\gamma) = p_j(\gamma)$.

Note that $\mathrm{dom}(p_i) \cap \mathrm{dom}(p_j) \subseteq F \subseteq \alpha$, so $\mathrm{dom}(p_i) - \alpha$ and $\mathrm{dom}(p_j) - \alpha$ are disjoint.

We leave it as an exercise to check that p is indeed a condition in P_ε, and $p_i \geq p$, $p_j \geq p$.

This is a contradiction, as $\langle p_i : i < \omega_1 \rangle$ was supposed to be an antichain.

3 Properness and countable support iteration

Definition 3.1: *Assume that $(N, \in)$ is a model of a fragment of ZFC (which should be large enough to develop the general theory of forcing). Let $P \in N$ be a forcing notion, and assume that for all $p_1, p_2 \in P \cap N$: $N \models p_1 \perp_P p_2 \Rightarrow V \models p_1 \perp_P p_2$. We say that G is N-generic for P iff:*

For all $A \in N$, if $N \models$ "$A \subseteq P$ is a maximal antichain of P", then $G \cap N \cap A \neq \emptyset$.

(So G is N-generic iff $G \cap N$ is N-generic.)

It is easy to see that we by replacing "maximal antichain" by "dense subset", "dense open subset" or "predense subset", we get an equivalent definition.

In fact, the following holds:

Fact 3.2: *Assume $p \in P \cap N$, $P \in N$, $p \leq q \in P$. Then the following are equivalent:*

(1) For all $D_1 \in N$: If $N \models$"D_1 is predense below p", then $D_1 \cap N$ is predense below q.
(2) For all $D_2 \in N$: If $N \models$"D_2 is predense $\subseteq P$," then $q \Vdash D_2 \cap N \cap G \neq \emptyset$.
(3) For all $D_3 \in N$: If $N \models$"D_3 is dense $\subseteq P$," then $q \Vdash D_3 \cap N \cap G \neq \emptyset$.
(4) For all $D_4 \in N$: If $N \models$"D_4 is a maximal antichain $\subseteq P$," then $q \Vdash D_4 \cap N \cap G \neq \emptyset$.
(5) For all $D_5 \in N$: If $N \models$"D_5 is open dense $\subseteq P$," then $q \Vdash D_5 \cap N \cap G \neq \emptyset$.

Proof: We will show
$$\begin{array}{ccccccccc} & & & & & (3) & & & \\ & & & & \nearrow & & \searrow & & \\ (1) & \rightarrow & (2) & & & & & (5) & \rightarrow (1). \\ & & & & \searrow & & \nearrow & & \\ & & & & & (4) & & & \end{array}$$

(2) $\Rightarrow$ (4), (2) $\Rightarrow$ (3), (3) $\Rightarrow$ (5) are trivial.

(1) $\Rightarrow$ (2): Let D_2 be predense. Then D_2 is predense below p, so $D_2 \cap N$ is predense below q. Hence $q \Vdash (D_2 \cap N) \cap G \neq \emptyset$.

(4) $\Rightarrow$ (5): Work in N. Assume that D_5 is open dense. Let D_4 be a maximal antichain $\subseteq D_5$ (i.e., let $D_4 \subseteq D_5$ be an antichain such that there is proper superset $D_4' \subseteq D_5$ that is also an antichain). It is enough to see that D_4 is a maximal antichain in P (in N): Assume not, and let r be incompatible with every element of D_4, let $s \leq r$ be in D_5, then also s is incompatible with every element of D_4, contradicting the maximality of D_4.

(5) $\Rightarrow$ (1): Let $D_1 \in N$, $N \models$"D_1 is predense below p." The following takes place in N:

Let $D_5 := \{r : \text{either } r \perp p, \text{ or } \exists d \in D_1\, r \leq d\}$. Then D_5 is open. We claim that D_5 is dense. To prove this, let s be any condition in P. If $s \perp p$, then $s \in D_1$.

Otherwise, there is a $t \in P$, $s \geq t$, $p \geq t$. t is compatible with some $d \in D_1$, so there is t' extending d, t, s, p. So $t \in D_5$.

Hence $N \models$"D_5 is dense", so $V \models q \Vdash D_5 \cap N \cap G \neq \emptyset$.

Finally we will to show that (in V), $D_1 \cap N$ is predense below q. Let $r \leq q$ be incompatible with all elements of $D_1 \cap N$. Let $r' \leq r$, and $r' \Vdash s \in D_5 \cap N \cap G$. Then $s \in D_5 \cap N$. Either $N \models s \perp p$, then $s \perp p$, which implies $s \perp r'$, a contradiction to $r' \Vdash s \in G$. Otherwise $N \models \exists d \in D_1\, s \leq d$. As r' and s are compatible, r' and d are compatible, so $D_1 \cap N$ is indeed predense below q.

Definition 3.3: *We say that $q \in P$ is N-generic (or (N, P)-generic) iff*

$$V \models q \Vdash_P \text{“}G_P \text{ is } N\text{-generic”}$$

(iff 3.2(2)–(5) hold).

Remark 3.4: *If q is N-generic, and $q' \leq q$, then also q' is N-generic.*

Notation 3.5: *For the following, we let χ be a “large enough” regular cardinal. “large enough” means that for all forcing notions P we consider, we have $\mathfrak{P}(P) \in H(\chi)$, i.e., the power set of P is hereditarily of size $< \chi$. Since all forcing notions we consider will be hereditarily countable, or countable support iterations of length $\leq \aleph_2$ of hereditarily countable forcing notion, and all the universes we consider satisfy GCH except for possibly $2^{\aleph_0} = \aleph_2$, we could choose $\chi := \aleph_3$. To be on the safe side, we let*

$$\chi := \beth_\omega^+$$

(So also in every extension that we consider, $\chi = \beth_\omega^+$).

We will consider countable elementary submodels of $(H(\chi), \in)$.

(The notions we will define below will depend on χ, but a careful examination [which we will not carry out here] shows that this dependence is only apparent.)

Note that all essential properties of P are absolute between V and $H(\chi)$, for example

$$V \models A \text{ is a maximal antichain of } P \quad \Leftrightarrow \quad H(\chi) \models A \text{ is a maximal antichain of } P$$

We also have the following fact:

Fact 3.6: *If $\Vdash \underset{\sim}{x} \in H(\chi)$, then there is a name $\underset{\sim}{\bar{x}} \in H(\chi)$ such that $\Vdash \underset{\sim}{x} = \underset{\sim}{\bar{x}}$.*

Proof: First note that for each element x of $H(\chi)$ there is $\lambda < \chi$ and a sequence $\langle x_i : i \leq \lambda \rangle$ of sets in $H(\chi)$ such that for all $i \leq \lambda$ $x_i \subseteq \{x_j : j < i\}$, and $x = x_\lambda$.

Applying this fact in $V[G]$ to $\underset{\sim}{x}$, we get a sequence $\langle \underset{\sim}{x}_i : i \leq \underset{\sim}{\lambda} \rangle$. We can find an ordinal λ such that $\Vdash \underset{\sim}{\lambda} \leq \lambda$, so wlog we may assume $\Vdash \underset{\sim}{\lambda} = \lambda$ (since we can define $\underset{\sim}{x}_i = \emptyset$ for $\underset{\sim}{\lambda} < i < \lambda$.

Now we define by induction $\underset{\sim}{\bar{x}}_i := \{\langle \underset{\sim}{\bar{x}}_j, p \rangle : p \Vdash \underset{\sim}{x}_j \in \underset{\sim}{x}_i\}$, and prove by induction $\Vdash \underset{\sim}{\bar{x}}_i = \underset{\sim}{x}_i$.

Definition 3.7: *A forcing notion P is called proper, if for some $x \in H(\chi)$, for all countable models $(N, \in) \prec (H(\chi), \in)$ that contain x and P, and for all $p \in P \cap N$ there exists a $q \leq p$ which N-generic.*

Remark 3.8: *The clause "for some x … " above is not essential, but it makes some proofs slightly simpler. For example, when we consider an iterated forcing notion $P_\varepsilon \in N$, we could reconstruct the iteration $\langle P_\alpha, \underset{\sim}{Q}_\alpha : \alpha < \varepsilon \rangle$ from P_ε (in N), but the above definition makes it unnecessary, as we can include "$\langle P_\alpha, \underset{\sim}{Q}_\alpha : \alpha < \varepsilon \rangle \in N$" into our assumption.*

We leave as an exercise to the reader to show the following fact. (We will not use it in the rest of the paper). Note that this fact is independent of χ.

Fact 3.9: *P is proper iff player II has a winning strategy in the following infinite game (see also [2]):*

In the first move, player I plays a condition p, and a maximal antichain A_1. Player II responds with a countable subset $B_1^1 \subseteq A_1$.

In the n-th move ($n > 1$), player I plays a maximal antichain A_n, and player II plays countable sets $B_1^n \subseteq A_1, \ldots, B_n^n \subseteq A_n$.

After ω many moves, player II wins iff there is a condition $q \leq p$ such that for all n, the set $B_n := \bigcup_{k \geq n} B_n^k$ is predense below q.

[Sketch of proof: If player II has a winning strategy, and player I plays all maximal antichains of the model N, then all responses of player II will be subsets of N. Conversely, player II can, in each move generate a Skolem hull N_n (inside $H(\chi)$) of everything played so far. At the end any condition generic for $N := \bigcup_n N_n$ will work. This definition of the strategy does not take place in $H(\chi)$, since $H(\chi)$ is needed as a parameter, but the resulting strategy itself IS in $H(\chi)$.]

Fact 3.10: *For N as above, $q \in P$ is N-generic iff for all $\underset{\sim}{\alpha} \in N$:*

$$\text{If } N \models \text{"}\underset{\sim}{\alpha} \text{ is a } P\text{-name for an ordinal", then } q \Vdash \underset{\sim}{\alpha} \in N$$

(Note that $q \Vdash \underset{\sim}{\alpha} \in N$ really means $q \Vdash \underset{\sim}{\alpha}[G_P] \in \widehat{N}$.)

Proof: Let $A \in N$, $N \models$"A is a maximal antichain." Working in N, we can find a cardinal κ, a function f from κ onto A, and a P-name $\underset{\sim}{\alpha}$ of an ordinal such that $\Vdash_P f(\underset{\sim}{\alpha}) \in G_P \cap A$. Then if $q \Vdash \underset{\sim}{\alpha} \in N$, also $q \Vdash f(\underset{\sim}{\alpha}) \in N$, so $q \Vdash N \cap A \cap G \neq \emptyset$. Hence q satisfies 3.2(4).

Conversely, let $\underset{\sim}{\alpha} \in N$ be a P-name of an ordinal. Then the set $D := \{r \in P : \exists \beta\, r \Vdash \underset{\sim}{\alpha} = \beta\}$ is dense. Let f be a function with domain D such that for all $r \in D$, $r \Vdash \underset{\sim}{\alpha} = f(r)\widehat{\ }$. Then $f \in N$, so if $q \Vdash \exists r \in D \cap N \cap G$, then also $q \Vdash \underset{\sim}{\alpha} \in N$, as $\Vdash$ "$r \in N \Rightarrow f(r) \in N$."

Remark 3.11: *A similar proof shows: q is N-generic iff*

$$\forall \underset{\sim}{x} \in N : \text{If } \underset{\sim}{x} \text{ is a } P\text{-name and } \Vdash \underset{\sim}{x} \in V, \text{ then } q \Vdash \underset{\sim}{x} \in N$$

(or in other words, $q \Vdash N[G] \cap V = N$).

Example 3.12: *If P satisfies the countable chain condition, then every condition (or equivalently, the condition $\mathbb{1}_Q$) is N-generic, for any countable model $N \prec H(\chi)$ containing P. Thus, any ccc forcing notion is proper.*

Proof: Note that if $A \in N$ is countable, then $A \subseteq N$, because A must be the range of a function $f \in N$ that has domain ω. As $f \in N$, and for all $n \in \omega$, $n \in N$, also $f(n) \in N$ for all n, so $A = \mathrm{rng}(f) \subseteq N$.

Let $A \in N$ be a maximal antichain, then $\Vdash_Q A \cap G \neq \emptyset$. So also $\mathbb{1}_Q \Vdash_Q A \cap N \cap G = A \cap G \neq \emptyset$. Hence $\mathbb{1}_Q$ satisfies condition 3.2(4).

(Alternatively, we could use 2.1 to show that ccc $\Rightarrow$ proper.)

Fact 3.13: *If Q is proper, and $A \in V[G_Q]$ a countable set of ordinals, then there is a countable set $B \in V$ of ordinals such that $A \subseteq B$.*

Proof: Let $\underset{\sim}{A}$ be a name for A, and let $\langle \underset{\sim}{\alpha}_n : n \in \omega \rangle \in V$ be a sequence of names for all ordinals in $\underset{\sim}{A}$. For each n, let A_n be a maximal antichain of conditions deciding $\underset{\sim}{\alpha}_n$. We will show that the set of conditions forcing "there exists a countable B in V covering A" is dense.

So fix a condition p. Let $N \prec H(\chi)$ be a countable elementary model containing p, Q and $\langle A_n : n \in \omega \rangle$ and let $q \leq p$ be N-generic.

Let $B := \bigcup_{n\in\omega} \{\beta : \exists r \in A_n \cap N : r \Vdash \alpha_n = \beta\}$. B is countable, and the genericity of q easily implies that $q \Vdash \underset{\sim}{A} \subseteq B$.

Corollary 3.14: *If Q proper and $cf(\delta) > \omega$, then $\Vdash_Q cf(\delta) > \omega$. In particular, $\aleph_1$ is not collapsed.*

Proof: Q cannot add a countable cofinal sequence in δ, by the previous fact. (Note that this proof works only for $\delta < \chi$, but for $\delta \geq \chi$ or indeed $\delta > |P|$ a much simpler argument shows that the cofinality of δ is preserved.)

Fact 3.15: *If $N \prec H(\chi)$, and $Q \in N$, then $\mathbb{1} \Vdash N[G] \prec H(\chi)^{V[G]}$.*

Proof: (Remember that χ is quite large compared to Q.) We will use the Tarski-Vaught criterion. So it is enough to see for all names $\underset{\sim}{a}$ in N:

$$(*) \qquad \mathbb{1}_Q \Vdash_Q \big(\exists x\, \varphi(x, \underset{\sim}{a})\big)^{H(\chi)} \;\Rightarrow\; \exists x \in N[G]\, \big(\varphi(x, \underset{\sim}{a})\big)^{H(\chi)}$$

By 0.5, there is a name $\underset{\sim}{\tau}$ such that

$$(**) \qquad \mathbb{1}_Q \Vdash_Q \underset{\sim}{\tau} \in H(\chi) \,\&\, \Big((\exists x\, \varphi(x, \underset{\sim}{a}))^{H(\chi)} \Rightarrow \varphi(\underset{\sim}{\tau}, \underset{\sim}{a}) \Big)^{H(\chi)}.$$

As this last statement can be rewritten as a statement in $H(\chi)$ (thanks to 3.6), we can find such $\underset{\sim}{\tau}$ in N.

Thus $\mathbb{1}_Q \Vdash_Q \underset{\sim}{\tau} \in N[G] \,\&\, \big((\exists x\, \varphi(x, \underset{\sim}{a}))^{H(\chi)} \Rightarrow \varphi(\underset{\sim}{\tau}, \underset{\sim}{a})^{H(\chi)} \big)$, which implies $(*)$.

Note that the same proof actually shows we even have $(N[G], N[G] \cap H(\chi), \in) \prec (H(\chi)^{V[G]}, H(\chi)^V, \in)$. So if we work with a proper forcing notion, by 3.11 we also get $(N[G], N, \in) \prec (H(\chi)^{V[G]}, H(\chi)^V, \in)$.

Lemma 3.16: *Assume that $Q_0 * \underset{\sim}{Q}_1$ is a composition of forcing notions. Then $(q_0, \underset{\sim}{q}_1)$ is N-generic, iff q_0 is N-generic for Q_0, and $q_0 \Vdash$ "$\underset{\sim}{q}_1$ is $N[G_0]$-generic for $\underset{\sim}{Q}_1[G_0]$."*

Similarly, if we have an iteration $\langle P_\alpha, Q_\alpha : \alpha < \varepsilon\rangle$, $\alpha \leq \varepsilon$, then $q \in P_{\alpha+1}$ is $(P_{\alpha+1}, N)$-generic iff:
$q{\restriction}\alpha$ is (P_α, N)-generic, and $q{\restriction}\alpha \underset{\alpha}{\Vdash}$ "$q(\alpha)$ is $(Q_\alpha, N[G_\alpha])$-generic."

Proof: (Recall that we identify $Q_0 * \underset{\sim}{Q}_1$-names with Q_0-names for $\underset{\sim}{Q}_1$-names)

We will use 3.10. For any name $\underset{\sim}{\alpha} \in N$ of an ordinal we have

$$(q_0, \underset{\sim}{q}_1) \Vdash_{Q_0 * \underset{\sim}{Q}_1} \underset{\sim}{\alpha}[G_{Q_0 * \underset{\sim}{Q}_1}] \in N \quad \text{iff} \quad q_0 \Vdash_{Q_0} \big(\underset{\sim}{q}_1 \Vdash_{\underset{\sim}{Q}_1[G_0]} \underset{\sim}{\alpha}[G_0][G_1] \in N\big)$$

This shows that properness is preserved under composition of forcing notions. The following two lemmata will show that properness is also preserved under countable support iteration. Note that the first one does not mention properness or countable models—even the fact that the iteration has countable support plays no role.

Preliminary Lemma 3.17:
Let $\langle P_\alpha, \underset{\sim}{Q}_\alpha : \alpha < \varepsilon\rangle$ be a countable support iteration.

Assume $\alpha_1 \leq \alpha_2 \leq \beta \leq \varepsilon$, $\underset{\sim}{p}_1$ is a P_{α_1}-name for a condition in P_ε. Let D be a dense open set of P_β.

Then $\mathbb{1}_{P_{\alpha_2}} \underset{\alpha_2}{\Vdash} \exists p_2\ \varphi(p_2)$, where $\varphi(p_2)$ is the conjunction of the following clauses:

(1) $p_2 \in \widehat{P}_\beta$, $p_2 \leq_\beta \underset{\sim}{p}_1$.
(2) $p_2 \in \widehat{D}$.
(3) If $\underset{\sim}{p}_1{\restriction}\alpha_2 \in G_{\alpha_2}$, then $p_2{\restriction}\alpha_2 \in G_{\alpha_2}$.

Remark 1: By the existential completeness lemma there is an α_2-name $\underset{\sim}{p}_2$ for a condition in P_β such that $\underset{\alpha_2}{\Vdash} \varphi(\underset{\sim}{p}_2)$.

Remark 2: The P_{α_1}-name $\underset{\sim}{p}_1$ corresponds naturally to a P_{α_2}-name, which we also call $\underset{\sim}{p}_1$. In other words, we may wlog assume that $\alpha_1 = \alpha_2$.

Proof of the lemma: Assume $\mathbb{1}_{P_{\alpha_2}} \not\Vdash_{\alpha_2} \exists p_2\ \varphi(p_2)$, then there exists a condition $r_2 \in P_{\alpha_2}$ such that

$$r_2 \Vdash_{\alpha_2} \text{there is no } p_2 \text{ satisfying (1)–(3).}$$

We may assume that r_2 decides what $\underset{\sim}{p}_1$ is, (i.e. $r_2 \underset{\alpha_2}{\Vdash}$ "$\underset{\sim}{p}_1 = \widehat{p}_1$" for some $p_1 \in V$), and r_2 also decides whether $p_1{\restriction}\alpha_2 \in G_{\alpha_2}$.

Case 1: $r_2 \underset{\alpha_2}{\Vdash} p_1{\restriction}\alpha_2 \notin G_{\alpha_2}$:

But then (3) is true for any p_2, so

$$r_2 \underset{\alpha_2}{\Vdash} \text{there is no } p_2 \text{ satisfying (1)–(2).}$$

which is a contradiction since D is dense open.

Case 2: $r_2 \underset{\alpha_2}{\Vdash} p_1{\restriction}\alpha_2 \in G_{\alpha_2}$. We may assume $r_2 \le_{\alpha_2} p_1{\restriction}\alpha_2$. Now let $r := p_1 \wedge r_2 = r_2 \cup p_1{\restriction}[\alpha_2, \beta) \le p_1$, and find $p_2 \in D$, $p_2 \le_\beta r$. Then

$$p_2{\restriction}\alpha_2 \underset{\alpha_2}{\Vdash} p_2 \text{ satisfies (1)–(3),}$$

again a contradiction, because $p_2{\restriction}\alpha_2 \le r_2$.

Induction Lemma 3.18: *Let $\bar{Q} := \langle P_\alpha, \underset{\sim}{Q}_\alpha : \alpha < \varepsilon \rangle$ be a countable support iteration, and assume that for all $\alpha < \varepsilon$, $\Vdash_\alpha$ "$\underset{\sim}{Q}_\alpha$ is a proper forcing notion." Let $N \prec H(\chi)$ be a countable model containing $\bar{Q}$. Then:*

For all $\beta \in N \cap \varepsilon$:

For all $\alpha \in N \cap \beta$, all $\underset{\sim}{p} \in N$:

Assume *$\underset{\sim}{p}$ is a P_α-name for a condition in P_β, and*

(a) $q \in P_\alpha$
(b) q is (P_α, N)-generic.
(c) $q \underset{\alpha}{\Vdash} \underset{\sim}{p}{\restriction}\alpha \in G_\alpha \cap N$

Then *there is a condition q^+:*

(a)$^+$ $q^+ \in P_\beta$, $q^+{\restriction}\alpha = q$
(b)$^+$ q^+ is (P_β, N)-generic
(c)$^+$ $q^+ \Vdash_\beta \underset{\sim}{p}{\restriction}\beta \in G_\beta \cap N$

The proof is by induction on β.

Successor step:

Let $\beta = \beta' + 1$. Since we can first use the induction hypothesis on α, β' to extend q to a condition $q' \in P_{\beta'}$ satisfying the appropriate version of (a)–(c), we may simplify the notation by assuming $\beta = \alpha + 1$.

Since $q \underset{\alpha}{\Vdash} N[G_\alpha] \prec H(\chi)^{V[G_\alpha]}$, we also have

$$q \underset{\alpha}{\Vdash} \text{"there is a } (Q_\alpha, N[G_\alpha])\text{-generic condition} \le \underset{\sim}{p}(\alpha)\text{"}$$

By "existential completeness", there is a P_α-name $q^+(\alpha)$ for it. By 3.16, we are done.

Limit step:

Let $\beta \in N$ be a limit ordinal, $\delta := \sup(\beta \cap N) = \bigcup_n \alpha_n$, $\alpha = \alpha_0 < \alpha_1 < \cdots$, $\alpha_n \in N$. Let $\langle D_n : n \in \omega \rangle$ enumerate all dense subsets of P_β that are in N.

First we will define a sequence $\langle \underset{\sim}{p}_n : n \in \omega\rangle$, $\underset{\sim}{p}_n \in N$, $\underset{\sim}{p}_0 = \underset{\sim}{p}$ such that the following will hold:

(0) $\underset{\sim}{p}_n$ is a P_{α_n}-name for a condition in P_β

(1) $\underset{\alpha_{n+1}}{\Vdash} \underset{\sim}{p}_{n+1} \leq \underset{\sim}{p}_n$

(2) $\underset{\alpha_{n+1}}{\Vdash} \underset{\sim}{p}_{n+1} \in D_n$.

(3) $\underset{\alpha_{n+1}}{\Vdash}$ "If $\underset{\sim}{p}_n \restriction \alpha_{n+1} \in G_{\alpha_{n+1}}$ then $\underset{\sim}{p}_{n+1} \restriction \alpha_{n+1} \in G_{\alpha_{n+1}}$".

For each n we thus get a name $\underset{\sim}{p}_n$ that is in N. For each n we can use the "preliminary lemma" (and remark 1 before its proof) in N to obtain $\underset{\sim}{p}_{n+1}$.

Now we define a sequence $\langle q_n : n \in \omega\rangle$, $q_n \in P_{\alpha_n}$, $q = q_0 \subseteq q_1 \subseteq \cdots$, $q_{n+1}\restriction \alpha_n = q_n$, and q_n satisfies (a), (b), (c) (if we write q_n for q, p_n for p, α_n for α.)

$q_{n+1} = q_n^+$ can be obtained by the induction hypothesis, applied to α_n, α_{n+1}, and $\underset{\sim}{p}_n \restriction \alpha_{n+1}$. By (c)$^+$ we know

$$q_n^+ \Vdash \underset{\sim}{p}_n \restriction \alpha_{n+1} \in G_{\alpha_{n+1}} \cap N$$

Hence by (3) we have

$$q_{n+1} \Vdash \underset{\sim}{p}_{n+1} \restriction \alpha_{n+1} \in G_{\alpha_{n+1}} \cap N$$

Since $q_{n+1}\restriction \alpha_n = q_n$, $q = \lim q_n$ exists and is $\leq q_n$ for all n.

We have to show that $q \Vdash p \in G_\beta \cap N$ and that q is generic. Let G_β be a generic filter containing q. We will write p_n for $\underset{\sim}{p}_n[G_{\alpha_n}]$. (Note that $p_n \in N$, because q_n was N-generic and $q_n \in G_{\alpha_n}$.)

Since $q_n \geq q \in G_\beta$, we have $p_n \restriction \alpha_n \in G_{\alpha_n} \cap N$ and $N \models p_n \leq p_{n-1} \leq \cdots \leq p_0$. Hence $p\restriction \alpha_n \in G_{\alpha_n} \cap N$ for all n, and so by 1.17, $p\restriction \delta \in G_\delta \cap N$. As $dom(p) \subseteq \delta$, $p\restriction\delta = p$, so $p \in G_\beta$. Similarly, $p_n \in G_\beta$ for all n.

Consider a dense set $D_n \subseteq P_\beta$. Since $q_{n+1} \Vdash$ "$\underset{\sim}{p}_{n+1} \in D_n$," we have $p_{n+1} \in G_\beta \cap D_n \cap N$.

Hence q is generic.

Corollary 3.19: *Let $\langle P_\alpha, \underset{\sim}{Q}_\alpha : \alpha < \varepsilon\rangle$ be a countable support iteration of proper forcing notions Q_α (as in 3.18). Then P_ε (the countable support limit of this iteration) is proper.*

Proof: Apply 3.18 with $\alpha = 0$, $\beta = \varepsilon$.

4 Quotient forcing

Let $\langle P_\alpha, Q_\alpha : \alpha < \varepsilon \rangle$ be either a countable support iteration of proper forcing notions, or a finite support iteration.

Definition 4.1: *We define P_ε/G_α to be a P_α-name for a forcing notion such that*

$$\Vdash_\alpha P_\varepsilon/G_\alpha = \{p \in \widehat{P}_\varepsilon : p{\restriction}\alpha \in G_\alpha\}$$

For $p, q \in P_\varepsilon/G_\alpha$ we let $p \geq_{P_\varepsilon/G_\alpha} q$ iff $p \geq_{P_\varepsilon} q$.

Notation 4.2: *We write $\leq_{\alpha\varepsilon}$, $\Vdash_{\alpha\varepsilon}$, ... , for $\leq_{P_\varepsilon/G_\alpha}$, $\Vdash_{P_\varepsilon/G_\alpha}$, etc.*

Fact 4.3: *$\mathbb{1}_{P_\alpha}$ forces the following:*

(1) $P_\varepsilon/G_\alpha \subseteq \widehat{P}_\varepsilon$.
(2) If $p, q \in P_\varepsilon/G_\alpha$, then $p \leq_{\alpha\varepsilon} q \Leftrightarrow p \leq_\varepsilon q$.
(3) If $p, q \in P_\varepsilon/G_\alpha$, then $p \perp_\varepsilon q$ implies $p \perp_{\alpha\varepsilon} q$. (However, note that $p \perp_{\alpha\varepsilon} q$ does not necessarily imply $p \perp_\varepsilon q$!)
(4) If $A \in V$, $A \subseteq P_\varepsilon$ is a maximal antichain, then $A \cap P_\varepsilon/G_\alpha$ is a maximal antichain in P_ε/G_α.

(1) and (2) are true by definition.

For (3), note that any $r \leq_{\alpha\varepsilon} p, q$ would also satisfy $r \leq_\varepsilon p, q$.

Proof of (4): By (3), $\Vdash_\alpha$"$A \cap P_\varepsilon/G_\alpha$ is an antichain." Assume that the conclusion is not true, then we can find a condition $r \in P_\alpha$ and a condition $p \in P_\varepsilon$ such that

$r \Vdash_\alpha$"$p \in P_\varepsilon/G_\alpha$, but for all $s \in A \cap P_\varepsilon/G_\alpha$ there is no $t \in P_\varepsilon/G_\alpha$ extending both p and s."

Since $r \Vdash$"$p{\restriction}\alpha \in G_\alpha$," r and $p{\restriction}\alpha$ are compatible, so we may wlog assume $r \leq p{\restriction}\alpha$. Let $\bar{p} := p \wedge r$, then $\bar{p} \leq p$ and $\bar{p}{\restriction}\alpha \leq r$. So

$\bar{p}{\restriction}\alpha \Vdash_\alpha$"$\bar{p} \in P_\varepsilon/G_\alpha$, and for all $s \in A \cap P_\varepsilon/G_\alpha$ there is no $t \in P_\varepsilon/G_\alpha$ extending $\bar{p}$ and s."

Let $s \in A$ be compatible with $\bar{p}$, and let $t \leq s, \bar{p}$. Then $t{\restriction}\alpha \Vdash t \in P_\varepsilon/G_\alpha \,\&\, t \leq s \,\&\, t \leq \bar{p}$, a contradiction, because $t{\restriction}\alpha \leq \bar{p}{\restriction}\alpha$.

Fact 4.4: *Assume $G_\alpha \subseteq P_\alpha$ is generic over V, and $G_{\alpha\varepsilon} \subseteq P_\varepsilon/G_\alpha$ is generic over $V[G_\alpha]$. Then*

(1) $G_{\alpha\varepsilon} \subseteq P_\varepsilon$ is a filter.
(2) $G_{\alpha\varepsilon}$ is generic for P_ε over V.
(3) $G_{\alpha\varepsilon} \supseteq G_\alpha$.
(4) $G_\alpha = G_{\alpha\varepsilon} \cap P_\alpha$.

Proof: (1) $G_{\alpha\varepsilon} \subseteq P_\varepsilon$ follows from $P_\varepsilon/G_\alpha \subseteq P_\varepsilon$. Any two conditions p, q in $G_{\alpha\varepsilon}$ have a common extension in $G_{\alpha\varepsilon}$, i.e., there is a $r \in G_{\alpha\varepsilon}$, $p \geq_{\alpha\varepsilon} r$, $q \geq_{\alpha\varepsilon} r$. Hence $p \geq_\varepsilon r$, $q \geq_\varepsilon r$.

(2): Let $A \subseteq P_\varepsilon$ be a maximal antichain in V. Then in V_α, $A \cap P_\varepsilon/G_\alpha$ is a maximal antichain by 4.3(3). So $G_{\alpha\varepsilon} \cap (A \cap P_\varepsilon/G_\alpha)$ and thus also $G_{\alpha\varepsilon} \cap A$ are nonempty.

Proof of (3): Assume $p \in G_\alpha$ but $p \notin G_{\alpha\varepsilon}$. Then there is a condition $q \in G_{\alpha\varepsilon}$ such that $q \perp_{\alpha\varepsilon} p$. As $p{\restriction}\alpha$, $q{\restriction}\alpha \in G_\alpha$, there is a condition $r \in G_\alpha$ such that $r \leq p{\restriction}\alpha, q{\restriction}\alpha$. Then $(q \wedge r){\restriction}\alpha = r \leq_\alpha p$, so $q \wedge r \leq_\varepsilon p$. We also have $q \wedge r \leq q$, and $q \wedge r \in P_\varepsilon/G_\alpha$. This contradicts $q \perp_{\alpha\varepsilon} p$.

(4): "$\subseteq$" follows from (3), and if $p \in G_{\alpha\varepsilon} \cap P_\alpha$, then $p \in P_\varepsilon/G_\alpha$, so $p = p{\restriction}\alpha \in G_\alpha$.

(Conversely, if $G_\varepsilon \subseteq P_\varepsilon$ is generic over V, and $G_\alpha := G_\varepsilon \cap P_\alpha$, then G_α is generic for over V, and $G_\varepsilon \subseteq P_\varepsilon/G_\alpha$ is generic over $V[G_\alpha]$.)

So we can write G_ε for $G_{\alpha\varepsilon}$.

Fact 4.5: *The map $i : P_\varepsilon \to P_\alpha * (P_\varepsilon/G_\alpha)$, defined by*

$$i(p) = \langle p{\restriction}\alpha, \widehat{p}\rangle$$

is a dense embedding. Hence, forcing with P_ε amounts to the same as first forcing with P_α, and then with the "quotient forcing" P_ε/G_α.

Proof: Note that $p{\restriction}\alpha \Vdash_\alpha$"$p \in P_\varepsilon/G_\alpha$", so $\mathrm{rng}(i) \subseteq P_\alpha * P_\varepsilon/G_\alpha$. i preserves the ordering, as $p \geq q$ clearly implies $q{\restriction}\alpha \Vdash \widehat{p} \geq \widehat{q}$.

To show that i is dense, let $\langle r, \underset{\sim}{s}\rangle \in P_\alpha * P_\varepsilon/G_\alpha$. So $r \Vdash_\alpha \underset{\sim}{s}{\restriction}\alpha \in G_\alpha$. Since $r \Vdash \underset{\sim}{s} \in P_\varepsilon$, we can find a stronger condition $r_0 \in P_\alpha$ and a condition $s_0 \in P_\varepsilon$ such that $r_0 \Vdash \underset{\sim}{s} = s_0$. As $r_0 \Vdash_\alpha s_0{\restriction}\alpha \in G_\alpha$, r_0 and $s_0{\restriction}\alpha$ are compatible, say $r_1 \leq r_0, s_0{\restriction}\alpha$.

Let $p := s_0 \wedge r_1$. Then $p \in P_\varepsilon$, $p \leq s_0$, so $\langle r, \underset{\sim}{s}\rangle \geq \langle r_0, \underset{\sim}{s}\rangle \geq \langle r_0, \widehat{s_0}\rangle \geq \langle r_1, \widehat{s_0}\rangle \geq \langle p{\restriction}\alpha, \widehat{p}\rangle$.

How can we describe the forcing notion P_ε/G_α without explicitly mentioning P_ε? For example, comparing the forcing notions P_α and $P_{\alpha+1}$, we have

$$P_{\alpha+1} \approx P_\alpha * Q_\alpha$$

and also

$$P_{\alpha+1} \approx P_\alpha * (P_{\alpha+1}/G_\alpha)$$

Similarly,

$$P_{\alpha+2} \approx P_\alpha * Q_\alpha * Q_{\alpha+1}$$

and also

$$P_{\alpha+2} \approx P_\alpha * (P_{\alpha+2}/G_\alpha)$$

This suggests that the forcing notions $P_{\alpha+\beta}/G_\alpha$ are isomorphic to iterations of length β. Theorem 4.6 below shows that this is indeed the case.

Theorem 4.6: *Assume $\langle P_\alpha, Q_\alpha : \alpha < \varepsilon\rangle$ is a countable support iteration of proper forcing notions, or a finite support iteration. Let $\alpha + \beta = \varepsilon$. Then there exists a P_α name $\langle \bar{P}_\gamma, \bar{Q}_\gamma : \gamma < \beta\rangle$ of a countable/finite support iteration of length β, such that*

$$\Vdash_\alpha \forall \gamma \leq \beta : \bar{P}_\beta \approx P_\varepsilon/G_\alpha$$

Proof: We will fix α, and proceed by induction on β. We will work in $V[G_\alpha]$, where $G_\alpha \subseteq P_\alpha$ is generic over V.

By induction on β, we will

(1) define a countable/finite support iteration $\langle \bar{P}_\gamma, \bar{Q}_\gamma : \gamma < \beta \rangle$
(2) define a map $i_{\alpha\beta}$ from $P_{\alpha+\beta}/G_\alpha$ into $\bar{P}_\beta$
(3) prove that $i_{\alpha\beta}$ is a dense* embedding.

Case 1: $\beta = 0$

(1) $\langle \bar{P}_\gamma, \bar{Q}_\gamma : \gamma < \beta \rangle$ is the empty sequence, and $\bar{P}_\beta = P_0 = \{\emptyset\}$
(2) $i_{\alpha\beta}$ is the constant map.
(3) Any two conditions in $P_{\alpha+0}/G_\alpha$ are compatible, as are any two conditions in $\bar{P}_0$. $p_1 \leq^* p_2$ is true for all $p_1, p_2 \in P_\alpha/G_\alpha$.

Case 2: $\beta = \beta' + 1$

(1) We have to define $\bar{Q}_{\beta'}$.
In V, $\underset{\sim}{Q}_{\alpha+\beta'}$ is a $P_{\alpha+\beta'}$-name for a forcing notion, which can be translated into a P_α-name for a $P_{\alpha+\beta'}/G_\alpha$-name. So in $V[G_\alpha]$, $\underset{\sim}{Q}_{\alpha+\beta'}$ is a $P_{\alpha+\beta'}/G_\alpha$-name, which by induction hypothesis can be translated to a $\bar{P}_{\beta'}$-name $\underset{\sim}{\bar{Q}}_{\beta'}$.
(2) For $p \in P_{\alpha+\beta}/G_\alpha$, we define $i_{\alpha\beta}(p) \in \bar{P}_\beta$ by defining $i_{\alpha\beta}(p) {\restriction} \beta'$ and $i_{\alpha\beta}(p)(\beta')$:
$$i_{\alpha\beta}(p) {\restriction} \beta' = i_{\alpha\beta'}(p {\restriction} (\alpha + \beta'))$$
If $p(\alpha + \beta')$ is undefined, we let $i_{\alpha\beta}(p)(\beta')$ be undefined. Otherwise, $p(\alpha + \beta')$ is (in $V[G_\alpha]$) a $P_{\alpha+\beta'}/G_\alpha$-name for an element of $Q_{\alpha+\beta'}$, which can be translated to a $\bar{P}_{\beta'}$-name for an element of $\bar{Q}_{\beta'}$. This will be $i_{\alpha\beta}(p)(\beta')$. Or, more sloppily: $i_{\alpha\beta}(p)(\beta' = p(\alpha + \beta')$. Note that $i_{\alpha\beta}$ extends $i_{\alpha\beta'}$.
(3) To see that $i_{\alpha\beta}$ is an embedding, check that for $p_1, p_2 \in P_{\alpha+\beta}$,
$$p_1 \geq^* p_2 \quad \text{iff} \quad p_1 {\restriction} \alpha{+}\beta' \geq^* p_2 {\restriction} \alpha{+}\beta' \ \& \\ \& \ p_2 {\restriction} \alpha{+}\beta' \Vdash p_1(\alpha{+}\beta') \geq^* p_2(\alpha{+}\beta')$$
which by induction hypothesis and using translation functions is equivalent to
$$p_1 \geq^* p_2 \quad \text{iff} \quad i_{\alpha\beta}(p_1) {\restriction} \beta' \geq^* i_{\alpha\beta}(p_2) {\restriction} \beta' \ \& \\ \& \ i_{\alpha\beta}(p_2) {\restriction} \beta' \Vdash i_{\alpha\beta}(p_1)(\beta') \geq^* i_{\alpha\beta}(p_2)(\beta')$$
i.e., $i_{\alpha\beta}(p_1) \geq^* i_{\alpha\beta}(p_2)$.
It is also easy to see that $i_{\alpha\beta}$ is dense*.

Case 3: β is a limit ordinal.

(1) We define $\bar{P}_\beta$ as the (countable/finite support) limit.
(2) We let $i_{\alpha\beta}(p) := \bigcup_{\gamma<\beta} i_{\alpha\gamma}$.
(3) As before, we can show $p \geq^* q$ iff $\forall \gamma < \beta$, $p {\restriction} \alpha + \gamma \geq^* q {\restriction} \alpha + \gamma$ iff $\forall \gamma < \beta$ $i_{\alpha\beta}(p) {\restriction} \gamma \leq^*$ etc.

The crucial point is the density condition 0.8(2). So let $\bar{p} \in \bar{P}_\beta$.

First we will deal with the case of a finite support iteration. So $\bar{p} \in \bar{P}_\gamma$, for some $\gamma < \beta$. By induction hypothesis we can find $p \in P_{\alpha+\gamma}/G_\alpha$ such that $i_{\alpha\gamma}(p) \leq^*_{\bar{P}_\gamma} \bar{p}$.

Then we also have $p \in P_{\alpha+\beta}/G_\alpha$, and $i_{\alpha\beta}(p) = i_{\alpha\gamma}(p) \leq_{\bar{P}_\beta} \bar{p}$.

Now assume that we are working with a countable support iteration of proper forcing notions. (If β has cofinality $> \omega$ in $V[G_\alpha]$, we can repeat the previous proof verbatim.)

In general, let $\bar{p} \in \bar{P}_\beta$. In $V[G_\alpha]$, $\mathrm{dom}(p)$ is a countable set of ordinals, whichis not necessarily an element of V.

Here we can use the fact that P_α is proper: By 3.13, $\mathrm{dom}(\bar{p})$ is covered by a countable set of ordinals that itself lies in V. So there is a countable set $F \in V$ such that $\mathrm{dom}(\bar{p}) \subseteq \{\gamma : \alpha + \gamma \in F\}$.

Find a name $\underset{\sim}{\bar{p}} \in V$ for $\bar{p}$. From $\underset{\sim}{\bar{p}}$ and F we can define a condition $p \in P_{\alpha+\beta}$, $p \in V$ as follows:

$\mathrm{dom}(p) = F$.

If $\alpha + \gamma \in F$, then let $p(\alpha+\gamma)$ be a name such that (in V)

$$\Vdash_{\alpha+\gamma} \text{“If } \underset{\sim}{\bar{p}}(\gamma) \in Q_{\alpha+\gamma}, \text{ then } p(\alpha+\gamma) = \underset{\sim}{\bar{p}}(\gamma)$$
$$\text{otherwise } p(\alpha+\gamma) = \mathbb{1}_{Q_{\alpha+\gamma}}.$$

(We use the translation function here.)

Because of F we know that p is indeed a function with countable domain. So $p \in P_\beta$, and in fact $\Vdash_\alpha p \in P_\beta/G_\alpha$.

Now we have to show that $i_{\alpha\beta}(p) \leq \bar{p}$. We have to prove by induction on γ that

$$\forall \gamma \leq \beta \ i_{\alpha\beta}(p){\restriction}\gamma = i_{\alpha\beta}(p{\restriction}\alpha+\gamma) \leq^* \bar{p}{\restriction}\gamma \tag{$*$}$$

We leave this as an exercise to the reader (use 1.7 and 1.8).

5 A general preservation theorem for countable support iteration

Context 5.1: We will consider functions f from ω to ω. In applications, these functions may actually be from ω to HF (the hereditarily finite sets), or from $\omega^{<\omega}$ to $\omega^{<\omega}$, etc. Since we can trivially (in a primitive recursive, absolute, ...) way code such functions by functions in ${}^\omega\omega$, all results from this section apply also to functions in ${}^\omega HF$, etc.

We fix a closed set $\mathbf{C} \subseteq {}^\omega\omega$. There is a tree $T \subseteq \omega^{<\omega}$ such that

$$\mathbf{C} = \{f \in {}^\omega\omega : \forall n\, f{\restriction}n \in T\}.$$

When we work in a universe V_1, we write $\mathbf{C}$ for the set $\{f \in {}^\omega\omega \cap V_1 : \forall n\, f{\restriction}n \in T\}$, i.e., we regard $\mathbf{C}$ not as a set per se, but as a formula defining a certain (closed) set.

Typical examples are $\mathbf{C} = {}^\omega\omega$, or $\mathbf{C} = \{f \in {}^\omega\omega : \forall n f(n) < H(n)\}$ for some $H \in {}^\omega\omega$.

$\langle \sqsubset_n : n \in \omega\rangle$ is an increasing sequence of two place relations on ${}^\omega\omega$. In general, we do not assume that these relations are transitive.

$\sqsubset_n$ will always be given by an arithmetical definition. Again, we consider $\sqsubset_n$ not as a set in itself, but rather as a definition of a certain arithmetical set, so if $\sqsubset_n$ is defined in a universe V_0, $f \sqsubset_n g$ makes sense even for $f, g \notin V_0$. These definitions will be absolute between any two $\in$-models, as the formula defining $\sqsubset$ is arithmetical.

(We do not require that the $\sqsubset_n$ are uniformly arithmetical, i.e., each $\sqsubset_n$ may be defined by a different formula. However, in all our applications there $f \sqsubset_n g$ is expressed by a single formula $\varphi(f, g, n)$.)

We let $\sqsubset = \bigcup_n \sqsubset_n$.

A typical example is given by $f \sqsubset_n g$ iff $\forall k \geq n\ f(n) < g(n)$. Then we have $f \sqsubset g$ iff $f \leq^* g$. (Actually, this example is not so "typical" since here the relations $\sqsubset_n$ are all transitive, which will not be true in general.)

Definition 5.2: *We will only consider forcing notions that add a real, or at least introduce a new ω-sequence of ordinals. If $p \Vdash \underset{\sim}{\tau} : \omega \to Ord$, we say that p decides $\underset{\sim}{\tau}{\restriction}n$, if for some $t : n \to Ord$, $p \Vdash \underset{\sim}{\tau}{\restriction}n = t$.*

Note that if $\Vdash_Q \underset{\sim}{\tau} : \omega \to Ord$ and $\Vdash_Q \underset{\sim}{\tau} \notin V$, then letting

$$E_n = \{p \in Q : p \text{ decides } \underset{\sim}{\tau}{\restriction}n\},$$

E_n is a dense open subset of Q and $\bigcap_n E_n$ is empty.

Assumption 5.3: *We assume that for all countable sets $a \subseteq {}^\omega\omega$ there is a g such that $\forall f \in a \cap \mathbf{C}\ f \sqsubset g$.*

We also assume that for every g and every n the set $\{f : f \sqsubset_n g\}$ is closed.

Definition 5.4: *We say that g ($\sqsubset$, $\mathbf{C}$)-covers) N if for all $f \in N \cap \mathbf{C}$ we have $f \sqsubset g$. (Usually we think of N as a countable elementary submodel of some $H(\chi)$.) When C and/or $\sqsubset$ are clear, we may just say "g covers N".*

We let χ be a "large enough" regular cardinal as in 3.5. We will consider countable elementary submodels of $\langle H(\chi), \in\rangle$.

(The notions we will define below will depend on χ, but a careful examination (which we do will not carry out here) shows that this dependence is only apparent.)

Definition 5.5: *We say that the forcing notion Q* **almost preserves** $\sqsubset$, *if*

WHENEVER $N \prec (H(\chi), \in)$ is a countable model containing Q, $\mathbf{C}$ and $\sqsubset$, g covers N, $p \in Q \cap N$,

THEN there exists an N-generic condition q stronger than p, such that $q \Vdash$ "g covers $N[G]$."

Fact 5.6: *If Q almost preserves $\sqsubset$, then $\Vdash_Q$ "$\forall f \in V[G]\, \exists g \in V\ f \sqsubset g$".*

Proof: Assume not, so there is a condition p and a name $\underset{\sim}{f}$ such that

$$p \Vdash \text{there is no } g \in V \text{ such that } \underset{\sim}{f} \sqsubset g.$$

Let $N \prec H(\chi)$ be a countable model containing $\underset{\sim}{f}$ and p, and let g $(\in V)$ cover N. Then we can find a condition $q \leq p$ forcing $\underset{\sim}{f} \sqsubset g$, a contradiction.

Example 5.7: *Let $\mathbf{C} = {}^\omega\omega$, and let (for $f, g \in {}^\omega\omega$)*

$$f \sqsubset_n g \ \Leftrightarrow\ \forall k \geq n\, f(k) < g(k)$$

and $f \sqsubset g$ iff for some n, $f \sqsubset_n g$. All the $\sqsubset_n$ are closed and transitive, and clearly for every countable set $a \subseteq {}^\omega\omega$ there is a g such that $\forall f \in a\ f \sqsubset g$.

A forcing notion Q is called ${}^\omega\omega$-bounding iff

$$\Vdash_Q \forall f \in {}^\omega\omega \cap V[G]\, \exists g \in {}^\omega\omega \cap V\ f \sqsubset g$$

Fact: A proper forcing notion Q almost preserves $\sqsubset$ iff it is ${}^\omega\omega$-bounding. (We will see later that we can omit the "almost" from this statement.)

Proof of the fact: If Q preserves $\sqsubset$, then Q is ${}^\omega\omega$-bounding by 5.6. Conversely, assume that Q is ${}^\omega\omega$-bounding, let $N \prec H(\chi)$, and assume that g covers N. Then $\Vdash \forall f \in N[G]\, \exists f' \in V : f \sqsubset f'$, so any N-generic condition q forces $\forall f \in N[G] \exists f' \in N : f \sqsubset f'$. Since for all $f' \in N$, $f' \sqsubset g$ (and $\sqsubset$ is absolute and transitive), for any N-generic q we have

$$q \Vdash \forall f \in N[G] : f \sqsubset g$$

So this definition 5.5 achieves exactly what we want. However, it is not clear whether this condition by itself is preserved under iteration. Before we give the definition we will actually use, we have recall the concept of "interpretation":

Definition 5.8: *Assume Q is a forcing notion, $\underset{\sim}{f}_0, \ldots, \underset{\sim}{f}_{k-1}$ are Q-names of functions in $\mathbf{C}$, $f^*_0, \ldots, f^*_{k-1}$ are functions in ${}^\omega\omega$, $\langle p_n : n < \omega\rangle$ an increasing sequence of conditions. We will write $\underset{\sim}{\bar f}$ for $\langle \underset{\sim}{f}_0, \ldots, \underset{\sim}{f}_{k-1}\rangle$ and $\bar f^*$ for $\langle f^*_0, \ldots, f^*_{k-1}\rangle$. A sequence $\langle \underset{\sim}{f}_0, \ldots, \underset{\sim}{f}_k\rangle$ will be written as $\langle \underset{\sim}{\bar f}, \underset{\sim}{f}_k\rangle$.*

We say that $\langle p_n : n < \omega\rangle$ interprets $\underset{\sim}{\bar f}$ as $\bar f^*$, if for all $i < k$, for all n, $p_n \Vdash \underset{\sim}{f}_i{\restriction}n = f^*_i{\restriction}n$.

(So when we say that $\langle p_n : n < \omega\rangle$ interprets $\underset{\sim}{\bar f}$ as $\bar f^*$, it is understood that the sequence $\langle p_n : n < \omega\rangle$ is increasing, etc.)

Remark 5.9: *If $\langle p_n : n < \omega\rangle$ interprets $\underset{\sim}{f}$ as f^*, where $\underset{\sim}{f}$ is a name for a function in $\mathbf{C}$, then f^* is a function in $\mathbf{C}$.*

Proof: $\mathbf{C} = \{f : \forall n\, f{\restriction}n \in T\}$ for some tree T. If $f^* \notin \mathbf{C}$, then for some n $f^*{\restriction}n \notin T$, so $p_n \Vdash \underset{\sim}{f}{\restriction}n \notin T$, a contradiction.

Definition 5.10: *We say that the sequence $\langle p_n : n < \omega\rangle \in {}^{\omega}Q$ is inconsistent, if there is no condition q such that $\forall n\, q \Vdash p_n \in G$ or equivalently, $\Vdash_Q \exists n : p_n \notin G_Q$. Note that if $\langle D_{Q,n} : n \in \omega\rangle$ is a sequence of dense open sets with $\bigcap_n D_{Q,n} = \emptyset$ and $\forall n\, p_n \in D_{Q,n}$, then $\langle p_n : n < \omega\rangle$ is inconsistent.*

(By our assumption in 5.2, for every forcing notion Q_α that we consider there is a sequence $\langle D_{Q,n} : n < \omega\rangle$ as above.)

Definition 5.11: *We say that the forcing notion Q* **preserves** $(\sqsubset, \mathbf{C})$*, if for some x:*

WHENEVER $N \prec H(\chi)$ is a countable model containing Q, x and $\sqsubset$, g covers N, $p_0 \in Q \cap N$, and whenever $\langle p_n : n < \omega\rangle \in {}^{\omega}Q \cap N$ (an increasing sequence of conditions) interprets $\underset{\sim}{\bar f} \in N$ as $\bar f^*$, such that for all $i < k$ $f^*_i \sqsubset_{n_i} g$,

THEN there is a condition $q \in Q$ such that

- **(a)** $q \le p$.
- **(b)** *q is N-generic.*
- **(c)** $q \Vdash \forall f \in N[G] \cap \mathbf{C}$ $f \sqsubset g$, *i.e.,* $q \Vdash$ *"g covers $N[G]$."*
- **(d)** $\forall i < k$: $q \Vdash \underset{\sim}{f}_i \sqsubset_{n_i} g$.

Note that (a)–(c) just say that Q almost preserves $\sqsubset$. Also note that (c) is equivalent to

- **(c')** $\forall \underset{\sim}{f} \in N$: If $\Vdash_Q \underset{\sim}{f} \in \mathbf{C}$, then $q \Vdash \underset{\sim}{f} \sqsubset g$.

When $\mathbf{C}$ is clear from the context (or irrelevant), we may say "Q preserves $\sqsubset$" instead of "Q preserves $(\sqsubset, \mathbf{C})$".

We call $x = x_Q$ the "witness" for the statement "Q preserves $\sqsubset$."

Lemma 5.12: *If Q_0 preserves $\sqsubset$, and $\Vdash_{Q_0}$ "$\underset{\sim}{Q}_1$ preserves $\sqsubset$", then $Q_0 * \underset{\sim}{Q}_1$ preserves $\sqsubset$.*

Idea of the proof: To use the assumptions on Q_0 and Q_1, we have to find a Q_0-name $\underset{\sim}{\bar f}'$ that "interpolates" between $\underset{\sim}{\bar f}$ and $\underset{\sim}{\bar f}^*$, i.e., $\underset{\sim}{\bar f}^*$ is an interpretation of $\underset{\sim}{\bar f}'$ which itself (in V^{Q_0}) is an interpretation of $\underset{\sim}{\bar f}$.

Proof: Let x_0 witness that Q_0 preserves $\sqsubset$, and $\Vdash$"$\underset{\sim}{x}_1$ witnesses that Q_1 preserves $\sqsubset$". $x := \langle x_0, \underset{\sim}{x}_1\rangle$ will witness that $Q_0 * Q_1$ preserves $\sqsubset$.

Assume N, $\langle p_n : n < \omega\rangle$, $\underset{\sim}{\bar f}$, $\bar f^*$, $n_0, \dots, n_{k-1}$ are as in 5.11, and g covers N. Each p_n is of the form $\langle p_n(0), p_n(1)\rangle$, where $p_n(0) \in Q_0$ and $p_n(0) \Vdash_{Q_0} p_n(1) \in \underset{\sim}{Q}_1$. We let $p_{-1} := \langle \mathbb{1}_{Q_0}, \mathbb{1}_{\underset{\sim}{Q}_1}\rangle$.

Let $G(0) \subseteq Q_0$ be generic, then we can define functions f'_i in $V[G(0)]$ and conditions $p'_n(1)$ in Q_1 for $i < k$ as follows:

> If for all $n \in \omega$ $p_n(0) \in G(0)$, then $f'_i := f^*_i$, $n^* := \omega$, $p'_n(1) := p_n(1)$.
> Otherwise, let $n^* := \max\{n \geq -1 : p_n(0) \in G(0)\}$, and find a sequence $\langle p'_n(1) : n \in \omega\rangle \in {}^{\omega}Q_1$ and functions f'_i for $i < k$ such that $p'_0(1) \geq p_{n^*}(1)$ and
> $$\langle p'_n(1) : n < \omega\rangle \text{ interprets } \underset{\sim}{\bar f} \text{ as } \bar f' \tag{$*1$}$$
> (Note that $(*1)$ will be true also if $n^* = \omega$.)

Then $f'_i{\restriction}n^* = f^*{\restriction}n^*$: This is clear if $n^* = \omega$. Otherwise, $p'_0(1) \Vdash_{Q_1}$ "$\underset{\sim}{f}_i{\restriction}n^* = f^*_i{\restriction}n^*$," so we also have $p'_{n^*}(1) \Vdash_{Q_1}$ "$\underset{\sim}{f}_i{\restriction}n^* = f^*_i{\restriction}n^*$."

Furthermore, $p'_{n^*}(1) \Vdash$ "$\underset{\sim}{f}_i{\restriction}n^* = f'_i{\restriction}n^*$", so $p'_{n^*}(1) \Vdash$ "$f'_i{\restriction}n^* = f^*_i{\restriction}n^*$."

Coming back to V, we can find Q_0-names $\underset{\sim}{f}'_i$, $\underset{\sim}{p}'_n(1)$, $\underset{\sim}{n}^*$ such that the above is forced. In particular,
$$p_n(0) \Vdash f'_i{\restriction}n = f^*_i{\restriction}n, \tag{$*0$}$$
as $p_n(0) \Vdash \underset{\sim}{n}^* \geq n$.

We can find these names $\underset{\sim}{f}'_i$ and $\underset{\sim}{p}'_n$ in N. Since Q_0 preserves $\sqsubset$, by $(*0)$ there is a generic $q(0) \in Q_0$ such that

(a_0) $q(0) \geq p_0(0)$.
(b_0) $q(0)$ is N-generic
(c_0) $q(0) \Vdash \forall f \in N[G(0)]\ f \sqsubset g$.
(d_0) $\forall i < k$: $q(0) \Vdash \underset{\sim}{f}'_i \sqsubset_{n_i} g$.

Working again in $V[G(0)]$, where $q(0) \in G(0)$, we note that $N[G(0)]$ is covered by g and $n^* \geq 0$, so since Q_1 preserves $\sqsubset$ and by $(*1)$, we can find a condition $q(1) \in Q_1$ such that

(a_1) $q(1) \leq p'_0(1) \leq p_0(1)$.
(b_1) $q(1)$ is $N[G(1)]$-generic
(c_1) $q(1) \Vdash \forall f \in N[G(0) * G(1)]\ f \sqsubset g$.
(d_1) $\forall i < k$: $q(1) \Vdash \underset{\sim}{f}_i \sqsubset_{n_i} g$.

We can find a name $\underset{\sim}{q}(1)$ such that the above is forced by $q(0)$. Now we let $q = \langle q(0), \underset{\sim}{q}(1)\rangle$. Then 5.11(a)–(d) holds.

Recall that $P_\beta/G_\alpha = \{p \in P_\beta : p{\restriction}\alpha \in G_\alpha\}$, and that the map $i : P_\beta \to P_\alpha * P_\beta/G_\alpha$, defined by $i(p) = \langle p{\restriction}\alpha, p\rangle$ is a dense embedding. So to each P_β name $\underset{\sim}{f}$ there is a naturally corresponding P_α-name for a P_β/G_α-name, which we also call $\underset{\sim}{f}$.

For the following, assume that $\langle P_\alpha, Q_\alpha : \alpha < \varepsilon\rangle$ is a countable support iteration $\sqsubset$-preserving forcing notions, i.e., for all $\alpha < \varepsilon$,

$$P_\alpha \Vdash \text{“}Q_\alpha \text{ preserves } \sqsubset (\text{witnessed by } \underset{\sim}{x}_\alpha.\text{”}$$

Also assume that each Q_α adds a new sequence of ordinals.

Our goal is to show that P_ε preserves $\sqsubset$ (witnessed by $x := \langle \underset{\sim}{x}_\alpha : \alpha < \varepsilon\rangle$. So we fix a countable elementary model N containing all relevant information.

Induction Lemma 5.13: *Let $\alpha \le \beta \le \varepsilon$. Assume that $\langle \underset{\sim}{p}_n : n < \omega\rangle \in N$ is a sequence of P_α-names for conditions in P_β/G_α such that*

$(\cdot)$ $\Vdash_\alpha \langle \underset{\sim}{p}_n : n < \omega\rangle$ interprets $\underset{\sim}{\bar f}$ as $\underset{\sim}{\bar f}^*$

(where $\underset{\sim}{\bar f} = \langle \underset{\sim}{f}_0, \dots, \underset{\sim}{f}_{k-1}\rangle$, $\underset{\sim}{\bar f}^* = \langle \underset{\sim}{f}^*_0, \dots, \underset{\sim}{f}^*_{k-1}\rangle$, the $\underset{\sim}{f}_i$ are P_β-names in N, and the $\underset{\sim}{f}^*_i$ are P_α-names). Furthermore, assume that $q \in P_\alpha$, and $\underset{\sim}{n}_0, \dots, \underset{\sim}{n}_{k-1}$ are P_α-names for integers such that for some g:

- *(a) $q \Vdash_\alpha \underset{\sim}{p}_0{\restriction}\alpha \in G_\alpha$. (This really follows from our assumption that $\Vdash_\alpha \underset{\sim}{p}_0 \in P_\beta/G_\alpha$.)*
- *(b) $q \in P_\alpha$ is N-generic.*
- *(c) $q \Vdash \forall f \in N[G_\alpha]\ f \sqsubset g$.*
- *(d) $\forall i < k$: $q \Vdash \underset{\sim}{f}^*_i \sqsubset_{\underset{\sim}{n}_i} g$.*

Then there exists $q^+ \in P_\beta$, satisfying

- *$(+)$ $q^+{\restriction}\alpha = q$*
- *$(a)^+$ $q^+ \Vdash_\beta \underset{\sim}{p}_0{\restriction}\beta \in G_\beta$.*
- *$(b)^+$ $q^+ \in P_\beta$ is N-generic (i.e., $q^+ \Vdash_\beta$ “$G_\beta \cap N$ is generic over N”.)*
- *$(c)^+$ $q^+ \Vdash \forall f \in N[G_\beta]\ f \sqsubset g$.*
- *$(d)^+$ $\forall i < k$: $q^+ \Vdash \underset{\sim}{f}_i \sqsubset_{\underset{\sim}{n}_i} g$.*

The reason for considering $\sqsubset$-preserving forcing notions is the following corollary:

Corollary 5.14: *If $\forall \alpha \Vdash_{P_\alpha}$ “Q_α preserves $\sqsubset$”, then P_ε preserves $\sqsubset$.*

Proof of the corollary: Use the induction lemma with with $\alpha = 0$, $\beta = \varepsilon$, $q = \mathbb{1}_{P_0}$. We can find $q^+ \in P_\varepsilon$ satisfying (a)$^+$–(d)$^+$. Since $q^+ \Vdash p_0 \in G_\varepsilon$, there is a condition $q^{++} \le q^+$, $q^{++} \le p_0$. Then q^{++} will satisfy 5.11(a)–(d).

To prove the case “β limit” of the induction lemma, we will need the following lemma. (Note that, as in the preliminary lemma for the proof of preservation of properness, this lemma does not mention properness, $\sqsubset$-preservation, etc. — not even countable models.)

Preliminary lemma 5.15: *Assume $\alpha_1 < \alpha_2 \le \beta$, $\langle \underset{\sim}{p}^1_n : n < \omega\rangle$ is a sequence of P_{α_1}-names for conditions in P_β/G_{α_1}. Let $D \subseteq P_\beta$ be a dense open set, $j \in \omega$.*

Assume $\langle \underset{\sim}{\bar f}, \underset{\sim}{f}_k\rangle = \langle \underset{\sim}{f}_0, \dots, \underset{\sim}{f}_k\rangle$ are P_β-names, and $\underset{\sim}{\bar f}^1 = \langle \underset{\sim}{f}^1_0, \dots, \underset{\sim}{f}^1_{k-1}\rangle$ are P_{α_1}-names, and

(1) $\Vdash_{\alpha_1} \langle \underset{\sim}{p}^1_n : n < \omega\rangle$ interprets $\underset{\sim}{\bar f}$ as $\underset{\sim}{\bar f}^1$.

(2) $\Vdash_{\alpha_1} \langle \underset{\sim}{p}^1_n(\alpha_1) : n < \omega\rangle$ is inconsistent.

THEN there are: P_{α_2}-names $\langle \underset{\sim}{\bar f}^2, \underset{\sim}{f}^2_k\rangle = \langle \underset{\sim}{f}^2_0, \dots, \underset{\sim}{f}^2_k\rangle$ and a sequence $\langle \underset{\sim}{p}^2_n : n < \omega\rangle$ of P_{α_2}-names for conditions in P_β/G_{α_2} such that

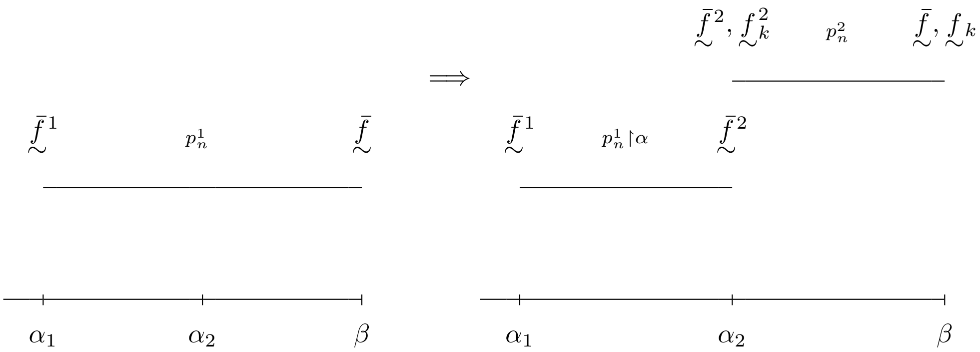
$(1)^+$ $\Vdash_{\alpha_2}$ "$\langle \underset{\sim}{p}^2_n : n < \omega\rangle$ interprets $\langle \underset{\sim}{\bar f}, \underset{\sim}{f}_k\rangle$ as $\langle \underset{\sim}{\bar f}^2, \underset{\sim}{f}^2_k\rangle$" and
$\Vdash_{\alpha_2}$ "$\underset{\sim}{p}^2_0$ decides $\underset{\sim}{f}_0{\restriction}j, \dots, \underset{\sim}{f}_k{\restriction}j$."

$(2)^+$ $\Vdash_{\alpha_2} \langle \underset{\sim}{p}^2_n(\alpha_2) : n < \omega\rangle$ is inconsistent.

$(3)^+$ $\Vdash_{\alpha_1} \langle \underset{\sim}{p}^1_n{\restriction}\alpha_2 : n < \omega\rangle$ interprets $\underset{\sim}{\bar f}^2$ as $\underset{\sim}{\bar f}^1$.

$(4)^+$ $\Vdash_{\alpha_2}$ If $\underset{\sim}{p}^1_n{\restriction}\alpha_2 \in G_{\alpha_2}$, then $\underset{\sim}{p}^2_0 \le \underset{\sim}{p}^1_n$.

$(5)^+$ $\Vdash_{\alpha_2} \underset{\sim}{p}^2_0 \in D$.



Proof of the preliminary lemma:

We will work in $V[G_{\alpha_2}]$. We write p^1_n for $\underset{\sim}{p}^1_n[G_{\alpha_1}]$. Let $p^1_{-1} := \mathbb{1}_{P_\beta}$. Let n^* be defined by

(i) $n^* := \max\{n \ge -1 : \underset{\sim}{p}^1_n[G_{\alpha_1}]{\restriction}\alpha_2 \in G_{\alpha_2}\}$.

(By 5.15(2), not all $p^1_n{\restriction}\alpha_2$ can be in G_{α_2}.)

Let $p^2_{-1} = p^1_{n^*} \in P_\beta/G_{\alpha_2}$. By assumption on Q_α, there is a sequence $\langle E_n : n < \omega\rangle$ of dense open subsets of Q_{α_2} such that $\bigcap_n E_n$ is empty. Let $\bar E_n := \{r \in P_\beta/G_{\alpha_2} : r(\alpha_2) \in E_n\}$. Then $\bar E_n$ is a dense open subset of P_β/G_{α_2}, and $\bigcap_n \bar E_n$ is empty.

For each n, let

$$D^n := D \cap \bar{E}_n \cap \{r \in P_\beta/G_{\alpha_2} : r \text{ decides } \underset{\sim}{f}_0 \restriction \max\{n,j\}, \dots, \underset{\sim}{f}_k \restriction \max\{n,j\}\}.$$

Then D^n is open dense in P_β/G_{α_2}. By induction we can now find a sequence of conditions p^2_n in P_β/G_{α_2} such that for all $n \in \omega$:

(ii) p^2_n in P_β/G_{α_2}
(iii) $p^2_{n-1} \geq p^2_n$. (So if $p^1_n \restriction \alpha_2 \in G_{\alpha_2}$, then $p^1_n \geq p^1_{n^*} = p^2_{-1} \geq p^2_n$.)
(iv) $p^2_n \in D^n$.

Since $D^n \subseteq E_n$, (iv) implies that $\langle p^2_n(\alpha_2) : n < \omega\rangle$ is inconsistent.

By (iii) and (iv), there are functions f^2_i such that for all $i \leq k$, all n:

(v) $p^2_n \Vdash_{\alpha_2,\beta} \underset{\sim}{f}_i \restriction \max\{n,j\} = f^2_i \restriction \max\{n,j\}$.

Coming back to V, we can find P_{α_2}-names $\underset{\sim}{p}^2_n$, $\underset{\sim}{f}^2_i$, $\underset{\sim}{n}^*$ such that (i)–(v) are forced by the empty condition of P_{α_2}.

Note that (v) implies that $5.15(1)^+$ will be satisfied, (iv) implies $5.15(2)^+$ and $5.15(5)^+$, and (iii) implies $5.15(4)^+$.

To show $5.15(3)^+$, let $G_{\alpha_1} \subseteq P_{\alpha_1}$ be any generic filter, $n \in \omega$, $i < k$. We will write p^1_n for $\underset{\sim}{p}^1_n[G_{\alpha_1}]$, f^1_i for $\underset{\sim}{f}^1_i[G_{\alpha_1}]$.

We claim that

$$V[G_{\alpha_1}] \models p^1_n \restriction \alpha_2 \Vdash_{\alpha_1,\alpha_2} \underset{\sim}{f}^2_i \restriction n = f^1_i \restriction n$$

Proof of the claim: Let $H \subseteq P_{\alpha_2}/G_{\alpha_1}$ be a $V[G_{\alpha_1}]$-generic filter containing $p^1_n \restriction \alpha_2$. Then $H \cap P_{\alpha_1} = G_{\alpha_1}$, and H is generic for P_{α_2} over V, so we will write G_{α_2} for H. Let $f^2_i = \underset{\sim}{f}^2_i[G_{\alpha_2}]$.

We have to check $V[G_{\alpha_2}] \models f^2_i \restriction n = f^1_i \restriction n$. It is enough to show that

$$V[G_{\alpha_2}] \models p^2_n \Vdash_{P_\beta/G_{\alpha_2}} f^2_i \restriction n = \underset{\sim}{f}_i \restriction n = f^1_i \restriction n.$$

The first equality is clear by the definition of $\underset{\sim}{f}^2_i$. To prove the second, let G_β be a $V[G_{\alpha_2}]$-generic filter on P_β/G_{α_2} containing p^2_n. Again, $G_\beta \supseteq G_{\alpha_2}$ is also generic for P_β over V, and it contains p^1_n. (Remember that $p^1_n \restriction \alpha_2 \in G_{\alpha_2}$, so by $5.15(4)^+$, $p^2_n \geq p^1_n$.) Hence $V[G_\beta] \models \underset{\sim}{f}_i[G_\beta] \restriction n = f^1_i \restriction n$.

Proof of the induction lemma: We proceed by induction on β.

The successor step is similar to the proof of 5.12:

Assume $\langle \underset{\sim}{p}_n : n \in \omega\rangle$ is a sequence of P_α-names for conditions in $P_{\beta+1}/G_\alpha$, interpreting $\underset{\sim}{\bar{f}}$ as $\underset{\sim}{\bar{f}}^*$ in $V[G_\alpha]$. (I.e., $\underset{\sim}{\bar{f}}^*$ is a P_α-name, and $\underset{\sim}{\bar{f}}$ is a $P_{\beta+1}$-name which we identify with the corresponding P_α-name for a $P_{\beta+1}$-name.)

Working in V_β, we can define $\bar{f}'$, n^*, $\langle p'_n : n \in \omega\rangle$ such that the following hold:

· $n^* = \sup(\{n \in \omega : p_n \restriction \beta \in G_\beta\} \cup \{-1\})$ (also $n^* = \omega$ is possible)
· $\langle p'_n : n \in \omega\rangle$ is an increasing sequence of conditions in Q_β, interpreting $\underset{\sim}{\bar{f}}$ as $\bar{f}'$.

· If $n^* \in \omega$, then $p'_0 \leq p_{n^*}(\beta)$.
· If $n^* = \omega$, then $\bar{f}' = \underset{\sim}{\bar{f}}^*$.

Coming back to V, we can find P_β-names $\underset{\sim}{n}^*$, etc. Now $\langle \underset{\sim}{p}_n\restriction\beta : n \in \omega\rangle$ is a sequence of names for conditions in P_β/G_α forced to interpret $\underset{\sim}{\vec{f}'}$ as $\underset{\sim}{\bar{f}}^*$, and $\langle \underset{\sim}{p}'_n : n \in \omega\rangle$ is a sequence of P_β-names for conditions in Q_β forced to interpret $\underset{\sim}{\bar{f}}$ as $\underset{\sim}{\vec{f}'}$. So we can use the induction hypothesis on α, β and the assumption on Q_β to obtain $q^+\restriction\beta$ and $q^+(\beta)$, respectively.

This ends the proof of the successor case.

Let β be a limit ordinal.

Let $\langle D_n : n < \omega\rangle$ enumerate all dense open subsets of P_β that are in N, where $D_0 = P_\beta$. Let $\delta := \sup(N \cap \beta)$, $\delta = \bigcup_j \alpha_j$, $\alpha = \alpha_0 < \alpha_1 < \cdots$, $\langle \alpha_j : j \in \omega\rangle \in N$. Fix P_β-names $\underset{\sim}{\bar{f}} = \langle \underset{\sim}{f}_0, \ldots, \underset{\sim}{f}_{k-1}\rangle$ of functions.

Assume that

$$(1)_0 \qquad \Vdash_\alpha \langle p_n : n < \omega\rangle \in {}^\omega(P_\beta/G_\alpha) \text{ interprets } \underset{\sim}{\bar{f}} \text{ as } \underset{\sim}{\bar{f}}^*$$

and let for $i < k$ $\underset{\sim}{n}_i$ be names of integers such that $\Vdash_\alpha \underset{\sim}{f}_i \sqsubset_{\underset{\sim}{n}_i} g$.

First claim: wlog we may assume that

$$(2)_0 \qquad \Vdash_\alpha \text{“}\langle p_n(\alpha) : n < \omega\rangle \in {}^\omega Q_\alpha \text{ is inconsistent”}$$

[Proof of first claim: It is enough to find a sequence $\langle p'_n : n < \omega\rangle$ satisfying $(1)_0$ and $(2)_0$ such that for all n, $\Vdash_\alpha p'_n \leq^* p_n$.

To find this sequence, we work in V_α. If $\langle p_n(\alpha) : n < \omega\rangle$ is inconsistent, we let $p'_n := p_n$. Otherwise, let $r_0 \in Q_\alpha$ be a condition such that for all n, $r_0 \leq^* p_n(\alpha)$, and let $r_0 \leq r_1 \leq \cdots$ be an increasing inconsistent sequence in Q_α. Now let $p'_n := p_n \wedge r_n$, i.e.,

$$p'_n(\gamma) = \begin{cases} r_n & \text{if } \gamma = \alpha \\ p_n(\gamma) & \text{if } \gamma \neq \alpha \end{cases}$$

Then $\langle p'_n : n < \omega\rangle$ satisfies the requirements. This proves the first claim.]

Let $\langle \underset{\sim}{f}_n : n < \omega\rangle$ enumerate all names $\underset{\sim}{f}$ in N satisfying $\Vdash_\beta \underset{\sim}{f} \in \mathbf{C}$.

We will construct sequences $\langle\langle \underset{\sim}{p}^j_n : n < \omega\rangle : j < \omega\rangle$ and $\langle\langle \underset{\sim}{f}^j_i : i < k+j\rangle : j < \omega\rangle$ satisfying the following for all $j \in \omega$:

($\star$) $\underset{\sim}{p}^j_n$ is a P_{α_j}-name for an element of P_β/G_{α_j}

($*$) $\underset{\sim}{f}^j_i$ a P_{α_j}-name for an element of ${}^\omega\omega$

(0) $\langle \underset{\sim}{p}^0_n : n < \omega\rangle = \langle \underset{\sim}{p}_n : n < \omega\rangle$, $\langle \underset{\sim}{f}^0_i : i < k\rangle = \langle \underset{\sim}{f}_i : i < k\rangle$.

$(1)_{j+1}$ $\Vdash_{\alpha_{j+1}}$ “$\langle \underset{\sim}{p}^{j+1}_n : n < \omega\rangle \in {}^\omega(P_\beta/G_{\alpha_{j+1}})$ interprets $\langle \underset{\sim}{f}_0, \ldots, \underset{\sim}{f}_{k+j}\rangle$ as $\langle \underset{\sim}{f}^{j+1}_0, \ldots, \underset{\sim}{f}^{j+1}_{k+j}\rangle$, and $\underset{\sim}{p}^j_0$ decides $\underset{\sim}{f}_0\restriction j, \ldots, \underset{\sim}{f}_{k+j}\restriction j$”

$(2)_{j+1}$ $\Vdash_{\alpha_{j+1}} \langle p_n^{j+1}(\alpha_{j+1}) : n < \omega \rangle$ is inconsistent.

$(3)_{j+1}$ $\Vdash_{\alpha_j}$ $\langle \underset{\sim}{p}{}_n^j \restriction \alpha_{j+1} : n < \omega \rangle \in {}^\omega(P_{\alpha_{j+1}}/G_{\alpha_j})$ interprets $\langle \underset{\sim}{f}{}_0^{j+1}, \dots, \underset{\sim}{f}{}_{k+j-1}^{j+1} \rangle$ as $\langle \underset{\sim}{f}{}_0^j, \dots, \underset{\sim}{f}{}_{k+j-1}^j \rangle$.

$(4)_{j+1}$ $\Vdash_{\alpha_{j+1}}$ If $\underset{\sim}{p}{}_0^j \restriction \alpha_{j+1} \in G_{\alpha_{j+1}}$, then $\underset{\sim}{p}{}_0^{j+1} \leq \underset{\sim}{p}{}_0^j$.

$(5)_{j+1}$ $\Vdash_{\alpha_{j+1}} \underset{\sim}{p}{}_0^{j+1} \in D_{j+1}$

Note that the statements $(1)_0$ and $(2)_0$ mentioned at the beginning of the proof are exactly $(1)_{j+1}$ and $(2)_{j+1}$ for $j = -1$.

We can obtain the sequences $\langle \underset{\sim}{p}{}_n^{j+1} : n < \omega \rangle$ and $\langle \underset{\sim}{f}{}_i^{j+1} : i < k + j + 1 \rangle$ from the given sequences $\langle \underset{\sim}{p}{}_n^j : n < \omega \rangle$, $\langle \underset{\sim}{f}{}_i^j : i < k + j + 1 \rangle$ and D_j by applying the preliminary lemma in N, so each $\langle \underset{\sim}{p}{}_n^j : n \in \omega \rangle$ will be in N.

Now we construct sequences $\langle q_j : j < \omega \rangle$ (where each q_j is an N-generic condition in P_{α_j}) and $\langle \underset{\sim}{n}_j : j < \omega \rangle$ ($\underset{\sim}{n}_j$ a P_{α_j}-name for an integer) satisfying

($\bullet$) $\Vdash_{\alpha_{j+1}}$: If $\exists n\, \underset{\sim}{f}{}_{k+j}^j \sqsubset_n g$, then $\underset{\sim}{f}{}_{k+j}^j \sqsubset_{\underset{\sim}{n}_{k+j}} g$.

(+) $q_{j+1} \restriction \alpha_j = q_j$.

(A) $q_{j+1} \Vdash \underset{\sim}{p}{}_0^j \restriction \alpha_{j+1} \in G_{\alpha_{j+1}}$.

(B) $q_{j+1} \in P_{\alpha_{j+1}}$ is N-generic.

(C) $q_{j+1} \Vdash \forall f \in N[G_{\alpha_{j+1}}]\ f \sqsubset g$.

(D) $q_{j+1} \Vdash \forall i < k + j + 1 : \underset{\sim}{f}_i \sqsubset_{\underset{\sim}{n}_i} g$.

We let $q_0 = q$. $\underset{\sim}{n}_0, \dots, \underset{\sim}{n}_{k-1}$ are already defined. By assumption (a)–(d) of the induction lemma, (A)–(D) are now satisfied for $j = -1$.

Given q_j and $\underset{\sim}{n}_0, \dots, \underset{\sim}{n}_{k+j-1}$, we can easily find $\underset{\sim}{n}_{j+k}$ by requirement ($\bullet$). Now apply the induction assumption to the sequences $\langle p_n^j : n < \omega \rangle$, $\langle \underset{\sim}{f}_i : i < k + j \rangle$, $\langle \underset{\sim}{f}{}_i^j : i < k + j \rangle$ to get q_{j+1}. This will show that q_{j+1} satisfies (+), (A), (B), (C). The induction assumption also implies that q_{j+1} will satisfy (D) for all $i < k + j$. Finally, ($\bullet$) and (C) imply that (D) is also satisfied for $i = k + j$.

Note that $q_{j+1} \Vdash \underset{\sim}{p}{}_0^j \restriction \alpha_{j+1} \in G_{\alpha_{j+1}}$, so by (4),

$$q_{j+1} \Vdash \underset{\sim}{p}{}_0^{j+1} \leq \underset{\sim}{p}{}_0^j$$

To conclude the proof of the Induction Lemma, let $q = \bigcup_j q_j$. Then $q \in P_\delta \subseteq P_\beta$. We have to check that this works.

Let G_β be any generic filter containing q. We write p_n^j for $\underset{\sim}{p}{}_n^j[G_{\alpha_j}]$, etc.

Clearly $\langle p_0^j : j < \omega \rangle$ is an increasing sequence of conditions. First we claim that $p_0^j \in D_j \cap N$. By (5), $p_0^j \in D_j$, and since q_j is generic, $p_0^j = \underset{\sim}{p}{}_0^j[G_{\alpha_j}] \in N$.

Next we note that for all k, $p_0^k \restriction \alpha_k \in G_{\alpha_k}$. Since $p_0^j \geq p_0^k$ for $j \leq k$, $p_0^j \restriction \alpha_k \in G_{\alpha_k}$ for all $k \geq j$, hence $p_0^j \in G_\delta$. Since $p_0^j \in N$, $\operatorname{dom}(p_0^j) \subseteq \delta$. So $p_0^j \in G_\beta$. This shows $G_\beta \cap D_j \cap N \neq \emptyset$. So q is generic.

For any i, $f_i^j \restriction j = f_i \restriction j$ (by the second clause in (1)). So $f_i = \lim_{j<\omega} f_i^j$. Since for all j, $f_i^j \sqsubset_{n_i} g$, and $\{f : f \sqsubset_{n_i} g\}$ is closed, $f_i \sqsubset_{n_i} g$. So g covers $N[G_\beta]$.

In particular, for $i < k$, $f_i \sqsubset_{n_i} g$, thus showing condition 5.11(d).

So we also finished the limit case.

6 Applications

The general strategy for preserving a property in limit stages of a countable support iteration is as follows:

(1) Find a stronger property that can be written as $Q \Vdash \forall f \in V[G]\, \exists g \in V\; f \sqsubset g$ for a Σ^0_2 relation $\sqsubset$. Let $\sqsubset = \bigcup_n \sqsubset_n$, where each $\sqsubset_n$ is closed.

(2) Prove that all Q_α preserve $\langle \sqsubset_n : n \in \omega\rangle$.

Then by the preservation theorem, for all α, $P_\alpha \Vdash \forall f \in V[G_\alpha]\, \exists g \in V\; f \sqsubset g$, and hence P_α has the property we wanted to preserve.

Fact 6.1: *If for all g and all n, the set $\{f : f \sqsubset_n g\}$ is closed* **and open**, *then:*

$$Q \textit{ almost preserves } \sqsubset \quad \Leftrightarrow \quad Q \textit{ preserves } \sqsubset$$

Proof: $\Leftarrow$ is clear.

$\Rightarrow$: Assume that g covers the countable elementary model N, $\bar{p} := \langle p_n : n \in \omega\rangle$ is an increasing sequence of conditions interpreting $\underset{\sim}{\bar{f}} = (\underset{\sim}{f}_0, \ldots, \underset{\sim}{f}_{k-1}) \in N$ as $\bar{f}^* = (f_0^*, \ldots, f_{k-1}^*)$, and let $f_i^* \sqsubset_{n_i} g$ for all $i < k$.

Fix $i < k$. Since the set $A_i := \{f : f \sqsubset_{n_i} g\}$ is open, there exists an integer n_i^* such that $[f_i^* \restriction n_i^*] \subseteq A_i$.

Let $n^* := \max\{n_i^* : i < k\}$. As Q almost preserves $\sqsubset$, we can find a generic condition $q \leq p_{n^*}$, satisfying 5.11(a)–(c).

For each i, we have $q \Vdash \underset{\sim}{f}_i \restriction n^* = f_i^* \restriction n^*$, thus

$$q \Vdash \underset{\sim}{f}_i \in [f_i^* \restriction n_i] \subseteq \{f : f \sqsubset_{n_i} g\}$$

so q also satisfies 5.11(d).

In general, "Q preserves $\sqsubset$" and "Q almost preserves $\sqsubset$" are not equivalent, as can be seen from the following example:

Example 6.2: *Let $\sqsubset = \langle \sqsubset_n : n \in \omega\rangle$ be defined by $f \sqsubset_n g \Leftrightarrow \forall k \geq n : f(k) < g(k)$, assume that Q is ${}^\omega\omega$-bounding (and thus preserves $\sqsubset$, see 6.5, below).*

Let $\sqsubset'_n = \sqsubset_n$ for $n > 0$, and let $f \sqsubset'_0 g$ iff for all k, $f(k) = 0$, and $\sqsubset' = \bigcup_n \sqsubset'_n$. Then for any f, g: $f \sqsubset g$ iff $f \sqsubset' g$, hence for any model N:

$$g \ \sqsubset\text{-}\textit{covers } N \quad \textit{iff} \quad g \ \sqsubset'\text{-}\textit{covers } N$$

and thus: Q almost preserves $\sqsubset$ iff Q almost preserves $\sqsubset'$.

Claim: If Q adds reals, then Q does not preserve $\sqsubset'$. (But if Q is ${}^\omega\omega$-bounding, then Q almost preserves $\sqsubset'$.)

Proof: Let $\underset{\sim}{x}$ be a name such that $\Vdash_Q \underset{\sim}{x} \in {}^\omega\omega \,\&\, \underset{\sim}{x} \notin V$. Let $\bar p := \langle p_n : n \in \omega\rangle$ be an increasing sequence of conditions interpreting $\underset{\sim}{x}$. Then

$$\Vdash_Q \exists n : p_n \notin G$$

Define a name $\underset{\sim}{f} \in {}^\omega 2$ by requiring $\Vdash_Q$ "$\underset{\sim}{f}(n) = 0 \Leftrightarrow p_n \in G_Q$." Let f^* be the function with $\forall n f^*(n) = 0$.

Then: $f^* \sqsubset'_0 g$, and $\langle p_n : n \in \omega\rangle$ interprets $\underset{\sim}{f}$ as f^*, but $\Vdash_Q \exists n \underset{\sim}{f}(n) \neq 0$, thus $\Vdash_Q \neg(\underset{\sim}{f} \sqsubset'_0 g)$.

Hence there is no condition satisfying 5.11(d).

Application 1: Preservation of ${}^\omega\omega$-bounding

Definition 6.3: *A forcing notion Q is called ${}^\omega\omega$-bounding iff*

$$\underset{Q}{\Vdash} \forall f \in {}^\omega\omega \cap V[G]\, \exists g \in {}^\omega\omega \cap V \ \forall n\ f(n) < g(n)$$

There is a natural way to translate this property into the framework of the "preservation theorem" in 5.11:

Definition 6.4: *We let (for $f, g \in {}^\omega\omega$)*

$$f \sqsubset_n^{\text{bound}} g \ \Leftrightarrow\ \forall k \geq n\, f(k) < g(k)$$

This is a closed relation. Letting $\sqsubset^{\text{bound}} = \bigcup_n \sqsubset_n^{\text{bound}}$, clearly Q is ${}^\omega\omega$-bounding iff

$$\underset{Q}{\Vdash} \forall f \in {}^\omega\omega \cap V[G]\, \exists g \in {}^\omega\omega \cap V \ f \sqsubset^{\text{bound}} g$$

Lemma 6.5: *A proper forcing notion Q preserves $\sqsubset^{\text{bound}}$ iff it is ${}^\omega\omega$-bounding.*

Proof: Assume Q preserves $\sqsubset^{\text{bound}}$. Then for any name $\underset{\sim}{f}$ and any condition p, let N be a model containing $\underset{\sim}{f}$ and p. Let g cover N. We can find a condition $q \leq p$ forcing that $\underset{\sim}{f} \sqsubset^{\text{bound}} g$. Now g is in the ground model, so Q is ${}^\omega\omega$-bounding.

Conversely, assume that Q is ${}^\omega\omega$-bounding, and consider a model N and a sequence $\langle p_n : n < \omega\rangle$ as in the hypothesis of 5.11. We also have names $\underset{\sim}{f}_0, \ldots, \underset{\sim}{f}_{k-1}$, and functions f_i^* such that $p_n \Vdash \underset{\sim}{f}_i \restriction n = f_i^* \restriction n$ for all $i < k$.

First we note that any N-generic q will force

$$\forall f \in N[G]\, \exists f' \in N\, \forall n\ f(n) < f'(n)$$

Since any such f' is eventually bounded by g, $q \Vdash \forall f \in N[G]\ f \sqsubset^{\text{bound}} g$.

We still have to deal with our fixed names $\underset{\sim}{f}_i$. Assume $f_i^* \sqsubset^{\text{bound}}_{n_i} g$. For each n we can find a condition $p'_n \leq p_n$ and functions $f'_{i,n} \in N$ $(i < k)$ such that $p'_n \Vdash \forall k\ \underset{\sim}{f}_i(k) < f'_{i,n}(k)$. Let for all $i < k$ f'_i be defined by

$$f'_i(k) = \max\{f'_{i,n}(k) : n \leq k\} + 1$$

We can find these sequences $\langle p'_n : n < \omega\rangle$ and $\langle f'_{i,n} : i < k, n < \omega\rangle$ in N, so there are n'_i such that $f'_i \sqsubset^{\text{bound}}_{n'_i} g$, $n'_i \geq n_i$. Let $n^* \geq n'_i$ for all i, and let $q \geq p'_{n^*}$ be a generic condition. Then

$$q \Vdash \forall k \geq n^*\ \underset{\sim}{f}_i(k) < f'_{i,n^*}(k) < f'_i(k) < g(k)$$
$$q \Vdash \forall k \in (n_i, n^*)\ \underset{\sim}{f}_i(k) = f^*_i(k) < g(k)$$

so $q \Vdash \underset{\sim}{f}_i \sqsubset^{\text{bound}}_{n_i} g$.

Corollary 6.6: *The countable support iteration of proper ${}^{\omega}\omega$-bounding forcing notions is (proper and) ${}^{\omega}\omega$-bounding.*

Application 2: Preserving outer measure one

Let Ω be the set of clopen subsets of ${}^{\omega}2$. Ω is a countable set. We will consider functions $f \in {}^{\omega}\Omega$ and functions $g \in {}^{\omega}2$.

$\mu(A)$ is the Lebesgue measure of any measurable set $A \subseteq {}^{\omega}2$, and we let $\mu^*(A)$ be the outer Lebesgue measure of any set $A \subseteq {}^{\omega}2$.

We let

$$\mathbf{C}^{\text{random}} := \{f \in {}^{\omega}\Omega : \forall n \in \omega\, \mu(f(n)) \leq 2^{-n}\}$$

This is a closed set (in the product topology of ${}^{\omega}\Omega$, where Ω is equipped with the discrete topology).

For $f \in \mathbf{C}^{\text{random}}$, we let

$$A_f := \bigcap_{n\in\omega} \bigcup_{k \geq n} f(k)$$

Fact 6.7:

(1) If $f \in \mathbf{C}^{\text{random}}$, then $A_f \subseteq {}^{\omega}2$ is a set of measure zero.
(2) If $H \subseteq {}^{\omega}2$ has measure zero, then there is $f \in \mathbf{C}^{\text{random}}$ such that $H \subseteq A_f$.

Proof: (1) follows from $\mu\big(\bigcup_{k\geq n} f(k)\big) \leq 2^{-n} + 2^{-n-1} + \cdots = 2^{-n+1}$.

(2): Let $\Gamma : \omega \times \omega \to \omega$ be a monotone bijection. For each m, H can be covered by an open set of measure $< 2^{-\Gamma(m,0)}$, say

$$H \subseteq \bigcup_{k\in\omega} [s^m_k] \qquad s^m_k \in 2^{<\omega} \qquad \mu\big(\bigcup_{k\in\omega} [s^m_k]\big) \leq 2^{-\Gamma(m,0)}$$

For each m, we can find a sequence $0 = k_0^m < k_1^m < \cdots$ of integers satisfying

$$\mu\Big(\bigcup_{k\geq k_j^m} s_k^m\Big) < 2^{-\Gamma(m,j)}$$

for all $j \in \omega$.

Define f by

$$f(\Gamma(m,j)) = \bigcup\{[s_k^m] : k_j^m \leq k k_{j+1}^m\}$$

This is a finite union of basic clopen sets, hence clopen. Also,

$$\mu(f(\Gamma(m,j))) \leq \mu\Big(\bigcup_{k\geq k_j^m} s_k^m\Big) \leq 2^{-\Gamma(m,j)}$$

so $f \in \mathbf{C}^{\text{random}}$.

Note that for all m, $\bigcup_{j\in\omega} f(\Gamma(m,j)) = \bigcup\{[s_k^m] : k \in \omega\} \supseteq H$. For all $n \in \omega$ there is $m \in \omega$ such that $\forall j : \Gamma(m,j) > n$, so

$$\forall n\, \exists m\; H \subseteq \bigcup_{j\in\omega} f(\Gamma(m,j)) \subseteq \bigcup_{k\geq m} f(k)$$

so $H \subseteq \bigcap_{m\in\omega}\bigcup_{k\geq m} f(k)$.

Definition 6.8: *For $f \in \mathbf{C}^{\text{random}}$, $g \in {}^{\omega}2$, $n \in \omega$ we let $f \sqsubset_n^{\text{random}} g$ iff $\forall k \geq n\; g \notin f(k)$.*

This is a closed relation since (for fixed k), "$g \in f(k)$" is clopen.

Fact 6.9: *$f \sqsubset^{\text{random}} g$ iff $g \notin A_f$.*

Fact 6.10: *For a countable model N of ZFC:*

$$g \sqsubset^{\text{random}}\text{-covers } N \quad\Leftrightarrow\quad g \text{ is random over } N$$

Proof: Assume that g is random over N. Fix a function $f \in \mathbf{C}^{\text{random}}$. Then as g is not an element of the null set A_f, we have $f \sqsubset^{\text{random}} g$.

Conversely, assume that $\forall f \in N\; f \sqsubset^{\text{random}} g$. Then for any measure zero set H in N we can find a sequence $f \in \mathbf{C}^{\text{random}}$ such that $H \subseteq A_f$. Since $f \sqsubset^{\text{random}} g$, $g \notin A_f$, so $g \notin H$.

Fact 6.11: *If Q preserves $\sqsubset^{\text{random}}$, then $\Vdash_Q \mu^*(V \cap {}^{\omega}2) = 1$.*

Proof: Assume $p \Vdash \mu(V \cap {}^{\omega}2) = 0$. Then there is a name $\underset{\sim}{f}$ such that $p \Vdash \underset{\sim}{f} \in \mathbf{C}^{\text{random}} \,\&\, V \cap {}^{\omega}2 \subseteq A_{\underset{\sim}{f}}$. Take any countable elementary model N containing p and $\underset{\sim}{f}$, and let g cover N. Then $p \Vdash g \in A_{\underset{\sim}{f}}$, but there is a condition $q \leq p$, $q \Vdash$ "g covers $N[G]$," so $q \Vdash g \notin A_{\underset{\sim}{f}}$, a contradiction.

This fact justifies the following definition:

Definition 6.12: *We say that a forcing notion Q "preserves outer measure one" iff Q preserves $\sqsubset^{\text{random}}$.*

In [7], the following property of forcing notions was considered:

Definition 6.13: *A forcing notion Q satisfies $*_4$ if for every countable $N \prec H(\chi)$, if $P \in N$, $\langle p_n : n \in \omega\rangle \in N$, each p_n in P, and $\langle \underset{\sim}{A}_n : n \in \omega\rangle \in N$, each $\underset{\sim}{A}_n$ a P-name, for every n, $p_n \Vdash$ "$\underset{\sim}{A}_n$ is a Borel set and $\mu(\underset{\sim}{A}_n) < \varepsilon_n$" and $\lim_{n\to\infty} \varepsilon_n = 0$ and $g \in {}^{\omega}2$ is random over N then there exists $q \in P$ such that*

(i) q is (N, P)-generic.
(ii) $q \Vdash$ "g is random over $N[G_Q]$."
(iii) there exists n such that $q \le p_n$ and $q \Vdash$ "$x \notin \underset{\sim}{A}_n$."

[7] also showed that Laver forcing has the property $*_4$.

Lemma 6.14: *If Q satisfies $*_4$, then Q "preserves outer measure one."*

Proof: Let $\langle p_n : n \in \omega\rangle$, $\bar{\underset{\sim}{f}} = \langle \underset{\sim}{f}_0, \dots, \underset{\sim}{f}_{k-1}\rangle$ be as in 5.11, let N be a countable elementary model, and let g cover N. Then g is random over N.

Define Q-names $\underset{\sim}{A}_n$ such that

$$\Vdash \underset{\sim}{A}_n = \bigcup_{i<k} \bigcup_{m \ge n} \underset{\sim}{f}_i(m)$$

Then $\Vdash \mu(A_n) \le \varepsilon_n$, where $\varepsilon_n := k \cdot 2^{-n+1}$.

Let $\langle p_n : n \in \omega\rangle$ interpret $\bar{\underset{\sim}{f}}$ as $\bar{f}^*$, and assume $f_i^* \sqsubset_{n_i}^{\text{random}} g$. Let $n^* := \max\{n_i : i < k\}$. Applying $*_4$ to the sequence $\langle p_n : n^* \le n < \omega\rangle$, we can find an $n^{**} \ge n^*$ and a generic condition $q \le p_{n^{**}}$ such that $q \Vdash$ "g is random over $N[G]$," and $q \Vdash g \notin A_{n^{**}}$.

As g is random over $N[G]$ iff g covers $N[G]$, q satisfies 5.11(a)–(c).

For each $i < k$ we have $q \Vdash g \notin \bigcup_{m \ge n^{**}} \underset{\sim}{f}_i(m)$. So to get 5.11(d), it suffices to show

$$q \Vdash g \notin \bigcup_{n_i \le m < n^{**}} \underset{\sim}{f}_i(m)$$

This follows easily from $q \le p_{n^{**}}$, as

$$p_{n^{**}} \Vdash g \notin \bigcup_{n_i \le m < n^{**}} f_i^*(m) = \bigcup_{n_i \le m < n^{**}} \underset{\sim}{f}_i(m)$$

Thus (by [7]) the Laver forcing "preserves outer measure one."

Lemma 6.15: *The random real forcing notion "preserves outer measure one."*

To prove this lemma, we first show the following claim:

Claim 6.16: *Let Q be the random real forcing, and let $\underset{\sim}{A}$ be a Q-name for a subset of $\mathbb{R}$, $p \in Q$, and assume that for some real c, $p \Vdash \mu^*(\underset{\sim}{A}) < c$. Then, letting*

$$A(p) := \{x \in \mathbb{R} : p \Vdash x \in \underset{\sim}{A}\},$$

we have $\mu^*(A(p)) \le c$.

Proof of the claim: Let $\underset{\sim}{B}$ be the name of a Borel set such that $p\Vdash$"$\underset{\sim}{A} \subseteq \underset{\sim}{B}$ & $\mu(\underset{\sim}{B}) < c$.

Notation: For any set $C \subseteq \mathbb{R}^2$, $t \in \mathbb{R}$, we let

$$C^t := \{x \in \mathbb{R} : (x,t) \in C\}$$
$$C_t := \{y \in \mathbb{R} : (t,y) \in C\}$$

There is a Borel set $C \subseteq \mathbb{R}^2$ such that $p\Vdash \underset{\sim}{B} = C_r$, where r is the canonical name of the random real. Wlog $C \subseteq p \times \mathbb{R}$. In V we have for almost all x $\mu(C_x) \le c$, so $\mu(C) \le c \cdot \mu(p)$.

Let $D := \{y : C^y =^* p\} = \{y : p \subseteq^* C^y\}$. (We write $X \subseteq^* Y$ iff $\mu(X - Y) = 0$.)

We claim $D = B(p)$. Proof: We have $y \in B(p) \Leftrightarrow p\Vdash y \in B \Leftrightarrow p\Vdash y \in C_r \Leftrightarrow p\Vdash (r,y) \in C \Leftrightarrow p\Vdash r \in C^y \Leftrightarrow \mu(p - C^y) = 0$.

Clearly $\mu(C) \ge \mu(D) \cdot \mu(p)$, so $\mu(D) \le \dfrac{\mu(C)}{\mu(p)} \le c$.

This ends the proof of 6.16.

Proof of 6.15:

We first show that Q almost preserves $\sqsubset^{\text{random}}$ and in fact any condition in N is generic and forces "g is random over $N[G]$," if g is random over N.

Every condition is generic, because Q satisfies ccc. Now assume that $\not\Vdash$ "g is random over $N[G]$." Then for some condition q, and some $\underset{\sim}{B} \in N$, $\Vdash_Q$"$\mu(\underset{\sim}{B}) = 0$" and $q\Vdash_Q$"$g \in \underset{\sim}{B}$." Let $C \in N$ such that $\Vdash$"$\underset{\sim}{B} = C_r$." So $q\Vdash r \in C^g$, which is impossible, since C is a set in N of measure zero, so C^g has measure zero. This shows that random real forcing almost preserves $\sqsubset^{\text{random}}, \mathbf{C}^{\text{random}}$.

Assume $f_i^* \sqsubset_{n_i}^{\text{random}} g$, $p_n\Vdash \underset{\sim}{f}_i \restriction n = f_i^* \restriction n$.

In N we can define a sequence $\langle B_n : n < \omega \rangle$ as follows:

$$B_n := \big\{x \in \mathbb{R} : p_n \Vdash x \in \bigcup_{i<k} \big(\bigcup_{m \ge n} \underset{\sim}{f}_i(m) \big) \big\}$$

By 6.16,

$$\mu^*(B_n) \le k \cdot \sum_{m \ge n} 2^{-m} \le \frac{2k}{2^n}$$

Since $\langle B_n : n \in \omega \rangle \in N$, and g is random over N, $g \notin \bigcap_n B_n$. So there exists a $n^* \ge \max(n_0, \ldots, n_{k-1})$ such that $g \notin B_n$. There exists a condition $q \le p_{n^*}$,

$$q \Vdash \forall i < k : g \notin \bigcup_{m \ge n^*} \underset{\sim}{f}_i(m)$$

Since for all $i < k$,

$$q \Vdash g \notin \bigcup_{n_i < m < n^*} f_i^*(m) = \bigcup_{n_i < m < n^*} \underset{\sim}{f}_i(m)$$

we have $q\Vdash \forall i < k\ \underset{\sim}{f}_i \sqsubset_{n_i}^{\text{random}} g$.

A similar proof shows that random real forcing satisfies $*_4$, proving a conjecture of Miller.

Application 3: Preserving nonmeager sets

We consider functions f from $\omega^{<\omega}$ to $\omega^{<\omega}$, and functions g from ω to ω.

Fact 6.17:

(1) For any $f : \omega^{<\omega} \to \omega^{<\omega}$, the set

$$A_f := \{g : \forall n\, g{\restriction}n{}^\frown f(g{\restriction}n) \not\subseteq g\}$$

is closed nowhere dense.

(2) Conversely, for every closed nowhere dense set H there is a function f such that $H \subseteq A_f$.

Proof: For every n, the set $\{g : g{\restriction}n{}^\frown f(g{\restriction}n) \subseteq g\} = \bigcup_{\eta\in {}^n\omega}[\eta^\frown f(\eta)]$ is clopen, so A_f is closed. A_f cannot contain any basic interval $[\eta]$, since A_f is disjoint from $[\eta^\frown f(\eta)]$.

Conversely, let H be a closed nowhere dense set in ${}^\omega\omega$. Then as H is closed, there is a tree $T \subseteq \omega^{<\omega}$ such that $H = \{f \in {}^\omega\omega : \forall n\, f{\restriction}n \in T\}$. As H is nowhere dense, for every $\eta \in \omega^{<\omega}$ there is an extension $\eta^\frown f(\eta)$ such that $H \cap [\eta^\frown f(\eta)] = \emptyset$.

Definition 6.18: *We let $\mathbf{C}^{\mathrm{Cohen}}$ be the set of all functions from $\omega^{<\omega}$ to $\omega^{<\omega}$.*
We define $\sqsubset^{\mathrm{Cohen}}$ by $f \sqsubset^{\mathrm{Cohen}}_k g$ iff

$f : \omega^{<\omega} \to \omega^{<\omega}$.
$g : \omega \to \omega$.
$\exists n \le k : g{\restriction}n{}^\frown f(g{\restriction}n) \subseteq g$.

Fact 6.19: $f \sqsubset^{\mathrm{Cohen}} g$ *iff* $g \notin A_f$.

Fact 6.20: g $\sqsubset^{\mathrm{Cohen}}$*-covers N iff g is Cohen over N.*

Proof: Recall that $g \in {}^\omega 2$ is Cohen over a model, iff it not contained in any meager set coded in the model, iff it is not contained in any closed nowhere dense set coded in the model.

Assume that g is Cohen over N. Fix a function $f \in \mathbf{C}^{\mathrm{Cohen}}$. Then as g is Cohen, g is not an element of the meager set A_f, so there is an n such that $f \sqsubset^{\mathrm{Cohen}}_n g$.

Conversely, assume that $\forall f \in N$ $f \sqsubset^{\mathrm{Cohen}} g$. Then for any closed nowhere dense set H in N we can find a sequence $f \in \mathbf{C}^{\mathrm{Cohen}}$ such that $H \subseteq A_f$. Since there is some n such that $f \sqsubset^{\mathrm{Cohen}}_n g$, $g \notin H$.

Fact 6.21: *If Q preserves $\sqsubset^{\mathrm{Cohen}}$, then $Q \Vdash$ "$V \cap {}^\omega\omega$ is not meager".*

Proof: Assume $p \Vdash V \cap {}^\omega 2$ is meager. Then there is a name $\underset{\sim}{f}$ such that $p \Vdash V \cap {}^\omega 2 \subseteq A_{\underset{\sim}{f}}$. Take any countable elementary model N containing p and $\underset{\sim}{f}$, and let g cover N. Then $p \Vdash g \in A_f$, a contradiction.

In [11, ch.18], a converse to this theorem is proved:

Theorem 6.22: *Assume that $(Q, \leq)$ is a Souslin proper forcing notion (see section 7), and*

$$(*) \qquad \Vdash_Q V \cap {}^\omega\omega \textit{ is not meager}$$

and moreover, Q has property $(*)$ in any extension of V (by set forcing). Then Q preserves $\sqsubset^{\text{Cohen}}$.

Example 6.23: *Cohen forcing preserves $\sqsubset^{\text{Cohen}}$.*

Proof: Note that each relation $\sqsubset_n^{\text{Cohen}}$ is clopen. By 6.1, it is enough to show that Cohen forcing almost preserves $\sqsubset^{\text{Cohen}}$.

We claim that if g covers N, then **any** condition forces "g covers $N[G]$." The proof is exactly the same as the proof in the previous section, with "measure zero" replaced by "meager", "random" replaced by "Cohen", etc.

Application 4: The Laver property

From now on until the end of this section, n will be a variable ranging over **positive** natural numbers.

Definition 6.24: *For a function $h : \omega \to \omega$, an "h-cone" is a sequence $\langle A_m : m \in \omega\rangle$ of finite subsets of ω with $|A_n| \leq h(n)$ for all $n > 0$.*

If $h, H : \omega \to \omega$, $\bar{A}$ an h-cone, we say that $\bar{A}$ is bounded by H if for all $n > 0$, $A_n \subseteq H(n)$.

$\mathbf{Q}_+$ is the set of nonnegative rationals. For $r \in \mathbf{Q}_+$, an r-cone is an h-cone where $h(m) = \lfloor 2^{mr} \rfloor$. ($\lfloor x \rfloor$ is the greatest integer $\leq x$.)

A "cone" is a sequence $\langle A_m : m \in \omega\rangle$ of finite subsets of ω with $|A_n| \leq 2^n$ (i.e., a 1-cone).

We say that $\bar{A} = \langle A_m : m \in \omega\rangle$ "covers" $f \in {}^\omega\omega$ if for all $n > 0$ $f(n) \in A_n$.

If $f \in {}^\omega([\omega]^{<\omega})$, we say that $\bar{A}$ covers f iff for all $n > 0$ $f(n) \subseteq A_n$.

For a function $H \in {}^\omega\omega$, we write $\prod H$ for the set $\{f \in {}^\omega\omega : \forall n > 0 \; f(n) < H(n)\}$. This is a closed subset of ${}^\omega\omega$.

Definition 6.25: *A forcing notion Q is said to have the Laver Property iff for every $H : \omega \to \omega$ in V,*

$$\textit{(Laver)}_H \qquad \Vdash_Q \text{“}\forall f \in \textstyle\prod H \cap V[G] : \exists \bar{A} \in V,\ \bar{A} \textit{ is a cone covering } f$$

Note that if $\forall n > 0\, H(n) \leq H'(n)$, then $\prod H \subseteq \prod H'$, so $(\text{Laver})_{H'}$ implies $(\text{Laver})_H$. So without loss of generality we may restrict ourselves to some dominating family of functions H, e.g. all increasing functions in ${}^\omega(\omega - \{0\})$.

Fact 6.26: *The following are equivalent for any two universes $V_0 \subseteq V_1$:*

(1) For all $H \in {}^\omega\omega \cap V_0$: If $f \in {}^\omega\omega \cap V_1$ is bounded by H, then f is covered by some cone of V_0.

(2) For all $H \in {}^\omega\omega \cap V_0$, for all functions $h_0 \in {}^\omega\omega \cap V_0$ that diverge to infinity:

If $f \in {}^\omega\omega \cap V_1$ is bounded by H, then f is covered by some h_0-cone of V_0.

(3) *For all $H \in {}^\omega\omega \cap V_0$: For all $h_0, h_1 \in {}^\omega\omega \cap V_0$: If for all n, $h_0(n) \leq h_1(n)$, and $h_0(n) = o(h_1(n))$, i.e., $\lim_{n\to\infty} \frac{h_1(n)}{h_0(n)} = \infty$, then for all h_0-cones $f \in V_1$:*

If f is bounded by H, then f is covered by some h_1-cone $F \in V_0$.

We will give the proof of this fact (a routine computation) below.

Definition 6.27: *If any/all of these conditions are satisfied, we say that V_1 has the Laver property over V_0.*

Note that by (1), a forcing notion Q has the Laver property iff

$$Q \Vdash V[G] \text{ has the Laver property over } V$$

Notation 6.28: *Fix some (recursive) map c from $[\omega]^{<\omega}$ onto the set of rational numbers in [0,1). If $c(x) = r$ we say "x codes r."*

Definition 6.29: *Let*

$$\mathbf{C}^H := \{f \in {}^\omega\big([\omega]^{<\omega}\big) : \text{If } f(0) \text{ codes } r_0, \text{ then } \forall n > 0 : f(n) \subseteq H(n) \text{ and } |f(n)| \leq 2^{nr_0}\}$$

Definition 6.30: *Let $\sqsubset^H = \langle \sqsubset^H_k : k < \omega\rangle$ be defined by $f \sqsubset^H_k g$ iff*

g is a cone bounded by H.
$f(0)$ codes some r_0.
$f \in \mathbf{C}^H$ (so $\forall n > 0$ $|f(n)| \leq 2^{nr_0}$)
$\forall n > k$ $f(n) \subseteq g(n)$.

We let $\sqsubset^H = \bigcup_k \sqsubset^H_k$.

Clearly the set $\{f : f \sqsubset^H_k g\}$ is closed for all g, and for any countable set a there is g such that $\forall f \in a \cap \mathbf{C}^H$ $f \sqsubset^H g$.

Definition 6.31: *Let Q be a forcing notion. If $H \in {}^\omega\omega$, $0 \leq r < s$, we write (Laver)$_{H,r,s}$ for the statement:*

(Laver)$_{H,r,s}$ $\Vdash_Q$ *"Every r-cone f in $V[G]$ that is bounded by H is covered by some s-cone $F \in V$"*

Thus, (Laver)$_H$ from 6.25 is equivalent to (Laver)$_{H,0,1}$.

Lemma 6.32: *Assume Q is a proper forcing notion. Then the following are equivalent:*

(1) *Q satisfies the Laver property.*
(2) *For all $H \in {}^\omega\omega$, all $0 \leq r < s$, Q satisfies (Laver)$_{H,r,s}$.*
(3) *For all $H \in {}^\omega\omega$, all $0 \leq r < 1$, Q satisfies (Laver)$_{H,r,1}$.*
(4) *For all $H \in {}^\omega\omega$, Q satisfies (Laver)$_{H,0,1}$.*

(5) *For all H, Q preserves $\sqsubset^H$.*
(6) *For all H, Q almost preserves $\sqsubset^H$.*

Proof: We will show

$$\begin{array}{ccccccc} (1) & \Rightarrow & (2) & \Rightarrow & (3) & \Rightarrow (4) \Rightarrow & (1). \\ & & \Downarrow & & \Uparrow & & \\ & & (5) & \Rightarrow & (6) & & \end{array}$$

(1) $\Rightarrow$ (2) follows from the characterisation 6.26(3), using the functions $h_0(n) = 2^{rn}$, $h_1(n) = 2^{sn}$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1) is clear, and also (5) $\Rightarrow$ (6) is clear.

(6) $\Rightarrow$ (3) follows from 5.6. (Note that in (3) we can restrict ourselves to **rational** numbers r.)

So it remains to show (2) $\Rightarrow$ (5).

Assume that Q satisfies $(\text{Laver})_{H,r,s}$ for all $0 \le r < s$. Let N be a countable elementary model, $H \in N$, and assume that g covers N.

Claim: If G is generic over V, and $G \cap N$ is generic over N, then g covers $N[G]$.

Proof of the claim: Let $f \in \mathbf{C}^H \cap N[G]$, and let $f(0)$ code r_0. Let $t_0 := (r_0 + 1)/2$.

So $\forall n > 0$ $f(n) \subseteq H(n)$ and $|f(n)| \le 2^{nr_0}$. By $(\text{Laver})_{H,r_0,t_0}$, there is $F \in V$, $\forall n > 0$ $|F(n)| \le 2^{nt_0}$ and $f(n) \subseteq F(n) \subseteq H(n)$. Since $G \cap N$ is generic over N, we can find this F in N. We may assume that $F(0)$ codes t_0. (So $f \in \mathbf{C}^H \cap N$.)

Since g covers N, there is a k such that $F \sqsubset^H_k g_s$. So for all $n \ge k$, $f(n) \subseteq F(n) \subseteq g_s(n)$. This ends the proof of the claim.

So every generic condition will force 5.11(a)–(c).

We still have to deal with condition (d) of 5.11. So assume that $\langle p_n : n < \omega \rangle$ interprets $\langle \underset{\sim}{f}_0, \ldots, \underset{\sim}{f}_{k-1} \rangle$ as $\langle f^*_0, \ldots, f^*_{k-1} \rangle$, and assume $\forall i < k$ $f^*_i \sqsubset_{n_i} g$. Let $f^*_i(0)$ code r_i, and let s_i, t_i be rationals such that $r_i < s_i < t_i < 1$.

By $(\text{Laver})_{H,r_i,s_i}$, for all $n > 0$ we can find a condition $p'_n \ge p_n$ such that for all $i < k$ there exists a function $F_{n,i}$ with $F_{n,i}(m) \subseteq H(m)$ and $|F_{n,i}(m)| \le 2^{ms_i}$ for all m, and

$$p'_n \Vdash \forall m > 0 \ \underset{\sim}{f}_i(m) \subseteq F_{n,i}(m)$$

We can find these sequences $\langle p'_n : n < \omega \rangle$ and $\langle F_{n,i} : n < \omega, i < k \rangle$ in N.

Now define for $i < k$ F_i as follows: $F_i(0)$ codes t_i, and for $n > 0$ let

$$F_i(n) = \bigcup \{ F_{m,i}(n) : m \le 2^{n(t_i - s_i)} \}$$

Then $F_i(n) \subseteq H(n)$, and $|F_i(n)| \le 2^{nt_i}$.

Furthermore, we have $F_{m,i}(n) \subseteq F_i(n)$, if $m \le 2^{n(t_i - s_i)}$.

Let m_0 be so large that

(i) For all $n \ge m_0$, all $i < k$, $2^{n(t_i - s_i)} > n$.
(ii) For all $i < k$, $F_i \sqsubset^H_{m_0} g$.
(iii) For all $i < k$, $m_0 > m_i$ (so $f^*_i \sqsubset^H_{m_0} g$.)

We now claim that for any generic $q \ge p'_{m_0}$,

$$q \Vdash \forall i < k \ \underset{\sim}{f}_i \sqsubset_{n_i} g$$

Proof: Let G be a generic filter containing q, fix some $i < k$, and let $f_i = \underset{\sim}{f}_i[G]$.

For $n \in [n_i, m_0)$ we have $f_i(n) = f_i^*(n)$ because $q \le p_{m_0}$, and $f_i^*(n) \subseteq g(n)$ because $f_i^* \sqsubset_{n_i}^H g$.

For $n \ge m_0$, we have

$f_i(n) \subseteq F_{m_0,i}(n)$ because $p_{m_0} \in G$,
$F_{m_0,i}(n) \subseteq F_i(n)$ because $m_0 \le n \le 2^{n(t_i - s_i)}$, and
$F_i(n) \subseteq g_{s_i}(n)$ because $F_i \sqsubset_{m_0}^H g$.

Hence for all $n \ge n_i$, $f_i(n) \subseteq g_{s_i}(n)$, so $f_i \sqsubset_{n_i} g$.

Corollary 6.33: *The Laver property is preserved under countable support iteration of proper forcing notions.*

Proof of 6.26:

Clearly (3) $\Rightarrow$ (2) $\Rightarrow$ (1).

To show that (1) $\Rightarrow$ (3), fix h_0, h_1 in V_0 and f in $\prod_n [\omega]^{<h_0(n)} \cap V_1$. We can find a function $\ell \in {}^\omega\omega \cap V_0$ such that for all n $\ell(\lfloor \log \frac{h_1(n)}{h_0(n)} \rfloor) > n$. (log is the logarithm to base 2.)

Let i be a bijection between HF (the hereditarily finite sets) and ω.

Let $H'(k) = 1 + \max\{i(\eta)(k) : |\eta| \le \ell(k), \forall n\, \eta(n) < H(n)\}$ and define $f' \in {}^\omega\omega$ by $f'(k) = i(f \restriction \ell(k))$. Then $H' \in V_0$, and $f'(k) < H'(k)$. Using (1) on f', we can get $F' \in V_0$, such that $\forall k\, f'(k) \in F'(k) \,\&\, |F'(k)| \le 2^k$. Clearly we may assume $i(\eta) \in F'(k) \Rightarrow \forall n \in \mathrm{dom}(\eta)\, |\eta(n)| \le h_0(n)$.

Let $F(n) = \bigcup\{\eta(n) : i(\eta) \in F'(\lfloor \log(h_1(n)/h_0(n)) \rfloor)\}$. So $|F(n)| \le h_0(n) \times \frac{h_1(n)}{h_0(n)} \le h_1(n)$.

Now we claim that for all n, $f(n) \subseteq F(n)$.

Proof: Fix n, and let $k = \left\lfloor \log \dfrac{h_1(n)}{h_0(n)} \right\rfloor$, $\eta = f \restriction \ell(k)$. Then $\ell(k) > n$, so $n \in \mathrm{dom}(\eta)$.

Since $i(\eta) = f'(k) \in F'(k)$, $f(n) = \eta(n) \subseteq F(n)$.

Application 5: The Sacks Property

Definition 6.34: *A forcing notion Q is said to have the Sacks Property iff*

$$\Vdash_Q \text{“}\forall f \in {}^\omega\omega \cap V[G] : \exists \bar{A} \in V,\ \bar{A} \textit{ is a cone covering } f$$

Fact and Definition 6.35: *The following are equivalent for any two universes $V_0 \subseteq V_1$:*

(1) Every $f \in V_1$ is covered by some cone of V_0.
(2) For all functions $h_0 \in {}^\omega\omega \cap V_0$ that diverge to infinity:
Every $f \in V_1$ is covered by some h_0-cone of V_0.

(3) *For all $h_0, h_1 \in {}^\omega\omega \cap V_0$: If for all n, $h_0(n) \le h_1(n)$, and $h_0(n) = o(h_1(n))$, then for all h_0-cones $f \in V_1$ there is an h_1-cone $F \in V_0$ covering f.*

If any/all these conditions are satisfied, we say that V_1 has the Sacks property over V_0. We say that a forcing notion Q has the Sacks property iff

$$Q \Vdash V[G] \text{ has the Sacks property over } V$$

Define $H^\infty(n) = \omega$ for all n, then all proofs about the Laver property can be translated into proofs for the corresponding facts about the Sacks property, by letting H range over $\{H^\infty\}$ instead of over all increasing functions.

In particular we get

(1) Q has the Sacks property iff Q satisfies $(\text{Laver})_{H,r,s}$ (where $H = H^\infty$) for all $r < s$ iff Q preserves $\sqsubset^{H^\infty}$.
(2) The Sacks property is preserved under countable support iteration of proper forcing notions.

(Alternatively, the fact that the Sacks property is preserved by countable support iteration follows fron the fact that Q has the Sacks property iff Q has the Laver property and is ${}^\omega\omega$-bounding.)

Application[2] — An Example:

We continue the example from the introduction.

Recall that we are trying to build a model where every set of $< \mathfrak{c}$ many functions is bounded, the set of reals cannot be covered by $< \mathfrak{c}$ many null sets, but there is a nonmeasurable set of size $< \mathfrak{c}$.

Starting in L we construct an countable support iteration $\langle P_\alpha, Q_\alpha : \alpha < \omega_2 \rangle$ by requiring

(1) If α is even, then $\Vdash_\alpha$ "$Q_\alpha =$ random real forcing". Let r_α be the name of the random real added by Q_α.
(2) If α is odd, then $\Vdash_\alpha Q_\alpha =$ Laver real forcing. Let f_α be the name of the Laver real added by Q_α.

Then we have:

(1) If $\underset{\sim}{B}$ is a P_β-name for a (code of a) Borel null set, and $\alpha > \beta$ is even, then
$$\Vdash_\varepsilon r_\alpha \notin \underset{\sim}{B}$$
(2) If $\underset{\sim}{f}$ is a P_β-name for a function in ${}^\omega\omega$, and $\alpha > \beta$ is odd, then
$$\Vdash_\varepsilon \underset{\sim}{f} <^* f_\alpha$$

(Note that P_{ω_2} is proper and satisfies the $\aleph_2$-cc, so no cardinals are collapsed.)

By 1.20 we know that every real in $V[G_{\omega_2}]$ in fact appears in some $V[G_\beta]$, for some $\beta < \omega_2$. So in V_{ω_2}, no set of size $\aleph_1$ of functions is unbounded, and no family of Borel null sets of size $\aleph_1$ covers all reals numbers.

Hence $V_{\omega_2} \models \mathfrak{b} = \mathfrak{d} \ + \ \mathbf{Cov}(\mathcal{N})$.

To conclude this example, we claim that

$$V_{\omega_2} \models \mathbb{R} \cap V_0 \text{ is not of measure zero}$$

To prove this claim it is by 6.11 enough to show that P_{ω_2} preserves $\sqsubset^{\text{random}}$. By the general preservation theorem it is enough to show that for all $\alpha < \omega_2$ we get $\Vdash_\alpha Q_\alpha$ preserves $\sqsubset^{\text{random}}$.

But we remarked already that both random forcing and Laver forcing preserve $\sqsubset^{\text{random}}$ (see 6.15 and [7]).

This finishes the application[2].

7. Souslin Proper forcing

We review the basic facts about iteration of Souslin proper forcing notions (from [5] and [3]).

Definition 7.1: *Assume that $(Q, \leq)$ is a forcing notion. We say that Q is a* **Souslin** *forcing notion iff Q, $\leq$, and the incompatibility relation $\perp_Q$ are analytic sets.*

(Note that in general incompatibility is a Π^1_1-relation (if $\leq$ is analytic), so the demand on $\perp$ really says that incompatibility is a Borel relation.)

Notation 7.2: *Using a universal Σ^1_1-set, we can associate with each real d two $\mathbf{\Sigma^1_1}$ relations $\leq_d$ and $\perp_d$ (subsets of $\mathbb{R} \times \mathbb{R}$) such that every analytic pair $\leq, \perp$ appears as some $\leq_d, \perp_d$, and the relations $x \leq_d y$ and $x \perp_d y$ are Σ^1_1.*

Definition 7.3: *We say that "d codes a Souslin forcing" iff*

(1) $Q_d := \langle \text{field}(\leq_d), \leq_d \rangle$ is a partial quasiorder. (We also write Q_d for the underlying set field($\leq_d$))
(2) For all $x, y \in Q_d$: $x \perp_d y \Leftrightarrow \neg\exists z \in Q_d : x \geq_d z \,\&\, y \geq_d z$, i.e., $\perp_d$ is equal to $\perp_{(Q_d, \leq_d)}$, the incompatibility relation for the forcing notion Q_d.

Remark 7.4: *Clearly these are Π^1_2 conditions on d.*

Definition 7.5: *Assume that d codes a Souslin forcing Q_d, and M is a transitive model model of ZFC* that contains d.*

(1) We let Q^M_d be the Souslin forcing coded by d in M.
(2) If $p \in Q^M_d$, $q \in Q_d$, we say that q is (p, M)-generic, iff

$$q \Vdash_{Q_d} \text{“}G_{Q_d} \cap M \text{ is } Q^M_d\text{-generic over } M \text{ and contains } p.\text{”}$$

(3) *We say that "Q_d is a Souslin Proper forcing" or "d codes a Souslin Proper forcing" if d codes a Souslin forcing, and*

$(*)$ *for all countable M as above, every $p \in Q_d^M$ there exists a (p,M)-generic $q \in Q_d$*

The property that $\perp$ is analytic is sometimes not necessary. We say that $(Q,\leq)$ is weakly Souslin, if $\leq$ (but not necessarily $\perp_Q$) is analytic. Similarly, we say that $(Q,\leq)$ is weakly Souslin proper if in addition, (3)$(*)$ holds.

Remark 7.6: *(1) "d codes a Souslin proper forcing" is a Π^1_3 statement about d. Hence (by Shoenfield's absoluteness theorem) if it holds in V, it holds in every submodel that contains all countable ordinals.*

(2) If $(M,\in)$ is a transitive model of a sufficiently large part of ZFC (M may be a class), and $M \models$ "$\chi := \beth_\omega^+$ exists, and $M_0 := H(\chi)^M$ is countable, then M_0 is a countable model of ZFC, and q is (p,M)-generic iff q is (p,M_0)-generic. So for all practical purposes we can pretend that M is countable. (In particular this is true if ω_1 is a inaccessible cardinal in M.)*

Proof: (1) Every countable model M is isomorphic to some well-founded (ω,R). If $x \in \mathbb{R}^M$, we also write x for its image under this isomorphism.

It is enough to show that "d codes a Souslin Proper forcing notion" is a Π^1_3 statement.

d codes a Souslin proper forcing iff <u>for all</u> $R \subseteq \omega\times\omega$

Either (ω,R) is not well-founded (i.e., <u>there exists</u> an R-descending sequence)
or $(\omega,R) \not\models ZFC^*$ (this is Δ^1_1)
or $d \notin (\omega,R)$
or for all $p \in Q_d^M$ <u>there is</u> $q \in Q_d$ such that <u>for all</u> $r \leq q$

(i) for all D such that $(\omega,R) \models$ "D is open dense in Q_d", there is an i, $(\omega,R) \models i \in D$, $r \not\perp i$
(ii) $r \not\perp p$.

((i) implies that $q \Vdash G\cap D \neq \emptyset$, and (ii) implies that $q \Vdash p \in G$.)

Proof of (2): M and M_0 contain the same dense sets of Q_d.

Context 7.7: *In this whole section, ε will be an ordinal $\leq \omega_2$. S will be a countable subset of ω_2 that is closed under immediate successors and predecessors, where the order type of S is in M. α and γ will stand for ordinals $\leq \varepsilon$ in S. M will be a countable transitive model of ZFC* (= a large enough fragment of ZFC), or an "essentially" countable model as in 7.6(2).*

For $\alpha \in S$, let α^S be the order type of $\alpha \cap S$.

$\vec{d}$ will be a sequence of length ε, and $\vec{c}$ will be a sequence of length ε^S, $\vec{c} \in M$.

Definition 7.8: *Given a sequence $\vec{d} = \langle d_\alpha : \alpha < \varepsilon\rangle$ we define a countable support iteration $\langle P_\alpha, Q_\alpha : \alpha < \varepsilon\rangle$, by letting Q_α be a P_α-name of $Q_{d_\alpha[G_\alpha]}$ (if this is Souslin proper), i.e. P_α forces the following:*

If $d_\alpha[G_\alpha]$ is a code for a Souslin proper forcing, then $Q_\alpha = Q_{d_\alpha}$ otherwise $Q_\alpha = \{\emptyset\}$ is the trivial forcing.

Notation and Remark 7.9: *We write $\bar{Q}_{\vec{d}}$ for the iteration defined above, and we write $P_{\vec{d}\restriction\alpha}$ for P_α, i.e., $P_{\vec{d}\restriction\alpha}$ is the α-th iteration stage obtained from the definition $\vec{d}$, and $Q_\alpha = Q_{d_\alpha}$ describes the successor extension. If $\vec{d} \in M \models ZFC^*$, then $P^M_{\vec{d}\restriction\alpha}$ is the α-th iteration stage, computed from the definition $\vec{d}$ in the model M.*

We say "$\vec{d}$ codes a Souslin proper iteration", if for all α, $\Vdash_{P_{\vec{d}\restriction\alpha}} d_\alpha$ codes a Souslin Proper forcing.

We will consider sequences $\vec{d}$ and $\vec{c}$, where $|\vec{d}| = \varepsilon$ and $|\vec{c}| = \varepsilon^S$, $\vec{c} \in M$.

Definition 7.10: *Assume M, $\vec{c}$, $\vec{d}$, S, ε are as in 7.7.*

By induction on $\alpha \in S \cap \varepsilon$, we will define a P_α-name $G_\alpha\restriction(S, M, \vec{c}\restriction\alpha^S, \vec{d}\restriction\alpha)$ (which we usually abbreviate to G'_α or $G_\alpha\restriction(S,M)$), by requiring that $\mathbb{1}_{P_\alpha}$ forces the following:

$G'_\alpha \subseteq P^M_{\vec{c}\restriction\alpha^S}$ and

- *If $\alpha = \beta + 1$, and*
  - *(a) G'_β is $(P^M_{\beta^S}, M)$-generic*
  - *(b) $d_\beta[G_\beta] = c_{\beta^S}[G'_\beta]$*
  - *(c) $d_\beta[G_\beta]$ codes a Souslin Proper forcing (in $V[G_\beta]$)*

  then $G'_\alpha = G'_\beta * \Big(G(\beta) \cap M[G'_\beta]\Big)$

  If $\alpha = \beta + 1$ and (a)–(c) does not hold, then $G'_\alpha = \emptyset$.
- *If α is a limit, then for $p \in P^M_{\vec{c}\restriction\alpha^S}$ we let*

$$p \in G'_\alpha \;\Leftrightarrow\; \forall \beta \in S \cap \alpha \; p\restriction\beta^S \in G'_\beta$$

Definition 7.11: *We call $\vec{d}$ and $\vec{c}$ "corresponding" sequences if for all $\alpha < \varepsilon$,*

$$\Vdash_\alpha \text{If } G'_\alpha \text{ is generic over } M, \text{ then } c_{\alpha^S}[G'_\alpha] = d_\alpha[G_\alpha]$$

Remark 7.12: *We will only consider sequences $\vec{d}$ that code a Souslin proper iteration (see 7.9), and we will only consider corresponding $\vec{c}$.*

Remarks and Notation 7.13:

(1) whenever the parameters M, S, $\vec{c}$, $\vec{d}$ are clear from the context we will write G'_α for $G_\alpha\restriction(S, M, \vec{c}\restriction\alpha^S, \vec{d}\restriction\alpha)$.

(2) If G_α is a $P_{\vec{d}\restriction\alpha}$-generic filter, then we also write G'_α for the evaluation of the name G'_α by G_α, similarly for μ_α.

(3) Let $M_\alpha = M[G'_\alpha]$.

(4) It might seem more natural to write $G_\alpha\restriction(S,M)$ as G'_{α^S} $(= (G')_{\alpha^S})$ instead of $G'_\alpha = (G_\alpha)'$, but this would only complicate the notation.

(5) "G'_α is generic" means of course "generic for the forcing $P^M_{\vec{c}\restriction\alpha^S}$ over the model M".

Remark 7.14: *Assume $\vec{c}$ and $\vec{d}$ are as in 7.12, and $M \models$ "$\underset{\sim}{p}$ is a $P^M_{\vec{c}\restriction\gamma^S}$-name of a condition in $P^M_{\vec{c}}$".*

We say that q is $\underset{\sim}{p}$-generic (or more precisely, $(\underset{\sim}{p}, \gamma, M, S, \underset{\sim}{c}, \underset{\sim}{d})$-generic) iff q forces:

(A) *G'_ε is generic over M.*
(B) *$\underset{\sim}{p}[G'_\gamma] \in G'_\varepsilon$.*

Theorem 7.15: *Let $\vec{d}$, $\vec{c}$, S, M, γ, ε be as in 7.7. Assume that $M \models$ "$\underset{\sim}{p}$ is a $P_{\vec{c}\restriction\gamma^S}$-name for a condition in $P_{\vec{c}}$" (so $\underset{\sim}{p}\restriction\gamma$ is the name for its restriction to γ, and there is a canonical $P_{\vec{c}}$-name which we also call $\underset{\sim}{p}$).*

Assume that $q \in P_{\vec{d}\restriction\gamma}$ is $(\underset{\sim}{p}\restriction\gamma, \gamma, M, S, \vec{c}\restriction\gamma^S, \vec{d}\restriction\gamma)$-generic. Then there exists a condition $q^+ \in P_{\vec{d}}$ such that $q^+\restriction\gamma = q$ and q^+ is $(\underset{\sim}{p}, \gamma, M, S, \vec{c}, \vec{d})$-generic.

Corollary 7.16: *Assume $M \models p \in P_{\vec{c}}$. Then there exists a (p, M)-generic condition $q \in P_{\vec{d}}$.*

Proof of 7.15:

The proof is by induction on ε.

Successor step: Here is the only place where we explicitly use Souslin properness: let $\varepsilon = \alpha + 1$.

Using the induction hypothesis on α, we get a $(\underset{\sim}{p}\restriction\alpha, \gamma)$-generic condition $q^+\restriction\alpha \in P_\alpha$. To find $q^+(\alpha)$, we will work in $V[G_\alpha]$, where G_α is an arbitrary generic filter containing $q^+\restriction\alpha$. Let $d := d_\alpha[G_\alpha]$ ($= c_{\alpha^S}[G'_\alpha]$, because $\vec{c}$ and $\vec{d}$ are corresponding sequences).

Since

(a) $G'_\alpha \subseteq P_{\vec{c}\restriction\alpha^S}$ is generic over M
(b) $V[G_\alpha] \models d$ codes a Souslin proper forcing,

and $p(\alpha)[G'_\alpha]$ is in the Souslin proper forcing Q_d, by definition we can find an $(p(\alpha)[G'_\alpha], M_\alpha)$-generic condition $q^+(\alpha)$.

Coming back to V, we use the existential completeness lemma to get a name (which we also call $q^+(\alpha)$) about which the above is forced by $q^+\restriction\alpha$.

Clearly this construction ensures that q^+ is generic, by 1.10.

Limit step: let $\langle \alpha_n : n < \omega \rangle$ be a cofinal sequence in $\varepsilon \cap S$, $\alpha_0 = \gamma$. Let $\langle D_n : n \in \omega \rangle$ enumerate all dense open subsets of $P^M_{\vec{c}}$ that are in M.

First we will define a sequence $\langle \underset{\sim}{p}_n : n \in \omega \rangle$, $\underset{\sim}{p}_n \in M$, $\underset{\sim}{p}_0 = \underset{\sim}{p}$, such that <u>in M</u> the following will hold:

(0) $\underset{\sim}{p}_n$ is a $P_{\vec{c}\restriction\alpha^S_n}$-name for a condition in $P_{\vec{c}}$
(1) $\Vdash_{\alpha^S_{n+1}} \underset{\sim}{p}_{n+1} \le \underset{\sim}{p}_n$

(2) $\Vdash_{\alpha^S_{n+1}} \underset{\sim}{p}_{n+1} \in D_n$.

(3) $\Vdash_{\alpha^S_{n+1}}$ If $\underset{\sim}{p}_n \restriction \alpha^S_{n+1} \in G_{\alpha^S_{n+1}}$ then $\underset{\sim}{p}_{n+1} \restriction \alpha^S_{n+1} \in G_{\alpha^S_{n+1}}$.

(Here, of course, G_β stands for the canonical name (in M) for the generic object of $P^M_{\vec{c}\restriction\beta}$, and $\Vdash_\beta$ is the forcing relation of $P^M_{\vec{c}\restriction\beta}$ in M.)

For each n we thus get a name $\underset{\sim}{p}_n$ that is in M. We use 3.17 and 0.5 (in M) to obtain $\underset{\sim}{p}_{n+1}$.

Now we define a sequence $\langle q_n : n \in \omega\rangle$, $q_n \in P_{\alpha_n}$, $q_0 = q$, such that for all n:

(a) $q_n \in P_{\alpha_n}$, $n \geq 1 \Rightarrow q_n \restriction \alpha_{n-1} = q_{n-1}$.

(b) q_n is generic for $\vec{d}\restriction\alpha$, $\vec{c}\restriction\alpha^S$, S, M.

(c) $q_n \Vdash \underset{\sim}{p}_n \restriction \alpha^S_n \in G'_{\alpha_n}$.

$q_{n+1} = q_n^+$ can be obtained by the induction hypothesis, applied to α^S_n, α^S_{n+1}, and $\underset{\sim}{p}_n \restriction \alpha^S_{n+1}$.

$q_{n+1} = q_n^+$ can be obtained by the induction hypothesis, applied to α_n, α_{n+1}, and $\underset{\sim}{p}_n \restriction \alpha_{n+1}$. By (c)$^+$ we know

$$q_n^+ \Vdash (\underset{\sim}{p}_n) \restriction \alpha^S_{n+1} \in G'_{\alpha_{n+1}} \cap M$$

Hence by (3) we have

$$q_{n+1} \Vdash (\underset{\sim}{p}_{n+1}) \restriction \alpha^S_{n+1} \in G'_{\alpha_{n+1}} \cap M$$

Since $q_{n+1} \restriction \alpha_n = q_n$, $q = \lim q_n$ exists and is $\leq q_n$ for all n.

We have to show that $q \Vdash p \in G'_\varepsilon \cap M$ and that q is generic. Let G_ε be a generic filter containing q. We will write p_n for $\underset{\sim}{p}_n[G'_{\alpha_n}]$. (Note that $p_n \in M$, because q_n was M-generic and $q_n \in G_{\alpha_n}$.)

Since $q_n \geq q \in G_\varepsilon$, we have $p_n \restriction \alpha^S_n \in G_{\alpha_n} \cap M$ and $M \models p_n \leq p_{n-1} \leq \cdots \leq p_0$.

Hence $p \restriction \alpha^S_n \in G'_{\alpha_n} \cap M$ for all n, and so by 1.17, $p \restriction \delta \in G_\delta \cap M$. As $dom(p) \subseteq \delta$, $p \restriction \delta = p$, so $p \in G_\varepsilon$. Similarly, $p_n \in G_\varepsilon$ for all n.

Consider a dense set $D_n \subseteq P_\varepsilon$. Since $q_{n+1} \Vdash$ "$\underset{\sim}{p}_{n+1} \in D_n$," we have $p_{n+1} \in G_\varepsilon \cap D_n \cap M$.

Hence q is generic.

Shelah has suggested the following modification of the definition of Souslin forcing:

Definition 7.17: *Assume $(Q, \leq)$ is a forcing notion, $Q \subseteq {}^\omega\omega$ and epd_Q ("effectively predense") is a relation on $Q \times [Q]^\omega$. $(Q, \leq, \mathrm{epd})$ is called a Souslin$^+$ forcing if*

(1) $\leq$ is analytic, and epd is a Σ^1_2 set.

(2) If $\mathrm{epd}(q, \{p_i : i < \omega\})$, then $\{p_i : i < \omega\}$ is predense below q.

$(Q, \leq, \mathrm{epd})$ is called Souslin$^+$ proper iff in addition

(3) Whenever $(M,\in) \models ZFC$ is a countable model, $Q \in M$ (i.e., the formulas defining $\leq_Q$ and epd_Q have parameters in M), and $p \in Q \cap M$, then there is a condition $q \leq p$ such that

$$(*) \qquad \begin{array}{c} \textit{For all } A \in M, \textit{ if } M \models A \textit{ is predense below } p \\ \textit{then } \mathrm{epd}(q, A \cap M) \end{array}$$

Definition 7.18: *We call a condition q satisfying $(*)$ "(p,Q,M)-effectively generic".*

Fact 7.19: *If q is (p,Q,M)-effectively generic, then q is (p,M)-generic.*

Note that for Souslin forcing the relation "q is (Q,M)-generic" is in general only Π^1_2, whereas the relation $(*)$ above is Σ^1_2.

Example 7.20: *Sacks forcing is Souslin$^+$ proper.*

Proof:

Recall that Sacks forcing $\mathbb{S}$ is the set of all perfect trees $\subseteq 2^{<\omega}$. (The incompatibility relation for Sacks forcing is not analytic, because to find a condition r extending p and q, one has to intersect the trees p and q (this is an arithmetical computation) and then find a perfect subtree $r \subseteq p \cap q$. The Cantor-Bendixson argument needed to obtain this tree r may take any countable number of steps, so an argument using the boundedness principle should show that "$p \perp q$" is not analytic.)

We let

$$epd(q, \{p_i : i < \omega\}) \Leftrightarrow \exists F \subseteq q, F \text{ is a front}, \forall \eta \in F\, \exists i\; p_i \geq q^{[\eta]}.$$

(Recall that $F \subseteq q$ is called a front, if F is an $\subseteq$-antichain that meets every branch of q, and $q^{[\eta]} = \{\nu \in q : \nu \subseteq \eta \vee \eta \subseteq \nu\}$.)

Clearly $epd(q, A)$ implies that A is predense below q.

To show that condition (3) is satisfied, we need a few definitions and a lemma about Sacks forcing $\mathbb{S}$.

Definition 7.21: *For $p \in \mathbb{S}$ we let*

$$\mathrm{split}(p) := \{\eta \in p : \eta^\frown 0 \in p \,\&\, \eta^\frown 1 \in p\}$$
$$\mathrm{split}_n(p) := \{\eta \in \mathrm{split}(p) : |\{\nu \in \mathrm{split}(p) : \nu \subset \eta\}| = n\}$$

$\mathrm{stem}(p)$ *is the unique element of* $\mathrm{split}_0(p)$*, i.e., the first node in p at which splitting occurs.*

If $p \in \mathbb{S}$, $D \subseteq \mathbb{S}$, we write $p \in^* D$ iff there is a pure extension r of p such that $r \in D$. (r is a pure extension of p if $\mathrm{stem}(p) = r$*).*

Lemma 7.22: *Assume $D \subseteq \mathbb{S}$ is an open dense set, and let $p \in \mathbb{S}$, $n \in \omega$. Then there is a condition $q \leq_n p$ and a front $F \subseteq q$ such that $\forall \eta \in F$, $q^{[\eta]} \in^* D$.*

Proof: Consider the following game $G(D,p,n)$: There are two players, who play ω many moves. We let $A_{-1} := \mathrm{split}_{n+1}(p)$.

In the nth move, player I plays $\eta_n \in A_{n-1}$, and player II responds by playing a set $A_n \subseteq p$ of pairwise incompatible elements of ${}^{\omega}2$, $|A_n| \geq 2$, and $\forall \nu \in A_n$, $\nu \supseteq \eta$.

Player I wins, if for some k, $p^{[\eta_k]} \in^* D$.

We claim that player II has no winning strategy. For assume that σ is a winning strategy for player II, then we can define

$$q := \{\eta \restriction l : l \le |\eta|, \eta \text{ appears as some } \eta_n \text{ in a play in which player II obeyed } \sigma\}$$

Clearly q is a tree. q is perfect: Assume $\eta \in q$. So there is an initial segment $\langle \eta_0, A_0, \dots, \eta_k, A_k\rangle$ of a play in which player II obeyed σ, and $\eta \subseteq \eta_k$. Let ν_0, ν_1 be two incompatible elements of A_k, then both ν_0 and ν_1 are in q. Hence $q \in \mathbb{S}$. It is also clear that $q \le_n p$.

But if σ was a winning strategy for player II, then for all $\eta \in q$, $q^{[\eta]} \notin^* D$, a contradiction.

The game is closed, so player I must have a winning strategy τ. Let q be defined as above, with "player II obeys σ" replaced by "player I obeys τ".

Again q is perfect: Let $\eta \in q$, so $\eta \subseteq \eta_k$, where $\langle \eta_0, A_0, \dots, \eta_k\rangle$ is an initial segment of a play in which player I obeyed τ. There are 4 incompatible extensions ν_0, ν_1, ν_2, ν_3 of η_k. Let

$$\langle \eta_0, A_0, \dots, \eta_k, \{\nu_0, \nu_1\}, \eta'\rangle \quad \text{and} \quad \langle \eta_0, A_0, \dots, \eta_k, \{\nu_2, \nu_3\}, \eta''\rangle$$

be initial segments of plays according to τ. Then $\eta' \in \{\nu_0, \nu_1\}$, $\eta'' \in \{\nu_2, \nu_3\}$, so η' and η'' are incompatible extensions of η in q. Since τ is a winning strategy, we can find a front as required.

Corollary 7.23: *If $D \subseteq \mathbb{S}$ is dense open, $p \in \mathbb{S}$, $n \in \omega$, then there is $p' \le_n p$ and a front $F \subseteq p'$ such that for all $\eta \in F$, ${p'}^{[\eta]} \in D$.*

Proof: FIrst we can get a condition q and a front $F \subseteq q$ as in the previous lemma. We may assume that the front F is above the n-th splitting level, i.e., $\forall \eta \in \mathrm{split}_n(q)\, \exists \nu \in F\, \eta \subseteq \nu$. Now thin out q at each $\eta \in F$ to get into D, i.e., for all $\eta \in F$ let $q_\eta \le q^{[\eta]}$ be a condition with stem η, $q_\eta \in D$, and let $p' := \bigcup_{\eta \in F} q_\eta$.

Proof of 7.20: Let $M \models ZFC$, $p \in P \cap M$, and let $\langle D_n : n \in m\rangle$ enumerate all sets D such that $M \models D$ is dense open below p. Using 7.23 we can find a sequence $\langle (p_n, F_n) : n \in \omega\rangle$, $p = p_0 \ge_0 p_1 \ge_1 \cdots$, $p_n, F_n \in M$, $F_n \subseteq p_n$ a front, and $\forall \eta \in F_n\, {q_n}^{[\eta]} \in D_n$. Let $q \le p_n$ for all n, then $F_n \cap q$ is a front in q, and for all $\eta \in F_n \cap q$ we have $q^{[\eta]} \in D$. Now it is easy to see that for all $A \in M$, if $M \models A$ predense below p, then $A \cap M$ is effectively predense below q.

Remark 7.24: *Similar arguments apply to many other forcing notions whose elements are perfect trees — Laver forcing, rational perfect sets, etc. A theorem analogous to 7.15 can be shown for Souslin$^+$ proper forcing notions.*

8 A simple preservation theorem for finite support iteration

Context 8.1: *Similar to 5.1.*

Assumption 8.2: *We assume that for all $f \in \mathbf{C}$ the set $\{g : f \sqsubset_n g\}$ is closed. (Often we even have that $\sqsubset_n$ itself is closed, in some reasonable topology.)*

Definition 8.3: *We say that M is $(\sqsubset, \omega)$-closed if*

$$\forall A \subseteq M \cap \mathbf{C} \text{ (A countable) } \exists f^* \in M :$$
$$\forall g : \; f^* \sqsubset g \Rightarrow \forall f \in A \cap \mathbf{C} \, f \sqsubset g \tag{$*$}$$

Remark: Note that $(*)$ is a Π^1_1-statement, hence absolute between any $\in$-models of ZFC.

Theorem 8.4: *Assume $M \subseteq V$ is $(\sqsubset, \omega)$-closed. Let $\langle P_\alpha, Q_\alpha : \alpha < \varepsilon \rangle$ (ε limit) be a finite support iteration of ccc forcing notions such that*

$$\forall \alpha < \delta \;\; \Vdash_\alpha \forall g \, \exists f \in M \cap \mathbf{C} \; f \not\sqsubset g$$

Then $\Vdash_\delta \forall g \, \exists f \in M \; f \not\sqsubset g$.

Proof: By 1.20, we may assume that $cf(\delta) = \omega$. Assume the conclusion is false, so there is a condition $p_0 \in P_\delta$ and a P_δ-name $\underset{\sim}{g}$ such that $p_0 \Vdash \forall f \in M \; f \sqsubset \underset{\sim}{g}$.

Let $\delta_0 < \delta_1 < \cdots$ be a sequence of ordinals conerging to δ. For each n, let $\underset{\sim}{g}_n$ be a P_{δ_n}-name such that

$\Vdash_{\delta_n} \underset{\sim}{g}_n$ interprets $\underset{\sim}{g}$ (considering $\underset{\sim}{g}$ as a P_{δ_n}-name for a P_δ / G_{δ_n}-name)

and let $\underset{\sim}{f}_n$ be a P_{δ_n}-name such that

$$\Vdash_{\delta_n} \underset{\sim}{f}_n \in M \,\&\, \underset{\sim}{f}_n \not\sqsubset h$$

Since P_δ is ccc (even properness is sufficient here), there is a countable set $F \in V$, $F \subseteq \mathbf{C}$ such that $\forall n \Vdash_{\delta_n} \underset{\sim}{f}_n \in F$.

By our assumption, M is $(\sqsubset, \omega)$-closed. So we can find a function $f^* \in M$ such that

$$\forall h \in {}^\omega\omega : f^* \sqsubset h \Rightarrow \forall f \in F \cap \mathbf{C} \, f \sqsubset g$$

Since $f^* \in M$, $p_0 \Vdash_\delta f^* \sqsubset \underset{\sim}{g}$. We can find a condition $p_1 \leq p_0$ and an integer n such that $p_1 \Vdash f^* \sqsubset_n \underset{\sim}{g}$. Since p_1 has finite support, there is an $m \in \omega$ such that $p_1 \in P_{\delta_m}$.

Now work in $V[G_{\delta_m}]$, where $p_1 \in G_{\delta_m}$. Clearly $\Vdash_{\delta_m, \delta} f^* \sqsubset_n \underset{\sim}{g}$. Since $f_m \not\sqsubset_n g_m$ (where $f_m = \underset{\sim}{f}_m[G_{\delta_m}]$, $g_m = \underset{\sim}{g}_m[G_{\delta_m}]$), we also have $f^* \not\sqsubset_m g_m$.

We consider the set $\{h : f^* \not\sqsubset_m h\}$. By assumption on $\sqsubset$, this is an open set, containing g_m. So there is $k \in \omega$ such that

$$\forall h : h{\restriction}k = g_m{\restriction}k \Rightarrow f^* \not\sqsubset_n h$$

Let $q \in P_\delta/G_{\delta_m}$ be a condition forcing $\underset{\sim}{g}\restriction k = g_m\restriction k$. Then q also forces $f^* \not\sqsubset_n \underset{\sim}{g}$, a contradiction.

Example 1 8.5: *Let*

$$f \sqsubset_n g \;\Leftrightarrow\; \forall k \geq n\, f(k) < g(k)$$

The "bounding" number $\mathfrak{b}$ is defined by

$$\mathfrak{b} := \{\min |B| : B \subseteq {}^\omega\omega, B \text{ is unbounded}\}$$

where "B is unbounded" means that there is no function g such that for all $f \in B$ we have $f \sqsubset g$. We are interested in preserving an unbounded family. It is easy to construct (by induction) an unbounded family B that is well-ordered by $\sqsubset$ and has order type $\mathfrak{b}$. It is also easy to see that the cofinality of $\mathfrak{b}$ must be uncountable.

Hence B is $(\sqsubset, \omega)$-closed. So we get

Fact 8.6: *Assume $\langle P_\alpha, Q_\alpha : \alpha < \varepsilon\rangle$ is a finite support iteration of ccc forcing notions such that for all α we have*

$$\Vdash_\alpha B \textit{ is unbounded in } {}^\omega\omega \cap V[G_\alpha]$$

Then

$$\Vdash_\varepsilon B \textit{ is unbounded in } {}^\omega\omega \cap V[G_\varepsilon]$$

Proof: Apply 8.4.

In particular, we get: If no Q_α adds a dominating function, and $V \models \mathfrak{b} = \kappa$, then $V[G_\varepsilon] \models \mathfrak{b} \leq \kappa$.

Example 2 8.7: *Let $\sqsubset^{\text{random}}$ be defined as in 6.8. It was proved above that $f \sqsubset g$ iff g is not in the null set coded by f. So, letting $M = V$ (which is clearly $(\sqsubset^{\text{random}}, \omega)$-closed, we get*

Fact 8.8: *If $\langle P_\alpha, Q_\alpha : \alpha < \varepsilon\rangle$ is a finite support iteration of ccc forcing notions such that for all α we have*

$$\Vdash_\alpha \textit{there is no random real over } V$$

then

$$\Vdash_\varepsilon \textit{there is no random real over } V$$

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