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A Taste of Proper Forcing

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Abstract

We review basic definitions and theorem in Shelah's theory of Proper Forcing.

Since these notes are meant to be an introduction to proper forcing, there are no new mathematical results here. I hope that these notes can

(a) serve as a "springboard" for diving into deeper literature (such as chapters III, X and XII of Shelah's books [9] and [8][1])

(b) inspire others to continue to expand this "secondary" or "talmudic" literature on proper forcing

(c) popularize the (conscious) use of the "alphabet convention" 1.2.

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1 Notation and Conventions

Claim 1.1. We use standard set theoretic notation, as it is found in the books of Jech [5] and Kunen [6].

This claim is of course wrong.

We use "upwards" notation for forcing, see below. However, we still call the generic subset of P "filter", although technically speaking it is an ideal.

For technical reasons all our forcing notions $\mathbb{P}$ come equipped with a weakest element $\emptyset_{\mathbb{P}}$.

Names in a forcing language (or variables ranging over such names) are usually (but not always) marked by a tilde, as in $\underset{\sim}{\alpha}$. Standard names for

[1] One of the main open problems in this field is the question when this book will appear.

objects in the ground model are in principle marked by a "check" accent (as in $\check{a}$), but we almost always omit it. G or $G_{\mathbb{P}}$ is the canonical name for a generic filter, but often also stands for a variable ranging over all V-generic filters.

$\underset{\sim}{a}[G]$ is the "evaluation" of the name $\underset{\sim}{a}$ by the filter G (sometimes written $val_G(\underset{\sim}{a})$)

We occasionally confuse the set of names, $V^{\mathbb{P}}$ with an arbitary generic extension $V[G_{\mathbb{P}}]$. We usually assume that V is the whole universe, but occasionally treat it as a countable model over which we can find generic filters.

We also confuse a forcing notion $\mathbb{P} = (P, \leq, \emptyset_P)$ with its carrier set P.

If $\mathbb{P}$ is a forcing notion, $q \in \mathbb{P}$, and $A \subseteq \mathbb{P}$ (typically, a maximal antichain), we write $A \restriction q$ be the set of elements of A "selected" by q, i.e., all $a \in A$ which are compatible with q.

dom(f) and ran(f) stand for the domain and range, respectively, of a function (or relation) f.

Quo Vadis?

Traditionally, there are two (contradictory) notations for interpreting a partial order as a forcing notion. A majority of set theorists (including the books by Kunen and Jech) uses the "Boolean" or "downwards" notation, where $q \leq p$ means that q is "stronger" than p (and in particular, $q \leq p \Rightarrow q \Vdash p \in \underset{\sim}{G}_{\mathbb{P}}$), citing the universal agreement on the standard order of a boolean algebra or a lattice: A conjunction $p \wedge q$ is traditionally considered to be smaller than its constituents.

The "Israeli" or "upwards" tradition (used not only by Shelah and some of his coauthors but also by Cohen in his original paper) expresses the same concept by $q \geq p$ (arguing that q has "more" information than p).

(A third possiblity is to abandon $\leq$ altogether and only use $q \Vdash p$ as an abbreviation for $q \Vdash p \in G$.)

We use here the Israeli notation, but to make it easier for the readers in the "Boolean" camp we in addition employ the alphabet convention, a notation which is compatible with the upwards as well as with the downwards interpretation.

Definition 1.2. The alphabet convention
Whenever two conditions are comparable, the notation is chosen so that the variable used for the stronger condition comes "lexicographically" later.

For example, we can have a condition q which is strictly stronger than p, but we try to avoid the converse situation. Similarly, a condition called p_2 or p_1' is allowed to be either stronger than p_1 or incompatible with p_1, but not (strictly) weaker.

Note that in some isolated cases the alphabet convention may be impossible or inconvenient to execute, for example if we work with quasiorders and have to establish $q \leq p$ and $p \leq q$, or if conditions are denoted by expressions such as $p \wedge q$ or $[\![\varphi(\underset{\sim}{r}, \underset{\sim}{s})]\!]$.

2 Proper Forcing

Imagine you want to construct a model of set theory in which $2^{\aleph_0} = \aleph_2$. A natural way to do this is to construct a model $V_1 = V_0^{\mathbb{P}}$, where V_0 is the "ground model" satisfying CH (this makes various combinatorics easier), and $\mathbb{P}$ is a forcing notion adding ${\aleph_2}^{V_0}$ many reals. This can of course only work if we can also ensure that ${\aleph_2}^{V_0} = {\aleph_2}^{V_1}$. A popular way to achieve this equality is to use a forcing notion that satisfies the countable chain condition. The ccc is helpful because of the following fact:

Fact 2.1. Let $\mathbb{P}$ satisfy the ccc, $\Vdash_{\mathbb{P}}$"$\underset{\sim}{A} \subseteq Ord$ is countable [or of cardinality $\leq \kappa$]", then there is a set $B \subseteq Ord$ which is countable [or of cardinality $\leq \kappa$] with $\Vdash_{\mathbb{P}} \underset{\sim}{A} \subseteq B$.

In particular, $\Vdash_{\mathbb{P}}$ "${\aleph_1}^V$ is uncountable and hence $= {\aleph_1}^{V^{\mathbb{P}}}$", since in $V^{\mathbb{P}}$ any countable subset of ${\aleph_1}^V$ is covered by a countable set from V, hence is bounded below ${\aleph_1}^V$.

However, in many cases ccc forcing is inappropriate. Propernes is a property of forcing notions which is weaker than ccc (and at the same time also a weakening of "σ-closed") and is still a condition sufficent for not collapsing ω_1. Instead of 2.1, a typical proper forcing construction will use the following fact (see 2.8 below):

Fact 2.2. If $\mathbb{P}$ is proper (defined below), then:

- whenever $p \Vdash \underset{\sim}{A} \subseteq Ord$ is countable, then there is $q \geq p$ and a countable set B such that $q \Vdash \underset{\sim}{A} \subseteq B$.
- If moreover $\mathbb{P}$ satisfies the $\aleph_2$-cc (i.e., any antichain of $\mathbb{P}$ has size at most $\aleph_1$), then $\mathbb{P}$ preserves all cardinalities and cofinalities.

We will give several equivalent definitions of properness:

Definition 2.3. Let $\mathbb{P}$ be a forcing notion, $p \in P$. The antichain game $G_{ac}(\mathbb{P}, p)$ is defined as follows: Player I plays a maximal antichain A_0 above p. Player II responds with a countable (i.e., at most countable) subset B_0^0. In the next move, player I again plays a maximal antichain A_1 above p, and player II is now allowed to play two countable sets: $B_0^1 \subseteq A_0$, $B_1^1 \subseteq A_1$.

In the n-th move, player I plays a maximal antichain A_n above p, and player II plays countable sets $B_0^n \subseteq A_0, \ldots, B_n^n \subseteq A_n$.

I	II
A_0	B_0^0
A_1	B_0^1, B_1^1
A_2	B_0^2, B_1^2, B_2^2
$\vdots$	$\vdots$
	$B_0, B_1, \dots$

After ω many moves, player II wins if there is a condition $q \geq p$ such that, letting $B_n := \bigcup_{k=n}^{\infty} B_n^k$,

$$\forall n : A_n \restriction q \subseteq B_n$$

(See the end of section 1 for the definition of $A \restriction q$)

Definition 2.4. $\mathbb{P}$ is proper iff for all $p \in \mathbb{P}$ player II has a winning strategy in the game $G_{ac}(\mathbb{P}, p)$.

Note that by this definition, ccc forcing notions are trivially proper, since we can play $B_n^n := A_n$.

Why are we so interested in antichains? Recall that one of our goals is not to collapse ω_1. So we have to deal with sequences $(\underset{\sim}{\alpha}_n : n \in \omega)$ of ordinals in the extension. Now it is well-known that the information in a name of an ordinal is really coded in an antichain, as follows:

Definition 2.5. Let $A \subseteq P$ be a maximal antichain above $p \in P$, and let $f : A \to Ord$ be a function. Then $\underset{\sim}{\alpha} := \underset{\sim}{\alpha}_{A,f} := \{(f(q), q) : q \in A\}$ is a name for an ordinal above p (i.e., $p \Vdash \underset{\sim}{\alpha}_{A,f} \in Ord$), and for each $q \in A$ we have $q \Vdash \underset{\sim}{\alpha} = f(q)$.

This definition shows us how to translate an antichain (plus an enumerating function) into a name of an ordinal. Conversely we can translate a name of an ordinal into a function defined on a maximal antichain:

Definition 2.6. Whenever $p \Vdash_{\mathbb{P}} \underset{\sim}{\beta} \in Ord$, we define $D_{\underset{\sim}{\beta}}$ (or more precisely, $D_{\underset{\sim}{\beta},p}$ as follows:

$$D_{\underset{\sim}{\beta}} := \{q \geq p : \exists \gamma_q \; q \Vdash \underset{\sim}{\beta} = \gamma\}$$

Clearly, $D_{\underset{\sim}{\beta}}$ will be dense open above p, and whenever $A \subseteq D_{\underset{\sim}{\beta}}$ is a maximal antichain in $D_{\underset{\sim}{\beta}}$ then the function f which maps each $q \in A$ to γ_q carries all interesting information about the name $\underset{\sim}{\beta}$: $p \Vdash \underset{\sim}{\beta} = \underset{\sim}{\alpha}_{A,f}$.

Using this correspondence we can now translate the antichain game to the following game:

Definition 2.7. Let $\mathbb{P}$ be a forcing notion, $p \in \mathbb{P}$. The (unrestricted) ordinal game $G_{or}(\mathbb{P}, p, \infty)$ is defined as follows:

In the n-th move, player I plays a $\mathbb{P}$-name $\underset{\sim}{\alpha}_n$ of an ordinal (above p, i.e., $p \Vdash \underset{\sim}{\alpha}_n \in Ord$). Player II reponds with a countable set $B_n \subseteq Ord$.

After ω many moves, player II wins if there is a condition $q \geq p$ such that, letting $B := \bigcup_{k \in \omega} B_n$ we have

$$q \Vdash_{\mathbb{P}} \forall n \; \underset{\sim}{\alpha}_k \in B$$

For any ordinal χ we also define the game $G_{or}(\mathbb{P}, p, \chi)$ which is similar to $G_{or}(\mathbb{P}, p, \infty)$, but player I has to play names $\underset{\sim}{\alpha}_n$ for which $p \Vdash \underset{\sim}{\alpha}_n < \chi$ holds, and player II responds with countable sets $B_n \subseteq \chi$.

Remarks 2.8. First, note that the set B is countable, so if player II has a winning strategy in $G_{or}(\mathbb{P}, p, \infty)$ then we have:

> If $p \Vdash \underset{\sim}{A} \subseteq Ord$ is countable, then there is $q \geq p$ and a countable set B such that $q \Vdash \underset{\sim}{A} \subseteq B$.
>
> In particular, $\Vdash_{\mathbb{P}} \aleph_1{}^V = \aleph_1{}^{V^{\mathbb{P}}}$, and more generally the property $cf(\alpha) > \aleph_0$ is preserved when passing from V to V^P.

Next, note that if player II has a winning strategy, then he also has a winning strategy in which all sets B_n are singletons (by dividing ω into countably many countable sets, and using a simple bookkeeping method). Furthermore, allowing player I to play countably many ordinals in each move does not change the existence of a winning strategy for player II, either.

Finally note that a winning strategy for player II in the game $G_{or}(\mathbb{P}, p, \chi)$ for any large enough χ (say, $\chi > |P|$) will give a winning strategy for player II in the antichain game, using the correspondence between names of ordinals and antichains discussed above. Conversely, a winning strategy for the antichain game gives a winning strategy for any $G_{or}(\mathbb{P}, p, \chi)$ (including $\chi = \infty$).

Also, note that if a forcing notion is σ-closed, we can easily describe a winning strategy for the game $G_{or}(\mathbb{P}, p)$: Player II will construct an increasing sequence $(p_n : n \in \omega)$, $p \leq p_0 \leq p_1 \leq \cdots$ and a sequence $(\beta_n : n \in \omega)$, such that $p_n \Vdash \underset{\sim}{\alpha}_n = \beta_n$. After ω many moves, any upper bound q of the sequence $p_n : n \in \omega$ will force $\forall n \, \underset{\sim}{\alpha}_n = \beta_n$.

We now give another characterisation of properness, which may at first look rather complicated, but turns out to often be easiest to verify in actual applications:

Definition 2.9. Let χ be a "large enough" regular cardinal. (It will turn out that the property we define will really not depend on χ). We write $H(\chi)$ for the family of sets whose transitive closure has cardinality $< \chi$. ($H(\chi)$ satisfies all of ZFC except posisbly for the power set axiom.) Let $(N, \in)$ be a countable elementary submodel of $(H(\chi), \in)$, and let $\mathbb{P} \in N$ be a forcing notion.

We say that $q \in \mathbb{P}$ is (N, P)-generic (or, when we are lazy: "N-generic" or "P-generic" or just "generic"), if the following two (equivalent) conditions hold

- Whenever $A \subseteq P$ is a maximal antichain, $A \in N$, then $A \restriction q \subseteq N$. (See the end of section 1 for the definition of $A \restriction q$)
- Whenever $\underset{\sim}{\alpha} \in N$, and $\Vdash_{\mathbb{P}} \underset{\sim}{\alpha} \in Ord$, then $q \Vdash_{\mathbb{P}} \underset{\sim}{\alpha} \in N$ (i.e.: Whenever $G \subseteq P$ is generic over V, $q \in G$, then $\underset{\sim}{\alpha}[G] \in N$), or in shorter notation:

$$q \Vdash N[G] \cap Ord = N \cap Ord$$

(The equivalence between those condition can again easily be shown using the correspondence between names of ordinals and antichains that we discussed above)

Since we have already defined properness, we will call the folowing proposition a "theorem". Alternatively, we could have used it as definition of properness.

Theorem 2.10. *$\mathbb{P}$ is proper iff: for all (or some) large enough regular χ, for all elementary countable submodels $(N, \in) \prec (H(\chi), \in)$ containing P and all $p \in P \cap N$ there is a condition $q \in \mathbb{P}$, $q \geq p$, such that q is $(N, \mathbb{P})$-generic.*

We leave the details of the proof to the reader, but we will give the following hints: If you have a strategy σ for player II in the game$G_{ac}(\mathbb{P}, p)$ then there will be such a strategy in N. Let player I play all antichains A which are elements of N — after all, there are only countably many! Player II will respond with countable subsets B_n which are in N (since σ as well as $A_0, \ldots, A_n$ are in N) Now since each B_n is countable and $\in N$, we must also have $B_n \subseteq N$.

Conversely, we can define a strategy by letting player II in the n-th step construct a countable model N_n containing $N_0, \ldots, N_{n-1}$ as well as $A_0, \ldots, A_n$ as elements, $N_n \prec H(\chi)$. After ω many steps, let $N = \bigcup_{n \in \omega} N_n$, then any (P, N)-generic condition will witness that player II has won.

We already know that σ-closed forcing notions are proper, as are all forcing notions satisfying the countable chain condition. Later we will see that properness is preserved under composition of forcing notions, so also any finite composition of forcing notions satisfying the ccc or σ-completeness (e.g., Mathias forcing) are proper.

The following example is essentially different from those "trivial" examples, and is quite typical for a large class of proper forcing notions:

Example 2.11. Let $\mathbb{P}$ be the forcing notion "adding a club to ω_1 with finite conditions". Conditions in $\mathbb{P}$ are finite strictly monotone partial functions p from ω_1 to ω_1. It is easy to see that the sets $\{p : \max \mathrm{dom}(p) > \alpha\}$ are dense for all $\alpha \in \omega_1$, so a generic filter G will add a monotone function g from some unbounded $B \subseteq \omega_1$ to ω_1.

Let C be the closure of $\mathrm{ran}(g)$, then C is a "generic" club set.

We claim that $\mathbb{P}$ is proper. So let $(N, \in) \prec (H(\chi), \in)$, $p \in N$. Let $\delta := N \cap \omega_1$. Clearly δ is an ordinal (recall that every countable element of N must be a subset of N). Note that $\mathrm{dom}(p)$ and $\mathrm{ran}(p)$ are finite sets in N, so they must be subsets of N and hence of δ.

Hence $q := p \cup \{(\delta, \delta)\}$ is a monotone function, so $q \in \mathbb{P}$. We claim that q is generic. So let $A \in N$ be a maximal antichain. We want to show that $A \restriction q \subseteq N$, so towards a contradiction assume that there is $r \in A \setminus N$, r compatible with q.

So for all $\alpha \in \mathrm{dom}(r)$, $r(\alpha) < \delta$ iff $\alpha < \delta$. Let $r' := r \cap N$, then also $r' \in \mathbb{P} \cap N$. Since $N \models A$ is a maximal antichain, we can find $a \in A \cap N$ which is compatible with r'. Now check that a must also be compatible with r, since for all pairs (α, β) in $r \setminus r'$ we have $\alpha, \beta > \delta$. But $a \in A \cap N$, $r \in A \setminus N$ and A is an antichain, so a and r cannot be compatible, a contradiction.

3 Variants of properness

It can happen that a forcing notion just "barely misses" being proper, but still has many of the good qualities enjoyed by proper forcing notions. In this section we will first give an example of a nonproper forcing notion that is "almost" proper, and moreover "almost" σ-complete, and then describe a variant of properness that is satisfied by this forcing notion.

Definition 3.1. Let $S \subseteq \omega_1$ be stationary. Define

$$\mathbb{P}_S := \{f : \exists \alpha < \omega_1 \ \mathrm{dom}(f) = \alpha + 1, f \text{ increasing continuous}, \mathrm{ran}(f) \subseteq S\}$$

with the natural ordering: $f \leq g$ iff $f \subseteq g$.

$\mathbb{P}_S$ is called "collapsing $\omega_1 \setminus S$ (or: shooting a club through S) with countable conditions". Indeed, it is easy to see that a generic filter on S will induce an increasing continous map f from ω_1 into S, so in $V^{\mathbb{P}_S}$ the set $\omega_1 \setminus S$ will be disjoint from the club set $\mathrm{ran}\, f$, hence nonstationary.

We will see below that $\mathbb{P}_S$ cannot be proper (at least if $\omega_1 \setminus S$ is stationary in the ground model). On the other hand, $\mathbb{P}_S$ is almost σ-complete, as the following remark shows:

Remark 3.2. Consider a sequence $f_0 \leq f_1 \leq f_2 \leq \cdots$ of conditions, let $\mathrm{dom}(f_n) = \alpha_n + 1$, and wlog assume $\alpha_n < \alpha_{n+1}$ for all n. Let

$$\alpha := \sup\{\alpha_n : n \in \omega\} \qquad \delta := \sup\{f_n(\alpha_n) : n \in \omega\}$$

It is now easy to see that

- If $\delta \in S$, then $(f_n : n \in \omega)$ has a (least) upper bound, namely, $\bigcup_n f_n \cup \{(\alpha, \delta)\}$
- If $\delta \notin S$, then $(f_n : n \in \omega)$ has no upper bound.

Thus, in "many" cases we have a version of σ-completeness.

We use this example to motivate the following definitions:

Definition 3.3. Let $S \subseteq \omega_1$. $\mathbb{P}$ a forcing notion. We say that P is S-proper iff:

> For all $(N, \in) \prec (H(\chi), \in)$, if N is countable with $N \cap \omega_1 \in S$, then for all $p \in \mathbb{P} \cap N$ there is $q \geq p$ which is N-generic.

Thus, we demand the existence of an $(N, \mathbb{P})$ generic condition not for ***all*** models N but only for a certain (stationary) subset of the set of countable elemetary submodels of $H(\chi)$.

Definition 3.4. 1. Let $(N, \in) \prec (H(\chi), \in)$, $\mathbb{P} \in N$ a forcing notion. We say that $q \in P$ is N-complete if for all dense open sets $D \in N$ there is $p \in N \cap D$, $p \leq q$, or in other words, if the set

$$\{p \in N : p \leq q\}$$

is an N-generic filter on $\mathbb{P}$.

2. Let $S \subseteq \omega_1$. $\mathbb{P}$ a forcing notion. We say that P is S-complete iff:

> For all $(N, \in) \prec (H(\chi), \in)$, if N is countable with $N \cap \omega_1 \in S$, then for all $p \in \mathbb{P} \cap N$ there is $q \geq p$ which is N-complete.

The following facts are immediate consequences of the definitions.

Fact 3.5. 1. If q is N-complete, then q is N-generic.

2. If $\mathbb{P}$ is S-complete then $\mathbb{P}$ is S-proper.

3. If $\mathbb{P}$ is σ-complete, then $\mathbb{P}$ is S-complete for every S.

4. If $\mathbb{P}$ is proper, then $\mathbb{P}$ is S-proper for any S. (More generally, properness $= \omega_1$-properness, and if $S \subseteq S'$ then S'-properness implies S-properness.)

Note that N-completeness is much stronger than N-genericity: An N-complete condition decides all names $\underset{\sim}{\alpha}$ of ordinals which are in N, whereas an N-generic condition merely forces that they will be interpreted somewhere in N.

We leave as an exercise to show that if $\mathbb{P}$ has the countable chain condition, then *every* condition $q \in P$ is N-generic (whereas typically N-complete conditions will not exist in such cases).

Finally we show that the notion of S-completeness is appropriate for the forcing notion we have defined above:

Fact 3.6. $\mathbb{P}_S$ is S-complete.

Proof. Let $N \prec H(\chi)$, $\delta := N \cap \omega_1$, $\delta \in S$, $p \in \mathbb{P} \cap N$. Let $(D_n : n \in \omega)$ list all dense open subsets of $\mathbb{P}_S$ which are in N. In particular, for every $\alpha < \delta$ this list will contain the set

$$E_\alpha := \{p \in \mathbb{P} : \max\mathrm{dom}(p) > \alpha\} \cap \{p \in \mathbb{P} : \max\mathrm{ran}(p) > \alpha\}$$

Define an increasing sequence $(f_n : n \in \omega)$ of conditions satisfying $p \leq p_0$ and $p_n \in D_n \cap N$ for all n. Let $\alpha_n := \max\mathrm{dom}\, f_n$, $\delta_n := \max\mathrm{ran}\, f_n$. Clearly $\delta_n < \delta$ (since $f_n \in N$), and the sequence $(\delta_n : n \in \omega)$ cannot be bounded below δ. By 3.2 the sequence $(f_n : n \in \omega)$ has an upper bound f. Check that f is N-complete. □

We now show that S-properness is still sufficient to ensure that ω_1 is not collapsed. Moreover, we show that all stationary subsets of S will remain stationary in $V^{\mathbb{P}}$ if $\mathbb{P}$ is S-proper.

(This will imply that $\mathbb{P}_S$ cannot be proper unless $\omega_1 \setminus S$ was nonstationary: If $\omega_1 \setminus S$ was stationary, then $\mathbb{P}_S$ does not preserve its stationarity.)

Theorem 3.7. *Assume that S is stationary, and $\mathbb{P}$ is S-proper. Then:*

1. $\Vdash_{\mathbb{P}} {\omega_1}^{V^{\mathbb{P}}} = {\omega_1}^V$

2. $\Vdash_{\mathbb{P}}$ *S is stationary.*

3. Whenever α is an ordinal with uncountable cofinality, then $\Vdash_{\mathbb{P}}$ "α has uncountable cofinality."

Proof. We prove only (2). (1) and (3) are easier.

So assume that $p \Vdash_{\mathbb{P}} \underset{\sim}{C} \cap S = \emptyset$, $\underset{\sim}{C}$ a closed unbounded set. We can find a name $\underset{\sim}{f}$ of a strictly increasing continuous function from ω_1 to ω_1 with $p \Vdash_{\mathbb{P}} \mathrm{ran}(\underset{\sim}{f}) = \underset{\sim}{C}$.

Now we choose a model $N \prec H(\chi)$ satisfying $N \cap \omega_1 \in S$, where N contains all necessary information, such as $\mathbb{P}$, $\underset{\sim}{f}$, p, etc. [Why is there such a model?

Remember that S is stationary. Start with a continous tower $(N_i : i < \omega_1)$ of elementary submodels and use the fact that $\{N_i \cap \omega_1 : i < \omega_1\}$ is a closed unbounded set]

Let $\delta := N \cap \omega_1$, so $\delta \in S$. By S-properness we can find $q \geq p$, q N-generic. We claim that $q \Vdash_{\mathbb{P}} \delta = \sup(\underset{\sim}{C} \cap \delta)$, which easily gives a contradiction.

If this were not the case, we could find $r \geq q$ and $\alpha < \delta$ (so $\alpha \in N$) such that $r \Vdash \underset{\sim}{C} \cap \delta \subseteq \alpha$. But as $\alpha \in N$, we can find a name $\underset{\sim}{\beta} \in N$ with $\Vdash_{\mathbb{P}} \underset{\sim}{f}(\alpha) = \underset{\sim}{\beta}$. Since q is N-generic, $q \Vdash_{\mathbb{P}} \underset{\sim}{\beta} \in N$. So $r \Vdash \alpha \leq \underset{\sim}{f}(\alpha) = \underset{\sim}{\beta} < \delta$, $\underset{\sim}{\beta} \in \underset{\sim}{C}$, a contradiction.

This concludes the proof.

□

(More generally, it is easy to check that $C \in N$ implies $N \cap \omega_1 \in C$ whenever $N < H(\chi)$ and $C \subseteq \omega_1$ is club. It is also true (for any generic extension) that $(N, \in) \prec (H(\chi)^V, \in)$ implies $(N[G], \in) \prec (H(\chi)^{V^P}, \in)$, if only χ is sufficiently large compared with P. [Here, $N[G]$ is defined to be $\{\underset{\sim}{x}[G] : \underset{\sim}{x} \in N\}$.]
Now for any V-generic filter G we have $\underset{\sim}{C}[G] \in N[G]$, so $N[G] \cap \omega_1 \in \underset{\sim}{C}[G]$. If G contains in addition the N-generic condition q, then we also have $N[G] \cap \omega_1 = N \cap \omega_1 = \delta \in S$, which leads to a contradiction with $\Vdash_{\mathbb{P}} \underset{\sim}{C} \cap S = \emptyset$)

Why are we so obsessed with this complicated property "properness" and its variations, when we are really mainly interested in the seemingly simpler properties of "not collapsing ω_1" or "preserving certain stationary subsets of ω_1"?

The reason is that in many applications we need to construct a forcing notion through (finite) composition and (transfinite) iteration of various simpler forcing notions. However, the property "not collapsing ω_1" may not be preserved in limit steps of such iterations. That is, if we define a sequence

$P_0 = \{\emptyset\}$

$P_1 = P_0 * \underset{\sim}{Q}_0$, where Q_0 is a P_0-name of a forcing notion

$P_2 = P_1 * \underset{\sim}{Q}_1$, where Q_1 is a P_1-name of a forcing notion

...

$P_{n+1} = P_n * \underset{\sim}{Q}_n$

...

then it is possible that all the forcing notions P_n (in V) and all the forcing notions $\underset{\sim}{Q}_n$ (in the respective V^{P_n}) preserve $\aleph_1$, but there is no conceivable "limit" $\tilde{P}_\omega$ which will also preserve $\aleph_1$.

Example 3.8. Partition ω_1 into ω many disjoint stationary sets $\omega_1 = S_0 \cup S_1 \cup \cdots$. Let Q_n be the forcing notion $\mathbb{P}_{\omega_1 \setminus S_n}$ in V^{P_n}, where P_n is defined as above. Then if $V' \supseteq V$ is any universe in which there are V-generic filters for all P_n, then

$V' \models \omega_1{}^V$ is a countable union of nonstationary sets, hence nonstationary

so in V' ω_1 is countable.

The *raison d'être* for properness is the following theorem:

Theorem 3.9. *Properness is preserved in countable support iteration.*
That is, assume that

1. *$(\mathbb{P}_\alpha, \mathbb{Q}_\alpha : \alpha < \delta)$ is a countable support iteration with CS limit P_δ. (I.e., each $\mathbb{P}_\alpha$ is the set of all partial functions with countable domain $\subseteq \alpha$ and $p{\restriction}\alpha \Vdash_{\mathbb{P}_\alpha} p(\alpha) \in \mathbb{Q}_\alpha$).*

2. *For all $\alpha < \delta$, $\Vdash_{P_\alpha} \underset{\sim}{\mathbb{Q}}_\alpha$ is proper.*

THen for all $\alpha < \delta$, $\mathbb{P}_\alpha$ is proper.

Shelah's original proof of this theorem can be found in [9, chapter III]. Alternative proofs are in [3] (repeated in [2]), or (using games) in [9, chapter XII] and [4].

Similar proofs show the analogous thereom with "proper" replaced by S-proper, for any $S \subseteq \omega_1$.

4 Semiproper iteration

Definition 4.1. Let $\mathbb{P}$ be a forcing notion, χ a sufficiently large regular cardinal, $(N, \in) \prec (H(\chi), \in)$, $P \in N$. We say that $q \in \mathbb{P}$ is (N, P)-semigeneric, if $q \Vdash_{\mathbb{P}} N[G] \cap \omega_1{}^V = N \cap \omega_1{}^V$, i.e., for all names $\underset{\sim}{\alpha}$ in N, if $\Vdash_{\mathbb{P}} \underset{\sim}{\alpha} \in \omega_1{}^V$, then $q \Vdash \underset{\sim}{\alpha} \in N$.

Definition 4.2. We say that $\mathbb{P}$ is semiproper iff for all $\mathbb{P}$ and N as above, for all $p \in \mathbb{P} \cap N$, there is $q \geq p$, q (P, N)-semigeneric.

Equivalently, $\mathbb{P}$ is semiproper if player II has a winning strategy in $G_{or}(\mathbb{P}, p, \aleph_1)$ for all $p \in \mathbb{P}$.

(The equivalence can be shown as in 2.10)

Why are we interested in semiproperness? Semiproperness is a weak version of properness which is still sufficeint to show that $\aleph_1$ is not collapsed (the same proof works). Moreover, semiproperness is preserved in some iterations. The difference between properness and semiproperness is that 3.7(3) is in general (for $\alpha > \omega_1$) not true for semiproper forcing, see 4.4(2).

There can be useful forcing notions which are semiproper but not proper, as the following example shows.

Example 4.3. Let κ be a measurable cardinal, D a normal ultrafilter on κ. Prikry forcing $\mathbb{P}_D$ is defined as

$$\mathbb{P}_D := \{(s,A) : s \in [\kappa]^{<\omega}, A \in D, \max s < \min A\}$$

We let $(s,A) \leq (t,B)$ (remember the alphabet convention) iff $s \subseteq t$, $B \subseteq A$, and $t \setminus s \subseteq A$ (so t is an end extension of s).

The following fact is well known:

Fact 4.4. 1. For all $\lambda < \kappa$, $\mathbb{P}_D$ does not add new subsets of λ.

2. $\Vdash_{\mathbb{P}_D}$ "κ is a cardinal of cofinality $\aleph_0$".

This implies that $\mathbb{P}_D$ does not collapse $\aleph_1$ but is not proper. We will show that $\mathbb{P}_D$ is semiproper.

Lemma 4.5. *Let D be a normal ultrafilter on κ. Let $\underset{\sim}{\alpha}$ be a $\mathbb{P}_D$-name for an ordinal, $(s_0, A) \in \mathbb{P}_D$, $(s_0,A) \Vdash_{\mathbb{P}_D} \underset{\sim}{\alpha} < \aleph_1$.*

Then there is a countable set B and a set $A' \subseteq A$, $A' \in D$ such that $(s_0, A') \Vdash_{\mathbb{P}_D} \underset{\sim}{\alpha} \in B$.

Proof. For notational simplicity we will assume that $s_0 = \emptyset$.

For each $s \in [\kappa]^{<\omega}$ let A_s, β_s be such that

- If there is $A^* \subseteq A, \beta^*$ such that $(s, A^*) \Vdash \underset{\sim}{\alpha} = \beta^*$, then (A_s, β_s) is such a pair.
- Otherwise $\beta_s = *$.

For $\alpha < \kappa$ let $A_\alpha := \bigcap_{s \in [\alpha+1]^{<\omega}} A_s$, and let A_κ be the diagonal intersection of all the A_α: $A_\kappa = \{i \in \kappa : (\forall \alpha{<}i)(i \in A_\alpha)\}$.

Then $A_\kappa \in D$ (because D is normal), and for all $\alpha < \kappa$, for all $s \in [\alpha+1]^{<\omega}$ we have $A_\kappa \setminus (\alpha+1) \subseteq A_\alpha \subseteq A_s$.

By Rowbottom's theorem we have the partition relation

$$\kappa \rightarrow (\kappa)^{<\omega}_{\aleph_1}$$

and moreover we can find the homogeneous set in D. (That is, given any function $f : [\kappa]^{<\omega} \rightarrow \omega_1$ there is $A' \in D$ such that for all $n \in \omega$, $f \restriction [A']^n$ is constant.)

So there is $A' \subseteq A_\kappa$, $A' \in D$, and for all $n \in \omega$ there is $\alpha_n \in \omega_1$ such that

$$\forall s \in [A']^n \ \alpha_s = \alpha_n$$

Let $B := \{\alpha_n : n \in \omega\}$. It is now easy to check that $(s_0, A') \Vdash \underset{\sim}{\alpha} \in B$. □

Unfortunately semiproperness is not preserved by countable support iteration. This has a very natural reason, which can informally be explained as follows: Consider a countable support iteration $(\mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha : \alpha < \kappa)$ with CS limit $\mathbb{P}_\kappa$. Assume that $cf(\kappa) > \aleph_0$, but that this property is "lost" during the iteration, say $\mathbb{P}_1 \Vdash cf(\kappa) = \aleph_0$. (E.g., if $\mathbb{Q}_0$ is Prikry forcing.) Now P_κ is the direct limit of $(P_\alpha : \alpha < \kappa)$ but if we want to construct a semigeneric condition for P_κ we would like to use a condition with *unbounded* support, i.e., we would like to put information on an unboudned (hence uncountable) set of coordinates.

Revised countable support (RCS) iteration circumvents this problem by (still informally speaking) allowing unbounded sets as supports, as long as they are at least countable in some intermediate universe. In other words, we allow the support of a condition to be not only a countable set, but even a *name* of a countable set.

(There is no need for such tricks when we deal with proper forcing, because countable sets in any proper forcing extension are covered by a countable set from the ground model.)

Definition 4.6. Let $\bar{\mathbb{P}} = (\mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha : \alpha < \delta)$ be an iteration, δ a limit ordinal. The RCS-limit of $\bar{\mathbb{P}}$ is defined as the set of all $p \in \text{proj}\lim \bar{\mathbb{P}}$ (= the projective or "full" limit of $\bar{\mathbb{P}}$) satisfying

$$\forall q \in \text{proj}\lim \mathbb{P},\ q \geq p \text{ implies that there is } \alpha < \delta \text{ and } r \geq q \restriction \alpha \text{ with}$$

$$r \Vdash \text{"}cf(\delta) = \aleph_0 \text{ or } \text{supp}(p) \cap (\alpha, \delta) = \emptyset.$$

Here, $\text{supp}(p) \cap (\alpha,\delta)$ is a P_α-name for the set $\{\beta \in (\alpha,\delta) : p \restriction (\alpha,\delta) \not\Vdash_{\mathbb{P}_{\alpha,\delta}} p(\beta) = \emptyset_{\mathbb{Q}_\alpha}$, and $P_{\alpha,\delta}$ is the "quotient forcing $P_\delta : P_\alpha$" (which can be shown to also be the result of an RCS iteration.

Definition 4.7. We call an iteration $\bar{\mathbb{P}} = (\mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha : \alpha < \delta)$ an RCS-iteration, if for all $\alpha < \delta$ we have $P_\alpha = RCS\lim(\bar{P} \restriction \alpha)$.

(For technical reasons we also define $RCS\lim \mathbb{P}$ if δ is a successor ordinal, $\delta = \beta+1$. In that case the RCS limit is essentially equal to $\mathbb{P}_\beta * \underset{\sim}{\mathbb{Q}}_\beta$ — formally we define it to be the set of all functions p with $\text{dom}(p) \subseteq \beta+1$ and either $p \in \mathbb{P}_\beta$ or $p \restriction \beta \in P_\beta$ and $p \restriction \beta \Vdash_{\mathbb{P}_\beta} p(\beta) \in \underset{\sim}{Q}_\beta$.

This definition of RCS iteration is from [7]. An equivalent definition (in the language of Boolean algebras) can be found in [1]. The original definition is in [9, chapter XII].

Theorem 4.8. *RCS iteration preserves semiproperness.*

Proof. See the papers quoted above. □

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Large cardinal properties of small cardinals

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1 Introduction

The fact that small cardinals (for example $\aleph_1$ and $\aleph_2$) can consistently have properties similar to those of large cardinals (for example measurable or supercompact cardinals) is a recurring theme in set theory. In these notes I discuss three examples of this phenomenon; stationary reflection, saturated ideals and the tree property.

None of the results discussed here is due to me unless I say so explicitly.

I would like to express my thanks to Joan Bagaria and Adrian Mathias for organising a very enjoyable meeting.

2 Large cardinals and elementary embeddings

We begin by reviewing the formulation of large cardinal properties in terms of elementary embeddings. See [40], [22] or [21] for more on this topic.

We will write "$j : V \longrightarrow M$" as a shorthand for the rather cumbrous assertion "M is transitive, j and M are classes of V and j is a non-trivial elementary embedding from V to M".

If $j : V \longrightarrow M$ then it is easy to see that j has a *critical point* κ. That is to say $j \restriction \kappa = \mathrm{id}_\kappa$ and $j(\kappa) > \kappa$. It turns out that many large cardinal properties can profitably be formulated in terms of elementary embeddings and their critical points.

The concept of a *measurable cardinal* was first considered by Ulam [42] in connection with problems in measure theory. Scott [35] initiated the study of elementary embedding formulations for large cardinals by proving

Theorem 2.1 (Scott [35]) *The following are equivalent.*

1. *κ is measurable (that is, there exists a normal measure on κ).*
2. *There exists $j : V \longrightarrow M$ such that $\mathrm{crit}(j) = \kappa$.*
3. *There exists $j : V \longrightarrow M$ such that $\mathrm{crit}(j) = \kappa$ and ${}^\kappa M \subseteq M$.*