A SINGLE BINARY FUNCTION IS ENOUGH

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THEOREM:[1]
Let X be any infinite set, and let $g_1, g_2, \dots$ be finitely or countably many operations on X. Then there is a binary operation f on X such that the clone generated by f contains each g_i.

Proof: Any k-ary operation $g : X^n \to X$ can be obtained from a 1-1 binary function $p(x, y)$ and a suitable unary function $u(x)$; nesting p will give a 1-1 function $p^{(n)} : X^n \to X$; find u such that $g = u \circ p^{(n)}$. (Due to Sierpiński?)
So we assume that all g_i are binary.
We may assume $X = Z \times \mathbb{N}$, where X and Z have the same cardinality. Let $i : X \to Z \times \{0\}$ be a bijection. For $x = (z, n) \in X$ write $x + 1$ for $(z, n + 1)$; similarly for $x + k$.
We can now find a function $f : X \times X \to X$ satisfying the following for all x, y:

(1) $f(x, x) = x + 1$
(2) $f(x, x + 1) = i(x)$.
(3) $f(i(x) + k, i(y)) = g_k(x, y)$ for $k = 1, 2, \dots$.

So the functions $x \mapsto x + 1$, i and g_k are all in the clone generated by f.

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[1] This theorem is probably well-known. Perhaps even with this particular proof? Agnes Szendrei has pointed out that for finite sets, this appears in *Webb, Donald L. Generation of any N-valued logic by one binary operation. Proc. Natl. Acad. Sci. USA 21, 252-254 (1935). Zentralblatt 0012.00103.* Webb generalized the Sheffer stroke to k-element sets as follows: $p|p = p+1 \pmod k$, $p|q = 0$ for $p \neq q$.