VU Diskrete Mathematik

Exercises for Dec 1, 2023

41) Find the homeomorphy classes of the set of topological spaces $\{B, C, D, O, Z, 1, 2, 8\}$, if the characters are regarded as point sets in $\mathbb{R}^2$ with the usual metric.

42) Which of the following sets are connected?

- the unit disk in $\mathbb{R}^2$;
- the set $\{(x, y) \in \mathbb{R}^2 \mid \exists\, r \in \mathbb{Q} : x^2 + y^2 = r^2\}$ in $\mathbb{R}^2$;
- the set $f(X)$, if $f : X \to Y$ is continuous and X is connected (X, Y being arbitrary topological spaces);
- the union of the intervals $(0, 1)$ and $(1, 2)$ in $\mathbb{R}$.

43) Let $X = \{1, 2, 3, 4, 5\}$ and $\mathcal{T} = \{\emptyset, \{1\}, \{1, 2\}, \{1, 2, 5\}, \{1, 3, 4\}, \{1, 2, 3, 4\}, X\}$. Show that $\mathcal{T}$ is a topology and find all closed subsets of X.

44) Let X be a topological space and $f : X \to [0, 1]$ be a continuous function, where $[0, 1]$ is equipped with the usual topology. Prove that for every $a \in [0, 1]$ the set $f^{-1}(\{a\})$ is a closed set. Assume that X is connected. Characterize all continuous functions $f : X \to [0, 1]$ for which $f^{-1}(\{a\})$ is open as well.

45–46) If M is a set, then $\overline{M}$, ∂M, M° denote the closure, the boundary, the interior of M, respectively. The closure of M is defined as the smallest closed set C with $M \subseteq C$. So, any subset of $\overline{M}$ is either not closed or does not contain M as a subset. The interior is the largest open set which is subset of M. It is the union of all open subsets of M. The boundary is defined by $\partial M = \overline{M} \setminus M^\circ$.

45) Let X and $\mathcal{T}$ be as in exercise 43. Determine the following sets: $\overline{\{1\}}, \overline{\{2\}}, \overline{\{3, 5\}}$

46) Let X and $\mathcal{T}$ be as in exercise 43. Determine the following sets: $\{1, 4, 5\}^\circ$, $\partial\{1, 2, 4, 5\}$.

47) Let (X, d) be a metric space and $A \subseteq X$ an open set. Prove that A is the union of ε-balls, i.e., there is an index set I and $x_i \in A$ as well as $\varepsilon_i > 0$ for all $i \in I$ such that $A = \bigcup_{i \in I} K(x_i, \varepsilon_i)$.

48) Let (X, d) be a metric space and $A \subseteq X$. For $x \in X$ define $d(x, A) := \inf\{d(x, a) \mid a \in A\}$. Prove that $f : X \to \mathbb{R}$, $x \mapsto d(x, A)$, is continuous. Then use the result of Exercise 44 to show that the set of zeros of f is closed. Finally, use what you have shown so far to prove that $d(x, A) = 0$ if and only if $x \in \bar{A}$.