VU Diskrete Mathematik

Exercises for Nov 10 – Session 2, 2023

33) Let $G = (V, E)$ be a graph and $u, v \in V$ two adjacent vertices of G. Define $G' = (V, E \setminus \{uv\})$ and G'' to be the graph obtained from G by merging u and v to a new vertex w that is adjacent to all $x \in \Gamma(u) \cup \Gamma(v) \setminus \{u, v\}$. For any graph H, let $f_H(n)$ denote the number of feasible vertex colourings of H that use only colours from the n-element colour set $\{c_1, \dots, c_n\}$. Prove that $f_G(n) = f_{G'}(n) - f_{G''}(n)$. Use this equation to show that $f_G(n)$ is a polynomial in n.

34) A block of a graph is a maximal (induced) subgraph which contains no cut vertex[1] and therefore no bridge as well. Every graph can be decomposed into blocks in a unique way and two different blocks have at most one vertex and no edge in common. The common vertex is always a cut vertex.
Given a connected graph G which has the blocks H_1, H_2,$\dots$,H_k. Assume that $\chi(H_i)$, $i = 1, 2, \dots, k$, is known. Compute $\chi(G)$.

35) Consider K_n, the complete graph with n vertices. A proper edge-coloring is a coloring of the edges such that any two edges that share a vertex receive different colors. Show that a proper edge-coloring of K_n with exactly $n - 1$ colors decomposes K_n into $n - 1$ perfect matchings.

36) Prove that K_n has a proper edge-coloring (see Exercise 35) if and only if n is even.

Hint: Exercises 28 and 35.

37) Show the following inequality for Ramsey numbers: If $r \geq 3$ then

$$R(n_1, \dots, n_{r-2}, n_{r-1}, n_r) \leq R(n_1, \dots, n_{r-2}, R(n_{r-1}, n_r))$$

Hint: Let $n = R(n_1, \dots, n_{r-2}, R(n_{r-1}, n_r))$ and consider an edge colouring of K_n with r colours, say $c_1, \dots, c_r$. Identify the colours c_{r-1} and c_r and apply the Ramsey property for $r - 1$ colours.

38) We abbreviate $R(3, 3, \dots, 3)$, where 3 is used k times, by R_k. Prove that $R_k \leq k(R_{k-1} - 1) + 2$.

39) Find all 2-colourings of $\{1, 2, \dots, 7\}$, say with colours red and blue, that do neither contain a red triple (x, y, z) (x, y, z not necessarily distinct!) with $x + y = z$ nor a blue triple (x, y, z) such that $z - y = y - x > 0$.

40) Prove that it is impossible to colour $\{1, 2, \dots, 8\}$ with colours red and blue in such a way that there is neither a red triple (x, y, z) (x, y, z not necessarily distinct!) with $x + y = z$ nor a blue triple (x, y, z) such that $z - y = y - x > 0$.

[1] A cut vertex is a vertex such that its removal increases the number of connected components. A bridge is an edge having this property. Thus the end vertices of a bridge are always cut vertices.