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DIPLOMARBEIT

Diverse families of
Random rooted trees
A compilation of characteristics

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Abstract

This diploma thesis deals with four big groups of random trees, namely Polya trees, simple generated trees, increasing trees and scale-free trees. Different characteristics, similarities and differences of these varieties are discussed, e.g. the limiting distribution of node-degrees. Most results are obtained using generating functions and methods of singulary analysis and stochastics.
In the first chapter the necessary background of stochastics and graph theory is given, which will become necessary throughout the work, knowledge of probability theory and analysis is favorable for the comprehension of the work. In the second chapter we discuss results tracing back to George Pólya and the year 1937. Based upon that we show that the limiting degree-distribution of Pólya trees is a normal distribution.
The third chapter adresses simply generated trees, a group whose generating function fulfills $a(z) = \varphi(a(z))$, for a power series φ with nonnegative coefficients. This group is equivalent to the group of Galton-Watson trees, which correspond to a Galton-Watson branching process. We can obtain interesting results on the structure of those trees in context of Brownian excursions.
In the fourth chapter we equip the trees with an additional parameter, namely the labelling of their nodes, and eye on those trees whose labellings along any path away from the root is increasing. For certain families of those increasing trees we can also find limiting degree distributions.
In the fifth and last chapter we define graphs and trees, which are similar no networks occuring in the real world, but were discovered only recently, the Scale free graphs and trees. The marcant property of these trees is the development through growth, the limiting degree distribution is exponential and independent of the beginning structure of the graph.

Zusammenfassung

Diese Diplomarbeit befasst sich mit vier großen Gruppen von Zufallsbäumen, den Pólya trees, simply generated trees, increasing trees, und der relativ neuen Struktur der Scale-free trees. Verschiedenste Charakteristiken, Gemeinsamkeiten und Unterschiede dieser Gruppen werden besprochen, wie zum Beispiel die Grenzverteilung der Knotengrade. Die Ergebnisse werden meist ausgehend von der erzeugenden Funktion der fraglichen Struktur unter Zuhilfenahme von Methoden aus der Stochastik und der Singularitätsanalyse gefunden.

Im ersten Kapitel werden diverse Begriffe aus Stochastik und Graphentheorie bereitgestellt, die im Verlauf der Arbeit benötigt werden. Zum Verständnis der folgenden Kapitel sind grundlegende Kenntnisse aus Wahrscheinlichkeitstheorie und Analysis von Vorteil.

Im zweiten Kapitel werden Ergebnisse besprochen, die auf George Pólya aus dem Jahre 1937 zurückgehen. Basierend auf diesen Ergebnissen wird gezeigt, dass die Grenzverteilung der Knotengrade eines Pólya-trees einer Normalverteilung entspricht.

Das dritte Kapitel befasst sich mit simply generated trees, einer Gruppe, deren erzeugende Funktion die Bedingung $a(z) = \varphi(a(z))$ erfüllt, für eine Potenzreihe φ mit nichtnegativen Koeffizienten. Diese Gruppe ist gleichzusetzen mit der Gruppe der Galton-Watson-Bäume, jene Bäume die einem Galton-Watson-Verzweigungsprozeß zugehörig sind. Wir können hier interessante Erkenntnisse über die Struktur der Bäume in Zusammenhang mit Brownschen Exkursionen gewinnen.

Im vierten Kapitel statten wir Bäume mit einem zusätzlichen Merkmal, nämlich der Markierung ihrer Knoten, aus und betrachten jene Bäume, deren Markierungen entlang jedes Pfades von der Wurzel weg aufsteigend verläuft. Für gewisse Gruppen dieser increasing trees können wir ebenfalls die Grenzverteilung der Knotengrade bestimmen.

Im fünften und letzten Kapitel schließlich definieren wir Graphen und Bäume, die den in der reellen Welt vorkommenden Netzwerken ähneln, jedoch erst kürzlich entwickelt worden sind, die Scale free trees und -Graphs. Das

markante Merkmal dieser Gruppe ist es, das der Graph durch Wachstum ensteht. Die Grenz-
verteilung der Knotengrade verläuft exponentiell und ist unabhängig von der Anfangsstruktur des Graphen.

Preface

Motivation

In my third year at university, the lecture on discrete mathematics of Professor Baron arouse my interest in this field and awakened the idea that this might be the field of mathematics I would want to specialize in later. Then, when spending an exchange-year at the university of Alicante, Spain, I concentrated more on lectures on logic and applications in computer science, not for reasons of interest but for lack of lectures on higher mathematics as the course of applied mathematics was not offered there, and also gained interest in this field. Still, when coming back from Spain, I remembered my 'old plan' and participated in Professor Gittenberger's seminar on discrete mathematics, where I held a presentation on Cayley's enumeration of trees. It was there that I decided that trees were going to be the theme of my diploma thesis and therefore asked Professor Gittenberger to supervise my work.
I decided to write this thesis in English as I am always searching to increase my foreign language skills, and as English is the most widespread language when coming to scientific literature.

Acknowledgements

I want to thank my supervisor Professor Gittenberger for the patience and freedom he gave me when needing half a year to finally get started with my work, and for the help he gave me when coming to an end. I'm very much indebted to my parents for enabling me to study and even supporting me in my idea of going to Spain for a year, and for not getting impatient as I studied some semesters longer as others might have.
I specially thank my mother and her excellent English skills for proofreading, and all my family and friends who supported me.

Veronika Kraus

Contents

Chapter 1

Methods and definitions

This thesis deals with random rooted trees. There are several families of random trees, provided with different restrictions and properties. This work explores the structure of those trees. Results are obtained using methods of probability theory and stochastics, just as the analysis of the asymptotic behaviour of generating functions. In this chapter the necessary background for the following is given.

We start with the definition of the structure we will describe, the following terms will be well-known to most readers:

1.1 Graph theory

Definition 1.1.1 (undirected graph)**.** *We call an ordered pair* $G = (V, E)$ *with*

- V *being a set, whose elements are called vertices or nodes,*
- E *being a set of unordered pairs of distinct vertices, called edges or lines.*

an undirected graph G*.*

Definition 1.1.2 (tree)**.** *We call the graph* $G = (V, E)$ *a tree* B *if it is connected (i.e. there exists a path between any pair of edges* $v, w \in V$*) and it is free of cycles (i.e. there exist no path without repeating edges starting and ending at the same node* $v \in V$*).*

This definition is equivalent to:

- *Any pair of nodes* $v, w \in V$ *is connected by a unique simple path.*

- *G has no cycles, and a simple cycle is formed if any edge is added to G.*
- *G is connected, and it is not connected anymore if any edge is removed from G.*
- *if $|V| < \infty$ and G is connected, then $|E| = |V| - 1$.*
- *if $|V| < \infty$ and G has no cycles, then $|E| = |V| - 1$.*

REMARK: An unconnected graph without cycles is called a forest. Each of it's components is a tree.

In this work, we will not work on a concrete tree, but on families of trees with a common characteristic:

Definition 1.1.3 (random tree)**.** *Let $\mathcal{T}$ be the set of all trees with a certain characteristic(e.g. all trees with n vertices). We choose any tree $B \in \mathcal{T}$ at random (every tree in $\mathcal{T}$ is chosen by a certain probability given by the definition of the tree family), and call B a random tree of the family $\mathcal{T}$.*

We will describe families of trees by ordinary or exponential generating functions:

$$\begin{aligned} T(z) &= \sum_{n\geq 0} T_n z^n \\ T(z) &= \sum_{n\geq 0} T_n \frac{z^n}{n!} \end{aligned}$$

where the coefficient T_n denotes the number of trees B_n of size n in the family $\mathcal{T}$. We need ordinary generating functions in the case of plane trees and exponential functions in the case of non-plane trees.
To examine the behaviour of a certain parameter of the family of trees $\mathcal{T}$ described by its generating function $T(z)$, we will construct bivariate or multivariate generating functions, containing information about these parameters in the variables $u_j, j = 1, \ldots, i$:

$$T(z, u_1 \ldots, u_i) = \sum_{n,m_1,\ldots,m_i \geq 0} T_{n,m_1,\ldots,m_i} z^n u_1^{m_1} \cdots u_i^{m_i},$$

e.g., in the bivariate generating function $T(z, u)$ the coefficient $T_{n,m}$ could denote the number of trees of size n with m leaves.

1.2 singularity analysis

Given a power series $T(z)$ with its expansion around a dominant singularity, Flajolet and Odlyzko [13] described a tool to examine the asymptotic order of growth of it's coefficients, in [15] this method is expanded. We will use this singularity analysis on the generating functions of families of trees, and will thereby obtain asymptotic results. The method of Flajolet and Odlyzko applies to functions f with a unique dominant singularity at $z = 1$ (through normalization, this assumption can be obtained for any function f with a unique dominant singularity) which, for some arbitrary $\alpha \in \mathbb{R}$ satisfy

$$f(z) \approx (1-z)^{\alpha} \qquad z \to 1,$$

The results are obtained using Cauchys integral formula

$$f_n = [z^n]f(z) = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{f(z)}{z^{n+1}} dz$$

and Hankel-like contours $\mathcal{C}$.

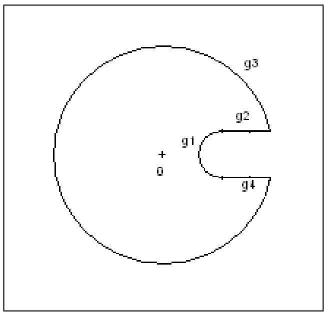

Figure 1.1: The Hankel-like contour to proof Theorem 1.2.1

Integrating along $\mathcal{C} = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ with

$$\begin{aligned}
\gamma_1 &= \{z = 1 - \frac{t}{n} | t = e^{i\phi}, \phi \in [-\frac{\pi}{2}, \frac{\pi}{2}]\} \\
\gamma_2 &= \{z = 1 + \frac{t+1}{n} | t \in [0, n]\} \\
\gamma_3 &= \{z | |z| = \sqrt{(4 + \frac{1}{n^2})}, \Re z \leq 2\} \\
\gamma_4 &= \{z = 1 + \frac{t-1}{n} | t \in [0, n]\}
\end{aligned}$$

as shown in Figure 1.1, leads to the following results on the asymptotic values of a power series $\sum f_n z^n$:

Theorem 1.2.1. *Let α and β be complex numbers $\alpha, \beta \in \mathbb{C} \backslash \mathbb{Z}_{\leq 0}$. ´The Taylor coefficients $f_n = [z^n]f(z)$ in*

$$f(z) = (1-z)^{-\alpha}\big(\frac{1}{z}\log(\frac{1}{1-z})\big)^{-\beta}$$

satisfy

$$f_n \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)}(\log n)^{-\beta}(1 + \sum_{k\geq 1}\frac{e_k^{(\alpha,\beta)}}{\log^k n}),$$

with

$$e_k^{(\alpha,\beta)} = (-1)^k\binom{-\beta}{k}\Gamma(\alpha)\frac{\partial^k}{\partial s^k}(\frac{1}{\Gamma(s)})\Big|_{s=\alpha}$$

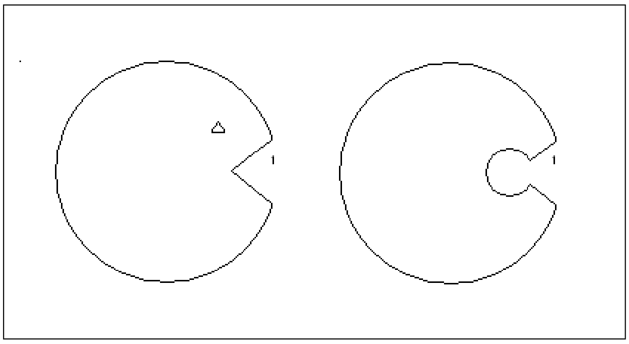

Figure 1.2: The domain and Hankel-like contour to proof Theorem 1.2.2

For functions f that fulfill $f(z) = \mathcal{O}(f(z) = (1-z)^{-\alpha}\big(\log(\frac{1}{1-z})\big)^{\beta})$ or $f(z) = o(f(z) = (1-z)^{-\alpha}\big(\log(\frac{1}{1-z})\big)^{\beta})$ a similar statement can be made. Therefore we define the domain $\Delta = \Delta(\Phi, R)$ by

$$\Delta(\Phi, R) = \{z \big| |z| < R, z \neq 1, |\mathrm{arg(z-1)}| > \Phi\}$$

and use the contour $\mathcal{C} = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ (cp Figure 1.2)

$$\begin{aligned}
\gamma_1 &= \{z||z-1| = \frac{1}{n}, |\mathrm{arg(z-1)}| \geq \Phi\} \\
\gamma_2 &= \{z|\frac{1}{n} \leq |z-1|, |z| \leq R, \mathrm{arg(z-1)} = \Phi\} \\
\gamma_3 &= \{z||z-1| = R, |\mathrm{arg(z-1)}| \geq \Phi\} \\
\gamma_4 &= \{z|\frac{1}{n} \leq |z-1|, |z| \leq R, \mathrm{arg(z-1)} = -\Phi\}
\end{aligned}$$

Then, the following theorem holds for f:

Theorem 1.2.2. *Let $\alpha, \beta \in \mathbb{R}$ be arbitrary real numbers and let $f(z)$ be a function that is analytic in Δ with the exception of the singulatity at $z = 1$.*

(i) Assume further that as z tends to 1 in Δ,

$$f(z) = \mathcal{O}((1 - z)^{-\alpha}(\log \frac{1}{1 - z})^{\beta})$$

Then the Taylor coefficients of $f(z)$ satisfy

$$f_n = [z^n]f(z) = \mathcal{O}(n^{\alpha-1}(\log n)^{\beta})$$

(ii) Assume that as z tends to 1 in Δ,

$$f(z) = o((1 - z)^{-\alpha}(\log \frac{1}{1 - z})^{\beta})$$

Then the Taylor coefficients of $f(z)$ satisfy

$$f_n = [z^n]f(z) = o(n^{\alpha-1}(\log n)^{\beta})$$

We will use these results and their conlusions throughout the work to determine the limiting behaviour of diverse generating functions, e.g. in Chapter/Section, and also use similar methods of proof, e.g. in Chapter

1.3 Probability theory and stochastics

Another field of mathematics we will use to obtain our results is the field of stochastic processes. The following can for instance be found in [2] and [19].

Definition 1.3.1 (Stochastic process)**.** *Let T be a subset of $\mathbb{R}$. A family of random variables $\{X(t)|t \in T\}$ with values in the state space Z is called a stochastic process. T can be a discrete time set or an interval, We thus speak of a discrete or continuous stochastic process.*

REMARK Observing the process $\{X(t)|t \in T\}$ through the whole time T and recording the values $X(t)$ for all $t \in T$, we obtain a real function $x = x(t), t \in T$, which we call the *trajectory* or *sample path* of the stochastic process.

A stochastic process can satisfy the following properties:

Definition 1.3.2 (independent increments). *A stochastic process* $\{X(t)|t \in T\}$ *has independent increments, if for any sequence* $t_1 < t_2 < \ldots < t_n, t_i \in T$ *the increments* $X(t_2) - X(t_1), X(t_3) - X(t_2), \ldots, X(t_n) - X(t_{n-1})$ *are independent, i.e., the increment the process takes in an interval does not influence it's increments in disjoint intervals.*

Definition 1.3.3 (stationary increments). *A stochastic process* $\{X(t)|t \in T\}$ *has stationary increments, if the increments* $X(t_2 + \tau) - X(t_1 + \tau)$ *have the same probability distribution for any* τ *with* $t_1 + \tau \in T$ *and* $t_2 + \tau \in T$*, for arbitrary but fixed* t_1, t_2.

Definition 1.3.4 (Markov chain). *A discrete stochastic process* $\{X_0, X_1, \ldots\}$ *with state space* Z *is called a Markov chain, if for any* $t = 1, 2, \ldots$ *and for any sequence* $x_0, x_1, \ldots, x_{t+1}, x_k \in Z$ *the following is true*

$$\mathbf{P}(X_{t+1} = x_{t+1}|X_t = x_t, \ldots, X_1 = x_1, X_0 = x_0) = \mathbf{P}(X_{t+1} = x_{t+1}|X_t = x_t)$$

i.e., given the present state, future states are independent of the past states, or, in other words, the present state captures all information that can influence the future of the process.

An example for a discrete Markov chain process are so called *branching processes*, which we will use in Chapter 3. In a branching process $T = \mathbb{N}_0$, the process models a population in which each individual in generation n produces some random number of individuals in generation $n+1$, according to a fixed probability distribution ξ that does not vary from individual to individual. We can create a tree according to a branching process by describing each individual by a node, the first individual $n = 0$ being the root and the offspring of every node being the adjacent nodes on the next level.

REMARK There exist also *continuous-time Markov processes* with the same definition as a Markov chain, but with a continuous index.

Definition 1.3.5 (Martingal). *A stochastic process* $\{X(t)|t \in T\}$ *with state space* Z *is called a martingale, if* $\mathbf{E}(X(t)) < \infty$ *for every* $t \in T$ *and for any time sequence* $t_1 < t_2 < \ldots < t_n < s < t$ *the following is true*

$$\mathbf{E}(X_t|X_s = x_s, \ldots, X_{t_1} = x_{t_1}, X_{t_0} = x_{t_0}) = x_s$$

REMARK We can define super- and submartingales with

$$\begin{aligned} \mathbf{E}(X_t|X_s = x_s, \ldots, X_{t_1} = x_{t_1}, X_{t_0} = x_{t_0}) &\leq x_s \text{ and} \\ \mathbf{E}(X_t|X_s = x_s, \ldots, X_{t_1} = x_{t_1}, X_{t_0} = x_{t_0}) &\geq x_s \text{ ,respectively.} \end{aligned}$$

In this work, we will use only discrete martingales, and even more precise, only martingales on $T = \mathbb{N}$. The information given by the past events can be processed in a filtration, that is:

Definition 1.3.6 (Filtration). *A family $(\mathcal{F}_t | t \in T)$ of sigma-algebras is called a Filtration, if $\mathcal{F}_s \subseteq \mathcal{F}_t$ for all $s < t$. For a stochastic process $(X(t), t \in T)$ let $\mathcal{F}_n$ be the sigma algebra induced by the random variables x_s with $s \leq n$, $(\mathcal{F}_n, n \in T)$ is then called the natural filtration of $X(s)$.*

With this notation, a martingal is given by the constraint

$$\mathbf{E}(X(t)|\mathcal{F}_n) = x_n$$

$\mathcal{F}_n$ can be any filtration, but throughout this work, $\mathcal{F}_n$ will denote the natural filtration of the given stochastic process.

An example for a continuous stochastic process is Brownian motion and Brownian excursion, which we will use in chapter 3.

Definition 1.3.7 (Brownian Motion). *A continuous stochastic process with state space $Z = \mathbb{R}$ and time $T = \mathbb{R}_0^+$ is called a Brownian motion process (especially in German literature often called Wiener process) if it fulfills*

(i) $W(0) = 0$

(ii) $\{X(t)|t \in T\}$ *has stationary and independent increments.*

(iii) $W(t) \sim \mathcal{N}(0, t)$ *for all $t \in T$, i.e. for any $t \in T$ the random variable $X(t)$ is normally distributed with mean value 0 and variance t.*

REMARK

- As the process has stationary increments, the difference $W_t - W_s$ is also normally distributed, i.e. $W_t - W_s \sim \mathcal{N}(0, t-s)$.
- As the process has independent increments, it is a Markov process.

Definition 1.3.8 (Brownian excursion). *Let $B(t), t \in \mathbb{R}_0^+$ be a Brownian motion process, and let its leftmost positive zero be at time t^*, w.l.o.g. $B(t) \geq 0$ for $t \leq t^*$. We define the associated Brownian excursion as the stochastic process $B_{ex}(t), t \in [0, 1]$ with*

(i) $B(0) = B_{ex}(0) = B_{ex}(1) = B(t^*) = 0$

(ii) $B_{ex}(t) = B(\frac{t}{t^*})$,

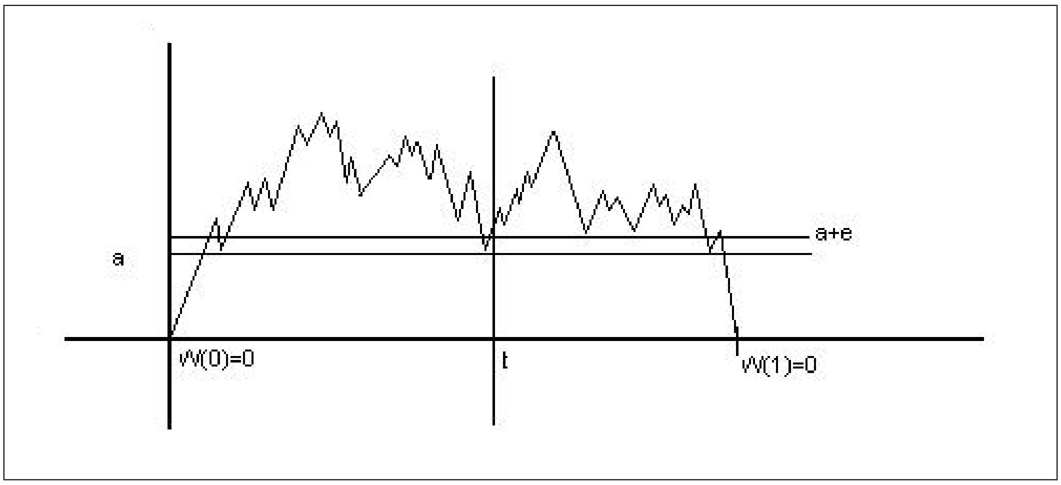

Figure 1.3: Brownian excursion local time

i.e. we rescale the part up to the first positive zero of $B(t)$ *on the interval* $[0, 1]$.

We want to know "How much time does the excursion spend on level a? Of course, the answer to this question would be 0, so we adapt the question and are interested in the time the excursion spends in the interval $[a, a + e]$, which is

$$L(a, a + e) = \int_0^1 \chi_{[a,a+e]}(B_{ex}(s))ds$$

Definition 1.3.9 (Brownian excursion local time)**.** *(cr. Figure 1.3)*
Let $\{B_{ex}(t)|t \in [0, 1]\}$ *be a brownian excursion, and* $L(a, a + e)$ *given by the above.*
Then, we call the function

$$l(a) := \frac{\partial}{\partial e} L(a, a + e)$$

the total local time at level a *of the brownian excursion* $B_{ex}(t)$*.*

REMARK Equivalently, we define the local time at level a at time t of $B_{ex}(t)$ using

$$L^{(t)}(a, a + e) = \int_0^t \chi_{[a,a+e]}(B_{ex}(s))ds$$

Then, the function

$$l(a,t) := \frac{\partial}{\partial e} L^{(t)}(a, a+e)$$

is called the local time at level a at time t of $B_{ex}(t)$.

Chapter 2

Pólya trees

I will start this work presenting results on Pólya trees. Those trees were first given attention by George Pólya in 1937 in his classical work [35]. Pólya trees are random trees with no restrictions on node degree, every tree B_n of size n is equally likely.

2.1 Introduction

In the following, we will denote by t_n the number of unrooted unlabeled nonplane trees and by T_n the number of rooted unlabeled nonplane trees of size n. Furthermore we define a planted tree to be a tree rooted at an endpoint and denote by P_n the number of planted trees, not counting the root. Obviously, $P_n = T_n$, and the degree of the root is increased by 1.

Further we introduce the generating functions

$$t(z) = \sum_{n\geq 1} t_n z^n \tag{2.1}$$

$$T(z) = \sum_{n\geq 1} T_n z^n \tag{2.2}$$

$$P(z) = \sum_{n\geq 1} P_n z^n \tag{2.3}$$

Rooted trees can be interpreted as a recursive structure, that is, $\mathcal{T}$ is a root followed by a set of rooted trees.Thus a tree of arbitrary size n^* can be constructed by choosing a set of trees B_{n_i} of sizes $n_i < n^*$, and connecting them by a new root. This arbitrary choice of trees of sizes n_i thus provide contributions $(1 + z + z^2 + \cdots)^{T_{n_i}}$ to the generating function, the new root provides a factor z, and thus

$$
\begin{aligned}
T(z) &= T_1 z + T_2 z^2 + T_3 z^3 + \cdots + T_n z^n + \cdots \\
&= z(1+z+z^2+\cdots)^{T_1}(1+z+z^2+\cdots)^{T_2}\cdots(1+z+z^2+\cdots)^{T_n}\cdots \\
&= z\frac{1}{(1-z)^{T_1}}\frac{1}{(1-z^2)^{T_2}}\frac{1}{(1-z^3)^{T_3}}\cdots\frac{1}{(1-z^n)^{T_n}}\cdots.
\end{aligned}
$$

Pólya showed in [35] that, interpreting the equation above as a functional equation for $T(z)$,

$$T(z) = ze^{\frac{T(z)}{1}+\frac{T(z^2)}{2}+\frac{T(z^3)}{3}+\cdots} \tag{2.4}$$

from where t_n and T_n can be derived as $n \to \infty$, which Pólya did in his work for trees where only node degrees 1 and 4 are allowed.

Further, Pólya showed that the radius of convergence ρ satisfies $0 < \rho < 1$, and that $z = \rho$ is the only singularity on the circle of convergence $|z| = \rho$, and stated the lemma

Lemma 2.1.1. *Let the power series*

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

have the finite radius of convergence $\alpha > 0$, with $x = \alpha$ the only singularity on its circle of convergence. Suppose also that $f(x)$ can be expanded near $x = \alpha$ in the form

$$f(x) = \frac{1}{(1-\frac{x}{\alpha})^s}g(x) + \frac{1}{(1-\frac{x}{\alpha})^t}h(x)$$

where $g(x)$ and $h(x)$ are analytic at $x = \alpha$, $g(\alpha) \neq 0$, s and t are real numbers, $s \neq 0, -1, -2, \ldots$, and either $t < s$ or $t = 0$. Then

$$a_n \sim \frac{g(\alpha)}{\Gamma(s)}\frac{n^{s-1}}{\alpha^n} \tag{2.5}$$

This lemma, in fact, is a special case of the results obtained in 1990 by Flajolet and Odlyzko [13]. Later, in 1948, Richard Otter expanded Pólyas work ([32]) and found that $T(\rho) = 1$, and $T(z)$ has the expansion

$$T(z) = 1 - b\sqrt{(\rho - z)} + c(\rho - z) + d\sqrt{(\rho - z)^3} + \cdots \tag{2.6}$$

By using the derivative of (2.4) he found the following recursion for the number of rooted trees

$$T_n = \frac{1}{n-1} \sum_{j=1}^{n-1} T_{n-j} \sum_{m|k} m T_m \tag{2.7}$$

for $n > 1$ and determined the exact values via the above expansion and Polyas lemma 2.1.1:

$$T_n \sim \frac{b\sqrt{\rho}}{2\sqrt{\pi}} \frac{1}{\sqrt{n^3}\rho^n} \tag{2.8}$$

.

Further, he constructed the relation

$$t(z) = T(z) - \frac{1}{2}T(z)^2 + \frac{1}{2}T(z^2), \tag{2.9}$$

by using a finite bound m for the maximum degree of nodes on t_n and obtaining:

$$t(z) = T^{(m)}(z) - \frac{1}{2}zT^{(m-1)}(z)^2 + \frac{1}{2}zT^{(m-1)}(z^2) \tag{2.10}$$

This equation is also valid for $m = \infty$, from where Otter derived the above result and set up a similar expansion as above for unrooted trees, from which he then derived the coefficients t_n. These are

$$t_n \sim \frac{b^3\sqrt{\rho^3}}{4\sqrt{\pi}} \frac{1}{\sqrt{n^5}} \frac{1}{\rho^n}$$

In 2004, equation (2.9), was reproved by Drmota [8], using a bijection:

Proof. Let $\mathcal{T}$ denote the set of rooted trees, $\sqcup$ the set of unrooted trees and further let $\mathcal{T}^{(p)}$ be the set of unordered pairs (B_1, B_2) of rooted trees of $\mathcal{T}$ with $B_1 \neq B_2$. We consider a pair (B_1, B_2) as a tree that is rooted by an *edge* connecting the roots of B_1 and B_2. Polyas theory indicates that the generating function of $\mathcal{T}^{(p)}$ is given by

$$T^{(p)}(z) = \frac{1}{2}T(z)^2 - \frac{1}{2}T(z^2)$$

By partitioning the three sets named above, we can show that there is a bijection between $\mathcal{T}$ and $\sqcup \cup \mathcal{T}^{(p)}$. If that bijection exists, then the result follows from

$$T(z) = t(z) + \frac{1}{2}T(z)^2 - \frac{1}{2}T(z^2)$$

□

2.2 The degree distribution of Polya trees

In this section, I will present results obtained by Robinson and Schwenk [36] in 1975 and by Drmota and Gittenberger [11] in 1999. It was shown that the mean value of the number of nodes of given degree k is almost proportional to the size of the tree, i.e. as $n \to \infty$ $\mathbf{E}X^(k)_n \sim \mu_k n$ for fixed k and for some $\mu_k > 0$ and that $D_{k,n}$ is asymptotically normally distributed.

To the generating functions introduced above we add the number of nodes of degree k as a second parameter. Thus, we have

$$
\begin{aligned}
t^{(k)}(z,u) &= \sum_{n,m\geq 1} t^{(k)}_{n,m} z^n u^m \\
T^{(k)}(z,u) &= \sum_{n,m\geq 1} T^{(k)}_{n,m} z^n u^m \\
P^{(k)}(z,u) &= \sum_{n,m\geq 1} P^{(k)}_{n,m} z^n u^m,
\end{aligned}
$$

where the coefficient $t^{(k)}_{n,m}/T^{(k)}_{n,m}/P^{(k)}_{n,m}$ is the number of unrooted/rooted/planted trees with n nodes (in the case of $P^{(k)}(z,u)$, n nodes others than the root), of which m have degree k.

If we set $u = 1$ in these series we ignore the special status of nodes of degree k and obtain the original series, i.e.

$$
\begin{aligned}
t^{(k)}(z,1) &= t(z) \\
T^{(k)}(z,1) &= T(z) \\
P^{(k)}(z,1) &= P(z) = T(z)
\end{aligned}
$$

Let $Z(S_k; x_1, \ldots, x_k)$ denote the cycle index of the symmetric group S_k of k elements, which has the form

$$
Z(S_k; x_1, \ldots, x_k) = \frac{1}{k!} \sum \prod_{i=1}^{k} x_i^{s_i},
$$

where the sum is over all permutations $s \in S$, and s_i is the number of cycles of length i in s, thus $\sum_{i=1}^{k} i s_i = k$ for every term.

Lemma 2.2.1. *The generating functions fulfill the following functional equations:*

$$
\begin{aligned}
P^{(k)}(z,u) &= ze^{\left(\sum_{i\geq 1}\frac{P^{(k)}(z^i,u^i)}{i}\right)} + z(u-1) \\
&\quad \times Z(S_{k-1}; P^{(k)}(z,u), P^{(k)}(z^2,u^2), \ldots, P^{(k)}(z^{k-1},u^{k-1})) \\
T^{(k)}(z,u) &= ze^{\left(\sum_{i\geq 1}\frac{P^{(k)}(z^i,u^i)}{i}\right)} + z(u-1) \\
&\quad \times Z(S_k; P^{(k)}(z,u), P^{(k)}(z^2,u^2), \ldots, P^{(k)}(z^k,u^k)) \\
t^{(k)}(z,u) &= T^{(k)}(z,u) - \frac{1}{2}P^{(k)}(z,u)^2 + \frac{1}{2}P^{(k)}(z^2,u^2)
\end{aligned}
$$

Proof. The proof of the first 2 equations is based on equation (2.4), with some modifications, one of them is of course adding the variables necessary to treat the number of points of degree k. The second change, the addition of the term $(zu - z)Z(S_{k-1}; P^{(k)}(z,u), P^{(k)}(z^2,u^2), \ldots, P^{(k)}(z^{k-1},u^{k-1})$ in planted trees and the term $(zu - z)Z(S_k; P^{(k)}(z,u), P^{(k)}(z^2,u^2), \ldots, P^{(k)}(z^k,u^k)$ in rooted trees, respectively, arises from the case where the node adjacent to the root resp. the root have degree k. The additional term needed for this modification is the named cycle index, because with Polyas equation (2.4) we see that

$$
Z(S_k; P^{(k)}(z,u), \ldots, P^{(k)}(z^k,u^k)) = [v^k]e^{\left(\sum_{i\geq 0} v^i \frac{P^{(k)}(z^i,u^i)}{i}\right)} \tag{2.11}
$$

i.e. it is the generating function of a forest consisting of exactly k planted trees.

Equation 3 is based on Otters result 2.9. Expressing this result in 2 variables, we have to involve $P(z,u)$ instead of $T(z,u)$, to be able to use the bijection we showed in the proof for (2.9). In order not to increase the degree of the root we have to use planted trees instead of rooted trees for the set $\mathcal{T}^{(p)}$, to avoid the additional root-edge to influence the degree. □

We now introduce two more generating functions

$$
\begin{aligned}
D^{(k)}(z) &= \sum_{n\geq 1} D_n^{(k)} z^n \\
d^{(k)}(z) &= \sum_{n\geq 1} d_n^{(k)} z^n
\end{aligned}
$$

where $D_n^{(k)}$ and $d_n^{(k)}$,respectively are the number of points of degree k occurring in all planted or unrooted trees, with n nodes.

From the definition of $D_n^{(k)}$ and $d_n^{(k)}$ it is obvious that $D_1^{(k)} = 0, \ldots, D_{k-1}^{(k)} = 0$ and $D_k^{(k)} = 1$, as there is only one planted tree with $k+1$ nodes which contains a node with degree k. Similarily, $d_0^{(k)} = 0, \ldots, d_k^{(k)} = 0, d_{k+1}^{(k)} = 1$, as a tree of n nodes has only $n-1$ edges and thus a maximum degree of $n-1$.

Further, the definition of the coefficients implies that:

$$\begin{aligned} D^{(k)}(z) &= P_u(z,1) \text{ and} \\ d^{(k)}(z) &= t_u(z,1), \end{aligned}$$

and $D^{(k)}(z)$ fulfills:

Lemma 2.2.2.

$$D^{(k)}(z) = T(z)\sum_{i\geq 1} D(z^i) + zZ(S_{k-1}; T(z), T(z^2), \ldots, T(z^{k-1})) \tag{2.12}$$

Proof. Differentiating the first equation of lemma 2.2.1 with respect to u leads to:

$$\begin{aligned} P_u(z,u) &= ze^{\left(\sum_{i\geq 1} \frac{P^{(k)}(z^i,u^i)}{i}\right)} \left[\sum_{i\geq 1} \frac{P^{(k)}(z^i,u^i)}{i}\right]_u \\ &\quad + zZ(S_{k-1}; T(z,u), T(z^2,u^2), \ldots, T(z^{k-1},u^{k-1})) \\ &\quad + (zu - z)[Z(S_{k-1}; T(z,u), T(z^2,u^2), \ldots, T(z^{k-1},u^{k-1})]_u \end{aligned}$$

Now we set $u = 1$ and apply the identities $D(z) = P_u(z,1)$ and $T(z,1) = T(z)$ and thus obtain the required result. □

REMARK We can find a similar equation for $d^{(k)}(z)$, using the third equation of lemma 2.2.1, and conducting similar computations:

$$\begin{aligned} d^{(k)}(z) &= D^{(k)}(z) - D^{(k)}(z)T(z) + D^{(k)}(z^2) + \\ &\quad zZ(S_k; T(z), T(z^2), \ldots, T(z^k)) - zZ(S_{k-1}; T(z), T(z^2), \ldots, T(z^{k-1})) \end{aligned}$$

and, using the description of $D^{(k)}(z)$ of lemma 2.2.2, this results in

$$d^{(k)}(z) = T(z)\sum_{i\geq 2} D(z^i) + zZ(S_k; T(z)+D(z^2), T(z^2)+D(z^4), \ldots, T(z^k)+D(z^{2k})) \tag{2.13}$$

The mean value

Also from lemma 2.2.2, we can derive

$$D_n^{(k)} = \sum_{k=1}^{n-1} T_{n-l} \sum_{m|l} D_m + [z^{n-1}]Z(S_{k-1}; T(z), T(z^2), \ldots, T(z^{k-1})) \tag{2.14}$$

for $n > k$, for $n = k$ the coefficient is 1 and for $n < k$ it is 0, as discussed above.

Similarly,

$$\begin{aligned} d_n^{(k)} &= D_n^{(k)} + D_{\frac{n}{2}}^{(k)} - \sum_{l=1}^{n-1} T_l D_{n-l}^{(k)} + \\ [z^{n-1}] & (z(S_k; T(z), T(z^2), \ldots, T(z^k)) - Z(S_{k-1}; T(z), T(z^2), \ldots, T(z^{k-1}))). \end{aligned}$$

Using the description of $D^{(k)}(z)$ from lemma 2.2.2 it is obvious that $D^{(k)}(z)$ has the same radius of convergence as $T(z)$, which is ρ, as except for $T(z)$ only higher powers occur. Thus $T(z)$ also has the only singularity at $z = \rho$ on the circle of convergence. The same argumentation holds for $d^{(k)}(z)$ with the description of 2.13.

We now alter the equation of lemma 2.2.2 to

$$D^{(k)}(z) - D^{(k)}(z)T(z) = T(z)\sum_{i\geq 2} D(z^i) + zZ(S_{k-1}; T(z), T(z^2), \ldots, T(z^{k-1}))$$

and thus can display $D^{(k)}(z)$ as

$$D^{(k)}(z) = \frac{T(z)\sum_{i\geq 2} D(z^i) + zZ(S_{k-1}; T(z), T(z^2), \ldots, T(z^{k-1}))}{1 - T(z)}$$

As $T(\rho) = 1$, at $z = \rho$ the numerator is

$$\sum_{i\geq 2} D^{(k)}(\rho^i) + \rho Z(S_{k-1}; T(\rho), T(\rho^2), \ldots, T(\rho^{k-1})),$$

while the denominator has the expansion

$$\begin{aligned} \frac{1}{1-T(z)} &= \frac{1}{1-(1-b\sqrt{(\rho-z)}+c(\rho-z)+\ldots)} \\ &= \frac{1}{b(\rho-z)^{\frac{1}{2}}}+\ldots \end{aligned} \qquad (2.15)$$

near $z=\rho$ by (2.6), the remaining terms being of higher order in $(\rho-z)$. Thus the preliminaries for lemma 2.1.1 are fulfilled and we get

$$\begin{aligned} D_n^{(k)} &\sim \frac{\sum_{i\geq 2} D^{(k)}(\rho^i) + \rho Z(S_{k-1}; T(\rho), T(\rho^2), \ldots, T(\rho^{k-1}))}{b\sqrt{\rho}\Gamma(\frac{1}{2})} \frac{1}{\sqrt{n}\rho^n} \\ &\sim \frac{\sum_{i\geq 2} D^{(k)}(\rho^i) + \rho Z(S_{k-1}; T(\rho), T(\rho^2), \ldots, T(\rho^{k-1}))}{b\sqrt{\rho\pi}} \frac{1}{\sqrt{n}\rho^n} \end{aligned}$$

(Note that $\Gamma(\frac{1}{2})=\sqrt{\pi}$).
With (2.8) we obtain for the ratio $X_n^{(k)} := \frac{D_n^{(k)}}{T_n}$, which stands for the mean value of nodes of degree k in trees of size n

$$X_n^{(k)} \sim n\frac{2}{b^2\rho}\Big(\sum_{i\geq 2} D^{(k)}(\rho^i) + \rho Z(S_{k-1}; T(\rho), T(\rho^2), \ldots, T(\rho^{k-1}))\Big) =: \mu_k n \qquad (2.16)$$

REMARKS

1. For the ratio $X_n'^{(k)} := \frac{d_n^{(k)}}{t_n}$ the analogous limit can be obtained with the following considerations:

   (2.6) raised to the power m results in

   $$T^m(z) = 1 - mb\sqrt{\rho-x} + (\binom{m}{2}b^2 + mc)(\rho-x) + \ldots$$

   and thus

   $$[z^n](T^m) \sim mT_n$$

With the help of Otter's result, $T(\rho) = 1$, we can write

$$[z^n](T^m) \sim T_n \frac{\partial}{\partial T}(T^m(z))\bigg|_{z=\rho}$$

and, since the factors of $T(z^i)$ are analytic at $z = \rho$ for $i > 1$, we can determine the asymptotic behaviour of the cycle index $Z(S_k; T(z), T(z^2), \ldots, T(z^k))$ by

$$\begin{aligned} Z(S_k; T(z), T(z^2), \ldots, T(z^k)) &\sim T(z)\frac{\partial}{\partial x_1} Z(S_k; T(\rho), T(\rho^2), \ldots, T(\rho^k)) \\ &\sim \sum \prod s_1 T(z) T(\rho)^{s_1 - 1} T(\rho^2)^{s_2} \cdots T(\rho^k)^{s_k}, \end{aligned}$$

where x_1 is the variable of $Z(S_k)$ which is replaced by $T(z)$. The partial derivative above is equal to $Z(S_{k-1})$, and thus, near $z = \rho$,

$$Z(S_k; T(z), T(z^2), \ldots, T(z^k)) \sim T(z) Z(S_{k-1}; T(z), T(z^2), \ldots, T(z^{k-1}))$$

Applying this result to (2.13), we obtain the same asymptotic value for $X_n'^{(k)}$ as for $X_n^{(k)}$.

2. In his paper [37] Schwenk examined the behaviour of $Z(S_k; T(z), T(z^2), \ldots, T(z^k))$ evaluated at $z = \rho$. He found that $Z(S_k; T(\rho), T(\rho^2), \ldots, T(\rho^k)) = C\rho^k$, where C is given by

$$C = e^{\left(\sum_{i \geq 1} \frac{1}{i}\left(\frac{T(\rho^i)}{\rho^i} - 1\right)\right)} \tag{2.17}$$

and that $\sum_{i \geq 2} D^{(k)}(\rho^i)$ decreases more rapidly than ρ^k. Therefore, by (2.16)

$$\mu_k \sim \frac{2C}{b^2 \rho} \rho^k \tag{2.18}$$

He further evaluated C:

$$C \approx 7.7581604 \cdots$$

.

The limiting distribution

Knowing the mean value $\mathbf{E}X_n^{(k)} = \mu_k n$, we will now determine the limiting distribution of $X_n^{(k)}$.

Therefore, we will first provide a set of propositions, which will give the required analytic background to determine the limiting distributions.

Theorem 2.2.3. *Suppose $F(z, u, y)$ is an analytic function around (z_0, u_0, y_0) such that*

$$\begin{aligned} F(z_0, u_0, y_0) &= y_0 \\ F_y(z_0, u_0, y_0) &= 1 \\ F_{yy}(z_0, u_0, y_0) &\neq 0 \\ F_z(z_0, u_0, y_0) &\neq 0 \end{aligned}$$

Then there exists a neighbourhood U of (z_0, u_0), a neighbourhood V of y_0, and analytic functions $g(z, u)$,$h(z, u)$ and $f(u)$, which are defined on U such that the only solutions $y \in V$ with $y = F(z, u, y)((z, u) \in U)$ are given by

$$y = g(z, u) \pm h(z, u)\sqrt{1 - \frac{z}{f(u)}}$$

Furthermore, $g(z_0, u_0) = y_0$ and $h(z_0, u_0) = \sqrt{\frac{2f(u_0)F_x(z_0,u_0,y_0)}{F_{yy}(z_0,u_0,y_0)}}$

Proof. see [9, Proposition 1] □

With the help of this theorem, the following lemmas can be derived. Proofs for lemma 2.2.4 and 2.2.5 can be found in [11].

Lemma 2.2.4. *Let k be a positive integer. Then there exist $\eta > 0$ and functions $g_1(z, u), g_2(z, u), h_1(z, u), h_2(z, u), f(u)$ with the following properties:*

(i) $g_1(z, u), g_2(z, u), h_1(z, u), h_2(z, u), f(u)$ are analytic for $|u - 1| < \eta$ and $|z - f(u)| < \eta$.

(ii) $g_i(\rho, 1) = 1, h_i(\rho, 1) = b\sqrt{\rho}, i = 1, 2,$ where b is given by (2.6) and $f(1) = \rho$.

(iii) $P^{(k)}(z, u)$ and $T^{(k)}(z, u)$ can be analytically continued to the region

$$R = \left\{(z, u) \in \mathbb{C}^2 : |u| \leq 1 + \frac{\eta}{2}, |z| \leq \rho + \frac{\eta}{2}, arg(z - f(u)) \neq 0\right\}$$

such that

$$P^{(k)}(z,u) = g_1(z,u) - h_1(z,u)\sqrt{1 - \frac{z}{f(u)}} \tag{2.19}$$

and

$$T^{(k)}(z,u) = g_2(z,u) - h_2(z,u)\sqrt{1 - \frac{z}{f(u)}} \tag{2.20}$$

for $(z,u) \in R$ *and* $|u-1| < \eta, |z - f(u)| < \eta$.

For $t^{(k)}(t,u)$ a similar proposition can be made:

Lemma 2.2.5. *Let k be a positive integer. Then there exist $\eta > 0$ and functions $g_3(z,u), h_3(z,u)$ with the following properties:*

(i) $g_3(z,u), h_3(z,u)$ *are analytic for* $|u-1| < \eta$ *and* $|z - f(u)| < \eta$*, with* $f(u)$ *from lemma 2.2.4.*

(ii) $g_3(\rho,1) > 0, h_3(\rho,1) = b^3/3 \neq 0$*, where b is given by* (2.6).

(iii) $t^{(k)}(z,u)$ *can be analytically continued to the region R defined by lemma 2.2.4, such that*

$$t^{(k)}(z,u) = g_3(z,u) - h_3(z,u)\sqrt{\left(1 - \frac{z}{f(u)}\right)^3} \tag{2.21}$$

for $(z,u) \in R$ *and* $|u-1| < \eta, |z - f(u)| < \eta$.

The following lemma is an application of Taylor's theorem and some results obtained by Flajolet and Odlyzko [13], and is also proven in [11].

Lemma 2.2.6. *Suppose that $y(z,u) = \sum y_{nm} z^n u^m$ is an analytic function with $y_{nm} \geq 0$ for all $n, m \in \mathbb{N}$ and that there exists $\eta > 0$ and functions $g(z,u), h(z,u), f(u)$, which are analytic for $|u-1| < \eta$ and $|x - \rho| < \eta$, where ρ is the radius of convergence of $y(z,1)$ such that $y(z,u)$ can be analytically continued to R and that*

$$y(z,u) = g(z,u) - h(z,u)\sqrt{1 - \frac{z}{f(u)}}$$

for $(z,u) \in R$, $|u-1| < \eta$ and $|z - f(u)| < \eta$. Then $y_n(u) = \sum_m y_{nm} u^m = [z^n] y(z,u)$ is asymptotically given by

$$y_n(u) = \frac{h(f(u),u)}{2\sqrt{\pi n^3}} f(u)^{-n+1} + \mathcal{O}\left(\frac{f(u)^{-n}}{\sqrt{n^5}}\right) \tag{2.22}$$

uniformly for $|u-1| < \eta$.
Similarly, if

$$y(z,u) = g(z,u) - h(z,u)\sqrt{\left(1 - \frac{z}{f(u)}\right)^3}$$

for $(z,u) \in R|u-1| < \eta$ *and* $|z - f(u)| < \eta$. *Then* $y_n(u) = \sum_m y_{nm}u^m = [z^n]y(z,u)$ *is asymptotically given by*

$$y_n(u) = \frac{2h(f(u),u)}{4\sqrt{\pi n^5}} f(u)^{-n+1} + \mathcal{O}\left(\frac{f(u)^{-n}}{\sqrt{n^7}}\right) \tag{2.23}$$

uniformly for $|u-1| < \eta$.

We will now study the random variable $X_n'^{(k)}$ with

$$\mathbf{P}(X'(n) = m) = \frac{t_{nm}^{(k)}}{t_n}$$

and determine its limiting distribution with the help of the lemmas stated so far.

Theorem 2.2.7. $X_n'^{(k)}$ *is asymptotically normally distributed with mean value* $\sim c_k n$ *and covariance* $\sim \sigma n$, *where*

$$\begin{aligned}
\mu_k &= \frac{f_u}{\rho} \\
\sigma &= \frac{f_u^2}{\rho^2} - \frac{f_{uu}}{\rho} - \frac{f_u}{\rho}
\end{aligned}$$

with

$$\begin{aligned}
f_u &= -\frac{F_u}{F_z}(\rho,1,1) \\
f_{uu} &= \Big[\frac{1}{F_{tt}F_z}\Big(\frac{F_uF_{tz}}{F_z} - F_{tu}\Big)^2 - \frac{1}{F_z}\Big(\frac{F_u^2F_{zz}}{F_z^2} - \frac{2F_uF_{zu}}{F_z} + F_{uu}\Big)\Big](\rho,1,1)
\end{aligned}$$

and

$$\begin{aligned} F(z,u,t) &= z e^t e^{\left(\sum_{i\geq 2} \frac{t^{(k)}(z^i,u^i)}{i}\right)} \\ &\quad + z(u-1) Z(S_{k-1}; t, t^{(k)}(z^2,u^2), \ldots, t^{(k)}(z^{k-1},u^{k-1})) \end{aligned}$$

Furthermore, for large k

$$\mu_k \sim \frac{2C}{b^2\rho}\rho^k \tag{2.24}$$

$$\sigma \sim \frac{2C}{b^2\rho}\rho^k \tag{2.25}$$

with C given by 2.17.

Proof. cp [11] First, we present a result based on [3, Theorem 1], which will be the base for the proof:

Proposition 2.2.8. *Suppose that $y_{n,m} \geq 0$ and that there exist functions $H(u), f(u)$ defined for $u = e^{it}, |t| < \epsilon, t$ real, such that $H(1) \neq 0$ and $H(u)$ is uniformly continuous and that $f(1) = \rho > 0$ and $f(e^{it})$ has continuous third derivates with*

$$y_n(u) = \sum_{m\geq 0} y_{n,m} u^m \sim a_n H(u) f(u)^{-n}$$

uniformly for $|t| < \epsilon$, for some sequence $a_n > 0$.Furthermore set

$$\begin{aligned} \mu &= i\frac{\partial}{\partial t} log f(e^{it})\big|_{t=0} \\ \sigma &= -\frac{\partial^2}{(\partial t)^2} log f(e^{it})\big|_{t=0} \end{aligned}$$

Then

$$\frac{X_n - n\mu}{\sqrt{n}} \to \mathcal{N}(0,\sigma),$$

i.e., X_n is asymptotically normal with mean value $\sim n\mu$ and covariance $\sim n\sigma$.

The parameters of interest, μ and σ, can be written as

$$\begin{aligned}\mu &= i^2 \frac{f_u(e^{it})}{f(e^{it})} e^{it}\Big|_{t=0} = \frac{f_u(1)}{f(1)} \\ \sigma &= \frac{f_u(1)^2 - f_{uu}(1) f(1)}{f(1)^2} - \frac{f_u(1)}{f(1)}.\end{aligned}$$

Altering u_0 in Theorem 2.2.3 implies that $y = y(f(u), u), z = f(u)$ are the solutions of the system of functional equations

$$y = F(z, u, y) \tag{2.26}$$

$$1 = F_y(z, u, y) \tag{2.27}$$

The partial derivative of (2.26) with respect to u is

$$\begin{aligned} y_u &= F_z f_u + F_u + F_y y_u \\ y_u \underbrace{(1 - F_y)}_{=0} &= F_z f_u + F_u \end{aligned}$$

by (2.27), therefore $F_z f_u + F_u \equiv 0$, and thus $f_u = -\frac{F_u}{F_z}$. Hence

$$\mu = \frac{F_u(z_0, 1, y_0)}{z_0 F_z(z_0, 1, y_0)}$$

where $z_0 = f(1)$ and $y_0 = y(z_0, 1)$. Another implicit differentiation of this equation leads to

$$\begin{aligned} f_{uu} = {} & \frac{1}{F_{yy} F_z} \left(\frac{F_u F_{yz}}{F_z} - F_{yu} \right)^2 - \\ & \frac{1}{F_z} \left(\frac{F_u^2 F_{zz}}{F_z^2} - \frac{2 F_u F_{zu}}{F_z} + F_u u \right) \end{aligned}$$

Now we will determine the partial derivatives of our function $F(z, u, t)$ in Theorem 2.2.7 and through this, examine the behaviour of σ for large k, while for μ we already know from above that it decreases geometrically in k. We use (2.26) and (2.27) and evaluate at $(\rho, 1, 1)$:

$$
\begin{aligned}
F_z &= F_{tz} = \frac{\overbrace{F(z,u,t)}^{=1}}{z} + z(\frac{F}{z})_z(z,u,t) \\
&= \frac{1}{\rho}\Big(1 + \sum_{l\geq 2} t_z(\rho^l,1)\rho^l\Big) \\
F_t &= F_{tt} = 1 \\
F_u &= \sum_{l\geq 2} t_u(\rho^l,1) + \rho Z(S_{k-1};1,t(\rho^2,1),\ldots,t(\rho^{k-1},1)) \\
F_{tu} &= \sum_{l\geq 2} t_u(\rho^l,1) + \rho Z(S_{k-2};1,t(\rho^2,1),\ldots,t(\rho^{k-2},1)) \\
F_{uu} &= \Big(\sum_{l\geq 2} t_u(\rho^l,1)\Big)^2 + \sum_{l\geq 2} l t_{uu}(\rho^l,1) + \sum_{l\geq 2} l(l-1)t_u(\rho^l,1) \\
&\quad +2\rho\frac{\partial}{\partial u} Z(S_{k-1};1,t(\rho^2,1),\ldots,t(\rho^{k-1},1)) \\
F_{zu} &= \frac{1}{\rho}\Big(1 + \sum_{l\geq 2} t_z(\rho^l,1)\rho^l\Big)\big(\sum_{l\geq 2} t_u(\rho^l,1)\big) + \sum_{l\geq 2} l t_{zu}(\rho^l,1)\rho^{l-1} \\
&\quad +Z(S_{k-1};1,t(\rho^2,1),\ldots,t(\rho^{k-1},1)) + \rho\frac{\partial}{\partial z} Z(S_{k-1};1,t(\rho^2,1),\ldots,t(\rho^{k-1},1)) \\
F_{zz} &= 2\sum_{l\geq 2} t_z(\rho^l,1)\rho^{l-1} + \sum_{l\geq 2}(l-2)t_z(\rho^l,1)\rho^{l-2}\sum_{l\geq 2} l t_{zz}(\rho^l,1)\rho^{2l-2}
\end{aligned}
$$

As discussed previously, $Z(S_k;1,t(\rho^2,1),\ldots,t(\rho^{k-1},1)) \sim C\rho^k$ and

$$\sum_{l\geq 2} t_u(\rho^l,1) = o(\rho^k)$$

as shown by Schwenk [37]. Using the same methods of proof,

$$
\begin{aligned}
\sum_{l\geq 2} l t_{zu}(\rho^l,1)\rho^{l-1} &= o(\rho^k) \\
\sum_{l\geq 2} l t_{uu}(\rho^l,1) &= o(\rho^{2k})
\end{aligned}
$$

can be obtained. Now, the terms left to examine are the ones containing derivatives of the cycle index. Therefore we first have to analyze the derivatives of the cycle index $Z(S_n;x_1,\ldots,x_n)$, for which we will use relation 2.11. From there, we see

$$\sum_{k\geq 0} Z(S_k; x_1, \dots, x_k) v^k = e^{\left(\sum_{l\geq 1} \frac{x_l}{l} v^l\right)}$$

and thus

$$\begin{aligned}\sum_{k\geq 0} \frac{\partial}{\partial x_i} Z(S_k; x_1, \dots, x_k) v^k &= e^{\left(\sum_{l\geq 1} \frac{x_l}{l} v^l\right)} \frac{v^i}{i} \\ &= \sum_{k\geq 0} Z(S_k; x_1, \dots, x_k) \frac{v^{k+i}}{i}.\end{aligned}$$

Hence, we obtain

$$\frac{\partial}{\partial a_i} Z(S_k; a_1, \dots, a_n) = \frac{1}{i} Z(S_{k-i}; a_1, \dots, a_{k-i}) \tag{2.28}$$

For the terms occurring in the derivatives of F, this results in

$$\begin{aligned}\frac{\partial}{\partial u} \quad & Z(S_k; t(\rho, 1), t(\rho^2, 1), \dots, t(\rho^k, 1)) \\ &= \sum_{l\geq 2} \frac{\partial}{\partial t_l} Z(S_k; t_1, \dots, t_k)\Big|_{t_m = t(\rho^m, 1), m=1,\dots,k} l t_u(\rho^l, 1) \\ &= \sum_{l\geq 2} Z(S_{k-l}; t(\rho, 1), t(\rho^2, 1), \dots, t(\rho^{k-l}, 1)) t_u(\rho^l, 1).\end{aligned}$$

Applying Schwenk's results on the cycle index, we obtain $Z(S_{k-l}; t(\rho, 1), t(\rho^2, 1), \dots, t(\rho^{k-l}, 1))$ $C\rho^{k-l}$ and $t_u(\rho^l) = o(\rho^{l+k})$, the latter arising from $t_u(\rho^l) \leq (2\rho^l)^k$,which implies $t_u(\rho^l) < (2\rho^2)^k \rho^{(l-2)k} = o(\rho^{(l-1)k})$ as $2\rho^2 < \rho$, and $k(l-1) \geq k + l - 2$ as $k \geq 1, l \geq 2$.

Hence,

$$\frac{\partial}{\partial u} Z(S_k; t(\rho, 1), t(\rho^2, 1), \dots, t(\rho^k, 1)) = o(\rho^{2k})$$

For the second term of that kind we have

$$\begin{aligned}&\frac{\partial}{\partial z} Z(S_k; t(\rho, 1), t(\rho^2, 1), \dots, t(\rho^k, 1)) \\ = &\sum_{l\geq 2} Z(S_{k-l}; t(\rho, 1), t(\rho^2, 1), \dots, t(\rho^{k-l}, 1)) t_z(\rho^l, 1) \rho^{l-1}\end{aligned} \tag{2.29}$$

$Z(S_{k-l}; t(\rho,1), t(\rho^2,1), \ldots, t(\rho^{k-l},1)) = C\rho^{k-l} + o(\rho^{k-l}$, and $t_z(y,1)$ is analytic at $y=0$ and, thus, $t_z(y,1) = 1 + o(y)$. This implies

$$\frac{\partial}{\partial z} Z(S_k; t(\rho,1), t(\rho^2,1), \ldots, t(\rho^k,1)) = \frac{C}{\rho} k\rho^k + o(\rho^k)$$

Applying these results, we get

$$\begin{aligned} f_{uu} &\sim \frac{1}{F_z}\underbrace{(F_u \overbrace{\frac{F_{tz}}{F_z}}^{=1} - F_{tu}}_{=0} \\ &\quad - \frac{1}{F_z}\left(\frac{\overbrace{F_u^2}^{=C^2\rho^{2k}} F_{zz}}{F_z^2} - \frac{\overbrace{2F_uF_{zu}}^{=(C^2/\rho)\rho^{2k}(2k)}}{F_z^2} + \overbrace{F_{uu}}^{=o(\rho^{2k})}\right) \end{aligned} \tag{2.30}$$

and for F_z

$$\begin{aligned} F_z &= \frac{1}{\rho}\big(1 + \sum_{l\geq 2} t_z(\rho^l,1)\rho^l\big) \\ &= \frac{1}{\rho}\big(\lim_{z\to\rho} \frac{zt_z(z,1)(1-t(z,1))}{t(z,1)} = \frac{1}{\rho}\frac{b^2\rho}{2}, \end{aligned}$$

because $t(z,1) = ze^{t(z,1)}e^{\sum_{i\geq 2}\frac{t(z^i,1)}{i}}$ and $t(z,1) = T(z)$, through differentiation and 2.6.

Therefore the dominating term in σ_k is $\frac{f_u}{\rho}$, and thus we get the required result

$$\mu_k \sim \sigma_k \sim \frac{2C}{b^2\rho}\rho^k$$

Applying the given theorems and lemmas, the proof of Theorem 2.2.7is complete. □

REMARKS

1. A similar conclusion as Theorem 2.2.7 holds for $t(z,u), T(z,u)$ and $P(z,u)$, and even for forests of n nodes.

2. The theorem can also be proven for multivariate distributions $X_{n\mathbf{k}} = (X_{nk_1}^{(1)}, \ldots, X_{nk_M}^{(M)})$.

3. If k grows to infinity as well, the distribution is either normal, Poisson or degenerated, depending on the behaviour of $\mathbf{E}(X_{n,k})$, as shown in [17].

Chapter 3

Simply generated trees

We will now discuss another group of trees, the so-called simple generated families of trees or Galton-Watson trees. These trees already provide some restrictions on their shape.

3.1 Introduction and node degree

Definition 3.1.1 (Simple generated tree)**.** *Let $\mathcal{A}$ denote a family of rooted trees, and $a(x) = \sum a_n x^n$ be its generating function. $\mathcal{A}$ is called a simply generated family of trees, if its generating function satisfies*

$$a(x) = x\varphi(a(x)), \qquad \varphi(t) = \sum_{i \geq 0} c_i t^i, \quad \varphi_i \geq 0, \varphi_0 > 0 \tag{3.1}$$

Definition 3.1.2 (Galton-Watson branching process)**.** *A Galton-Watson process is a stochastic process X_t, more precisely a branching process (see for example [20]), with:*

1. $X_0 = 1$ *(We start with a single individual)*

2. *At time $t + s$, every particle that existed at time t will have a number of successors distributed like X_s, the number of successors of different particles will be independent of each other and independent of the time before t.*

That is, in simple words, the number of offspring of an individual in the process is a copy of ξ, where ξ is a random variable.

We call a Galton-Watson process critical, if $\mathbf{E}(\xi) = 1$, that is, if every individual is expected to have exactly one son.

Definition 3.1.3 (conditioned Galton-Watson Tree)**.** *Let T_n be a random rooted tree of size n. We call T_n a conditioned Galton-Watson tree if it has the same degree distribution as the family tree of a Galton-Watson branching process with some offspring distribution ξ, conditioned to have total progeny n.*

To start this chapter, we will demonstrate that the families of trees defined by Definition 3.1.1 are the same families than those defined by Definition 3.1.3:

We assign a weight to every tree T of a simply generated family of trees $\mathcal{T}$ by

$$w(t) = \prod_{v \in V_T} \varphi_{d(v)}$$

V_T being the set of nodes of T, $d(v)$ the out-degree of node v and φ_k the k-th coefficient of the power series $\varphi(t)$ in the definition of simply generated trees. This function induces a probability distribution, the likelihood of a tree of size n being B is proportional to $w(T)$.

Now we consider a Galton-Watson branching process X, without loss of generality we may assume that the offspring distribution ξ is given by

$$\mathbf{P}(\xi = k) = \frac{\tau^k \varphi_k}{\varphi \tau}$$

for some sequence $\varphi_k, k \geq 0$ of non-negative integers such that the power series $\sum_{k \geq 0} \varphi_k t^k$ has a positive or infinite radius of convergence R, and for some positive number τ within R. Then, the distribution of X conditioned on the total progeny $|X|$ is determined by $\mathbf{P}(X = T||X| = n)$ and that is the same as the probability distribution induced by the weight function above.

Thus, the families of trees created through 3.1.1 are the same as those created by 3.1.3. Thus, the degree distribution of a simplygenerated tree or Galton-Watson tree is implicitly given by its offspring distribution ξ.

REMARK Many interesting random trees are Galton Watson trees, for example:

- labelled trees, with an Poisson offspring distribution $\xi \sim Po(1), \sigma^2 = 1$, and with generating function

$$a(x) = xe^{a(x)} = \sum_{n \geq 1} n^{n-1} \frac{x^n}{n!}$$

- plane trees, with $\mathbf{P}(\xi = k) = 2^{-(k+1)}, \sigma^2 = 2$ and

$$a(x) = \frac{x}{1 - a(x)} = \sum_{n \geq 1} \binom{2n-2}{n-1} \frac{x^n}{n}$$

- binary trees, with $\xi \sim Bi(2, \frac{1}{2}), \sigma^2 = \frac{1}{2}$ and

$$a(x) = x(1 + a(x))^2$$

- strict binary trees, with $\mathbf{P}(\xi = 0) = \mathbf{P}(\xi = 2) = \frac{1}{2}, \sigma^2 = 1$ and

$$a(x) = x(1 + a^2(x))$$

3.2 The Generating function of simplygenerated trees

The structure of simply generated families of trees is probably the best explored under all families of random trees.

We will now explore some properties of its generating function.

Theorem 3.2.1. *Suppose $\varphi(t) = 1 + c_1 t + c_2 t^2 + \cdots$ is a regular function of t when $|t| < R \leq \infty$ and let*

$$a = a(x) = x + a_2 x^2 + a_3 x^3 + \cdots$$

denote the solution of $a(x) = x\varphi(a(x))$ in the neighbourhood of $x = 0$. If

(i) $c_1 > 0$ and $c_j > 0$ for some $j \geq 2$,

(ii) $c_i \geq 0$ for $i \geq 2$, (a precondition already mentioned in the definition of simplygenerated trees), and

(iii) $\tau\varphi'(\tau) = \varphi(\tau)$ for some τ, where $0 < \tau < R$.

Then τ is unique, and $a(x)$ is regular in the disk $|x| \leq \rho = \frac{\tau}{\varphi(\tau)}$ except at $x = \rho$, i.e. ρ is the only singularity of $a(x)$. Furthermore $a(x)$ has an expansion in the neighbourhood of ρ of the form

$$a(x) = \tau - b(\rho - x)^{\frac{1}{2}} - b_2(\rho - x) \cdots \tag{3.2}$$

where $b = \rho^{-1}(\frac{2\tau}{\varphi''(\tau)})^{\frac{1}{2}}$

Proof. (cp [29])

We define

$$f(t) = t\varphi'(t) - \varphi(t) \tag{3.3}$$

$f(t)$ is a strictly increasing function for $0 \leq t \leq R$, because

$$\begin{aligned} f(0) &= -1. \\ f'(t) &= t\varphi''(t) > 0 \text{ for } 0 < t < R \text{ because of (i) and (ii),} \end{aligned}$$

and thus, τ is unique.

From (iii) it follows that $t\varphi'(t) - \varphi(t) < 0$ for $0 \leq t \leq \tau$.

We now consider the functional relation $F(x, a) \equiv a - x\varphi(a) = 0$. Then $F_a = 1 - x\varphi'(a)$, and the observations above imply that $F_a \neq 0$ when $|x| < \rho = \frac{\tau}{\varphi(\tau)}$ and $|a| < \tau$.

Since $F_a(\rho, \tau) = 0$, it follows from the implicit function theorem (see for example [21]) that $a = a(x)$ is regular for $|x| < \rho$, that $a(\rho) = \tau$ and that $x = \rho$ is a singularity of $a(x)$.

We consider the case $|x| = \rho$ but $x \neq \rho$: From $a_1 = 1, a_2 = c_1 > 0$ ((ii)) it follows that $|a(x)| < a(\rho) = \tau$; and so $|\varphi'(a(x))| < \varphi'(\tau) = 1/\rho$, by (i) and (ii).

Hence $|x\varphi'(a(x))| < 1$ if $|x| = \rho$ but $x \neq \rho$.

We now have

$$F_a(x, a(x)) \neq 0 \text{ except when } x = \rho$$

Since $F_x \neq 0, F_a = 0$ and $F_{aa} \neq 0$ at (ρ, τ), if follows that $a(x)$ is regular for $|x| \leq \rho$ except at $x = \rho$. Using the Taylor series near (ρ, τ)

$$\begin{aligned} F(x, a) &= \underbrace{F(\rho, \tau)}_{=0} + F_x(x - \rho) + \underbrace{F_a(a - \tau)}_{=0} + \\ &\quad + F_{xx}\frac{(x - \rho)^2}{2} + F_{xa}(x - \rho)(a - \tau) + F_{aa}\frac{(a - \tau)^2}{2} + \cdots, \end{aligned}$$

for $x \to \rho$ and $a(x) \to \tau$ the terms of lowest order of magnitude have to be asymptotically equal, that is

$$(a - \tau)^2 \sim \frac{2F_x}{F_{aa}}(x - \rho)$$

and thus

$$a \sim \tau \pm \sqrt{\frac{2F_x}{F_{aa}}(x-\rho)}$$

Hence, using a so called Puiseux-series $\sum_n b_n x^{\frac{n}{k}}$, a has the expansion (3.2) around $x = \rho$.

□

3.3 The profile and contour processes

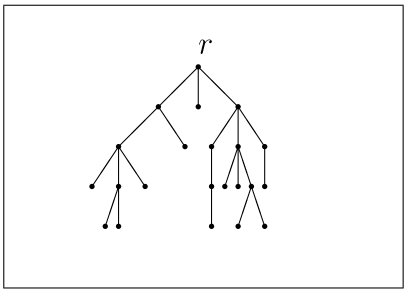

Figure 3.1: A sample tree

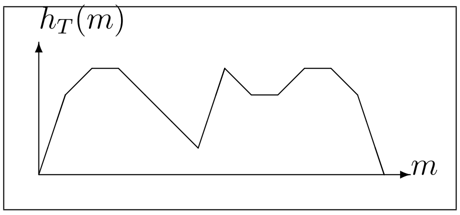

Figure 3.2: The contour of the above tree

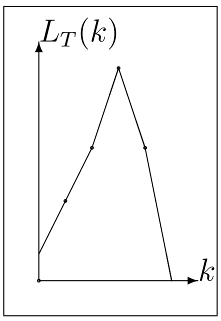

Figure 3.3: The profile of the above tree

In the following, we will deal with two processes describing the shape of the tree, the contour process and the profile.

Let T be a tree of size n, with its leaves ordered (in the plane case, we can order the leaves from left to right, in the non-plane case the offspring distribution ξ induces an order).

The height $h_T(x)$ of a node x in T is defined by the number of edges on the unique path from the root to x. As the trees T are equipped with a probability distribution within the set of trees of size n, the heights of the leaves are also randomly distributed and are denoted by $\hat{H}_n(m)$. By linear interpolation, we get a continuous stochastic process:

$$\hat{H}_n(t) = (\lfloor t \rfloor + 1 - t)\hat{H}_n(\lfloor t \rfloor) + (t - \lfloor t \rfloor)\hat{H}_n(\lfloor t \rfloor + 1)$$

Definition 3.3.1 (contour process)**.** *The scaled process*

$$\hat{C}_n(t) = \frac{1}{\sqrt{n}}\hat{H}_n(tn), \quad 0 \leq t \leq 1$$

is called the contour process of the family of trees $\mathcal{T}$.

REMARK: With $sup_{x\geq 0}\hat{H}_n(x) =: H_n$ we denote the height of the tree.

By $L_T(k)$ we denote the number of nodes at height k. Also $L_T(k)$ is a random variable as T is a random tree, and so we again create a continuous stochastic process by linear interpolation:

$$L_n(t) = (\lfloor t \rfloor + 1 - t)L_n(\lfloor t \rfloor) + (t - \lfloor t \rfloor)L_n(\lfloor t \rfloor + 1), \quad t \geq 0$$

Definition 3.3.2 (Profile)**.** *We call the scaled process*

$$l_n(t) = \frac{1}{\sqrt{n}}L_n(t\sqrt{n}) \quad t \geq 0$$

the profile of the simplygenerated family of trees $\mathcal{T}$.

REMARK The maximum of $L_T(k)$ is called the width of the tree T, and is denoted by W.

In the following, we will see that these two processes stand in close connection with Brownian excursions.

Theorem 3.3.3. *Let $W^+(t)$ denote Brownian excursion of duration* 1 *(for definitions see 1). Further assume that $\varphi(t)$ has a positive or infinite radius of convergence R and $d = gcd(k|\varphi_k > 0) = 1$, and suppose that the equation*

$$t\varphi'(t) = \varphi(t) \tag{3.4}$$

has a minimal positive solution $\tau < R$. Define the offspring distribution ξ of the corresponding Galton-Watson tree by $\mathbf{P}(\xi = k) = \frac{\tau^k \varphi_k}{\varphi(\tau)}$ as mentioned in the introduction of this chapter, and let σ^2 be its variance, given by

$$\sigma^2 = \frac{\tau^2 \varphi''(\tau)}{\varphi(\tau)} \tag{3.5}$$

Then the contour process $\hat{C}_n(t)$ converges weakly to Brownian excursion, i.e.,

$$\hat{C}_n(\frac{\varphi_0}{\varphi(\tau)} t) \xrightarrow{w} \frac{2}{\sigma} W^+(t) \tag{3.6}$$

in $\mathcal{C}[0,1]$.

REMARK If Theorem 3.3.3 is true, then the distribution of the height $h_n(t) = \max_{t \geq 0} \hat{H}_n(t) = \sqrt{n}\hat{C}_n(t)$ also converges against $\frac{2}{\sigma}\sqrt{n}\sup_{0 \leq t \leq 1} W(t)$, and the moments $\mathbf{E}(h_n^p)$ converge against the moments of Bronwian excursion local time, as stated in [14].

Theorem 3.3.4. *Again, let $W^+(t)$ be Brownian excursion of duration 1, and let $l(t)$ be its (total) local time at level t, i.e.,*

$$l(t) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^1 \mathbf{I}_{[t,t+\epsilon]}(W(s)) ds \tag{3.7}$$

Under the same premises as in Theorem 3.3.3, the process $l_n(t)$ converges weakly to Brownian excursion local time, i.e.,

$$l_n(t) \xrightarrow{w} \frac{\sigma}{2} l(\frac{\sigma}{2} t)$$

in $\mathcal{C}[0,\infty)$, as $n \to \infty$.

REMARK If Theorem 3.3.4 is true, then the width of Galton-Watson trees $w_n = \max_{t \geq 0} L_n(t) = \sqrt{n}\sup_{t \geq 0} l_n$ also converges against $\frac{\sigma}{2}\sqrt{n} sup_{t \geq 0}$, and even convergence of moments is given, as stated in [12].

PROOFS

Proofs for Theorem 3.3.3 and Theorem 3.3.4 work along the same plan, and can be found in [18] and in [10], respectively. In this work, we will show the general idea and draw an outline for the proof of Theorem 3.3.4, diverse calculation steps are omitted in favor of clarity, the reader is asked to consult the according paper for details. The proof is accomplished in two parts:

1. Weak convergence of the finite-dimensional distributions is shown with the help of Cauchy's integral formula.

2. Tightness of the sequences are to be shown.

Together this is sufficient to show weak convergence of distributions.

The main idea of the first part is the following:

Let $\mathcal{T}$ be a family of simplygenerated trees, and let $(\circ)$ denote a node. $\mathcal{T}$ fulfills the symbolic recursion:

$$\mathcal{T} = \varphi_0 \cdot (\circ) \cup \varphi_1 \cdot (\circ) \times \mathcal{T} \cup \varphi_2 \cdot (\circ) \times \mathcal{T} \times \mathcal{T} \cup \cdots =: \Phi(\mathcal{T})$$

.

Translating the operators $\cup$ and $\times$ into sum and product in the corresponding GFs, we obtain the characteristic functional equation of simplygenerated trees

$$a(x) = x\varphi(a(x))$$

Now we mark all substructures of a tree T which fulfill a characteristic $\phi(T)$ in which we are interested (in the case of the profile this will be all nodes on level d, for the contour it would be all leaves), and denote a marked node by $\bullet$. This is equivalent to introducing a new variable in the generating function and thus creating a bivariate GF:

$$a(x, u) = \sum_{m,n \geq 0} a_{mn} x^n u^m$$

The distribution of the characteristic we are interested in is then given by:

$$\mathbf{P}\{\phi(T) = m \| T| = n\} = \frac{a_{mn}}{a_n}$$

where a_{mn} is the coefficient of $x^n u^m$ in $a(x, u)$.

With the help of the above recursion and the correspondence

$$\begin{array}{ccc} \circ & \leftrightarrow & x \\ \bullet & \leftrightarrow & ux \end{array}$$

we can determine the exact shape of the GF.

In terms of the profile and the number of nodes on level d, this is:

Let $a_d(x, u) = \sum_{m,n \geq 0} a_{dmn} x^n u^m$ be the GF of nodes on level d, and let $\hat{\mathcal{T}}$ be the family of trees with marked nodes on level d. Then:

$$\hat{\mathcal{T}} = \Phi^d((\bullet) \times \mathcal{T})$$

and

$$a_d(x,u) = y_d(x, ua(x)) \tag{3.8}$$

where

$$\begin{aligned} y_0(x,u) &= u \\ y_{i+1}(x,u) &= x\varphi(y_i(x,u)), \quad i \geq 0 \end{aligned} \tag{3.9}$$

Further, the distribution of $L_n(d)$ is given by

$$\mathbf{P}\{L_n(d) = m||T| = n\} = \frac{a_{dmn}}{a_n}$$

In order to show weak convergence of the fdds of $l_n(k)$, it is enough to show pointwise convergence in $(-\epsilon, \epsilon)$, with arbitrary $\epsilon > 0$, of the characteristic functions $\chi_X(t) = \mathbf{E}(e^{itX})$, as convergence in characteristic functions implies convergence in distributions if the limit is continuous in $t = 0$, which for Brownian excursion local time is true (cp [27][p. 189ff]).

The characteristic function of $\frac{1}{\sqrt{n}} L_n(k)$ is

$$\chi_{kn}(t) = \frac{1}{a_n}[x^n] y_k(x, e^{\frac{it}{\sqrt{n}}} a(x))$$

and that of the finite-dimensional distributions $\left(\frac{1}{\sqrt{n}} L_n(k_1), \ldots, \frac{1}{\sqrt{n}} L_n(k_p)\right)$ is given by

$$\chi_{k_1,\ldots,k_p n}(t_1, \ldots, t_p) = \frac{1}{a_n}[x^n] y_{k_1}\left(x, e^{\frac{it_1}{\sqrt{n}}} y_{k_2-k_1}(x, \ldots, y_{k_p-k_{p-1}}(x, e^{\frac{it_p}{\sqrt{n}}} a(x)) \cdots)\right)$$

Now, recursion 3.9 will be analyzed in detail to find a suitable contour for using Cauchy's integral formula, with the help of the new recursive series:

$$w_i = w_i(x,u) = y_i(x,u) - a(x)$$

As we have seen earlier in this chapter, $a(x)$ has one singularity at $x_0 = \frac{\tau}{\varphi(\tau)}$ and around it a local expansion of the form:

$$a(x) = \tau - \frac{\sqrt{2}\tau}{\sigma}\sqrt{1 - \frac{x}{x_0}} + \mathcal{O}\left(\left|1 - \frac{x}{x_0}\right|\right)$$

The assumption $d = 1$ implies that $|x\varphi'(a(x))| < 1$ for $|x| = x_0, x \neq x_0$, and hence, by the implicit function theorem, $a(x)$ has an analytic continuation to the region $|x| < x_0+\delta, \arg(\mathrm{x}-\mathrm{x}_0) \neq 0$ for some $\delta > 0$, and the function $\alpha = x\varphi'(a(x))$ has similar analytic properties and the local expansion

$$\alpha = 1 - \sigma\sqrt{2}\sqrt{1 - \frac{x}{x_0}} + \mathcal{O}\left(\left|1 - \frac{x}{x_0}\right|\right) \tag{3.10}$$

With this information, we can state the following lemma:

Lemma 3.3.5. *Set $\alpha = x\varphi'(a(x))$ and suppose that $w_0 = u - a(x) = \mathcal{O}(1)$ and $\frac{1}{2} \leq |\alpha| \leq 1 + \mathcal{O}(w_0|)$. If $i = \mathcal{O}(|w_0|^{-1})$, then*

$$w_i = \mathcal{O}(w_0\alpha^i)$$

Proof. This lemma can be shown using an induction on i on the local Taylor expansion

$$\begin{aligned} y_{i+1}(x,u) &= x\varphi(y_i(x,u)) \\ &= x\varphi(a(x) + w_i) \\ &= a(x) + x\varphi'(a(x))w_i + x\varphi''(a(x) + \theta_i)\frac{w_i^2}{2} \\ &= a(x) + \alpha w_i + x\varphi''(a(x) + \theta_i)\frac{w_i^2}{2}. \end{aligned}$$

□

We now set $x = x_0(1 + \frac{z}{n})$, and assume that $|w_0| = |u - a(x)| = \mathcal{O}(\frac{1}{\sqrt{n}})$ and $\frac{z}{n} \to 0$ in such a way that $|arg(-z)| < \pi$ and

$$|1 - \sqrt{\frac{-z}{n}}| \leq 1 + \frac{C}{\sqrt{n}}$$

are satisfied. We further have $\alpha = 1 + \mathcal{O}(\frac{1}{\sqrt{n}})$ and can apply Lemma 3.3.5 for $i = \mathcal{O}(\sqrt{n})$.

The asymptotic relation

$$w_{i+1} = \alpha w_i + \beta w_i^2 + \mathcal{O}(|w_i|^3),$$

where $\beta = x\varphi''(a(x))/2$,

leads to

Lemma 3.3.6. *Under the given premises, $y_k(x,u)$ from recursion 3.9 admits the local representation*

$$y_k(x,u) = a(x) + \frac{(u-a(x))\alpha^k}{\frac{\sqrt{\frac{-z}{n}} + \frac{\sigma(\tau-u)}{\tau\sqrt{2}}}{2\sqrt{\frac{-z}{n}}} + \frac{\sqrt{\frac{-z}{n}} - \frac{\sigma(\tau-u)}{\tau\sqrt{2}}}{2\sqrt{\frac{-z}{n}}}\alpha^k + \mathcal{O}\Big(\sqrt{\frac{|z|}{n}}\Big)}$$

uniformly for $k = \mathcal{O}(\sqrt{n})$.

because rewriting the relation and setting $q_i = \frac{\alpha^i}{w_i}$ leads to

$$q_i = \frac{1}{w_0} - \frac{\beta}{\alpha}\frac{1-\alpha^i}{1-\alpha} + \mathcal{O}\Big(|w_0|\Big|\frac{1-\alpha^{2i}}{1-\alpha^2}\Big|\Big)$$

and, with $x = x_0(1+\frac{z}{n})$

$$\begin{aligned} w_0 &= u - a(x) = u - \tau + \frac{\tau\sqrt{2}}{\sigma}\sqrt{-\frac{z}{n}} + \mathcal{O}(\frac{|z|}{n}) \\ \beta &= \frac{x_0\varphi''(\tau)}{2}\Big(1+\mathcal{O}\Big(\sqrt{\frac{|z|}{n}}\Big)\Big) = \frac{\sigma^2}{2\tau}\Big(1+\mathcal{O}\Big(\sqrt{\frac{|z|}{n}}\Big)\Big). \end{aligned}$$

Combining these results leads to the above statement.
The results obtained so far can be used to show the following theorem:

Theorem 3.3.7. *Let $k_i = \kappa_i\sqrt{n}, i = 1,\ldots,p$ where $0 < \kappa_1 < \cdots < \kappa_p$. Then the characteristic function $\chi_{\kappa_1\cdots\kappa_p}(t_1,\ldots,t_p) = lim_{n\to\infty}\chi_{k_1\cdots k_p n}(t_1,\ldots,t_p)$ of the limiting distribution of $(\frac{1}{\sqrt{n}}L_n(k_1),\ldots,\frac{1}{\sqrt{n}}L_n(k_p))$ satisfies*

$$\chi_{\kappa_1\cdots\kappa_p}(t_1,\ldots,t_p) = 1 + \frac{\sigma}{i\sqrt{2\pi}}\int_\gamma f_{\kappa_1,\cdots,\kappa_p,\sigma}(x,t_1,\ldots,t_p)e^{-x}dx \qquad (3.11)$$

where

$$\begin{aligned} f\ &_{\kappa_1,\cdots,\kappa_p,\sigma}(x,t_1,\ldots,t_p) = \\ &= \Phi_{\kappa_1,\sigma}\big(x, it_1 + \Phi_{\kappa_2-\kappa_1,\sigma}\big(\cdots\Phi_{\kappa_{p-1}-\kappa_{p-2},\sigma}(x, it_{p-1} + \Phi_{\kappa_p-\kappa_{p-1},\sigma}(x,it_p)\big)\big) \qquad (3.12) \end{aligned}$$

with

$$\Phi_{\kappa\sigma}(x,t) = \frac{t\sqrt{-x}e^{-\kappa\sigma\sqrt{\frac{-x}{2}}}}{\sqrt{-x}e^{\kappa\sigma\sqrt{\frac{-x}{2}}} - t\frac{\sigma}{\sqrt{2}}\sinh\left(\kappa\sigma\sqrt{-\frac{x}{2}}\right)}$$

and γ is the Hankel-like contour $\gamma_1 \cup \gamma_2 \cup \gamma_3$ defined by

$$\begin{aligned}
\gamma_1 &= \{s||s| = 1 \text{ and } \Re s \leq 0\} \\
\gamma_2 &= \{s|\Im s = 1 \text{ and } \Re s \geq 0\} \\
\gamma_3 &= \overline{\gamma_2}
\end{aligned}$$

Proof. The proof of this theorem is made stepwise. Let k and h be non-negative integers, and let $\chi_{k,k+h,n}(s,t)$ be the characteristic function of the joint distribution of $\frac{1}{\sqrt{n}}L_n(k)$ and $\frac{1}{\sqrt{n}}L_n(k+h)$. Denote by $\chi_{\kappa,\kappa+\eta}(s,t) = \lim_{n\to\infty}\chi_{k,k+h,n}(s,t)$ the characteristic function of the limiting distribution of $(\frac{1}{\sqrt{n}}L_n(k), \frac{1}{\sqrt{n}}L_n(k+h))$. Then it can be shown that $\chi_{\kappa,\kappa+\eta}(s,t)$ fulfills the proposition of Theorem 3.3.7, using Cauchy's integral formula and a truncated Hankel contour $\gamma' = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ around the singularity x_0 closed by a circular arcΓ_4:

$$\begin{aligned}
\Gamma_1 &= \left\{x = x_0\big(1 + \frac{z}{n}\big)\big|\Re z \leq 0 \text{and } |z| = 1\right\} \\
\Gamma_2 &= \left\{x = x_0\big(1 + \frac{z}{n}\big)\big|\Im z = 1 \text{ and } 0 \leq \Re z \leq log^2 n\right\} \\
\Gamma_3 &= \overline{\Gamma_2} \\
\Gamma_4 &= \left\{x\big||x| = x_0\left|1 + \frac{log^2 n + i}{n}\right| \text{ and } arg\big(1 + \frac{log^2 n + i}{n}\big) \leq |arg(z)| \leq \pi\right\}
\end{aligned}$$

where it will be found that the contribution of Γ_4 is negligibly small and that the substitution of γ' by γ is justified by the dominated convergence theorem.

Then the steps of the proof for dimension 2 can be iterated and thus the theorem can be proofed. □

Now the next step is taken from the other side, determining the fdds of Brownian excursion local time. Those can be shown to be:

Theorem 3.3.8. *Let $\overline{\chi}_{\kappa_1\cdots\kappa_p}(t_1,\ldots,t_p)$ denote the characteristic function of the joint distribution of $(l(\kappa_1),\ldots,l(\kappa_p))$. Then we have*

$$\overline{\chi}_{\kappa_1\cdots\kappa_p}(t_1,\dots,t_p) = 1 + \frac{\sqrt{2}}{i\sqrt{\pi}}\int_\gamma f_{\kappa_1,\cdots,\kappa_p,2}(x,t_1,\dots,t_p)e^{-x}dx \tag{3.13}$$

where $f_{\kappa_1,\cdots,\kappa_p,2}(x,t_1,\dots,t_p)$ *is given by the same definitions as in 3.3.7.*

The last missing tile in the proof of Theorem 3.3.4 is the proof of tightness of the sequence of random variables $l_n(t) = \frac{1}{\sqrt{n}}L_n(t\sqrt{n}), t \geq 0$ in $C[0,\infty)$. As a sequence of stochastic processes $X_n(t), t \geq 0$ is tight in $C[0,\infty)$ if and only if $X_n(t), 0 \leq t \leq T$ is tight in $C[0,T]$ for all $T > 0$, it is enough to show tightness on a finite intervall, i.e. it is enough to show tightness of $L_n(t), 0 \leq t \leq A\sqrt{n}$ for some real constant $A > 0$.

According to [5], and estimate of the form

$$\mathbf{P}\{|L_n(\rho\sqrt{n}) - L_n((\rho+\theta)\sqrt{n}| \geq \epsilon\sqrt{n}\} \leq C\frac{\theta^\alpha}{\epsilon^\beta} \tag{3.14}$$

for some $\alpha > 1, \beta \geq 0, C > 0$ uniformly for $0 \leq \rho \leq \rho+\theta \leq A$, together with tightness of $L_n(0)$, which is obviously satisfied, imply tightness of the demanded sequence.

So the proposition to obtain is 3.14, which can be derived from

Lemma 3.3.9. *There exists a constant* $C > 0$ *such that*

$$\mathbf{E}(L_n(r) - L_n(r+h))^4 \leq Ch^2n$$

holds for all nonnegative integers n, r, h,

which can be shown through calculation of the expected value and singularity analysis.

Putting all pieces together, finally the weak convergence

$$l_n(t) \xrightarrow{w} \frac{\sigma}{2}l(\frac{\sigma}{2}t)$$

of Theorem 3.3.4 is shown.

The joint distribution of height and width

For the distributions of the height h_n and the width w_n of any Galton Watson tree of total progeny n, even the following theorem is true

Theorem 3.3.10. *For any conditioned Galton Watson tree T_n*

$$\frac{1}{\sqrt{n}}(h_n, w_n) \xrightarrow{w} (\frac{H}{\sigma}, \sigma W)$$

as $n \to \infty$, where

$$\begin{aligned} H &= \int_0^1 \frac{1}{B(t)} dt \\ W &= \max_{t \in [0,1]} B(t) \end{aligned}$$

and $B(t)$ is a normalized Brownian excursion.

REMARK The joint distribution (H, W) given above is equal in distribution to $(2 \max_t B(t), \frac{1}{2} \max_{x \geq 0} l(x))$, where $l(x)$ is the local time of $B(t)$, thus $H \xrightarrow{w} 2W$.

The above theorem is proved for binary trees in [7], in [23] joint moments are calculated.

3.4 Conditioned Galton-Watson trees do not grow

We will show through an counter example, that the families of trees discussed in this chapter can in general not be obtained by adding vertices one by one, i.e. there exist simply generated families of trees and at least one $n \in \mathbb{N}$, for which the tree resulting from adding a new leaf to B_n by some random procedure does not have the distribution of B_{n+1}, which is a major difference to the graphs discussed in chapter 5, which are created by adding leaves one by one.

Now what does the property mentioned above mean?

Property 3.4.1. *It is possible to define B_n and B_{n+1} on a common probability space such that $B_n \subset B_{n+1}$. Or equivalently:*
It is possible to construct $B_1, B_2, B_3, \ldots$ as a Markov chain where at each step a new leaf is added.

Let $W_k(B)$ denote the number of vertices of distance k from the root. If Property 3.4.1 holds, then also:

Property 3.4.2. *For every $k \geq 0$ and $n \geq 1$,*

$$\mathbf{E} W_k(B_n) \leq \mathbf{E} W_k(B_{n+1}).$$

Theorem 3.4.3. *Conditioned Galton-Watson trees do not (necessarily) fulfill Property 3.4.2, and hence do not (necessarily) fulfill Property 3.4.1.*

Proof. The follwing example was found in [22]. Consider the following Galton Watson process:

Let $\epsilon > 0$ be a small number and let the offspring distribution be given by:

$$\mathbf{P}(\xi = 0) = \frac{1-\epsilon}{2} \quad \mathbf{P}(\xi = 1) = \epsilon \quad \mathbf{P}(\xi = 2) = \frac{1-\epsilon}{2}$$

We have $\mathbf{E}\xi = 1$ and $\sigma^2 := Var\xi = 1 - \epsilon$.

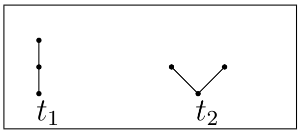

Figure 3.4: The trees with three vertices

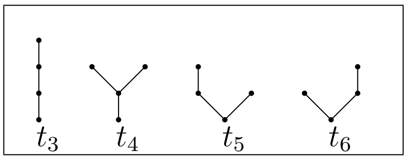

Figure 3.5: The trees with four vertices

Let B be the Galton-Watson tree according to ξ. For $n = 3$ we have two possible trees, see Figure 3.4, with corresponding probabilities:

$$\begin{aligned} \mathbf{P}(B = t_1) &= p_1^2 p_0 = \epsilon^2 \frac{1-\epsilon}{2} = \frac{1}{2}\epsilon^2 + \mathcal{O}(\epsilon^3) \\ \mathbf{P}(B = t_2) &= p_2 p_0^2 = \big(\frac{1-\epsilon}{2}\big)^3 = \frac{1}{8} + \mathcal{O}(\epsilon), \end{aligned}$$

where $p_j := \mathbf{P}(\xi = j)$. Thus, conditioning on $|B| = 3$, we have:

$$\begin{aligned} \mathbf{P}(B_3 = t_1) &= \frac{\mathbf{P}(B = t_1)}{\mathbf{P}(B = t_1) + \mathbf{P}(B = t_2)} = 4\epsilon^2 + \mathcal{O}(\epsilon^3) \\ \mathbf{P}(B_3 = t_2) &= \frac{\mathbf{P}(B = t_2)}{\mathbf{P}(B = t_1) + \mathbf{P}(B = t_2)} = 1 - 4\epsilon^2 + \mathcal{O}(\epsilon^3), \end{aligned}$$

For $n = 4$ we have the four possibilities in Figure 3.5 and the probabilities:

$$\begin{aligned}
\mathbf{P}(B=t_3) &= p_1^3p_0 = \epsilon^3\frac{1-\epsilon}{2} = \frac{1}{2}\epsilon^3 + \mathcal{O}(\epsilon^4)\\
\mathbf{P}(B=t_4) &= p_1p_2p_0^2 = \epsilon\big(\frac{1-\epsilon}{2}\big)^3 = \frac{1}{8}\epsilon + \mathcal{O}(\epsilon^2)\\
\mathbf{P}(B=t_5) &= \mathbf{P}(B=t_6) = p_2p_1p_0^2 = \mathbf{P}(B=t_4),
\end{aligned}$$

and thus, conditioned on $|B| = 4$, that is

$$\begin{aligned}
\mathbf{P}(B_4=t_3) &= \frac{\mathbf{P}(B=t_3)}{\mathbf{P}(B=t_3) + 3*\mathbf{P}(B=t_4)} = \mathcal{O}(\epsilon^2)\\
\mathbf{P}(B_4=t_4) &= \mathbf{P}(B_4=t_5) = \mathbf{P}(B_4=t_6) = \frac{1}{3} + \mathcal{O}(\epsilon^2).
\end{aligned}$$

Now we consider $W_1(B_n)$ and get:

$$\begin{aligned}
\mathbf{E}W_1(B_3) &= 1*\mathbf{P}(B_3=t_1) + 2*\mathbf{P}(B_3=t_2) = 2 + \mathcal{O}(\epsilon^2)\\
\mathbf{E}W_1(B_4) &= 1*\mathbf{P}(B_4=t_3) + 1*\mathbf{P}(B_4=t_4) + 4*\mathbf{P}(B_4=t_5) = \frac{5}{3} + \mathcal{O}(\epsilon^2),
\end{aligned}$$

and hence, if ϵ is small enough,

$$\mathbf{E}W_1(B_3) > \mathbf{E}W_1(B_4).$$

So for the conditioned Galton-Watson tree with offspring distribution ξ Property 3.4.2 fails and hence Theorem 3.4.3 is true.

□

REMARK

Not every family of simply generated trees fails Property 3.4.2 (for example, random d-ary trees hold it, as investigated in [28]. Those families which hold the Property, are called *very simply generated trees*.

Chapter 4

Increasing trees

In this chapter we introduce trees with an additional characteristic, which is their labeling. We will establish a connection to the families discussed in the previous chapter and find some interesting results on these very well examined families.

4.1 Introduction

Definition 4.1.1 (labelled tree). *Let B be a tree with n nodes, and I be the set of integers $\{1, \dots, n\}$.*
B is called a labelled tree if each node of V_B is given a unique label $i \in I$.

Definition 4.1.2 (increasing tree). *Let B_n be a labelled tree of size n.*
B_n is called an increasing tree if the sequence of labels along any branch of B_n, starting at the root, is increasing.
(Obviously, the root is always labelled with 1).

Definition 4.1.3 (degree-weight function). *Let $\varphi_{k\geq 0}$ be a sequence of non-negative integers with $\varphi_0 > 0$ and assume there exists at least one $k \geq 2$ with $\varphi_k > 0$. This sequence assigns a weight to every node of degree k. The sequence $\varphi(k)$ is called degree-weight sequence. Its generating function $\varphi(t) = \sum_{k\geq 0} \varphi_k t^k$ is the degree-weight function of the family of trees considered.*

REMARK In the plane case φ_k can be interpreted as the sorts of nodes of outdegree k, in the non-plane case the division by $n!$ eliminates the factor of ordering subtrees.

Definition 4.1.4 (Family of increasing trees). *A family of increasing trees is the collection of all plane/non-plane increasing trees with φ_k sorts of nodes of outdegree k.*

Simple families of increasing trees

Note that we can generate an increasing tree by taking any unlabelled rooted tree and provide it with a valid increasing labeling. We consider increasing trees derived from simple generated trees (as described in chapter 3). We call these families of trees *simple families of increasing trees.* Simple families of increasing trees can be described via their degree-weight function $\varphi(t)$(cp Definition 4.1.3): We then define the weight $w(T)$ of any tree T by $w(T) = \prod_v \varphi_{d(v)}, v \in V_T, d(v)$ being the outdegree of node v. $\mathcal{L}(T)$ denotes the set of possible increasing labellings for T, and $L(T) = |\mathcal{L}|$ its cardinality. We can then define the EGF of the family by

$$T(z) = \sum_{n\geq 1} T_n \frac{z^n}{n!} \qquad T_n := \sum_{|T|=n} w(T)L(T)$$

Alternatively, simple families of increasing trees can also be describes via the formal recursive equation:

$$\mathcal{T} = \mathbf{1} \times \left(\varphi_0 \cdot \{\epsilon\} \dot\cup \varphi_1 \cdot \mathcal{T} \dot\cup \varphi_2 \cdot \mathcal{T} * \mathcal{T} \dot\cup \varphi_3 \cdot \mathcal{T} * \mathcal{T} * \mathcal{T} \dot\cup \cdots\right) = \mathbf{1} \times \varphi(\mathcal{T}) \quad (4.1)$$

where $\mathbf{1}$ denotes the node labelled with 1, $\times$ the cartesian product,$*$ the partition product for labelled objects ans $\varphi(\mathcal{T})$ the substituted structure.

The three most interesting increasing families are the following:

1. **Recursive trees** are the family of non-plane increasing trees such that all node degrees are allowed. Hence, the degree weight function is:

$$\varphi(t) = \sum_{k\geq 0} \frac{1}{k!} t^k = e^t \quad (4.2)$$

   Solving 4.8 we obtain the EGF

$$T(z) = log\left(\frac{1}{1-z}\right) \text{ and } T_n = (n-1)! \text{ for } n \geq 1 \quad (4.3)$$

2. **Plane-oriented recursive trees** or **Heap ordered trees**are the same as recursive trees, but in the plane case. Thus, the degree weight function is

$$\varphi(t) = \sum_{k\geq 0} t^k = \frac{1}{1-t} \quad (4.4)$$

In this case, 4.8 leads to

$$T(z) = 1 - \sqrt{1-2z} \text{ and } T_n = \frac{(n-1)!}{2^{n-1}}\binom{2n-2}{n-1} \tag{4.5}$$
$$= 1 \cdot 3 \cdot 5 \cdots (2n-3) = (2n-3)!! \text{ for } n \geq 1$$

3. **Binary increasing trees** are plane trees where each node has $0, 1$ or 2 sons, and thus, as we have to differ between left and right sons if $k = 1$, the degree weight function is

$$\varphi(t) = (1+t)^2 \tag{4.6}$$

(in the case of strict binary trees, where only outdegrees 0 and 2 are allowed, the degree-weight function would be $\varphi(t) = 1 + t^2$)

Applying 4.8 we get

$$T(z) = \frac{z}{1-z} \text{ and } T_n = n! \text{ for } n \geq 1 \tag{4.7}$$

Some simple increasing families hold the following property:

Property 4.1.5 (Insertion process). *We consider a family of trees $\mathcal{T}$. For every tree $T' \in \mathcal{T}$ of size $n-1$ with vertices $v_1, \ldots, v_{n-1}$ there exist probabilities $p_{T'}(v_1), \ldots, p_{T'}(v_{n-1})$. By choosing a vertex v_i in a random tree T' of size $n-1$, according to the probabilities $p_{T'}(v_i)$, and attaching a new node with label n to it, we obtain a random tree $T \in \mathcal{T}$ of size n. We say, the family $\mathcal{T}$ can be constructed via an insertion process or a probabilistic rule.*

We call those families *grown simple families of increasing trees.* A rule for these families will be found in 4.1.6, and we will see that the families named above are examples of such grown simple families of increasing trees. The following theorem was stated and proved in [33] and in [34], the proof is omitted here.

Theorem 4.1.6 (Grown simple families of increasing trees). *The following three properties of a simple family of increasing trees $\mathcal{T}$ are equivalent:*

(i) The total weights T_n of trees of size n of $\mathcal{T}$ satisfy the equation

$$\frac{T_{n+1}}{T_n} = c_1 n + c_2$$

with fixed constants c_1, c_2, for all $n \in \mathbb{N}$.

(ii) *Starting with a random increasing tree T of size $n \geq j$ of $\mathcal{T}$ and removing all nodes with labels larger than j we obtain a random increasing tree T' of size j of $\mathcal{T}$.*

(iii) *The family $\mathcal{T}$ can be constructed via an insertion process (resp. a probabilistic growth rule), as discussed in 4.1.5.*

The family $\mathcal{T}$ satisfies these (equivalent) properties and is thus a very simple family of increasing trees if and only if the degree-weight generating function $\varphi(t) = \sum_{k\geq 0} \varphi_k t^k$ is given by one of the following formulas, where c_1, c_2 are the constants appearing in property (i).

$$\begin{aligned}
\textbf{\textit{Case A}}: \varphi(t) &= \varphi_0 e^{\frac{c_1 t}{\varphi_0}}, \ \textit{for } \varphi_0 > 0, c_1 > 0 (\Rightarrow c_2 = 0)\\
\textbf{\textit{Case B}}: \varphi(t) &= \varphi_0 (1 + \frac{c_2 t}{\varphi_0})^d, \ \textit{for } \varphi_0 > 0, c_2 > 0, d := \frac{c_1}{c_2} + 1 \in \{2,3,4,\dots\}\\
\textbf{\textit{Case C}}: \varphi(t) &= \frac{\varphi_0}{(1 + \frac{c_2 t}{\varphi_0})^{-\frac{c_1}{c_2} - 1}}, \ \textit{for } \varphi_0 > 0, 0 < -c_2 < c_1
\end{aligned}$$

REMARK Referring to the families of trees we introduced above, *Recursive trees* are *Case A* for $\varphi_0 = 1$ and $c_1 = 1$; *binary increasing trees* are *Case B* for $\varphi_0 = 1, c_1 = 1, c_2 = 2$ and thus $d = 2$; and *heap ordered trees* are *Case C* for $\varphi_0 = 1, c_1 = 2$ and $c_2 = -1$.
*C*ase B -trees are, more generally said, d-ary increasing trees.

Let $\mathcal{T}$ be a family of increasing trees with the degree-weight function $\varphi(t)$, and let T_n be the total number of trees of size n in the variety. Then we can state the following lemma for the family's exponential generating function:

Lemma 4.1.7. *The EGF of the family of increasing trees defined by $\varphi(t)$*

$$T(z) = \sum_{n=0}^{\infty} T_n \frac{z^n}{n!}$$

fulfills the autonomous first order differential equation

$$T'(z) = \varphi(T(z)), \qquad T(0) = 0 \tag{4.8}$$

Proof. The following proof is based on a proof found in [4]

Forming a forest of l trees corresponds to the EGF $T^l(z)$ (or $T^l(z)/l!$ respectively, if the forest is unordered, illustrating the non-plane case). Adding a node with a minimal label (the root), connecting the trees of a forest with l components, enumerated by $W(z)$, corresponds to the EGF $\int_0^z W(u)du$.

Thus we obtain:

$$T(z) = \int_0^z \left(\sum_{k=0}^{\infty} \varphi_k T^k(u) \right) du$$

and from there we can derive the desired result. □

.

Definition 4.1.8 (Polynomial families)**.** *Let $\mathcal{T}$ be a family of increasing trees. IF $\varphi(t)$ is a function of t^p for some $p \geq 2$, so that $\varphi(t) = \psi(t^p)$ for some power series ψ, we call $\varphi(t)$ periodic, the maximum possible p its period and the according family of increasing trees a polynomial family of increasing trees. Otherwise $\varphi(t)$ is aperiodic and we take $t = 1$.*

For increasing trees, we can also make a statement about the singularities of its generating function:

Theorem 4.1.9. *(cp [4]) Given a degree function $\varphi(t)$ that is polynomial or entire, the dominant real positive singularity of the function $T(z)$, solution to $T'(z) = \varphi(T(z))$ and $T(0) = 0$, is*

$$\rho = \int_0^{\infty} \frac{1}{\varphi(u)} du$$

Further, if $\varphi(t)$ is nonperiodic, then ρ is the only dominant singularity of $T(z)$.

Proof. First we have to reformulate the differential equation of Lemma 4.1.7 and obtain the equivalent equation

$$z = \int_0^{T(z)} \frac{1}{\varphi(t)} dt$$

For t on the positive real axis, $\varphi(t)$ does not vanish and increases with t^2 as $t \to \infty$, as $\varphi_i \geq 0$, $\varphi_0 \neq 0$ and $\varphi_i \neq 0$ for some $i \geq 2$, thus the integral is clearly defined. For any real $0 < y < \infty$, the integral $\int_0^y \frac{1}{\varphi(t)} dt$ is analytic and it's derivative is not equal to 0, therefore it is invertible. Therefore, due to the identity above, $T(z)$ is analytic for all real z with $0 < z < \rho$, but obviously, for $z \to \rho_-$, $T(z) \to \infty$, and therefore ρ is a singularity. Let $z_0 = r_0 e^{is}$ with $r_0 < \rho$. As $T(z)$ has only positive Taylor coefficients, we can use the triangular inequality and get $|T(z_0)| \leq T(r_0)$. Now we can use the following lemma:

Lemma 4.1.10. *With the premises given above, for $s \neq 0$ equality $|T(z_0)| = T(r_0)$ is only possible if $T(z) = z^a T^*(z^p)$ for some integers a and $p \geq 2$, in which case $s = \frac{2m\pi}{p}$.*

This implies that equality is only possible if $\varphi(t)$ is periodic. Thus, in the non-periodic case, $|T(z_0)| < T(r_0)$. We choose a positive real r_1 with $|T(z_0)| = T(r_1)$. As $T(z)$ is increasing on the positive real axis, $r_1 < r_0$. We define a function ψ which fulfills $\psi'(z) = \varphi(\psi(z))$ and $\psi(r_0) = T(r_1)$. The system of differential equations is autonomous (i.e. it is independent on the independent variable, in our case z), and thus $\psi(z)$ and $T(z)$ are related by $\psi(z) = T(z - r_0 + r_1)$. Since $\varphi(t)$ has only non-negative coefficients and $r_1 < r_0$, we have $|T(z)| \leq \psi(|z|)$, and hence $|T(z)| \leq T(|z| - r_0 + r_1$. This induced that $T(z)$ exists along any ray with angle $s \neq 0$, for $|z| < \rho - r_0 + r_1$, and is analytic there.

For periodic $\varphi(t)$ the argument has to be slightly altered but still applies, and the other singularities are to be found at angles $s = \frac{m\pi}{p}$. □

For polynomial φ, we can further determine the following exact formula for $T(z)$, using the expansion of $\frac{1}{\varphi(t)}$ as $t \to \infty$, and integration:

Lemma 4.1.11. *(cp [4]) Let $\varphi(t) = \varphi_0 + \cdots + \varphi_p t^p$ be a polynompial degree weight function with degree $p \geq 2$. Then, in a complex neighborhood of ρ, the solution $T(z)$ of 4.1.7 is of the form*

$$T(z) = \frac{1}{\Delta(z)} H(\Delta(z)) \quad \textit{where } \Delta(z) = \eta(\frac{1-z}{\rho})^{\delta}$$

where

$$\delta = \frac{1}{p-1} \qquad \eta = (\frac{\varphi_p \rho}{\delta})^{\delta}$$

and $H(t) = \sum_{m \geq 0} h_m t^m$ is analytic at $t = 0$,

$$h_0 = 1, \quad h_1 = -\frac{\varphi_{p-1}}{p\varphi_p}, \quad h_2 = -\frac{2p\varphi_p\varphi_{p-2} - (p-1)\varphi_{p-1}^2}{2p(p+1)\varphi_p^2}$$

4.2 The Profile

As in the previous chapter, let $L_l^{(n)}$ be the expected number of nodes at level l of all trees B_n of size n in a family $\mathcal{T}$. (The depth of the root is defined to be 0). For fixed n the sequence $(L_l^{(n)})_{l=0}^n$ describes the *mean profile* of trees in the family. We define the bivariate generating function

$$L(z,u) = \sum_{n\geq 0}\sum_{l\geq 0} L_l^{(n)}\frac{z^n}{n!}u^l$$

For $L(z,u)$ we can show the following theorem:

Theorem 4.2.1. *(cp [4, Theorem 8] The bivariate generating function $L(z,u)$ satisfies*

$$L(z,u) = (T'(z))^u \int_0^z (T'(t))^{1-u} dt \qquad (4.9)$$

Further, let D_n be the height of a random node in a random tree of $\mathcal{T}$ with size n, i.e.,

$$\mathbf{P}(D_n = l) = \frac{L_l^{(n)}}{\sum_{k\geq 0} L_l^{(n)}}$$

For a polynomial variety of degree d, the mean value μ_n and the variance σ_n^2 of D_n satisfy

$$\begin{aligned}\mu_n &= (\delta+1)\log n + \mathcal{O}(1)\\ \sigma_n^2 &= (\delta+1)\log n + \mathcal{O}(1)\end{aligned}$$

and the distribution is asymptotically normal,

$$\frac{D_n - \mu_n}{\sigma_n} \xrightarrow{w} \mathcal{N}(0,1)$$

REMARK We will find limiting distributions for D_n for other families of increasing trees in section 4.4

Proof. We define the level polynomial of the tree by $s(T) := \sum_{v\in V(T)} u^{h(v)}$, $h(v)$ being the height of node v in the tree T. Let us denote by $T' \triangleleft T$ that T' is a subtree of T, that is, one of the trees that remain if we eliminate the root of T. Then $s(T)$ is inductively defined by

$$s(T) = \begin{cases} 1 & \text{if } |T| = 1\\ 1 + u\sum_{T'\triangleleft T} s(T') & \text{otherwise}\end{cases}$$

Thus, the generating polynomial $L^{(n)}(u) = \sum_{l\geq 0} L_l^{(n)} u^l$ behaves like the expectation of $s(T)$. We can use the same line of reasoning as in the proof of Lemma 4.1.7 and thus obtain the equation

$$L(z,u) = T(z) + u\int_0^z L(t,u)\varphi'(T(t))dt$$

which gives the differential equation

$$\frac{\partial}{\partial z}L(z,u) = T'(z) + uL(z,u)\frac{\varphi'(T(z))T'(z)}{\varphi(T(z))}, \quad L(0,u) = 1$$

When integrating the homogeneous equation first, we get the solution

$$e^{u\log T'(z)} = (T'(z))^u.$$

The integral form of $L(z,u)$ is then obtained by the variation-of-parameter method (see, for example, [38, p.28f].

To show Gaussian distribution in case of polynomial φ, we need the identity of $T(z)$ from Lemma 4.1.11, and obtain

$$\log(T'(z)) = (\delta+1)\log\frac{1}{1-\frac{z}{\rho}} + C + \mathcal{O}((1-\frac{z}{\rho})^{2\delta})$$

We can now use the following theorem by Flajolet and Soria (cp [16, Theorem 1]):

Theorem 4.2.2. *Let $\mathcal{P}$ and $\mathcal{C}$ be two classes of combinatorial structures, such that*

$$P(z,u) = e^{uC(z)}.$$

Let Ω_n be the number of components in a random $\mathcal{P}$-structure of size n, with probability distribution

$$\mathbf{P}(\Omega_n = k) = \frac{P_{n,k}}{\sum_l P_{n,l}} \quad \textit{with } P_{n,k} = n![u^k z^n]e^{uC(z)}.$$

If $C(z)$ is a logarithmic function, then Ω_n, once normalized, converges weakly to a limiting Gaussian distribution:

$$\mathbf{P}(a < \frac{\Omega_n - \mu_n}{\sigma_n} < b) \to \frac{1}{2\pi}\int_a^b e^{-\frac{t^2}{2}}dt.$$

The theorem applies here to $(T'(z))^u$, for $L(z,u)$ we have to prove that the integral $\int_0^z (T'(z))^{1-u}$ does not alter the result. The integral is convergent for u in a complex neighborhood of 1 and $|z| \le \rho$ so that it is an unessential perturbation, and thus the theorem can be applied. □

4.3 Node degree

In this section, limiting distributions of the out-degree of nodes in simple families of increasing trees will be given. Important results and proofs are carried out in this thesis. Some calculations are omitted, in these cases exact references are given.

We start by introducing a random variable $X_{n,j}, 1 \leq j \leq n$, which counts the out-degree of the node with label j in a random increasing tree of size n. We will develop a trivariate function $N(z, u, v)$, which is the generating function of the probabilities $\mathbf{P}\{X_{n,j} = m\}$ in a simple family of increasing trees with degree-weight generating function $\varphi(t)$.

For grown simple families we will compute explicit formulas for the probability $\mathbf{P}\{X_{n,j} = m\}$ and for the moments $\mathbf{E}(X_{n,j}^{\underline{s}})$ ($x^{\underline{s}}$ meaning the falling factorials $x(x-1)\cdots(x-l+1)$).

The following part of this work largely follows [26].

We start at the root $j = 1$ and introduce a bivariate generating function for its root-degree:

$$M(z, v) := \sum_{n \geq 1} \sum_{m \geq 0} \mathbf{P}\{X_{n,1} = m\} T_n \frac{z^n}{n!} v^m \tag{4.10}$$

For this function, we can easily show the following lemma:

Lemma 4.3.1. *The bivariate generating function of the root-degree is given by*

$$M(z, v) = \int_0^z \varphi(vT(t))dt \tag{4.11}$$

Proof. The exponential generating function of trees with root degree m is

$$\varphi_m \int_0^z T^m(t)dt$$

according to [4]; and so, clearly, equation 4.11 is true.

□

Now we consider all other nodes with $2 \leq j \leq n$. Suppose the increasing tree of size n has root degree r and its r subtrees have sizes $k_1, \ldots, k_r$ and are enumerated. Further suppose our considered node j lies in the first subtree and is the i-th smallest node there, then we can reduce the computation of $\mathbf{P}\{X_{n,j} = m\}$ to $\mathbf{P}\{X_{k_1,i} = m\}$, and get as a factor the total weight of the

r subtrees and the root node $\varphi_r T_{k_1} \cdots T_{k_r}$ divided by the total weight T_n of trees of size n, multiplied by the number of relabellings, which preserve the order of the r subtrees. We can choose: The $i-1$ labels smaller than j in the leftmost subtree from $2, 3, \ldots, j-1$, the $k_1 - i$ labels larger than j in the same subtree from $j+1, \ldots, n$ and then we distribute the remaining $n-1-k_1$ labels to the other subtrees. This results in:

$$\binom{j-2}{i-1}\binom{n-j}{k_1-i}\binom{n-1-k_1}{k_2, k_3, \ldots, k_r}$$

We now have to sum up over all choices for the rank i of label j in its subtree, over the subtree sizes $k_1, \ldots, k_r$ and over the root degrees r and before that, we have to consider symmetry and include a factor r, if the node j is not in the leftmost, but in the second, third,... subtree.

Finally, we get

$$\begin{aligned} \mathbf{P}\{X_{n,j} = m\} &= \sum_{r \geq 1} r\varphi_r \sum_{\substack{k_1 + \cdots + k_r = n-1, \\ k_1, \ldots, k_r \geq 1}} \frac{T_{k_1} \cdots T_{k_r}}{T_n} \times \\ &\times \sum_{i=1}^{min\{k_1, j-1\}} \mathbf{P}\{X_{k_1,i} = m\} \binom{j-2}{i-1}\binom{n-j}{k_1-i}\binom{n-1-k_1}{k_2, k_3, \ldots, k_r} \end{aligned} \tag{4.12}$$

This recurrence can be expressed via the trivariate generating function

$$N(z,u,v) := \sum_{k \geq 0} \sum_{j \geq 1} \sum_{m \geq 0} \mathbf{P}\{X_{k+j,j} = m\} T_{k+j} \frac{z^{j-1}}{(j-1)!} \frac{u^k}{k!} v^m \tag{4.13}$$

when setting $n := k + j, k \geq 0$. This interpretation is admissible as (4.12) leads to the following differential equation when multiplying with $T_{k+j} \frac{z^{j-2}}{(j-2)!} \frac{u^k}{k!} v^m$ and summing up over $k \geq 0, j \geq 2$ and $m \geq 0$, just as 4.13 does,

$$\frac{\partial}{\partial z} N(z,u,v) = \varphi'(t(z+u)) N(z,u,v) \tag{4.14}$$

and they fulfill the initial condition:

$$N(0,u,v) = \sum_{k \geq 0} \sum_{m \geq 0} \mathbf{P}\{X_{k+1,1} = m\} T_{k+1} \frac{u^k}{k!} v^m = \frac{\partial}{\partial u} M(u,v) = \varphi(vT(u)) \tag{4.15}$$

For the trivariate function $N(z,u,v)$ we can show the following theorem:

Theorem 4.3.2. *The function $N(z,u,v)$, that gives the probability that the node with label j in a randomly chosen tree of size n of a simple family of increasing trees has exactly m sons, is given by the formula:*

$$N(z,u,v) = \frac{\varphi(vT(u))\varphi(T(z+u))}{\varphi(T(u))} \tag{4.16}$$

Proof. A general solution of (4.14) has the following form:

$$N(z,u,v) = C(u,v)e^{(\int_0^z \varphi'(T(t+u))dt)}$$

with some function $C(u,v)$. We now make use of the initial condition (4.15) and get:

$$N(z,u,v) = \varphi'(v(T(u))e^{(\int_0^z \varphi'(T(t+u))dt)} \tag{4.17}$$

.

which leads to the result of the theorem, considering the simplifications of the integral which can be made on the basis of the equation $T'(z) = \varphi(T(z))$:

$$\begin{aligned}\int_0^z \varphi'(T(t+u))udt &= \int_0^z \frac{\varphi'(T(t+u))T'(t+u)}{\varphi(T(t+u))}dt \\ &= \int_{T(u)}^{T(z+u)} (log\varphi(w))'dw = \log\left(\frac{\varphi(T(z+u))}{\varphi(T(u))}\right)\end{aligned}$$

□

We now have a general formula for $\mathbf{P}\{X_{n,j} = m\}$, from this formula we will derive exact results for grown simple increasing families. For these families we obtained exact formulas for $\varphi(t)$ and $T(t)$ at the beginning of this chapter. They are:

$$\begin{aligned}\varphi(t) &= \varphi_0 e^{\frac{c_1 t}{\varphi_0}} \quad T(z) = \frac{\varphi_0}{c_1}\log(\frac{1}{1-c_1 z}) \text{ in Case A} \\ \varphi(t) &= \varphi_0(1+\frac{c_2 t}{\varphi_0})^d \quad T(z) = \frac{\varphi_0}{c_2}(\frac{1}{(1-(1-d)c_2 z)^{\frac{1}{1-d}}} - 1) \text{ Case B} \\ \varphi(t) &= \frac{\varphi_0}{(1+\frac{c_2 t}{\varphi_0})^{-\frac{c_1}{c_2}-1}} \quad T(z) = \frac{\varphi_0}{c_2}(\frac{1}{(1-c_1 z)^{\frac{c_2}{c_1}}} - 1) \text{ Case C}\end{aligned}$$

Inserting these results in the formula named in theorem 4.3.2 in case A results in:

$$\begin{aligned} N(z,u,v) &= \frac{\varphi_0 e^{\frac{c_1}{\varphi_0}\frac{\varphi_0}{c_1} v \log(\frac{1}{1-c_1 u})} \varphi_0 e^{\frac{c_1}{\varphi_0}\frac{\varphi_0}{c_1} \log(\frac{1}{1-c_1(z+u)})}}{\varphi_0 e^{\frac{c_1}{\varphi_0}\frac{\varphi_0}{c_1} \log(\frac{1}{1-c_1 z})}} \\ &= \frac{\varphi_0}{(1-c_1u)^v(1-\frac{c_1 z}{1-c_1 u})} \end{aligned}$$

By extracting coefficients following probabilities are obtained:

$$\begin{aligned} \mathbf{P}\{X_{n,j}=m\} &= \frac{1}{c_1^{n-1}\binom{n-1}{j-1}}[u^{n-j}v^m]\frac{c_1^{j-1}}{(1-c_1u)^{v+j-1}} \\ &= \frac{1}{\binom{n-1}{j-1}}\sum_{k=0}^{n-j}[u^{n-j-k}]\frac{1}{(1-c_1u)^{j-1}}[u^k v^m]\frac{c_1^{j-1}}{(1-c_1u)^v} \end{aligned}$$

This, together with the generating function identity for the Stirling numbers of first kind

$$\sum_{n\geq 0}\sum_{m=0}^{m} s_{n,k}\frac{z^n}{n!}v^m = \frac{1}{(1-z)^v}$$

results in

$$\mathbf{P}\{X_{n,j}=m\} = \frac{1}{\binom{n-1}{j-1}}\sum_{k=m}^{n-j}\binom{n-k-2}{j-2}\frac{s_{k,m}}{k!} \tag{4.18}$$

for $m \geq 1$.

Carrying out similar computations with the explicit formulas for Case B and Case C, following results can be obtained:

Theorem 4.3.3.

$$\mathbf{P}\{X_{n,j}=m\} = \begin{cases} \frac{1}{\binom{n-1}{j-1}}\sum_{k=m}^{n-j}\binom{n-k-2}{j-2}\frac{s_{k,m}}{k!} & \textit{Case A} \\ \binom{d}{m}\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}\frac{\Gamma(n-1+\frac{k}{d-1})\Gamma(j+\frac{1}{d-1})}{\Gamma(j-1+\frac{k}{d-1})\Gamma(n+\frac{1}{d-1})} & \textit{Case B} \\ \binom{m-2-\frac{c_1}{c_2}}{m}\sum_{k=0}^{m}\binom{m}{k}(-1)^k\frac{\Gamma(n-1+k\frac{c_2}{c_1})\Gamma(j+\frac{c_2}{c_1})}{\Gamma(j-1+k\frac{c_2}{c_1})\Gamma(n+\frac{c_2}{c_1})} & \textit{Case C} \end{cases}$$

Theorem 4.3.4. *The s-th factorial moments of the probability distribution discussed are:*

$$\mathbf{E}((X_{n,j})^{\underline{s}}) = \begin{cases} \frac{s!}{\binom{n-1}{j-1}} \sum_{l=0}^{n-j} \binom{n-l-1}{j-1} \frac{s_{l,s}}{l!} \; \textit{Case A} \\ \frac{d^{\underline{s}} \binom{\frac{1}{d-1}+j-1}{j-1} (n+\frac{s}{d-1}-2)^{\underline{n-j}}}{\binom{n-1}{j-1}\binom{n-1+\frac{1}{d-1}}{n-1}(n-j)!} \; \textit{Case B} \\ \frac{\Gamma(s-1-\frac{c_1}{c_2})}{\Gamma(-1-\frac{c_1}{c_2})} \sum_{k=0}^{s} \binom{s}{k} (-1)^k \frac{\Gamma(n-\frac{c_2}{c_1}(s-1-k))\Gamma(j+\frac{c_2}{c_1})}{\Gamma(j-\frac{c_2}{c_1}(s-1-k))\Gamma(n+\frac{c_2}{c_1})} \; \textit{Case C} \end{cases}$$

Proofs of 4.3.4 and Case B and Case C of 4.3.3 work through straightforward calculations and are omitted here. They can be found in [26].

From Theorems 4.3.3 and 4.3.4, a full characterization of the distributional behaviour of $X_{n,j}$ as $n \to \infty$ can be derived, for all cases of grown simple families of increasing trees. The distribution changes depending on the growth of $j = j(n)$ compared to n.

I will state the according theorem and carry out the proof for d-ary increasing trees, i.e. for Case B, as results are very interesting in that case. The distributions for the remaining cases can be found in table 4.1. For the interested reader, the according theorems and proofs can be found in [26], just as the following.

Theorem 4.3.5 (The distribution of node-degrees in d-ary increasing trees)**.** *The limiting distribution of the random variable $X_{n,j}$ in a randomly chosen tree of a grown simple family of increasing trees of Case B of size n, as given by 4.1.6, is, for $n \to \infty$ and depending on the growth of j, given as follows:*

(i) The region for j small: $j = o(n)$:

$$\mathbf{P}\{X_{n,j}\} = d \to 1,$$

e.g. the node-degree converges almost surely, that is, with probability 1, *to the maximal degree d.*

(ii) The central region for $j : j \to \infty$ such that $j = \rho n$,with $0 < \rho < 1$. The random variable $X_{n,j}$ is asymptotically binomially distributed $B(n,p)$ with parameters $n = d$ and $p = 1 - \rho^{\frac{1}{d-1}}$.

$$X_{n,j} \xrightarrow{(d)} X_\rho \text{ with } \mathbf{P}\{X_\rho = m\} = \binom{d}{m} \left(1 - \rho^{\frac{1}{d-1}}\right)^m \left(\rho^{\frac{1}{d-1}}\right)^{d-m}$$

(iii) The region for j large: $l := n - j = o(n)$:

$$\mathbf{P}\{X_{n,j}\} = 0 \to 1,$$

with rate of convergence $\mathcal{O}(\frac{l}{n})$*, e.g. the node-degree converges almost surely towards* 0*.*

Proof. (i) We start with the region for j small, $j = o(n)$, and, with $\Gamma(x) = (x-1)!$, obtain

$$\begin{aligned}
\mathbf{P}\{X_{n,j} = d\} &= \binom{d}{d}\sum_{k=0}^{d-1}\binom{d}{k}(-1)^{d-k}\frac{\Gamma(n-1+\frac{k}{d-1})\Gamma(j+\frac{1}{d-1})}{\Gamma(n+\frac{1}{d-1})\Gamma(j-1+\frac{k}{d-1})} + \\
&\quad \binom{d}{d}\binom{d}{d}(-1)^{d-d}\frac{\Gamma(n+\frac{1}{d-1})\Gamma(j+\frac{1}{d-1})}{\Gamma(n+\frac{1}{d-1})\Gamma(j+\frac{1}{d-1})} \\
&= 1 + \sum_{k=0}^{d-1}\binom{d}{k}(-1)^{d-k}\frac{\binom{j-1+\frac{1}{d-1}}{j-2\frac{k}{d-1}}}{\binom{n-1+\frac{1}{d-1}}{n-2\frac{k}{d-1}}} \qquad (4.19)
\end{aligned}$$

For $0 \le k \le d-1$ we can form the following inequation:

$$\begin{aligned}
\frac{\binom{j-1+\frac{1}{d-1}}{j-2\frac{k}{d-1}}}{\binom{n-1+\frac{1}{d-1}}{n-2\frac{k}{d-1}}} &= \frac{(n-2+\frac{k}{d-1})^{\underline{n-j}}}{(n-1+\frac{1}{d-1})^{\underline{n-j}}} \qquad (4.20) \\
&\le \frac{(n-1)^{\underline{n-j}}}{(n-1+\frac{1}{d-1})^{\underline{n-j}}} = \frac{(n-1)!(j-1+\frac{1}{d-1})!}{(n-1+\frac{1}{d-1})!(j-1)!}
\end{aligned}$$

Splitting the region $j = o(n)$ into two cases $j \le \log(n)$ and $j > \log(n)$ we obtain following results from (4.20):

- For $j \le \log(n)$, for $j \ge 2$ (the first bound also holds for $j = 1$)

$$\frac{(j-1+\frac{1}{d-1})!}{(j-1)!} = \frac{(j-1+\frac{1}{d-1})!j}{j!} \le j \le log(n) = \mathcal{O}(log(n))$$

and

$$\frac{(n-1+\frac{1}{d-1})!}{(n-1)!} = n^{\frac{1}{d-1}}\big(1+\mathcal{O}(\frac{1}{n})\big),$$

and thus, together,

$$\frac{(n-1)!(j-1+\frac{1}{d-1})!}{(n-1+\frac{1}{d-1})!(j-1)!} = \mathcal{O}\big(\frac{log(n)}{n^{\frac{1}{d-1}}}\big).$$

For the probabilities $\mathbf{P}\{X_{n,j} = d\}$ we obtain

$$\sum_{k=0}^{d-1} \binom{d}{k} (-1)^{d-k} \frac{\binom{j-1+\frac{1}{d-1}}{j-2\frac{k}{d-1}}}{\binom{n-1+\frac{1}{d-1}}{n-2\frac{k}{d-1}}} = \mathcal{O}\big(\frac{log(n)}{n^{\frac{1}{d-1}}}\big). \tag{4.21}$$

- For$j > \log(n)$ we use

$$\frac{(j-1+\frac{1}{d-1})!}{(j-1)!} = j^{\frac{1}{d-1}}\big(1 + \mathcal{O}(\frac{1}{j})\big).$$

With this, we obtain

$$\frac{(n-1)!(j-1+\frac{1}{d-1})!}{(n-1+\frac{1}{d-1})!(j-1)!} = (\frac{j}{n})^{\frac{1}{d-1}}\big(1 + \mathcal{O}(\frac{1}{j}) + \mathcal{O}(\frac{1}{n})\big).$$

Thus, for the probabilities $\mathbf{P}\{X_{n,j} = d\}$ we

$$\sum_{k=0}^{d-1} \binom{d}{k} (-1)^{d-k} \frac{\binom{j-1+\frac{1}{d-1}}{j-2\frac{k}{d-1}}}{\binom{n-1+\frac{1}{d-1}}{n-2\frac{k}{d-1}}} = \mathcal{O}\big((\frac{j}{n})^{\frac{1}{d-1}}\big). \tag{4.22}$$

The combination of the two cases with equations (4.21) and (4.22) leads from (4.19) to

$$\mathbf{P}\{X_{n,j}\} = 1 + \mathcal{O}\big((\frac{j}{n})^{\frac{1}{d-1}}\big) + \mathcal{O}\big(\frac{log(n)}{n^{\frac{1}{d-1}}}\big) \to 1 \tag{4.23}$$

So we obtained the desired result and showed that for $n \to \infty$ the outdegree of nodes with small label j converges towards d in a d-ary increasing tree.

(ii) For the region $j \to \infty$ with $j = \rho n$ and $0 < \rho < 1$ we need the help of Stirling's formula for the Gamma-function:

$$\Gamma(x) = \sqrt{\frac{2\pi}{x}}(\frac{x}{e})^x\big(1 + \mathcal{O}(\frac{1}{x})\big) \tag{4.24}$$

and get:

$$\begin{aligned}
\mathbf{P}\{X_{n,j}=m\} &= \binom{d}{m}\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}\frac{\Gamma(n-1+\frac{k}{d-1})\Gamma(\rho n+\frac{1}{d-1})}{\Gamma(\rho n-1+\frac{k}{d-1})\Gamma(n+\frac{1}{d-1})} \\
&= \binom{d}{m}\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}\rho^{1-\frac{k-1}{d-1}}\big(1+\mathcal{O}(\frac{1}{n})\big) \\
&= \binom{d}{m}\rho^{1+\frac{1}{d-1}}\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}\rho^{-\frac{k}{d-1}}) + \mathcal{O}(\frac{1}{n}) \\
&= \binom{d}{m}\rho^{1+\frac{1}{d-1}}(\rho^{-\frac{1}{d-1}}-1)^{m} + \mathcal{O}(\frac{1}{n}) \\
&= \binom{d}{m}\big(1-\rho^{\frac{1}{d-1}}\big)^{m}(\rho^{\frac{1}{d-1}})^{d-m} + \mathcal{O}(\frac{1}{n}) \qquad (4.25)
\end{aligned}$$

which, for $n \to \infty$, obviously is a binomial distribution with parameters d and $1-\rho^{\frac{1}{d-1}}$.

(iii) For the region $l := n - j = o(n)$ we use Stirling's formula 4.24 again and obtain:

$$\begin{aligned}
\mathbf{P}\{X_{n,j}=0\} &= \frac{\Gamma(n-1)\Gamma(j+\frac{1}{d-1})}{\Gamma(j-1)\Gamma(n+\frac{1}{d-1})} \\
&= \frac{\Gamma(n-1)\Gamma(n+\frac{1}{d-1}-l)}{\Gamma(n-1-l)\Gamma(n+\frac{1}{d-1})} = 1+\mathcal{O}(\frac{l}{n}) \qquad (4.26)
\end{aligned}$$

So, we also showed that for very large j, the node degree will be 0 with probability 1 for $n \to \infty$, and thus, we completed the proof of the theorem.

□

The following table contains a full characterization of the limiting distribution of the random variable $X_{n,j}$ for all families of grown simple families of increasing trees, which are: recursive trees (Case A), d-ary increasing trees (Case B) and generalized plane-oriented trees (Case C). We examine all sequences (n, j) and differ on the growth of $j = j(n)$ compared to n.

REMARK The degenerated distribution in the region for j large, $l := n-j = o(n)$ are the same for all three cases, namely this is: $\mathbf{P}\{X_{n,j} = 0\} \to 1$ with rate of convergence $\mathcal{O}(\frac{l}{n})$.

	(Case A)	(Case B)	(Case C)
j fixed	Gaussian	degenerate	characterized via moments
$j \to \infty, j = o(n)$			Gamma
$j \sim \rho n, 0 < \rho < 1$	Poisson	binomial	negative binomial
$l := n - j = o(n)$	degenerate	degenerate	degenerate

Table 4.1: limiting distribution behavior

4.4 The expected level of nodes

Using similar methods as in the previous section, we will derive results for the expected height of node j. For the known families of simple increasing trees, we will evaluate exact limiting distributions.

Therefore, let D_n denote a random variable that counts the level of the node labelled with n, and $D_{n,j}$ denotes the random variable counting the level of node $j \leq n$.

To set up a generating function for $D_{n,j}$, we create special tricolored increasing trees, colored as the following: Exactly one node is colored *white*, which is the node labelled j and thus the node the height of which we are interested in. All nodes having smaller labels than that of j are then colored *black* and all remaining nodes with labels $j+1, \ldots, n$ are colored *blue*. We assume that the white node is not the root, and that the considered tricolored tree has outdegree $r \geq 1$. Further, we assume that the white node is located in the first subtree of the root. Then, the $r-1$ remaining subtrees $B_2, \ldots, B_r$ are only bicolored with j_i blue nodes and k_i black nodes, while the first subtree B_1 is still tricolored with j_1 blue and k_1 black nodes, $2 \leq i \leq r$, $0 \leq j_i \leq |B_i|$, $k_i = |B_i| - j_i$. The subtrees can be relabelled, preserving the order of the nodes, and are thus increasing trees by themselves. There are $\binom{j_1+\cdots+j_r}{j_1,\ldots,j_r}\binom{k_1+\cdots+k_r}{k_1,\ldots,k_r}$ different labellings as the $j_1+\cdots+j_r$ resp $k_1+\cdots+k_r$ labels of the black resp. blue nodes are distributed over the black and blue nodes in $B_1, \ldots, B_r$ in an order-preserving way. To describe the above via generating functions we have to use exponential generating functions, i.e., $f(z,u) = \sum_{j\geq 0}\sum_{k\geq 0} f_{j,k}\frac{z^j}{j!}\frac{u^k}{k!}$and $f(z,u,v) = \sum_{j\geq 0}\sum_{k\geq 0}\sum_{m\geq 0} f_{j,k,m}\frac{z^j}{j!}\frac{u^k}{k!}v^m$, where v marks the height of the white node, for sequences $f_{j,k}, f_{j,k,m}$.

Then the total weight of all suitable trees of the above shape, with j black and k blue nodes, where the white node is located on level m is given by $\mathbf{P}(D_{k+j+1,j+1} = m)T_{k+j+1}$ and thus its generating function is

$$\sum_{j\geq 0}\sum_{k\geq 0}\sum_{m\geq 0} \mathbf{P}(D_{k+j+1,j+1} = m)T_{k+j+1}\frac{z^j}{j!}\frac{u^k}{k!}v^m =: N(z,u,v)$$

and the generating function of all suitable bicolored trees with j black and k blue nodes is

$$\sum_{j\geq 0}\sum_{k\geq 0} T_{k+j}\frac{z^j}{j!}\frac{u^k}{k!} = T(z+u)$$

We now want to put the r subtrees together again and obtain $T(z+u)^{r-1}N(z,u,v)$ and a factor v, as the white node is one level higher in the complete tree than it is in its subtree. Furthermore, the possibility that the white node can also be in the second, third, $\ldots$, r-th subtree leads to a factor r, and, according to 4.1, the fact that the root has degree r includes the factor φ_r. Summing up over all possible root-degrees r leads to

$$\sum_{r\geq 0} r\varphi_r vT(z+u)^{r-1}N(z,u,v) = v\varphi'(T(z+u))N(z,u,v)$$

and, as the node labelled 1 is definitely colored black, equation (4.1) further leads to the following differential equation:

$$\frac{\partial}{\partial z}N(z,u,v) = v\varphi'(T(z+u))N(z,u,v) \tag{4.27}$$

REMARK Equation (4.27) can also be derived using exactly the same arguments as in the derivation of equation (4.12) in the previous section, that is, we suppose the tree has root degree r and its r subtrees have sizes $k_1,\ldots,k_r$, and node j lies in the leftmost subtree, then the computation of $\mathbf{P}(D_{n,j}=m)$ reduces to the computation of $\mathbf{P}(D_{k_1,i}=m-1)$, as the additional level of the root has to be considered. With the same arguments as in Section 4.3, we obtain the same factors and thus obtain for the probability of node j being located at level m

$$\begin{aligned}\mathbf{P}(D_{n,j}=m) &= \sum_{r\geq 1} r\varphi_r \sum_{\substack{k_1+\cdots+k_r=n-1,\\ k_1,\ldots,k_r\geq 1}} \frac{T_{k_1}\cdots T_{k_r}}{T_n}\times \\ &\times \sum_{i=1}^{min\{k_1,j-1\}} \mathbf{P}(D_{k_1,i}=m-1)\binom{j-2}{i-1}\binom{n-j}{k_1-i}\binom{n-1-k_1}{k_2,k_3,\ldots,k_r}\end{aligned} \tag{4.28}$$

Setting $n=k+j$ with $k\geq 0$ and setting up the trivariate generating function

$$N(z,u,v) := \sum_{k\geq 0}\sum_{j\geq 1}\sum_{m\geq 0}\mathbf{P}(D_{k+j,j}=m)T_{k+j}\frac{z^{j-1}}{(j-1)!}\frac{u^k}{k!}v^m$$

also leads to (4.27), when multiplying (4.28) with $T_{k+j}\frac{z^{j-2}}{(j-2)!}\frac{u^k}{k!}v^m$ and summing up over $k \geq 0, j \geq 2$ and $m \geq 0$.

The general solution of (4.27) is given by

$$N(z,u,v) = C(u,v)e^{v\int_0^z \varphi'(T(t+u))dt}$$

with an unknown function $C(u,v)$. But since $\mathbf{P}(D_{n,1} = 1) = 1$

$$\begin{aligned} N(0,u,v) &= \sum_{k\geq 0}\sum m \geq 0\mathbf{P}(D_{k+1,1} = m)T_{k+1}\frac{u^k}{k!}v^m \\ &= \sum_{k\geq 0} T_{k+1}\frac{u^k}{k!} = T'(u) = \varphi(T(u)) \end{aligned}$$

and thus, we obtain

$$N(z,u,v) = \varphi(T(u))e^{v\int_0^z \varphi'(T(t+u))dt}.$$

Again using $T'(z) = \varphi(T(z))$ we get

$$\begin{aligned} N(z,u,v) &= \varphi(T(u))e^{v\int_0^z \frac{\varphi'(T(t+u))T'(t+u)}{\varphi(T(t+u))}dt} \\ &= \varphi(T(u))e^{v\int_0^z \log(\varphi(T(t+u)))'dt} \\ &= \varphi(T(u))e^{v(\log(\varphi(T(z+u)))-\log(\varphi(T(u))))} \end{aligned} \quad (4.29)$$

and thus obtain the following exact formula for the trivariate generating function $N(z,u,v)$

$$N(z,u,v) = \varphi(T(u))\Big(\frac{\varphi(T(z+u))}{\varphi(T(u))}\Big)^v = T'(u)\Big(\frac{T'(z+u)}{T'(u)}\Big)^v. \quad (4.30)$$

Along the same line of reasoning, a bivariate generating function $N(z,v)$ can be derived for the height of the node labelled n, where only black and white nodes are necessary. $N(z,v)$ is then

$$N(z,v) = \varphi_0\Big(\frac{\varphi(T(z))}{\varphi_0}\Big)^v \quad (4.31)$$

We will need that generating function to derive exact distributions in the following.

Results for simple families of increasing trees

Applying the general result obtained above to the generating function of the tree families known earlier in this chapter, we can determine the exact probabilities $\mathbf{P}(D_{n,j} = m)$ for the special cases of recursive, heap ordered and binary trees:

- **recursive trees:** We have $\varphi(t) = e^t, T(z) = \log(\frac{1}{1-z})$, $T_n = (n-1)!$, and thus

$$N(z,u,v) = \frac{1}{1-u}(\frac{1-u}{1-(u+z)})^v = \frac{1}{(1-u)(1-\frac{z}{1-u})^v}$$

Extracting coefficients, using $[x^n]\frac{1}{(1-x)^\alpha} = \binom{\alpha+n-1}{n}$ leads to the probability generating function

$$\begin{aligned}
\sum_{m\geq 0} \mathbf{P}(D_{k+j,j} = m)v^m &= \frac{(j-1)!k!}{T_{k+j}}[z^{j-1}u^k]N(z,u,v) \\
&= \frac{(j-1)!k!}{(k+j-1)!}\binom{v+j-2}{j-1}[u^k]\frac{1}{(1-u)^j} \\
&= \frac{(j-1)!k!}{(k+j-1)!}\binom{v+j-2}{j-1}\underbrace{\binom{k+j-1}{k}}_{=\frac{(j-1)!k!}{(k+j-1)!}} \\
&= \binom{v+j-2}{j-1}
\end{aligned}$$

Thus the probability distribution of $D_{k+j,j}$ is independent of k and is thus equal to D_n. With (4.31)

$$N(z,v) = \frac{1}{(1-z)^v}$$

and thus, with the Stirling number identity $[x^k z^n](1+x)^u = \frac{s(n,k)}{n!}$,

$$\mathbf{P}(D_n = m) = \frac{(n-1)!}{T_n}[z^{n-1}v^m]\frac{1}{(1-z)^v} = \frac{1}{(n-1)!}s(n-1,m)$$

With singularity analysis the distribution of D_n can be shown to be asymptotically Gaussian with

$$\mathbf{E}(D_n) = H_{n-1} \text{ and } \mathbf{V}(D_n) = H_{n-1} - H_{n-1}^{(2)},$$

$H_n^{(m)}$ being the harmonic number $H_n^{(m)} = \sum_{k=1}^{n} \frac{1}{k^m}$.

- **heap ordered trees:** We have $\varphi(t) = \frac{1}{1-t}$, $T(z) = 1 - \sqrt{1-2z}$ and $T_n = \frac{(n-1)!}{2^{n-1}}\binom{2n-2}{n-1}$, and thus

$$N(z,u,v) = \frac{\sqrt{(1-2u)^v}}{\sqrt{1-2u}\sqrt{(1-2(z+u))^v}} = \frac{1}{\sqrt{1-2u}(1-\frac{2z}{1-2u})^{\frac{v}{2}}}$$

and

$$\begin{aligned}\sum_{m\geq 0} \mathbf{P}(D_{k+j,j} = m)v^m &= \frac{(j-1)!k!}{T_{j+k}} 2^{k+j-1} \binom{\frac{v}{2}+j-2}{j-1}\binom{k+j-\frac{3}{2}}{k} \\ &= \frac{(j-1)!k!\binom{\frac{v}{2}+j-2}{j-1}\binom{k+j-\frac{3}{2}}{k}}{(k+j-1)!\binom{k+j-\frac{3}{2}}{k+j-1}} \\ &= \frac{\binom{\frac{v}{2}+j-2}{j-1}}{\binom{j-\frac{3}{2}}{j-1}} = \prod_{i=1}^{j-1} \frac{2i-2+v}{2i-1}\end{aligned}$$

Again we find the distribution of $D_{k+j,j}$ independent of k and thus we can evaluate D_n, which is by (4.31)

$$N(z,v) = \frac{1}{(1-2z)^{\frac{v}{2}}}$$

and

$$\mathbf{P}(D_n = m) = \frac{(n-1)!}{T_n}[z^{n-1}v^m]\frac{1}{(1-2z)^{\frac{v}{2}}} = \frac{2^{n-1-m}}{(2n-3)!!}s(n-1,m)$$

which, again via singularity analysis, can be shown to be Gaussian distributed with

$$\begin{aligned}\mathbf{E}(D_n) &= H_{2n-2} - \frac{1}{2}H_{n-1} \\ \mathbf{V}(D_n) &= H_{2n-2} - \frac{1}{2}H_{n-1} - H_{2n-2}^{(2)} - \frac{1}{4}H_{n-1}^{(2)} \qquad (4.32)\end{aligned}$$

- **binary increasing trees:** We have $\varphi(t) = (1+t)^2$, $T(z) = \frac{z}{1-z}$ and $T_n = n!$. Thus

$$N(z,u,v) = \frac{1}{(1-z)^2} \frac{(1-u)^{2v}}{(1-(u+z))^{2v}} = \frac{1}{(1-u)^2(1-\frac{z}{1-u})^{2v}}$$

and

$$\begin{aligned} \sum_{m\geq 0} \mathbf{P}(D_{k+j,j} = m)v^m &= \frac{(j-1)!k!}{(k+j)!} \binom{2v+j-2}{j-1} \binom{k+j}{k} \\ &= \frac{1}{j} \binom{2v+j-2}{j-1} \end{aligned}$$

Thus, the distribution of $D_{k+j,j}$ is independent of k also in binary increasing trees, the distribution of D_n is

$$\begin{aligned} N(z,v) &= \frac{1}{(1-z)^{2v}} \\ \mathbf{P}(D_n = m) &= \frac{2^m}{n!} s(n-1,m) \end{aligned}$$

which can be shown to be Gaussian as well with

$$\begin{aligned} \mathbf{E}(D_n) &= 2H_n - 2 \\ \mathbf{V}(D_n) &= 2H_n + 2 - 4H_n^{(2)} \end{aligned}$$

Recapitulatory, we find that the depth of node j for fixed j in recursive, heap ordered and binary increasing trees is independent of the size n of the tree and is asymptotically Gaussian distributed. The question arises, whether there are more increasing trees for which the property $D_{n,j} = D_j$ holds. In [33] the answer to this question is given. The solution are all those trees which fulfill a *randomness preserving property*, which means that, starting with a random tree of size n and removing all nodes larger than j, we obtain a random tree of size j. In Theorem 4.1.6 we saw that this property is equivalent to property 4.1.5, and grown simple families are the families which hold the properties.

Chapter 5

Scale Free Graphs and Trees

5.1 The Scale Free Model

The Scale Free model for graphs was considered first by *Albert* and*Barabasi* [1]in 1999. It evolved from the fact that most of today's large real world networks could not be adequately described by the present graph model. The Scale Free Model is based on two simple but novel ideas:

1. *Growth*: Instead of starting with a fixed number of nodes N which are then randomly connected or rewired, we start with a small number (m_0) of vertices. At every step we add a new vertex and connect it with $m(\leq m_0)$ different vertices already present in the system by m new edges.

2. *Preferential attachment*: The likelihood of connecting to an existing node depends on the node's degree. We assume that the probability P_i that a new vertex is connected to vertex i is proportional to the degree k_i of vertex i.

The sense of the considerations named above can easily be illuminated by means of the probably most present real world network: the *world wide web*: The WWW grows by the addition of new web pages, and a webpage will more likely include hyperlinks to popular documents with already high degree than to those with a small number of links.

Numerical simulations indicated that the Scale Free network evolves into a scale-invariant state with the probability that a node has k edges following a power-law distribution, a property which we will show in 5.3.1 for Scale Free Trees. For a general scale free graph G, assume that the probability that a new node is connected to a present node in step $n+1$ is $\frac{\lambda_1 k+\lambda_0}{S_n}$, where

λ_1, λ_0 are nonnegative parameters and S_n is the sum of degrees after n steps. The following theorem holds for G, which is shown in [31].

Theorem 5.1.1. *The asymptotic degree distribution of a Scale Free graph with probability of the new node being connected to a present node of degree k given by the above, is a power law distribution with exponent λ*

$$\lambda = 2 + \sqrt{1 + \frac{2\lambda_0}{\lambda_1^2}}$$

The question arises, whether both growth and preferential attachment are necessary for power law scaling or is just one of the properties enough? The answer to this question is yes, both are necessary. To achieve this answer, we consider two different models, containing only one of the two properties each:

Model A . We eliminate the factor of preferential attachment, while the characteristic of growth is held up. That is, we start with a small number of nodes, m_0, and add a new node with $m \leq m_0$ edges at each timestep n, with equal probability of the new node connecting to a present node i being $\mathbf{P}(i) = \frac{1}{m_0+n+1}$, independent of i. It can then be shown that for $n \to \infty$, the degree distribution decays exponentially, i.e. the probability that a node has degree k is

$$\mathbf{P}(k) = \frac{e}{m} e^{(-\frac{k}{m})}$$

Thus, without preferential attachment, a non-scale-free degree distribution can occur.

Model B . We keep the factor of preferential attachment, but eliminate growth. That is, we start with a fixed number of nodes N and no edges. At each time step a node is selected randomly and connected to a node i in the system with probability $\mathbf{P}(i) = \frac{k_i}{\sum_j k_j}$, where k_i denotes the degree of node i. As N is constant, after $T \simeq N^2$ timesteps all nodes in the system are connected. It can be shown that at the beginning of the system, there is power-law scaling, but $\mathbf{P}(k)$ is not stationary, that is, it can be shown that, at time n, the degrees have reached

$$k_i(n) \simeq \frac{2}{N} n$$

for large n, thus $\mathbf{P}(k)$ becomes Gaussian distributed.

We have now seen that the power law scaling observed in real-world networks can only be guaranteed with the two characteristics *growth* and *preferential attachment* used simultaneously.

5.2 The diameter of a Scale Free Graph

Let $\mathcal{G}_n^{(m)}$ be the family of trees constructed along the model described above with $m \geq 1$ after n steps.

Definition 5.2.1 (Diameter)**.** *Let G be an undirected graph and $u, v \in V_n(G)$. We call $\max_{(u,v)\in V^2}(d(u,v))$ the diameter of the graph G, where $d(u,v)$ is the distance between u and v, i.e. the diameter of a graph is the 'longest shortest path' between two of its nodes. We denote the diameter of graph G by* $\mathrm{diam(G)}$.

In their work, [6], Bollobás and Riordan showed the following property for a Scale free Graph:

Theorem 5.2.2. *For fixed $m \geq 2$ and a positive real number ϵ, almost every graph $G_n^{(m)} \in \mathcal{G}_n^{(m)}$ is connected and has diameter $diam(G_n^{(m)})$ satisfying*

$$\frac{(1-\epsilon)\log n}{\log\log n} \leq \mathrm{diam(G_n^{(m)})} \leq \frac{(1+\epsilon)\log \mathrm{n}}{\log\log \mathrm{n}}$$

REMARK

For $m = 1$ the resulting graph is free of cycles, and thus is called a Scale free tree (see section 5.3), it can be shown that its diameter is asymptotically $\log n$.

5.3 Scale Free Trees

To build a scale free tree, we start with a single edge. At every further step we start one new edge from one of the vertices created so far, the other endpoint of that edge is a new vertex. Adjusted to the scale free model described at the beginning of this chapter, we create a scale free graph with $m_0 = 2$ and $m = 1$.

We can generalize the model by creating a non-decreasing sequence of positive numbers $\{\varphi(k), k \geq 1\}$, the probability of a vertex with degree k being chosen in the n-th step is proportional to $\varphi(k)$, that is, with probability $\varphi(k)/S_n$, where S_n is the sum of $\varphi(k)$ over all vertices of the tree with n edges.

In this thesis we will concentrate on the case where $\varphi(k) = k+\beta$, because then we have S_n non-random: $S_n = 2n + \beta(n+1)$, with $\beta > -1$.

5.3.1 Scale free trees have a power law degree distribution

By using martingales (for definition see Chapter 1) we will show the following theorem:

Theorem 5.3.1. *In a scale free tree, the proportion of vertices of degree k converges almost surely to a limit c_k, which, as a function of k, decreases at the rate $k^{-(3+\beta)}$.*

Proof. We will divide the proof into two parts: First we will determine the value of c_k, assuming its existence, and then we will show the convergence. The following proof is based on T.F.Móri's proof in [30].

NOTATION: After the n-th step, we denote V_n and E_n as the sets of vertices and edges ordered by their time of construction, and $d_{n,i}$ as the degree of vertex i at that time (obviously $d_{0,n} + \cdots + d_{n,n} = 2n$). Now, as mentioned above, we assign the weight $\varphi(k) = k + \beta$ to every vertex and get the sum of all weights $S_n = \varphi(d_{n,0}) + \cdots + \varphi(d_{n,n})$. Let $\mathcal{F}_n = \sigma(e_1, \ldots, e_n)$ be the natural filtration, then $\mathbf{P}(e_{n+1} = (i, n+1) \mid \mathcal{F}) = \varphi(d_{n,i})/S_n$.

Let $a_{n,k}$ be the number of nodes of degree k after the n-th step, and $\mathbf{a}_n = (a_{n,1}, a_{n,2}, \ldots)$, $\mathbf{f}_0 = (1, 0, 0, \ldots)$, and $\mathbf{f}_i = (0, \ldots, 0, -1, 1, 0, \ldots), i \geq 1$, where -1 stands on position i. Finally, let X_n be the degree of the starting point of e_{n+1}. With this notations, we have:

$$\sum_{k\geq 1} a_{n,k} = n + 1 \quad \sum_{k\geq 1} k a_{n,k} = 2n$$

$$\mathbf{P}(X_n = i \mid \mathcal{F}_n) = \frac{\varphi(i) a_{n,i}}{S_n} =: \pi_{n,i}$$

and the recursion

$$\mathbf{a}_1 = 2f_0, \quad a_{n+1} = a_n + f_0 + f_{X_n} \tag{5.1}$$

Inserting $\pi_{n,i}$ in 5.1, we obtain for the coordinates of a_{n+1}:

$$\begin{aligned} \mathbf{E}(a_{n+1,1} \mid \mathcal{F}_n) &= a_{n,1} + 1 - \pi_{n,1} = \Big(1 - \frac{\varphi(1)}{S_n}\Big) a_{n,1} + 1 \\ \mathbf{E}(a_{n+1,i} \mid \mathcal{F}_n) &= a_{n,i} + \pi_{n,i-1} - \pi_{n,i} \\ &= \Big(1 - \frac{\varphi(i)}{S_n}\Big) a_{n,i} + \frac{\varphi(i-1)}{S_n} a_{n,i-1} \end{aligned} \tag{5.2}$$

Now we suppose that, as $n \to \infty$, $\frac{a_{n,k}}{n}$, which is the proportion of vertices with degree k converges to a limit c_k for $k = 1, 2 \ldots$. Then

$$\lim_{n\to\infty} \pi_{n,k} = \frac{\varphi(k)c_k}{S}, \quad S = \lim_{n\to\infty} \frac{S_n}{n} = \sum_{i=1}^{\infty} \varphi(i)c_i$$

We can now apply the law of large numbers (see for example [27]) known in probability theory, and get:

$$c_1 = \frac{S}{S+\varphi(1)}, \quad c_i = c_{i-1}\frac{\varphi(i-1)}{S+\varphi(i)}.$$

and further

$$c_i = \prod_{j=1}^{i-1} \frac{\varphi(j)}{S+\varphi(j)} \cdot \frac{S}{S+\varphi(i)} = \prod_{j=1}^{i-1} \frac{\varphi(j)}{S+\varphi(j)} - \prod_{j=1}^{i} \frac{\varphi(j)}{S+\varphi(j)}. \tag{5.3}$$

Using this result in our formula for S, together with the fact that $\sum_{i=1}^{\infty} c_i = 1$ whenever $\sum_{i=1}^{\infty} \varphi(i)^{-1} = \infty$ we get:

$$S = \sum_{i=1}^{\infty} \varphi(i)c_i = S\sum_{i=1}^{\infty}\prod_{j=1}^{i} \frac{\varphi(j)}{S+\varphi(j)}$$

which allows us to derive S from the following equation:

$$\sum_{i=1}^{\infty}\prod_{j=1}^{i} \frac{\varphi(j)}{S+\varphi(j)} = 1 \tag{5.4}$$

The i-th term of this series is $c_{i+1} + c_{i+2} + \ldots$. Therefore (5.4) is equal to $\sum_{i=1}^{\infty} ic_i = 2$. That means, in our considerations, where $\varphi(i) = i + \beta$, that $S = \sum_{i=1}^{\infty} ic_i + \beta\sum_{i=1}^{\infty} c_i = 2 + \beta$. Further, using B- and Γ-functions, for c_i we get:

$$\begin{aligned} c_i \quad &= \quad \frac{2+\beta}{i+\beta}\prod_{j=1}^{i}\frac{j+\beta}{j+2+2\beta} = \frac{2+\beta}{i+\beta}\frac{\binom{i+\beta}{i}}{\binom{i+2+2\beta}{i}} \sim \\ &\sim (2+\beta)e^{-\beta}\frac{\Gamma(2\beta+3)}{\Gamma(\beta+1)}\frac{1}{i^{\beta+3}} \qquad (i\to\infty). \end{aligned} \tag{5.5}$$

The first part of the proof is now finished, it is shown that if the proportion of vertices with degree k converge to a limit c_k, then this value decreases at the rate $k^{-(3+\beta)}$. We will now use martingales to show that, in fact, c_k always exists.

We will start introducing the centered variables $b_{n,i} = a_{n,i} - \frac{S_n}{2+\beta}c_i$, where $S_n = 2n + \beta(n+1)$, and define

$$\begin{aligned} q_{n,i} &:= \prod_{k=i}^{n-1}\left(1 - \frac{i+\beta}{S_k}\right) = \prod_{k=i}^{n-1}\frac{k - \dfrac{i}{2+\beta}}{k + \dfrac{\beta}{2+\beta}} = \\ &= \frac{\Gamma\left(n - \dfrac{i}{2+\beta}\right)\Gamma\left(i + \dfrac{\beta}{2+\beta}\right)}{\Gamma\left(n + \dfrac{\beta}{2+\beta}\right)\Gamma\left(\dfrac{1+\beta}{2+\beta}i\right)} \sim \frac{\Gamma\left(i + \dfrac{\beta}{2+\beta}\right)}{\Gamma\left(\dfrac{1+\beta}{2+\beta}i\right)} \cdot n^{-\frac{i+\beta}{2+\beta}} \qquad (5.6) \end{aligned}$$

Now we substitute $a_{n+1,i}$ for $b_{n+1,i}$ in 5.2 and get

$$\begin{aligned} \mathbf{E}(b_{n+1,1} \mid \mathcal{F}_n) &= \big(1 - \frac{1+\beta}{S_n}\big)\big(b_{n,1} + \frac{S_n}{3+2\beta}\big) + 1 - \frac{S-n+1}{3+2\beta} \\ &= \big(1 - \frac{1+\beta}{S_n}\big)b_{n,1} \qquad (5.7) \end{aligned}$$

and, for $i > 1$

$$\begin{aligned} \mathbf{E}(b_{n+1,i} \mid \mathcal{F}_n) &= \big(1 - \frac{i+\beta}{S_n}\big)\big(b_{n,i} + \frac{S_n}{2+\beta}c_i\big) + \\ &\quad + \frac{i-1+\beta}{S_n}\big(b_{n,i-1} + \frac{S_n}{2+\beta}c_{i-1}\big) - \frac{S_{n+1}}{2+\beta}c_i \\ &= \big(1 - \frac{i+\beta}{S_n}\big)b_{n,i} + \frac{i-1+\beta}{S_n}b_{n,i-1} \qquad (5.8) \end{aligned}$$

By $q_{n+1,i} = q_{n,i} \cdot (1 - \frac{i+\beta}{S_n})$ and (5.7) it is obvious that $(\frac{b_{n,1}}{q_{n,1}}, \mathcal{F}_n)$ is a martingale, through straightforward calculations we can also verify the general statement:

Lemma 5.3.2. *For every $i = 1, 2, \ldots$ the sequence*

$$Z_n^{(i)} := \frac{1}{q_{n,i}} \sum_{j=0}^{i-1} (-1)^j \binom{i-1+\beta}{j} b_{n,i-j}, \quad n \geq i$$

is a martingale with respect to the filtration $\mathcal{F}_n$.

Proof. Let $b_{n,0} = 0$ (which makes (5.7) a special case of (5.8)), and let us apply (5.8):

$$\begin{aligned}
&\mathbf{E}\left(\sum_{j=0}^{i-1} (-1)^j \binom{i-1+\beta}{j} b_{n+1,i-j} \mid \mathcal{F}_n \right) = \\
&= \sum_{j=0}^{i-1} (-1)^j \binom{i-1+\beta}{j} \left[(1 - \frac{i-j+\beta}{S_n}) b_{n,i-j} + \frac{i-j-1+\beta}{S_n} b_{n,i-j-1} \right] \\
&= \sum_{j=0}^{i-1} (-1)^j b_{n,i-j} \underbrace{\left[(1 - \frac{i-j+\beta}{S_n}) \binom{i-1+\beta}{j} - \frac{i-j+\beta}{S_n} \binom{i-1+\beta}{j-1} \right]}_{\binom{i-1+\beta}{j} \left(1 - \frac{i+\beta}{S_n}\right)} \\
&= \frac{q_{n+1,i}}{q_{n,i}} \sum_{j=0}^{i-1} (-1)^j \binom{i-1+\beta}{j} b_{n,i-j}
\end{aligned}$$

□

We can also express $Z_n^{(i)}$ directly in terms of $a_{n,i}$ in the following way:

$$Z_n^{(i)} = \frac{1}{q_{n,i}} \left(\sum_{j=0}^{i-1} (-1)^j \binom{i-1+\beta}{j} a_{n,i-j} + (-1)^i \binom{i-1+\beta}{i-1} \frac{S_n}{i+2+2\beta} \right) \tag{5.9}$$

because

$$\begin{aligned}
&\sum_{j=0}^{i-1}(-1)^{j-1}\binom{i-1+\beta}{j}\frac{S_n}{2+\beta}c_{i-j} = \\
&= S_n\frac{(1+\beta)(2+\beta)\cdots(i-1+\beta)}{(3+2\beta)(4+2\beta)\cdots(i+2+2\beta)}\sum_{j=0}^{i-1}(-1)^j\binom{i+2+2\beta}{j} \\
&= S_n\frac{(1+\beta)(2+\beta)\cdots(i-1+\beta)}{(3+2\beta)(4+2\beta)\cdots(i+2+2\beta)}(-1)^{i-1}\binom{i+1+2\beta}{i-1} \\
&= (-1)^{i-1}\binom{i-1+\beta}{i-1}\frac{S_n}{i+2+2\beta}
\end{aligned}$$

Lemma 5.3.3. *The variables $b_{n,i}$ can be expressed in terms of the martingales $Z_n^{(i)}$:*

$$b_{n,i} = \sum_{j=1}^{i}\binom{i-1+\beta}{i-j}q_{n,i}Z_n^(i). \tag{5.10}$$

Proof. With the help of the following identity:

$$q_{n,i}Z_n^i = \sum_{j=1}^{i}\binom{-j-\beta}{i-j}b_{n,j}$$

we can apply the Vandermonde convolution formula to the right-hand-side of (5.10) as follows:

$$\begin{aligned}
\sum_{j=1}^{i}\binom{i-1+\beta}{i-j}q_{n,j}Z_n^{(j)} &= \sum_{j=1}^{i}\binom{i-1+\beta}{i-j}\sum_{k=1}^{j}\binom{-k-\beta}{j-k}b_{n,k} \\
&= \sum_{k=1}^{i}b_{n,k}\sum_{j=k}^{i}\binom{i-1+\beta}{i-j}\binom{-k-\beta}{j-k} \\
&= \sum_{k=1}^{i}b_{n,k}\binom{i-1-k}{i-k} = b_{n,i}
\end{aligned}$$

□

We arrive at the final step of our proof and will show by means of the present knowledge, that, with probability 1,

$$\lim_{n\to\infty}\frac{a_{n,i}}{n} = c_i \quad i = 1, 2, \ldots$$

Let $I(P)$ be the indicator function of the event P. Recursion (5.1) and equation (5.9) imply

$$\sum_{j=1}^{i}(-1)^j\binom{i-1+\beta}{i-j}a_{n+1,j}=\sum_{j=1}^{i}(-1)^j\binom{i-1+\beta}{i-j}a_{n,j}-\binom{i-1+\beta}{i-1}-$$

$$-(-1)^{X_n}\Big[\binom{i-1+\beta}{i-X_n}I(X_n\le i)+\binom{i-1+\beta}{i-X_n-1}I(X_n<i)\Big]$$

$$=\sum_{j=1}^{i}(-1)^j\binom{i-1+\beta}{i-j}a_{n,j}-\binom{i-1+\beta}{i-1}-(-1)^{X_n}\binom{i+\beta}{i-X_n}I(X_n\le i)$$

Therefore

$$\begin{aligned}\mathbf{var}(q_{n+1,i}Z_{n+1}^{(i)}\mid\mathcal{F}_n) &= \mathbf{var}\Big(\sum_{j=1}^{i}(-1)^j\binom{i-1+\beta}{i-j}a_{n+1,j}\mid\mathcal{F}_n\Big)\\ &= \mathbf{var}\Big((-1)^{X_n}\binom{i+\beta}{i-X_n}I(X_n\le i)\mid\mathcal{F}_n\Big)\\ &= \sum_{j=1}^{i}\binom{i+\beta}{i-j}^2\pi_{n,j}-\Big(\sum_{j=1}^{i}(-1)^j\binom{i+\beta}{i-j}\pi_{n,j}\Big)^2\end{aligned}$$

The sum of the probabilities $\pi_{n,j}$ does not exceed 1, so

$$\mathbf{var}(q_{n+1,i}Z_{n+1}^{(i)}\mid\mathcal{F}_n)\le\max_{1\le j\le i}\binom{i+\beta}{i-j}^2$$

and with this result, we obtain the estimate

$$\begin{aligned}\mathbf{var}(Z_n^{(i)}) &= \mathbf{E}\Big(\sum_{m=1}^{n}\mathbf{var}(Z_m^{(i)}\mid\mathcal{F}_{m-1})\Big)\\ &= \mathcal{O}\Big(\sum_{m=1}^{n}q_{m,i}^{-2}\Big)=\mathcal{O}\big(n^{1+2\frac{i+\beta}{2+\beta}}\big)\end{aligned}\qquad(5.11)$$

Hence, using the estimate for $q_{n,i}$ in (5.6), $\mathbf{var}(q_{n,i}Z_n^{(i)})=\mathcal{O}(n)$ as $n\to\infty$. Through (5.10) this implies that $\mathbf{E}b_{n,i}^2=\mathcal{O}(n)$. This further means that the series

$$\sum_{n=1}^{\infty}\mathbf{E}(n^{-2}b_{n^2,i})^2$$

is convergent. Thus with probability 1

$$\lim_{n\to\infty} \frac{a_{n^2,i}}{n^2} = c_i.$$

For $n^2 \leq m < (n+1)^2$, we can write $|a_{m,i} - a_{n^2,i}| \leq |m - n^2| \leq 2n$ and $a_{m,i} \leq 2m$, therefore

$$\Big|\frac{a_{n^2,i}}{n^2} - \frac{a_{m,i}}{m}\Big| \leq \frac{a_{m,i}}{m}\Big(\frac{m}{n^2} - 1\Big) + \frac{|a_{m,i} - a_{n^2,i}|}{n^2} \leq \frac{6}{n}$$

Finally, we achieved our goal and showed the property in Theorem 5.3.1. □

Móri further posed the question whether the degree distribution is the same on all levels. The answer to this question is yes on sufficiently high levels, as we will see in the following theorem stated and proved in [25], while on lower levels the degree distribution is still power law, but with exponent -2, independent of the parameter β and the level l. Sufficiently high levels are those which contain most of the vertices, as we will see in the following section those are located around $l = \frac{1+\beta}{2+\beta}\log n$.

Theorem 5.3.4. *With any constants* $0 < l_1 < l_2$*, for* $l_1\sqrt{\log n} < l - \frac{1+\beta}{2+\beta}\log n < l_2\sqrt{\log n}$ *the ratio of vertices with degree* k *converges almost surely to a limit* c_k *on level* l *and* c_k *is equal to the limit of the ratio of* k*-degree vertices in the whole tree.*

Proof. The proof of the theorem runs along similar lines as the derivation of the width of the tree, which will be conducted in 5.3.2, thus we will only present the general idea of this proof.

The generating function used is

$$G^{(n)}_{\geq k}(z) = \sum_{l\geq 0} X_{\geq k}[n, l+1]z^l \tag{5.12}$$

where $X_{\geq k}[n, l+1]$ is the number of nodes of degree at least k after n steps on level $l+1$. It can then be shown that

$$\begin{aligned}
\mathbf{E}(G^{(n+1)}(z)|\mathcal{F}_n) &= \frac{2n+1+z}{2n}G^{(n)}(z) \text{ for k=1} \\
\mathbf{E}(G^{(n+1)}_{\geq k+1}(z)|\mathcal{F}_n) &= \frac{2n-k}{2n}G^{(n)}_{\geq k+1}(z) + \frac{k}{2n}G^{(n)}_{\geq k}(z)
\end{aligned}$$

and that

$$\frac{G^{(n)}(z)}{\mathbf{E}(G^{(n)}(z))}$$

and

$$W_k^{(n)}(z)\prod_{i=1}^{n}\frac{2i}{2i+1-k}$$

for $k \geq 2$ are martingales with respect to the filtration $\mathcal{F}_n$ for any fixed $z \in \mathbb{C}$, where $W_k^{(n)}(z)$ is a linear combination of functions $U_k^{(n)}(z)$, which are given by

$$U_k^{(n)}(z) = G_{\geq k}^{(n)}(z) - c_k(z)G^{(n)}(z)$$

With similar arguments as in Theorem 5.3.2, uniform convergence of the martingales can be shown and via integration the theorem can be proved, cp [25]. □

5.3.2 The width of a Scale Free tree

In this section we will determine the width of a scale free tree, as $n \to \infty$, a parameter we already got to know in chapter 3, along with the level where it occurs, again using the theory of martingales. Therefore, let us denote by $L_l^{(n)}$ the number of nodes at the l-th level after the n-th step, and by $W_n := \max(L_l^{(n)}|l \geq 1)$ the width of the tree in question. As mentioned in the beginning of this chapter, we consider only weight functions $\varphi(k) = k + \beta, b > -1$, where k is the degree of a node, and set

$$\alpha = \frac{1+\beta}{2+\beta}.$$

We start by defining a new sequence to count the weight on level l. That is:

$$\begin{aligned} Y_0^{(n)} &= L_1^{(n)} + \beta \\ Y_l^{(n)} &= L_{l+1}^{(n)} + (1+\beta)L_l^{(n)} \end{aligned}$$

as the sum over all degrees on level l is $L_{l+1}^{(n)} + L_l^{(n)}$.

We now introduce a series of generating functions

$$G^{(n)}(z) = \sum_{k \geq 0} Y_l^{(n)} z^l \tag{5.13}$$

For this function, we can show that:

Lemma 5.3.5. *For any fixed $z \in \mathbb{C}$, and for*

$$E^{(n)}(z) = \prod_{j=1}^{n-1} \frac{s_j + 1 + (1+\beta)z}{s_j}$$

the sequence

$$M^{(n)}(z) := \frac{G^{(n)}(z)}{E^{(n)}(z)}$$

is a martingale with respect to the filtration $\mathcal{F}^{(n)}$, where $\mathcal{F}^{(n)}$ denotes the natural σ-field generated by the first n steps of $G^{(n)}(z)$.

Proof. Calculating the expectations of $Y_l^{(n)}$ we get

$$\begin{aligned}
\mathbf{E}(Y_0^{(n+1)}|\mathcal{F}^{(n)}) &= \frac{Y_0^{(n)}}{s_n}(Y_0^{(n)} + 1) + (1 - \frac{Y_0^{(n)}}{s_n})Y_0^{(n)} \\
&= Y_0^{(n)} \frac{s_n + 1}{s_n}
\end{aligned}$$

and for $k > 0$

$$\begin{aligned}
\mathbf{E}(Y_l^{(n+1)}|\mathcal{F}^{(n)}) &= (\frac{Y_l^{(n)}}{s_n} + \frac{Y_{l-1}^{(n)}(1+\beta)}{s_n})(Y_l^{(n)} + 1) \\
&\quad + Y_l^{(n)}(1 - \frac{Y_l^{(n)}}{s_n} - \frac{Y_{l-1}^{(n)}(1+\beta)}{s_n} \\
&= Y_l^{(n)} \frac{s_n + 1}{s_n} + Y_{l-1}^{(n)} \frac{1+\beta}{s_n}
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\mathbf{E}(G^{(n+1)}(z)|\mathcal{F}^{(n)}) &= \frac{s_n + 1}{s_n} G^{(n)}(z) + \frac{1+\beta}{s_n} z G^{(n)}(z) \\
&= \frac{s_n + 1 + (1+\beta)z}{s_n} G^{(n)}(z)
\end{aligned}$$

and hence, since $G^{(1)}(z) = (1+\beta)(1+z)$, we obtain

$$\begin{aligned}\mathbf{E}(G^{(n)})(z) &= (1+\beta)(1+z)\prod_{j=1}^{n-1}\frac{s_j+1+(1+\beta)z}{s_j} \\ &= (1+\beta)(1+z)E^{(n)}(z)\end{aligned}$$

Hence, $M^{(n)}(z)$ is a martingale. □

With the help of this result, we can state the following lemma on the asymptotics of the expectation:

Lemma 5.3.6. *For any compact set of complex numbers $C \subset \mathbb{C}$*

$$\begin{aligned}\mathbf{E}(G^{(n)}(z)) &= \frac{n^{1+\alpha(z-1)}(1+\beta)(1+z)\Gamma(2\alpha)}{\Gamma(1+\alpha(1+z))} + \mathcal{O}(n^{\alpha\Re(z-1)}) \\ E^{(n)} &= \frac{n^{1+\alpha(z-1)}\Gamma(2\alpha)}{\Gamma(1+\alpha(1+z))} + \mathcal{O}(n^{\alpha\Re(z-1)})\end{aligned}$$

uniformly for $z \in C$ as $n \to \infty$.

Proof. With $s_n = (2+\beta)n+\beta)$ and with the information obtained above, we have

$$\begin{aligned}\mathbf{E}(G^{(n)})(z) &= (1+\beta)(1+z)\prod_{j=1}^{n-1}\frac{s_j+1+(1+\beta)z}{s_j} \\ &= (1+\beta)(1+z)\prod_{j=1}^{n-1}\frac{j+\alpha(1+z)}{j+2\alpha-1} \\ &= (1+\beta)(1+z)\frac{\Gamma(n+\alpha(1+z))}{\Gamma(1+\alpha(1+z))}\frac{\Gamma(2\alpha)}{\Gamma(n+2\alpha-1)}\end{aligned}$$

We now use the fact that, over any compact set,

$$\frac{\Gamma(n'+z')}{\Gamma(n')} = (n')^{z'} + \mathcal{O}(n'^{\Re(z'-1)})$$

for $n' = n+2\alpha-1$ and $z' = \alpha(z-1)+1$, and hence obtain the required result as $n \to \infty$. □

For the study of the convergence of $M^{(n)}(z)$, we first need to determine the covariance function of $G^{(n)}(z)$:

Lemma 5.3.7. *For every pair* $z_1, z_2 \in \mathbb{C}$

$$\begin{aligned}
\mathbf{C}_G^{(n+1)}(z_1,z_2) &:= \mathbf{E}(G^{(n+1)}(z_1)G^{(n+1)}(z_2)) \\
&= \sum_{j=1}^{n}\big(b_j(z_1,z_2)\prod_{k=j+1}^{n} a_k(z_1,z_2)\big) \\
&\quad +(1+\beta)^2(1+z_1)(1+z_2)\prod_{j=1}^{n} a_j(z_1,z_2)
\end{aligned}$$

with

$$\begin{aligned}
a_k(z_1,z_2) &= 1+\frac{2+(1+\beta)(z_1+z_2)}{s_k} \\
b_k(z_1,z_2) &= \frac{(1+z_1+z_1\beta)(1+z_2+z_2\beta)}{s_k}\mathbf{E}(G^{(k)}(z_1,z_2)) \qquad (5.14)
\end{aligned}$$

Proof. Inserting a node new node in step $n+1$, the weight of the level above this new node, denoted by l_n is increased by one, while the weight of the level of the new node itself (l_n+1) is increased by $(1+\beta)$. Thus, $G^{(n+1)}(z) - G^{(n)}(z) = z^{l_n}(1+z(1+\beta))$. Hence,

$$\begin{aligned}
& C_G^{(n+1)}(z_1,z_2) \\
= \ & \mathbf{E}\big[\mathbf{E}\big((G^{(n)}(z_1)+z_1^{l_n}(1+z_1+\beta z_1))(G_G^{(n)}(z_2)+z_2^{l_n}(1+z_2+\beta z_2))|\mathcal{F}_n\big)\big] \\
= \ & C_G^{(n)}(z_1,z_2)+\mathbf{E}\big[\mathbf{E}\big(G^{(n)}(z_1)z_2^{l_n}(1+z_2+z_2\beta)+z_1^{l_n}(1+z_1+z_1\beta)G^{(n)}(z_2) \\
& +z_1^{l_n}z_2^{l_n}(1+z_1+z_1\beta)(1+z_2+z_2\beta)|\mathcal{F}_n\big)\big]
\end{aligned}$$

The conditional distribution of the level l_n with respect to $\mathcal{F}_n$ is

$$\mathbf{P}(l_n = l|\mathcal{F}_n) = \begin{cases} \frac{Y_l^{(n)}}{s_n} & \text{if } l>0, \\ \frac{Y_0^{(n)}}{s_n} & \text{if } l=0. \end{cases}$$

Hence the conditional expectations are

$$\begin{aligned}
\mathbf{E}\big(G^{(n)}(z_1)z_2^{l_n}(1+z_2+z_2\beta)|\mathcal{F}_n\big) &= \frac{1+z_2+z_2\beta}{s_n}G^{(n)}(z_1)G^{(n)}(z_2) \\
\mathbf{E}\big(G^{(n)}(z_2)z_1^{l_n}(1+z_1+z_1\beta)|\mathcal{F}_n\big) &= \frac{1+z_1+z_1\beta}{s_n}G^{(n)}(z_1)G^{(n)}(z_2) \\
\mathbf{E}\big(z_1^{l_n}z_2^{l_n}(1+z_1+z_1\beta)(1+z_2+z_2\beta)|\mathcal{F}_n\big) &= \frac{(1+z_1+z_1\beta)(1+z_2+z_2\beta)}{s_n}G^{(n)}(z_1z_2)
\end{aligned}$$

And thus

$$\begin{aligned} C_G^{(n+1)}(z_1,z_2) &= \left(1+\frac{2+(1+\beta)(z_1+z_2)}{s_n}\right)C_G^{(n)}(z_1,z_2) \\ &\quad + \frac{(1+z_1+z_1\beta)(1+z_2+z_2\beta)}{s_n}\mathbf{E}(G^{(n)}(z_1z_2)) \end{aligned}$$

With $C_G^{(1)}(z_1,z_2) = (1+\beta)^2(1+z_1)(1+z_2)$ this implies the lemma through induction. □

With the information obtained so far, together with some known results, convergence of $M^{(n)}(z)$ can be shown via approximations and some helpful simplifications, such as

$$\prod_{i=1}^{n}\frac{i+v}{i+w} = n^{\Re(v-w)}\left(\frac{\Gamma(1+w)}{\Gamma(1+v)} + \mathcal{O}(\frac{1}{n})\right)$$

The according results are stated here, proofs and necessary references to other literature can be found in [24].

Corollary 5.3.8. *The set of martingales $\{M^{(n)}(z) : n \in \mathbb{N}\}$ is bounded in L^2 for any fixed $|z-1| < \sqrt{\frac{1}{\alpha}}$, where L^p is the space of p-power integrable functions. Thus there exists a random variable $M(z) \in L^2$ such that $M^{(n)}(z) \to M(z)$ a.s. in l^2, as $n \to \infty$, for $z \in \mathcal{H} := \{w \in \mathbb{C} \big| |w-1| < \sqrt{\frac{1}{\alpha}}\}$.*

Corollary 5.3.9. *The martingale $M^{(n)}(z)$ and all its derivatives converge uniformly on all compact subsets of $\mathcal{H}$.*

Secondly, we need some more information on the asymptotics of $G^{(n)}(z)$ in order to calculate $L_l^{(n)}$.

Lemma 5.3.10. *For every $\delta > 0$ and z such that $|z-1| \le \sqrt{\frac{1}{\alpha}} - \delta$,*

$$\mathbf{E}(|G^{(n)}(z)|^2) = \mathcal{O}(n^{2(1+\alpha(\Re z-1))}).$$

For any z such that $\sqrt{\frac{1}{\alpha}} - \delta \le |z-1| \le \sqrt{\frac{1}{\alpha}}$, we obtain

$$\mathbf{E}(|G^{(n)}(z)|^2) = \mathcal{O}(n^{2(1+\alpha(\Re z-1))}\log n),$$

with uniform error terms as $n \to \infty$. Furthermore, for any compact $C \subseteq \mathbb{C} - \mathcal{H}$, we obtain

$$\mathbf{E}(|G^{(n)}(z)|^2) = \mathcal{O}(n^{1+\alpha(|z|^2-1)}\log n)$$

uniformly for $z \in C$.

Lemma 5.3.11. *For every* $0 < |z| < 2$*, we have*

$$|G'^{(n)}(z)| = \mathcal{O}(\frac{1}{|z|}(\log n)n^{(1-\alpha)(1+|z|+|z|\beta)}) \ a.s.$$

Lemma 5.3.12. *We have to distinguish*

Case A $\beta \neq 0$*. For any* $K > 0$*, there exists a* $\delta > 0$ *such that*

$$\sup_{|z|=1, |z-1| \geq \sqrt{\frac{1}{\alpha}}-\delta} |G^{(n)}(z)| = \mathcal{O}(\frac{n}{(\log n)^K})$$

a.s.,as $n \to \infty$*.*

Case B If $\beta = 0$*, then the above holds for the function* $\frac{|G^{(n)}(z)|}{|1+z|}$ *on*

$$\gamma(\delta) := \{z \big| |z| = 1, |z-1| \geq \sqrt{2}-\delta, \Re z > -0.9\} \cup \{z \big| \Re z = -0.9, |z| \leq 1\}$$

For any $K > 0$*, there exists a* $\delta > 0$ *such that*

$$\sup_{\gamma(\delta)} \left| \frac{G^{(n)}(z)}{1+z} \right| = = \left(\frac{n}{(\log n)^K} \right)$$

a.s.,as $n \to \infty$*.*

Width of a Scale Free tree

Theorem 5.3.13 (Width of a Scale Free Tree)**.** *With probability* 1*, the size of level* l *of a Scale free tree after the* n*-th step is*

$$L_l^{(n)} = \frac{n}{\sqrt{2\alpha\pi \log n}} e^{\left(-\frac{(l-\alpha \log n)^2}{2\alpha \log n} \right)} + \mathcal{O}(\frac{n}{\log n}) \qquad (5.15)$$

as $n \to \infty$*, where the error term is uniform for all* $l \geq 0$*, and the width of the tree is*

$$W^{(n)} = \frac{n}{\sqrt{2\alpha\pi \log n}} (1 + \mathcal{O}(\frac{1}{\sqrt{\log n}})) \qquad (5.16)$$

almost surely as $n \to \infty$*, and is reached approximately at a level of* $\alpha \log n$

Proof. By definition,

$$G^{(n)}(z) = \sum_{l\geq 0} Y_l^{(n)} z^l = L_1^{(n)} + \beta \quad + \quad ((1+\beta)L_1^{(n)} + L_2^{(n)})z$$
$$+ \quad ((1+\beta)L_2^{(n)} + L_3^{(n)})z^2 + \cdots$$

and therefore

$$\frac{G^{(n)}(z) - \beta}{1 + (1+\beta)z} = \sum_{l\geq 0} L_{l+1}^{(n)} z^l$$

if $z \neq \frac{-1}{1+\beta}$. This exception does not matter if $b \neq 0$, because then $|\frac{1}{1+\beta}| < 1$ and the function can be extended to this point regularly. We can use Cauchy's formula to extract $L_l^{(n)}$:

- If $\beta \neq 0$ then

$$\begin{aligned} L_{l+1}^{(n)} &= \frac{1}{2\pi i}\int_{|z|=1} \frac{G^{(n)}(\xi) - \beta}{(1 + (1+\beta)\xi)\xi^{l+1}} d\xi \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{G^{(n)}(e^{it}) - \beta}{1 + (1+\beta)e^{it}} e^{-lit} dt \end{aligned}$$

  and we split the integral into two parts. Let $\kappa = \min(\pi, \arccos(1 - \frac{1}{2\alpha}))$ and let

$$I_1 := \frac{1}{2\pi}\int_{|t|\leq \kappa-\delta} \frac{G^{(n)}(e^{it}) - \beta}{1 + (1+\beta)e^{it}} e^{-lit} dt \tag{5.17}$$

$$I_2 := \frac{1}{2\pi}\int_{\pi\geq|t|\geq \kappa-\delta} \frac{G^{(n)}(e^{it}) - \beta}{1 + (1+\beta)e^{it}} e^{-lit} dt \tag{5.18}$$

- If $\beta = 0$, instead of $|z| = 1$ we integrate on

$$\gamma = \{\xi \big| |\xi| = 1, \Re\xi > -0.9\} \cup \{\xi \big| |\xi| \leq 1, \Re\xi = -0.9\}$$

  Let I_1 be the same as above and let

$$I_2 := \frac{1}{2\pi i}\int_{\gamma(\delta)} \frac{G^{(n)}(\xi)}{(1+\xi)\xi^{l+1}} d\xi \tag{5.19}$$

where δ is the same as in Lemma 5.3.12, first or second case, respectively.

In both cases, for any $K > 0$, we can approximate the second integral as follows, as $|e^{lit}| < 1$, and by Lemma 5.3.12:

$$|I_2| \leq \frac{1}{2\pi} \int |\frac{G^{(n)}(\xi) - \beta}{1 + (1+\beta)\xi}| d\xi \ll \frac{n}{(\log n)^K}$$

where the integral is on

- $$\{\xi \| \xi| = 1, |\xi - 1| \geq \sqrt{\frac{1}{\alpha}} - \delta\}$$

  for $\beta \neq 0$ and on
- $\gamma(\delta)$ for $\beta = 0$.

For $|t| \leq \kappa - \delta$, we can use Corollary 5.3.9 and Lemma 5.3.6 to derive the following approximation:

$$\begin{aligned}
|G^{(n)}(e^{it})| &= |M^{(n)}(e^{it})E^{(n)}(e^{it})| \ll n^{(1-\alpha)(1+(1+\beta)\Re e^{it})} \\
&= n^{1+\alpha((2+\beta)\cos t - (1+\beta)\cos t - 1)} \\
&= n n^{\alpha(\cos t - 1)} = n e^{(\log n)\alpha(\cos t - 1)} \ll n e^{-c' t^2 (\log n)}
\end{aligned}$$

for some constant $c' > 0$. Through this, with a sufficiently small $\vartheta > 0$, we obtain

$$\begin{aligned}
\frac{1}{2\pi} \int_{(\log n)^{-\frac{(1-\vartheta)}{2}} \leq |t| \leq \kappa - \delta} |G^{(n)}(e^{it})| dt &\ll n \int_{(\log n)^{-\frac{1-\vartheta}{2}}}^{\infty} e^{-c' t^2 \log n} dt \\
&\ll n e^{-c' (\log n)^{\vartheta}}
\end{aligned}$$

and a remaining integral of

$$I_0 := \frac{1}{2\pi} \int_{|t| \leq (\log n)^{-\frac{1-\vartheta}{2}}} \frac{G^{(n)}(e^{it})}{1 + (1+\beta)e^{it}} e^{-lit} dt$$

.

For this, we again use Lemma 5.3.6 and get

$$\begin{aligned}
G^{(n)}(z) &= E^{(n)}(z) M^{(n)}(z) = \mathbf{E}(G^{(n)}(z)) \frac{M^{(n)}(z)}{(1+\beta)(1+z)} \\
\mathbf{E}(G^{(n)}(z)) &= n^{(1-\alpha)(1+z(1+\beta))} \frac{(1+\beta)(1+z)\Gamma(2\alpha)}{\Gamma(2\alpha)} + \mathcal{O}(n^{(\Re z - 1)\alpha}) \\
&= n \underbrace{n^{\alpha((2+\beta)z - 1 - z - \beta z)}}_{= n^{\alpha(z-1)}} \Big(\frac{(1+\beta)(1+z)\Gamma(2\alpha)}{\Gamma(1 + \alpha(1+z))} + \mathcal{O}(\frac{1}{n}) \Big)
\end{aligned}$$

uniformly. If $t \to 0$, such that $|t| \leq (\log n)^{-\frac{1-\vartheta}{2}}$, we can use Taylor series and get, with $\Gamma(z+1) = z\Gamma(z)$ and other simplifications,

$$\begin{aligned}
\frac{\mathbf{E}(G^{(n)}(e^{it}))}{1+(1+\beta)e^{it}} &= ne^{(\log n)\alpha(e^{it}-1)}\big(\frac{(1+\beta)(1+e^{it})\Gamma(2\alpha)}{(1+(1+\beta)e^{it})\Gamma(1+\alpha(1+e^{it}))} + \mathcal{O}(\frac{1}{n})\big) \\
&= ne^{-\alpha\frac{t^2}{2}\log n + (it\alpha)\log n} \cdot \\
&\times \Big(1 - it\big(\alpha - \frac{1}{2} + 2\alpha^2\Gamma'(1+2\alpha)\big) - \frac{\alpha t^3}{6} i \log n + \mathcal{O}(t^2 + t^4 \log n)\Big).
\end{aligned}$$

On the other hand, $M^{(n)}(1) = 2(1+\beta)$, and hence, with another taylor series

$$\frac{M^{(n)}(e^{it})}{(1+\beta)(1+e^{it})} = 1 + it\frac{M'^{(n)}(1) - (1+\beta)}{2(1+\beta)} + \mathcal{O}(t^2)$$

So, combining these two series in the equation above,

$$\begin{aligned}
\frac{G^{(n)}(e^{it})e^{-lit}}{1+(1+\beta)e^{it}} &= ne^{-\alpha\frac{t^2}{2}\log n + it(\alpha\log n - l)} \cdot \\
&\times \Big(1 - it\big(\alpha - \frac{1}{2} + 2\alpha^2\Gamma'(1+2\alpha) - \frac{M'^{(n)}(1) - (1+\beta)}{2(1+\beta)}\big) \\
&- \frac{\alpha t^3}{6} i \log n + \mathcal{O}(t^2 + t^4 \log n)\Big),
\end{aligned}$$

uniformly with respect to l. For the same reason as in the first part of I_1, we have

$$\int_{|t| \geq (\log n)^{-\frac{1-\vartheta}{2}}} e^{-t^2 \log n}(1 + t + t^3 \log n) \ll e^{-(\log n)^{\vartheta}}$$

Thus,

$$\begin{aligned}
\frac{I_0}{n} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-(\alpha\frac{t^2}{2})\log n + it(\alpha\log n - l)} \cdot \\
&\times \Big(1 - it\big(\alpha - \frac{1}{2} + 2\alpha^2\Gamma'(1+2\alpha) - \frac{M'^{(n)}(1) - (1+\beta)}{2(1+\beta)}\big) - \frac{\alpha t^3}{6} i \log n\Big) dt \\
&+ \mathcal{O}((\log n)^{-\frac{3}{2}}).
\end{aligned}$$

Integration leads to

$$\begin{aligned}
\frac{I_0}{n} &= \frac{1}{\sqrt{2\alpha\pi\log n}} e^{-\frac{((\log n)\alpha - l)^2}{2\alpha\log n}} \\
&\times \Big(1 + \frac{((\log n)\alpha - l)}{2\alpha\log n} - \frac{((\log n)\alpha - l)^3}{6\alpha^2(\log n)^2} \\
&+ \frac{(\log n)\alpha - l}{\alpha\log n}(\alpha - \frac{1}{2} + 2\alpha^2\Gamma'(1+2\alpha) - \frac{M'^{(n)}(1) - (1+\beta)}{2(1+\beta)})\Big) \\
&+ \mathcal{O}((\log n)^{-\frac{3}{2}})
\end{aligned}$$

and from there, we obtain

$$\begin{aligned}
\frac{L_l^{(n)}}{n/\sqrt{2\alpha\pi\log n}} &= e^{-\frac{((\log n)\alpha - l)^2}{2\alpha\log n}} \\
&\times \Big(1 + \frac{((\log n)\alpha - l)}{2\alpha\log n} - \frac{((\log n)\alpha - l)^3}{6(\alpha\log n)^2} \\
&+ \frac{(\log n)\alpha - l}{\alpha\log n}(\alpha - \frac{1}{2} + 2\alpha^2\Gamma'(1+2\alpha) - \frac{M'^{(n)}(1) - (1+\beta)}{2(1+\beta)})\Big) \\
&+ \mathcal{O}(\frac{1}{\log n})
\end{aligned} \tag{5.20}$$

a.s., with an error term uniform in l. This completes he first part of the proof.

It only remains to find the maximum of (5.15). The derivative is

$$\frac{n}{\sqrt{2\alpha\pi\log n}} e^{\left(-\frac{(l-\alpha\log n)^2}{2\alpha\log n}\right)} \left(\frac{2(l-\alpha\log n)}{2\alpha\log n}\right)$$

and the maximum is reached where:

$$\begin{aligned}
\frac{2(l-\alpha\log n)}{2\alpha\log n} &= 0 \\
l &= \alpha\log n
\end{aligned}$$

The width of the tree is thus given by (5.16). □

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