COMPUTABILITY AND UNCOUNTABLE LINEAR ORDERS I: COMPUTABLE CATEGORICITY

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ABSTRACT. We study the computable structure theory of linear orders of size $\aleph_1$ within the framework of admissible computability theory. In particular, we characterize which of these linear orders are computably categorical.

1. INTRODUCTION

Effective properties of countable linear orderings have been studied extensively since the 1960's. This line of research, surveyed in Downey [1], is part of a broader program of understanding the information content of mathematical structures. Among the notions central to this theory are the notions of *computable categoricity* and the *degree spectrum* of a structure. A computable structure is said to be computably categorical if it is effectively isomorphic to any of its computable copies. The degree spectrum of a structure is the collection of Turing degrees which contain a copy of the structure. Examples of major results concerning the effective properties of linear orderings are the Dzgoev [3] and Remmel [15] characterization of the computably categorical linear orderings as those with finitely many successivities and the Richter [16] theorem that the computable order-types are the only ones whose degree spectrum contains a least element.

Traditionally, the domain of computability theory consists of hereditarily finite objects (for example the natural numbers, finite sequences and sets of natural numbers, and so on). For this reason, effectiveness considerations have mostly been applied only to countable mathematical structures. Early on, though, generalizations of the theory of computable functions on domains of larger cardinality were considered. Takeuti [19] and [20] generalized recursion theory to the class of all ordinals. Kreisel and Sacks [9] and [10], following work of Kreisel [8], developed

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metarecursion theory, which is the study of computability on the computable ordinals, or equivalently, on their notations. These two approaches were unified by Kripke [11] and Platek [14] in the study of recursion theory on admissible ordinals.

Greenberg and Knight [7] initiated the application of admissible computability theory to the study of effectiveness properties of uncountable structures. Under the assumption that all reals are constructible, they investigate the analogues of classical results about fields and vector spaces, results from pure computable model theory such as the relationship between Scott families and computable categoricity, and results about linear orderings.

A main interest in these investigations is the contrast between the countable and the uncountable case. Some results from classical computability theory, about countable structures, generalize to the uncountable case, albeit with sometimes different proofs. Other classical results fail in the uncountable setting. For example, Greenberg and Knight show that Richter's result mentioned above fails for uncountable cardinals; in fact, in the uncountable setting, every degree is the least degree of the spectrum of some linear ordering. In either case, the examination of classical results in new surroundings sheds light on the classical theory, often by highlighting essential assumptions that go without notice if generalizations are not considered, and by separating notions that happen to coincide in the countable setting.

A major theme arising from this work is the importance of the notion of true finiteness. In ω_1-computability, the correct analogue for "finite" is "countable". For example, ω_1-computations take countably many steps and manipulate hereditarily countable objects. Yet, true finiteness has some inherent properties which do not generalize. Lerman and Simpson [12] and Lerman [13] exhibited the effects of the difference between true finiteness and its generalization on the lattice of c.e. sets under inclusion; Greenberg [5] has exhibited the effects on the c.e. degrees. For linear orderings, the important observation is that while a finite set can determine only finitely many cuts in a linear ordering, a countable set may determine uncountably many cuts in a linear ordering. This difference underlies the failure of Richter's theorem for ω_1, as well as many of the other differences we shall see.

In this paper, we continue the investigation started by Greenberg and Knight [7], concentrating on linear orderings of size $\aleph_1$. We again assume that $\mathbb{R} \subset L$, the pertinent effect of which is that L_{ω_1} is amenable in V; that is, L_{ω_1} coincides with H_{ω_1}, the collection of hereditarily countable sets. This assumption implies the continuum hypothesis in a strong sense: It gives a $\Delta_1(H_{\omega_1})$ bijection between 2^ω and ω_1. Also, amenability and the regularity of ω_1 imply that the results of this paper hold when relativized to any subset of ω_1.

Again, true finiteness plays a central role. However, we observe new aspects of working with linear orderings of size $\aleph_1$. We uncover hidden effectiveness conditions which become vacuous when working with countable linear orders. We also rely heavily on the Hausdorff analysis of countable linear orders. Unlike all other results so far, the results in this paper do not easily generalize to cardinalities beyond $\aleph_1$.

In this paper we investigate the Dzgoev-Remmel characterization of the computably categorical linear orderings mentioned above. In Theorem 3.1, we find the correct analogue of this characterization for linear orderings of size $\aleph_1$. We begin though (Theorem 2.4) with the easier case of uniform effective categoricity. In the sequel to this paper [6] we study degree spectra, both of linear orders and of the successor relation on computable linear orders.

1.1. **Notation, Terminology, Background.** We refer the reader to Sacks [18] for additional background on admissible computability, and to Greenberg and Knight [7] for specific background on ω_1-computability theory, for definitions and basic facts on ω_1-computable model theory, and for effectiveness properties of linear orderings of size $\aleph_1$ in particular. In order to distinguish computability in the countable case from computability on the admissible ordinal ω_1, we will usually denote computability in the former case as ω-*computability* and the latter case as ω_1-*computability*; in this paper, though, when we omit the prefix, we mean ω_1-computability. For the entire paper, we assume that every real is constructible.

We also refer the reader to Rosenstein [17] for additional background on order-types and linear orders. We use the following notation and terminology for linear orders.

Definition 1.1. Let $\mathcal{L} = (L, <_{\mathcal{L}})$ be a linear order.

(1) A subset X of L is *convex*, or an $\mathcal{L}$-*interval*, if for all $x, y \in X$ and $z \in L$, if $x <_{\mathcal{L}} z <_{\mathcal{L}} y$ then $z \in X$.
(2) If $A, B \subseteq L$, then we write $A <_{\mathcal{L}} B$ if $a <_{\mathcal{L}} b$ for all $a \in A$ and $b \in B$; in this case we let $(A, B)_{\mathcal{L}}$ be the $\mathcal{L}$-*interval* determined by A and B, be the convex set $\{x \in L : A <_{\mathcal{L}} x <_{\mathcal{L}} B\}$. If $A = \{a\}$, we also write $a <_{\mathcal{L}} B$ and $(a, B)_{\mathcal{L}}$; if $A = \emptyset$ we write $(-\infty, B)_{\mathcal{L}}$, and so on.
(3) If $Q \subseteq L$, then a *cut* of Q is a partition of Q into subsets Q_1 and Q_2 such that $Q_1 <_{\mathcal{L}} Q_2$.
(4) If $Q \subseteq L$, then a Q-*interval* of $\mathcal{L}$ is an $\mathcal{L}$-interval determined by some cut of Q.
(5) A *block* of $\mathcal{L}$ is a nonempty convex subset X of L such that for all $a, b \in X$, the interval $(a, b)_{\mathcal{L}}$ is finite. Note that every block is at most countable in size.
(6) Let $\mathcal{L}$ be a linear ordering. A pair of elements $a <_{\mathcal{L}} b$ in $\mathcal{L}$ are *adjacent* if $(a, b)_{\mathcal{L}}$ is empty. We say that a is the *predecessor* of b (in $\mathcal{L}$) and b is the *successor* of a (in $\mathcal{L}$).

An *order-type* is an isomorphism class of linear orderings, although we often identify the order-type of a well-ordering with the unique ordinal it contains. We write $\mathrm{otp}(\mathcal{L})$ for the order-type of a linear ordering $\mathcal{L}$. An element of an order-type λ is also called a *presentation* of λ.

If P is a property of linear orderings, then we say that an order-type λ has property P if some presentation of λ has property P. Hence, we say that λ is *computable* if it has a computable presentation, and that the *size* of λ is some cardinal κ if the presentations of λ have cardinality κ.

Remark 1.2. In this paper, we assume that the universe of any linear ordering is a subset of H_{ω_1}. So each order-type λ is a set (rather than a proper class). If its presentations are of size $\aleph_1$, then the set λ has size $2^{\aleph_1}$. Nonetheless we say that the size of λ is $\aleph_1$, as we only care about the size of the presentations of λ.

We fix notation for some order-types which will appear often in this paper.

Definition 1.3. We denote the order-type of the natural numbers by ω and of the least uncountable ordinal by ω_1.

We denote the order-type of the rational numbers by η (also by η_0). This is the saturated countable order-type. We denote the saturated order-type of size $\aleph_1$ by η_1.

We denote the order-type of the integers $\mathbb{Z}$ by ζ and the order-type of the real numbers $\mathbb{R}$ by ρ.

We note the existence of η_1 follows from the continuum hypothesis. By Cantor's argument, a linear order $\mathcal{L}$ of size $\aleph_1$ is saturated if and only if for any at most countable sets $A, B \subset L$ such that $A <_{\mathcal{L}} B$, the interval $(A, B)_{\mathcal{L}}$ is nonempty. Note that as A or B may be empty, this implies that $\mathcal{L}$ has uncountable coinitiality and cofinality.

It will often be important whether a linear order has a subset of order-type η_0.

Definition 1.4. A linear order is *nonscattered* if it has a subset of order type η_0 and *scattered* otherwise.

We use standard sum and product notation: $\mathcal{A}+\mathcal{B}$ for appending $\mathcal{B}$ to the right of $\mathcal{A}$ and $\mathcal{A}\cdot\mathcal{B}$ for replacing every point of $\mathcal{B}$ by a copy of $\mathcal{A}$. As these operations are invariant under isomorphisms, we extend the notation to order-types as well.

We also use restrictions of linear orderings.

Definition 1.5. Let $\mathcal{A} = (A, <_{\mathcal{A}})$ be a linear ordering. If $B \subseteq A$, then we let $\mathcal{A} \upharpoonright B$ be the linear ordering $(B, <_{\mathcal{A}}\upharpoonright B^2)$. If $A \subseteq \omega_1$ and $\alpha < \omega_1$, then we let $\mathcal{A} \upharpoonright \alpha$ be $\mathcal{A} \upharpoonright (A \cap \alpha)$, recalling that a von Neumann ordinal is the collection of its predecessors. We also denote $\mathcal{A} \upharpoonright \alpha$ by $\mathcal{A}_\alpha$.

We recall the basic definitions of ω_1-computability. We work with the structure $(H_{\omega_1}; \in)$ enriched by constants naming all the elements of the structure. Note that under the assumption $\mathbb{R} \subset L$, $H_{\omega_1} = L_{\omega_1}$. A formula (with parameters from H_{ω_1}) is $\Delta_0(H_{\omega_1})$ if all of its quantifiers are bounded. A formula is $\Sigma_1(H_{\omega_1})$ if it is of the form $\exists \bar{x}\, \varphi$ where φ is $\Delta_0(H_{\omega_1})$.

Definition 1.6. A relation $R \subseteq (H_{\omega_1})^n$ is ω_1-*computably enumerable* if it is definable by a $\Sigma_1(H_{\omega_1})$ formula. A relation $R \subseteq (H_{\omega_1})^n$ is ω_1-*computable* if both it and its complement $H_{\omega_1}^n \setminus R$ are ω_1-c.e. A partial function $f\colon (H_{\omega_1})^n \to H_{\omega_1}$ is *partial* ω_1-*computable* if its graph $\{(\bar{a}, f(\bar{a})) : \bar{a} \in \operatorname{dom} f\}$ is an ω_1-c.e. relation. An ω_1-*computable* function $f\colon (H_{\omega_1})^n \to H_{\omega_1}$ is a partial ω_1-computable function whose domain is ω_1-computable.

The main tool of computability is recursion.

Proposition 1.7. *Let $I\colon H_{\omega_1} \to H_{\omega_1}$ be a computable function. Then there is a unique computable function $f\colon \omega_1 \to H_{\omega_1}$ such that for all $\alpha < \omega_1$, $f(\alpha) = I(f \upharpoonright \alpha)$.*

Defining computable functions by recursion allows us to view them dynamically, as is common in countable computability. Processes of computation, described by $\Delta_0(L_{\omega_1})$ formulas, for instance, and taking only countably many steps, can be used to define computable functions.

Familiar facts about computability in the countable setting lift to ω_1-computability with identical reasoning. For example, a set $A \subseteq H_{\omega_1}$ is ω_1-computable if and only if its characteristic function is ω_1-computable. There is an ω_1-computable bijection between ω_1 and H_{ω_1}. The standard well-ordering of $L_{\omega_1} = H_{\omega_1}$ of order-type ω_1 is ω_1-computable. A nonempty subset A of H_{ω_1} is ω_1-c.e. if and only if it is the domain of a partial ω_1-computable function if and only if it is the range of an ω_1-computable function. We can effectively list (in order-type ω_1) all $\Sigma_1(H_{\omega_1})$

formulas and so can effectively list all partial ω_1-computable functions (as $\langle \Phi_e \rangle_{e<\omega_1}$) and ω_1-c.e. sets (as $\langle W_e \rangle_{e<\omega_1}$). This means that the set $\{(e,x) : x \in W_e\}$ is ω_1-c.e.; we often denote it by $\emptyset'$. There is an effective, uniform enumeration of all ω_1-c.e. sets: a uniformly ω_1-computable double sequence $\langle W_{e,s} \rangle_{s,e<\omega_1}$ such that for all e, $W_e = \bigcup_s W_{e,s}$. With a standard proof, the Fixed Point Theorem (Recursion Theorem) holds: If $f : \omega_1 \to \omega_1$ is ω_1-computable then there is some $e < \omega_1$ such that $\Phi_e = \Phi_{f(e)}$.

An intuition for informal definitions of such computable objects develops with experience. As an example we observe the following

Fact 1.8. The collection of countable scattered linear orderings is ω_1-computable. We mean the subset of H_{ω_1} which consists of binary relations which define a linear ordering of some set (of course, also an element of H_{ω_1}). To see this, first note that the collection of linear orderings is definable by a formula only using bounded quantifiers.

We observe that given a countable linear ordering $\mathcal{L}$, the collection of embeddings of $\mathbb{Q}$ into $\mathcal{L}$ is defined by a bounded formula. This shows that that the collection of nonscattered linear orderings is defined by an existential formula and so is ω_1-c.e.

However we can give a "decision procedure" for the set of scattered linear orders. We observe that the Hausdorff analysis of scattered linear orderings can be defined by effective recursion. Given a countable linear order $\mathcal{L}$, we let $\mathcal{L}'$ be the linear order obtained by identifying points which are finitely far apart. The graph of the function $\mathcal{L} \mapsto \mathcal{L}'$ is definable by a bounded formula. By effective recursion, we can now iterate the Hausdorff derivative (transfinitely if necessary, taking direct limits at limit steps) until we get either a dense or empty linear ordering. Which is the case can be observed effectively.

Since we focus on linear orderings, in this paper we do not need the general definition of an ω_1-computable structure. A linear ordering $\mathcal{L}$ of ω_1 is ω_1-computable if it is ω_1-computable as a relation (a set of pairs). As in the countable context, we can effectively list ω_1-computable order-types:

Fact 1.9. There is a uniformly ω_1-computable list $\langle \mathcal{L}_\beta \rangle_{\beta<\omega_1}$ of ω_1-computable linear orderings such that for any ω_1-computable linear ordering $\mathcal{A}$ there is some $\beta < \omega_1$ such that $\mathcal{A} \cong \mathcal{L}_\beta$.

2. Uniform ω_1-Computable Categoricity

In this paper, we characterize the ω_1-computably categorical and uniformly ω_1-computably categorical linear orders. We recall the appropriate definitions.

Definition 2.1. Fix a cardinal $\kappa \in \{\omega, \omega_1\}$.

A κ-computable order-type λ is *κ-computably categorical* if for all κ-computable $\mathcal{A}, \mathcal{B} \in \lambda$ there is a κ-computable isomorphism $f : \mathcal{A} \cong \mathcal{B}$.

A κ-computable order-type λ is *uniformly κ-computably categorical* if there is a κ-computable function mapping a pair of indices of two κ-computable presentations $\mathcal{A}, \mathcal{B} \in \lambda$ to an index of a κ-computable isomorphism between them.

If $\mathcal{L}$ is a κ-computable linear order, then we say that $\mathcal{L}$ is *(uniformly) κ-computably categorical* if its order-type is (uniformly) κ-computably categorical.

As mentioned in the introduction, in the countable framework, Dzgoev [3] (see also [4]) and Remmel [15] independently showed that a computable linear ordering is computably categorical if and only if it has finitely many adjacencies (see Definition 1.1). Equivalently, a computable linear ordering $\mathcal{L}$ is computably categorical if and only if there is a finite set $C \subseteq \mathcal{L}$ such that every $\mathcal{L}$-interval determined by C is either finite or has order-type η_0.

While these two characterizations are equivalent for ω-computable linear orderings, their generalizations to uncountable linear orderings are not. The first does not generalize to a characterization of ω_1-computably categorical linear orderings. Naïvely, one would guess that an ω_1-computable linear ordering is ω_1-computably categorical if and only if it has only countably many adjacencies (or perhaps countably many countable intervals). The next two examples show that these conditions are neither necessary nor sufficient for ω_1-computable categoricity.

Example 2.2. The order-type $2 \cdot \rho$ is ω_1-computably categorical. To see this, fix computable presentations $\mathcal{A}$ and $\mathcal{B}$ of $2 \cdot \rho$. We may fix the "dense" countable subsets $2 \cdot \eta$ of $2 \cdot \rho$ in both $\mathcal{A}$ and $\mathcal{B}$ as a parameter. Then for any point in $\mathcal{A}$ or $\mathcal{B}$, we can determine whether it is the "left" or "right" point of its pair simply by waiting until both have shown up in the same interval determined by the copy of $2 \cdot \eta$.

Example 2.3. The order-type $\eta \cdot \omega_1$ is not ω_1-computably categorical. To see this, we construct computable presentations $\mathcal{A}$ and $\mathcal{B}$ of $\eta \cdot \omega_1$ meeting the requirement

$$\mathcal{R}_e : \text{The function } \Phi_e \text{ is not an isomorphism from } \mathcal{A} \text{ to } \mathcal{B}.$$

for all $e \in \omega_1$.

In order to satisfy $\mathcal{R}_e$, we wait for Φ_e to be completely defined on some copy of η in $\mathcal{A}$, where the image of this copy in $\mathcal{B}$ is greater than the *restraint*. We then add an extra point to $\mathcal{B}$ within the image and move the *restraint* (for $\mathcal{R}_j$ with $j > e$) to a point in $\mathcal{B}$ greater than the image of this copy.

As a step towards characterizing the ω_1-computably categorical linear orderings, we treat the uniform case.

Theorem 2.4. *An order-type λ is uniformly ω_1-computably categorical if and only if λ is finite or $\lambda = \eta_1$.*

Remark 2.5. We note that not only is the order-type η_1 uniformly ω_1-computably categorical, the effective back-and-forth argument demonstrating uniform ω_1-computable categoricity shows that if $\mathcal{A}$ and $\mathcal{B}$ are computable presentations of η_1, we can effectively extend any countable partial embedding $\psi : \mathcal{A} \to \mathcal{B}$ to an isomorphism between $\mathcal{A}$ and $\mathcal{B}$. This is uniform given ψ and ω_1-computable indices for $\mathcal{A}$ and $\mathcal{B}$.

Proof of Theorem 2.4. Every finite order-type is clearly uniformly ω_1-computably categorical. An effective back-and-forth argument of length ω_1 shows that η_1 is uniformly ω_1-computably categorical. This establishes one direction of the theorem.

In order to prove the other direction, let λ be an infinite, uniformly ω_1-computably categorical order-type, and let $\mathcal{L}$ be a computable presentation of λ. We show that $\mathcal{L}$ is $\aleph_1$-saturated. To do this, given countable subsets A and B of $\mathcal{L}$ such that $A <_{\mathcal{L}} B$, we "force" $\mathcal{L}$ to enumerate a point between A and B.

This is done by building an auxiliary ω_1-computable linear ordering $\mathcal{K}$. We ensure that $\mathcal{K}$ is isomorphic to $\mathcal{L}$. By the Fixed Point Theorem, we know an ω_1-computable index for $\mathcal{K}$ during the construction, and with the uniform categoricity of $\mathcal{L}$, we obtain an ω_1-computable isomorphism $\Phi\colon \mathcal{K} \to \mathcal{L}$. Controlling $\mathcal{K}$, we can bend $\mathcal{L}$'s shape to our wishes.

In greater detail, the Fixed Point Theorem is applied as follows. For each $e < \omega_1$, we perform a separate construction. The e^{th} construction observes the e^{th} partial ω_1-computable function Φ^e and builds an ω_1-computable linear order $\mathcal{K}^e$. We ensure (even in the case that Φ^e is not total) that $\mathcal{K}^e$ is isomorphic to $\mathcal{L}$. Since the construction of $\mathcal{K}^e$ is effective, uniformly in e, we obtain an ω_1-computable function $f\colon \omega_1 \to \omega_1$ such that for all e, $\Phi^{f(e)}$ is an isomorphism from $\mathcal{K}^e$ to $\mathcal{L}$. By the Fixed Point Theorem, there is some $e^* < \omega_1$ such that $\Phi^{e^*} = \Phi^{f(e^*)}$. The construction we eventually use is given by this e^*. Letting $\mathcal{K} = \mathcal{K}^{e^*}$ and $\Phi = \Phi^{e^*}$, during this "real" construction, we build $\mathcal{K}$ while knowing the ω_1-computable function Φ, which is an isomorphism from $\mathcal{K}$ to $\mathcal{L}$.

However, we need to ensure that each $\mathcal{K}^e$ is isomorphic to $\mathcal{L}$. To do this, we define, for each e and each $s < \omega_1$ an isomorphism from $\mathcal{K}^e_s$ (our stage s approximation to $\mathcal{K}^e$) to $\mathcal{L}_s = \mathcal{L} \restriction s$. We will ensure that if Φ^e is not an isomorphism from $\mathcal{K}^e$ to $\mathcal{L}$, then the sequence $\langle F^e_s \rangle_{s<\omega_1}$ reaches a limit F^e which we will ensure is an isomorphism from $\mathcal{K}^e$ to $\mathcal{L}$. From the point of view of the correct construction e^*, at each stage s we need to define an isomorphism $F_s = F^{e^*}_s$ from $\mathcal{K}_s = \mathcal{K}^{e^*}_s$ to $\mathcal{L}_s$, even though we do not need the maps F_s to converge to an isomorphism from $\mathcal{K}$ to $\mathcal{L}$.

We restrict ourselves now to the correct construction e^* and go back to explaining how the auxiliary order $\mathcal{K}$ and the isomorphism Φ are used to control the structure of $\mathcal{L}$. At some stage s, we observe countable subsets C and D of $\mathcal{L}_s$ with $C <_{\mathcal{L}} D$ and $(C,D)_{\mathcal{L}_s} = \emptyset$. The plan is to add a point z to $\mathcal{K}_{s+1}$ between $\Phi^{-1}(C)$ and $\Phi^{-1}(D)$. If we do this, since Φ is indeed an isomorphism from $\mathcal{K}$ to $\mathcal{L}$, $\Phi(z)$ must be a point in $\mathcal{L}$ between C and D. If we keep track correctly, we can thus treat any pair $C <_{\mathcal{L}} D$ of countable sets and so show that $\mathcal{L}$ is $\aleph_1$-saturated.

Recall, however, that we need to define an isomorphism $F_{s+1}\colon \mathcal{K}_{s+1} \to \mathcal{L}_{s+1}$. The eventual point $\Phi(z)$ may be enumerated into $\mathcal{L}$ much later. As a result, we need to find an embedding $g\colon \mathcal{K}_s \cup \{z\} \to \mathcal{L}_{s+1}$ and then add more points to $\mathcal{K}_s \cup \{z\}$ to define $\mathcal{K}_{s+1}$ such that g can be extended to the desired F_{s+1}. The map g (and so F_{s+1}) may disagree with Φ. See Figure 1.

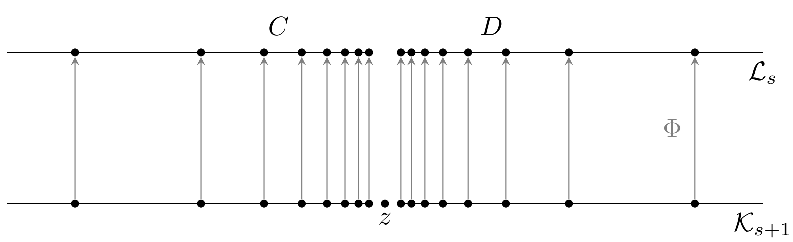

FIGURE 1. Saturating $\mathcal{L}$. The point z is added to $\mathcal{K}_{s+1}$, and the result is embedded by F_{s+1} into $\mathcal{L}_{s+1}$; F_{s+1} does not agree with Φ.

Unfortunately, we cannot always guarantee the existence of an embedding of $\mathcal{K}_s \cup \{z\}$ into $\mathcal{L}_{s+1}$. If $\mathcal{L}_s$ is nonscattered, then such an embedding is ensured as every countable linear order is embeddable into any nonscattered linear order. If $\mathcal{L}_{s+1}$ is scattered, then there may not be an embedding g as desired. Thus, before executing the above strategy, we work towards guaranteeing that $\mathcal{L}$ is nonscattered.

Again, we utilize the auxiliary order $\mathcal{K}$. If $\mathcal{L}_s$ is infinite and scattered, there is necessarily an infinite block B in $\mathcal{L}_s$. We then add a point between any adjacent points in $\Phi^{-1}(B)$. Again, since $\Phi\colon \mathcal{K} \to \mathcal{L}$ is an isomorphism, this means that B is not really a block of $\mathcal{L}$. If we keep books wisely, we will be able to arrange that $\mathcal{L}$ does not have infinite blocks and so will be nonscattered. In turn, this would mean that for some s, $\mathcal{L}_s$ is nonscattered, and so eventually we could return to the strategy described earlier for making $\mathcal{L}$ saturated. Again, we need to define F_{s+1}; here we let F_{s+1} agree with Φ outside $\Phi^{-1}[B]$, but can "correct" Φ on $\Phi^{-1}[B]$ together with the new points in $\mathcal{K}_{s+1}$ to an isomorphism with B; this depends on the shape of the block. See Figure 2 for the case that $B \cong \mathbb{Z}$.

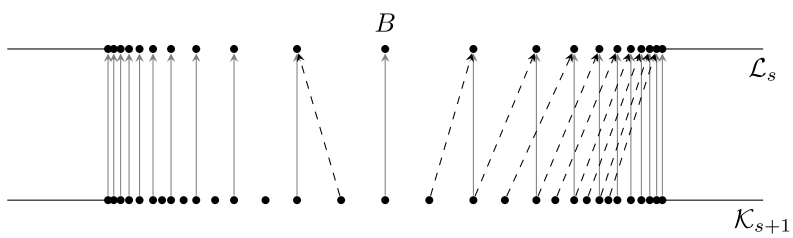

FIGURE 2. Descattering $\mathcal{L}$. B is a block of $\mathcal{L}_s$ of order-type ζ. Points are added between adjacent points in $\Phi^{-1}[B]$. F_{s+1} (dashed) is an isomorphism between the new ζ in $\mathcal{K}_{s+1}$ and B.

The construction is thus split into two phases: a *descattering* phase and a *saturating* phase. We employ the descattering strategy while $\mathcal{L}_s$ is scattered; once $\mathcal{L}_s$ becomes nonscattered, we follow the saturating strategy. Note that both strategies above rely on the fact that at stage s we have access to $\Phi \restriction \mathcal{K}_s$ and that $\Phi \restriction \mathcal{K}_s$ is an isomorphism from $\mathcal{K}_s$ to $\mathcal{L}_s$. The regularity of ω_1 implies the existence of a closed and unbounded set of stages s at which this is the case, and so we restrict our action to these stages. While we are waiting for the next such stage (which can be forever in the e^{th} construction for some $e \neq e^*$), we need to ensure that $\mathcal{K}_s$ is isomorphic to $\mathcal{L}_s$. This can be done without changing the values of F, and so on intervals of stages t on which we don't act we will obtain an increasing sequence of isomorphisms F_t. This allows us to define F_s for all limit stages s (and in particular ensure that $\mathcal{K}_s \cong \mathcal{L}_s$ for all limit stages). Either F_t stabilizes below s and the union map F_s is an isomorphism; or we act cofinally in s, in which case s, too, is a stage at which we see $\Phi \restriction \mathcal{K}_s$ to be an isomorphism from $\mathcal{K}_s$ to $\mathcal{L}_s$, in which case we can simply let $F_s = \Phi \restriction \mathcal{K}_s$. If $e \neq e^*$ is a "failed" construction then we eventually cease changing F^e, which will ensure that F^e is an isomorphism from $\mathcal{K}^e$ to $\mathcal{L}$ as required.

Construction e: Since $\mathcal{L}$ is infinite, we may assume that $\mathcal{L}_\omega$ is infinite. We define an increasing and continuous sequence $\langle \mathcal{K}^e_s \rangle_{\omega \leq s < \omega_1}$ of countable linear orderings; and for each s with $\omega \leq s \leq \omega_1$, an isomorphism $F^e_s\colon \mathcal{K}_s \to \mathcal{L}_s$. We start with $\mathcal{K}^e_\omega := \mathcal{L}_\omega$ and $F^e_\omega := \mathrm{id}_{\mathcal{K}^e_\omega}$.

Let $s < \omega_1$ be infinite, and suppose that $\mathcal{K}^e_s$ and F^e_s are already defined. We first define an embedding $\hat{F}^e_s$ of $\mathcal{K}^e_s$ into $\mathcal{L}_s$. After $\hat{F}^e_s$ is defined, we let $\mathcal{K}^e_{s+1}$ be an extension of $\mathcal{K}_s^e$ to a countable linear ordering such that we can extend $\hat{F}_s^e$ to an isomorphism $F_{s+1}^e\colon \mathcal{K}_{s+1}^e \to \mathcal{L}_{s+1}$; this will conclude stage s.

We define $\hat{F}_s^e$. Let Φ_s^e be the function Φ^e, restricted to the inputs x such that $\Phi^e(x)$ converges before stage s. At stage s, we check if Φ_s^e is an isomorphism from $\mathcal{K}_s^e$ to $\mathcal{L}_s$. If not, then we let $\hat{F}_s^e := F_s^e$. This means that in this case, F_{s+1}^e will extend F_s^e.

Suppose that $\Phi_s^e\colon \mathcal{K}_s^e \to \mathcal{L}_s$ is an isomorphism. There are two cases, depending on whether $\mathcal{L}_s$ is scattered or not.

- *Descattering:* If $\mathcal{L}_s$ is scattered, we let B_s be the $<_{\omega_1}$-least infinite block of $\mathcal{L}_s$. Since the order-type of B_s is either ω, ω^*, or ζ, there is a self-embedding f_s of $\mathcal{L}_s$ (which we can take to be the identity outside B_s, though this is unimportant) such that for all adjacent $a <_{\mathcal{L}} b$ in B_s, $f_s(a)$ and $f_s(b)$ are not adjacent in $\mathcal{L}_s$. We pick some such embedding f_s. We let $\hat{F}_s^e := f_s \circ \Phi_s^e$.
- *Saturating*: If $\mathcal{L}_s$ is nonscattered, we let (C_s, D_s) be the $<_{\omega_1}$-least pair of countable sets $C, D \subseteq \mathcal{L}_s$ such that $C <_{\mathcal{L}} D$ and $(C,D)_{\mathcal{L}_s}$ is empty. Since $\mathcal{L}_s$ is nonscattered, there is a self-embedding f_s of $\mathcal{L}_s$ such that $(f_s[C_s], f_s[D_s])_{\mathcal{L}_s}$ is nonempty (add a point to $\mathcal{L}_s$ between C_s and D_s and embed the result into $\mathcal{L}_s$). We let $\hat{F}_s^e := f_s \circ \Phi_s^e$.

To complete the construction, we need to define F_s^e for limit stages s, since we already stipulated that $\mathcal{K}_s^e := \bigcup_{t<s} \mathcal{K}_t^e$ for limit s. Let J^e be the set of stages t such that Φ_t^e is an isomorphism from $\mathcal{K}_t^e$ to $\mathcal{L}_t$. Let s be a limit stage. If $J^e \cap s$ is bounded below s, then (by induction) for all $r < t$ in the interval $(\sup(J^e \cap s), s)$, we have $F_r^e \subset F_t^e$. It then follows that $F_s^e := \bigcup_{t \in (\sup(J^e \cap s), s)} F_t^e$ is an isomorphism between $\mathcal{K}_s^e$ and $\mathcal{L}_s$. If $J^e \cap s$ is unbounded below s, then $s \in J^e$ and so we let $F_s^e := \Phi_s^e$.

Verification: Let $\mathcal{K}^e := \mathcal{K}_{\omega_1}^e = \bigcup_{s<\omega_1} \mathcal{K}_s^e$. We first show that $\mathcal{K}^e$ and $\mathcal{L}$ are isomorphic (for all e). One point is that J^e is unbounded in ω_1 if and only if Φ^e is total and is an isomorphism from $\mathcal{K}^e$ to $\mathcal{L}$; in the right-to-left direction we use the fact that ω_1 is regular and that the sequences $\langle \mathcal{K}_s^e \rangle$ and $\langle \mathcal{L}_s \rangle$ are continuous. So if J^e is unbounded in ω_1 then Φ^e witnesses that $\mathcal{K}^e$ and $\mathcal{L}$ are isomorphic. On the other hand, if J^e is bounded below ω_1, then after stage $\sup(J^e)$, no action is taken to change F_s^e, and so $F^e := \bigcup_{s>\sup(J^e)} F_s^e$ is an isomorphism between $\mathcal{K}^e$ and $\mathcal{L}$. Hence in either case $\mathcal{K}^e$ and $\mathcal{L}$ are isomorphic.

Now that we know that $\mathcal{K}^e$ and $\mathcal{L}$ are isomorphic for all e, we can carry out the plan using the Fixed Point Theorem. We obtain e^* such that $\Phi^{e^*}\colon \mathcal{K}^{e^*} \to \mathcal{L}$ is an isomorphism. From now we only consider the e^*-construction. Let $\mathcal{K} = \mathcal{K}^{e^*}$, $J = J^{e^*}$, and so on. We know that J is unbounded in ω_1.

We show that $\mathcal{L}$ is nonscattered. Suppose, for a contradiction, that $\mathcal{L}$ is scattered. Hence for all s, the order $\mathcal{L}_s$ is scattered. Now we observe that if $s < t$ are both in J, then $B_s \neq B_t$ (where recall that B_s is the $<_{\omega_1}$-least infinite block of $\mathcal{L}_s$). For let $a, b \in B_s$ be adjacent in B_s. The definition of $\hat{F}_s$ and the fact that $f_s(a)$ and $f_s(b)$ are not adjacent in $\mathcal{L}$ means that $\mathcal{K}_{s+1}$ contains a point z between $\Phi^{-1}(a)$ and $\Phi^{-1}(b)$. Then $z \in \mathcal{K}_t$. Since Φ_t is an isomorphism between $\mathcal{K}_t$ and $\mathcal{L}_t$ (and, of course, Φ_t extends Φ_s), we see that a and b cannot be adjacent in $\mathcal{L}_t$. In particular, B_s is not a block of $\mathcal{L}_t$, so $B_s \neq B_t$.

Now the fact that J is unbounded in ω_1 shows that $\mathcal{L}$ is nonscattered. For if $\mathcal{L}$ is scattered, then it contains an infinite block. Let B be the $<_{\omega_1}$-least infinite block of $\mathcal{L}$. Being an infinite block of $\mathcal{L}$ is a Π^0_1 property; this and the regularity of ω_1 implies that for all but countably many $s \in J$, B is the $<_{\omega_1}$-least infinite block of $\mathcal{L}_s$, i.e., $B_s = B$. This contradicts the fact that J is unbounded and the fact that $s < t$ in J implies $B_s \neq B_t$.

Let s_0 be the least stage such that $\mathcal{L}_{s_0}$ is nonscattered. We now show that $\mathcal{L}$ is $\aleph_1$-saturated. The proof is similar. First we observe that if $s_0 \le s < t$ and $s, t \in J$, then $(C_s, D_s) \neq (C_t, D_t)$. For the definition $\hat{F}_s = f_s \circ \Phi_s$ and the property of f_s imply that the interval $\left(\Phi^{-1}C_s, \Phi^{-1}D_s\right)_{\mathcal{K}_{s+1}}$ is nonempty, and so as Φ_t is an isomorphism from $\mathcal{K}_t$ to $\mathcal{L}_t$, the interval $(C_s, D_s)_{\mathcal{K}_t}$ is nonempty. We can then show that no pair (C, D) can be the $<_{\omega_1}$-least pair of countable subsets $C <_{\mathcal{L}} D$ such that $(C, D)_{\mathcal{L}}$ is empty, as this would contradict that J is unbounded in ω_1; again, the property defining the pair (C, D) is Π^0_1. Hence $\mathcal{L}$ is $\aleph_1$-saturated, which completes the proof. □

3. ω_1-Computable Categoricity

We turn to the main result of this paper, the characterization of ω_1-computably categorical linear orderings. During earlier work on this subject, trying to generalize the Remmel-Dzgoev criterion, Knight conjectured that a linear ordering $\mathcal{L}$ is ω_1-computably categorical if and only if there is a countable subset Q of $\mathcal{L}$ and a number n such that every Q-interval of $\mathcal{L}$ is either empty, contains exactly n points, or is $\aleph_1$-saturated. While not quite correct, this conjecture does contain an important ingredient which is correct: If $\mathcal{L}$ is ω_1-computably categorical, then there is some countable subset Q of $\mathcal{L}$ such that every Q-interval is either finite or has order-type η_1.

The added ingredient is effectiveness. An ordering $\mathcal{L}$ with a countable subset Q can be ω_1-computably categorical, witnessed by Q, even if $\mathcal{L}$ contains finite Q-intervals of different sizes. However, for each n, we need to effectively enumerate those cuts of Q that define intervals that may have size n. This added ingredient sheds light on the countable case as well. The characterization below of ω_1-computable categoricity is a correct characterization of ω-computable categoricity if we replace "countable" by "finite". The special properties of the cardinal ω make the effectiveness condition redundant in the countable case. The uncountable case allows us to recover this important aspect of the criterion, which is invisible if one only sees the countable context.

The effectiveness condition of Theorem 3.1 implies another difference between countable and uncountable linear orderings. Given the theorem (and relativizing it), it is easy to construct an order-type λ of size $\aleph_1$ with ω_1-computable presentations which is not ω_1-computably categorical but is *relatively ω_1-computably categorical above* $\mathbf{d}$: There is a degree $\mathbf{d}$ such that any two presentations $\mathcal{L}_1, \mathcal{L}_2 \ge \mathbf{d}$ of λ are $(\mathcal{L}_1 \oplus \mathcal{L}_2)$-$\omega_1$-computably isomorphic. There are no such countable order-types: If λ is a countable order-type with ω-computable elements that is not ω-computably categorical, then for every ω-Turing degree $\mathbf{d}$ there are $\mathbf{d}$-computable presentations $\mathcal{L}_1$ and $\mathcal{L}_2$ of λ which are not isomorphic by any $\mathbf{d}$-computable isomorphism.

Theorem 3.1. *An ω_1-computable linear order $\mathcal{L}$ is ω_1-computably categorical if and only if there are a countable set $Q \subset \mathcal{L}$ and a collection $\{V_n : 0 < n < \omega\}$ of pairwise disjoint ω_1-c.e. sets of cuts of Q with the following properties:*

(1) Every Q-interval of $\mathcal{L}$ is either finite or has order-type η_1.
(2) For any cut (Q_1, Q_2) of Q, if the Q-interval $(Q_1, Q_2)_{\mathcal{L}}$ has size $n > 0$, then $(Q_1, Q_2) \in V_n$.

Note that since the c.e. sets V_n are pairwise disjoint, it follows that if $(Q_1, Q_2) \in V_n$ then the interval $(Q_1, Q_2)_{\mathcal{L}}$ is either empty, has size n, or is $\aleph_1$-saturated.

Proof. ($\Longleftarrow$) Let $\mathcal{L}$ be an ω_1-computable linear order, equipped with sets Q and $\{V_n\}$ as described in the theorem. To show that $\mathcal{L}$ is ω_1-computably categorical, let $\mathcal{K}$ be an ω_1-computable linear order which is isomorphic to $\mathcal{L}$, and let $g\colon \mathcal{L} \to \mathcal{K}$ be an arbitrary (not necessarily effective) isomorphism. We define an ω_1-computable isomorphism $f\colon \mathcal{L} \to \mathcal{K}$ by starting with $g \upharpoonright Q$. We extend $g \upharpoonright Q$ to a map f on $\mathcal{L}$ by defining f on every Q-interval. Let $A := (Q_1, Q_2)_{\mathcal{L}}$ be a Q-interval; let $B := g[A] = (g[Q_1], g[Q_2])_{\mathcal{K}}$. If A is empty, we do not need to define f on A. If A is nonempty, we wait for a stage s at which either $A_s := (Q_1, Q_2)_{\mathcal{L}\upharpoonright s}$ is infinite; or $(Q_1, Q_2) \in V_n$ at stage s, $|A_s| = n$, and $B_s := (g[Q_1], g[Q_2])_{\mathcal{K}\upharpoonright s}$ also has size n for some positive $n < \omega$. At least one of the two has to happen. Here we use the fact that since the collection of sets $\{V_n\}$ is countable, the sequence $\langle V_n \rangle$ is uniformly c.e.

Now in the latter case, we define f to be the order-preserving bijection between A_s and B_s. In the former case, we know that both A and B are $\aleph_1$-saturated, so an ω_1-computable isomorphism between A and B can be built uniformly from our indices $(Q_1, Q_2)_{\mathcal{L}}$ and $(g[Q_1], g[Q_2])_{\mathcal{K}}$ for A and B. If we first see that $|A_s| = n = |B_s|$ and $(Q_1, Q_2) \in V_n$ and define f on A_s, and then more points are added to A, it must be that A and B have order-type η_1. The map $f \upharpoonright A_s$ can be uniformly extended to an ω_1-computable isomorphism between A and B.

($\Longrightarrow$) Let $\mathcal{L}$ be an ω_1-computable, ω_1-computably categorical linear order. We want to find sets Q and V_n as in the theorem. We attempt to emulate the proof of Theorem 2.4. To show that $\mathcal{L}$ has the desired form, we construct an auxiliary ω_1-computable linear ordering $\mathcal{K}$ isomorphic to $\mathcal{L}$ and use an ω_1-computable isomorphism between $\mathcal{K}$ and $\mathcal{L}$ in order to force $\mathcal{L}$ to add points in locations we choose. Since the ω_1-computable categoricity of $\mathcal{L}$ may fail to be uniform, this time we only have one construction (we construct one $\mathcal{K}$ rather than ω_1 many $\mathcal{K}^e$); but we need to guess which ω_1-computable function is the ω_1-computable isomorphism between $\mathcal{K}$ and $\mathcal{L}$. Let $\langle \Phi_j \rangle_{j<\omega_1}$ list all partial ω_1-computable functions. The *guess* R_j guesses that Φ_j is an isomorphism from $\mathcal{K}$ to $\mathcal{L}$.

As in the previous proof, we build $\mathcal{K}$ as the union of an increasing, continuous, ω_1-computable sequence $\langle \mathcal{K}_s \rangle$ of countable linear orders. When $\Phi_{j,s}$ is an isomorphism between $\mathcal{K}_s$ and $\mathcal{L}_s$, we guess that R_j is correct. If we succeed in making $\mathcal{K}$ isomorphic to $\mathcal{L}$ then some R_j will be correct. On the stages at which we guess this R_j is correct we would like to implement the strategy employed for proving Theorem 2.4.

It is more difficult to ensure that $\mathcal{K}$ is indeed isomorphic to $\mathcal{L}$. As before we construct maps $F_s\colon \mathcal{K}_s \to \mathcal{L}_s$. The aim is that in the contradictory event that no R_j is correct, $\lim_{s\to\omega_1} F_s$ exists and is an isomorphism from $\mathcal{K}$ to $\mathcal{L}$. We need to consider the possibility that eventually, each R_j does not appear correct any more, but the stages at which *some* R_j appears correct are unbounded in ω_1. For notational convenience, we will define (Definition 3.4) for each $j < \omega_1$ a set of stages J_j, a subset of the set of stages s such that $\Phi_{j,s}$ is an isomorphism from $\mathcal{K}_s$

to $\mathcal{L}_s$. Let $s_j = \sup J_j$ and for simplicity suppose that $s_0 < s_1 < s_2 < \cdots$ are all countable but $\sup_{j<\omega_1} s_j = \omega_1$. If at stage s_j we naïvely define $F_{s_j} = \Phi_{j,s}$ then there is no guarantee that the sequence $\langle F_s \rangle$ converges pointwise. For this reason we view the construction as a priority construction, with R_j assigned a stronger priority than R_i if $j < i$. If a guess R_j *receives attention* at some stage t and defines F_{t+1}, and a weaker guess R_i wishes to define F_{s+1} at a later stage s, then unless R_j acted between stages t and s, R_i is required to let F_{s+1} extend F_{t+1} (see Claim 3.3).

The restraint imposed by stronger guesses complicates the individual strategy of each guess. Suppose that R_j is the strongest guess which is correct. For all $i < j$, we eventually stop believing that R_i is correct. If $s^* - 1$ is the last stage at which any R_i for $i < j$ receives attention (or s^* is the limit of the stages at which any R_i for $i < j$ receives attention) then at all stages $s \geq s^*$ we are required to let F_s extend F_{s^*}. This means that during stages $s \in J_j$ beyond s^*, the guess R_j must play its strategy on each $\mathcal{K}_{s^*}$-interval separately (as $\mathcal{K}_{s^*} = \operatorname{dom} F_{s^*}$). The set $\operatorname{range} F_{s^*}$ is a first approximation of the desired set of parameters Q.

This approximation to the definition of Q is not quite correct because of an annoying fact: Φ_j can be an isomorphism from $\mathcal{K}$ to $\mathcal{L}$ which does not extend F_{s^*}. Fix a cut (S_1, S_2) of $\mathcal{K}_{s^*}$. For $s \geq s^*$ (including $s = \omega_1$) we let $A_s = (S_1, S_2)_{\mathcal{K}_s}$; $B_s = (F_{s^*}[S_1], F_{s^*}[S_2])_{\mathcal{L}_s} = (F_s[S_1], F_s[S_2])_{\mathcal{L}_s}$; and $C_s = (\Phi_j[S_1], \Phi_j[S_2])_{\mathcal{L}_s}$. Disagreement between Φ_j and F_{s^*} could cause B_s and C_s to be distinct. If $s \in J_j$ then $\Phi_{j,s} \restriction A_s$ is an isomorphism from A_s to C_s. Since F_s extends F_{s^*}, the map $F_s \restriction A_s$ is an embedding of A_s into B_s, but we will not always be able to ensure that it is onto B_s. See Figure 3.

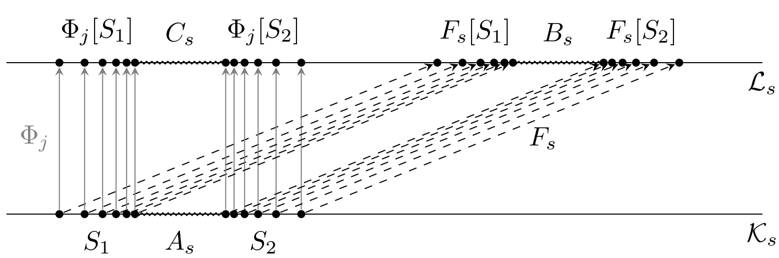

FIGURE 3. The intervals A_s, B_s and C_s.

Let $s \geq s^*$ be in J_j, and suppose, for example, that we want to force our opponent to enumerate a point in $\mathcal{L}$ between some points x and y in C_s (in order to make C_{ω_1} nonscattered, for example, or saturated). The only thing we can do is to enumerate a point in $\mathcal{K}_{s+1}$ between $\Phi_j^{-1}(x)$ and $\Phi_j^{-1}(y)$. However, we are required to let F_{s+1} map A_{s+1} into B_{s+1}, not C_{s+1}, and there may be no way to do that. In the two cases which occur in the construction for Theorem 2.4, we arrange the following.

- If C_s is nonscattered (and we are trying to saturate it), and B_s is nonscattered as well, then we can always embed A_{s+1} into B_{s+1}. So we just need to ensure that when we try to saturate C_s, B_s is nonscattered as well.
- If C_s is scattered, then we will ensure (see Claim 3.9 and the discussion following it) that C_s and B_s are isomorphic, indeed that $F_s \circ \Phi_j^{-1}$ gives an isomorphism from C_s to B_s (equivalently that $F_s \restriction A_s$ is onto B_s). In

this case, we could imagine that C_s and B_s are identical and carry out the scattering strategy of the previous construction.

Here we have two related tasks. The first is defining F_s at stages $s > s^*$ which are limit points of J_j.[1] The second is indeed ensuring that if A_s is scattered then $F_s \restriction A_s$ maps A_s onto B_s. Consider the difficulty of making F_s onto B_s. In the previous construction this issue was skirted by defining $F_s = \Phi_s$ at such stages s. In the current construction we cannot do this because of the restraint imposed on R_j, that F_s must extend F_{s^*}. Suppose for example that $s^* < s_0 < s_1 < \dots$ are stages in J_j, that C_{s_0} is nonscattered, and that at each stage s_n we enumerate points into A_{s_n+1} in order to make C_{ω_1} saturated. Let $s = \sup_n s_n$. At each stage s_n we use a self-embedding of B_{s_n} to redefine $F_{s_n+1} \restriction A_{s_n+1}$. There is no reason to believe that $F_{s_n} \restriction A_{s_n}$ reaches a limit. Indeed, again the only thing we can do at stage s is to notice that B_s is nonscattered and so let $F_s \restriction A_s$ be an arbitrarily chosen embedding of A_s into B_s; A_s and B_s may fail to be isomorphic, in which case F_s will not be onto B_s.

In the case that B_s is scattered, we need to ensure that $F_{s_n} \restriction A_{s_n}$ reaches a limit. If each F_{s_n} is onto B_{s_n}, then the limit $F_s \restriction A_s$ will be onto B_s. In other words, both tasks – defining F_s and ensuring it is onto B_s – will be successfully performed if we ensure that the maps $F_{s_n} \restriction A_{s_n}$ reach a limit. Indeed, we arrange that for $x \in A_{s_m}$, $F_{s_n}(x)$ changes only two or three times at stages $s_n > s_m$.

To do this we need to consider not only the intervals A_s, B_s and C_s, but also their *j-conjugates*. The idea is to ensure that there is a cut (S_1', S_2') such that $C_s(S_1', S_2') = B_s(S_1, S_2)$ and also a cut (S_1'', S_2'') such that $B_s(S_1'', S_2'') = C_s(S_1, S_2)$ and this is repeated. This is not automatically so, as Φ_j may not map $\mathcal{K}_{s^*}$ isomorphically onto $\mathcal{L}_{s^*}$. For this reason we need to increase the sets Q and $S = \Phi_j^{-1}[Q] = F_s^{-1}[Q]$ of parameters. Thus, the guess R_j needs to wait after stage s^* for a stage $t \in J_j$ at which $\mathcal{K}_{s^*}$ is contained in the range of $\Phi_j^{-1} \circ F_t$, $(\Phi_j^{-1} \circ F_t)^2$, and so on. To ensure that F_s eventually stabilizes to give this containment, in the meantime R_j may need to impose further restraint on weaker guesses. Once we have a set $S \subseteq \mathcal{K}_t$ which contains $\mathcal{K}_{s^*}$ and is invariant under $\Phi_j^{-1} \circ F_t$, we can fix both S and $F \restriction S$, let $Q = F_t[S] = \Phi_j[S]$, and define, for any cut (S_1, S_2) of S and $n \in \mathbb{Z}$, the conjugate cuts $(S_1, S_2)^n = \big((\Phi_j^{-1} \circ F_t)^n[S_1], (\Phi_j^{-1} \circ F_t)^n[S_2]\big)$. For $s \geq t$, letting $A_{n,s} = (S_1, S_2)^n_{\mathcal{K}_s}$ and similarly defining $B_{n,s}$ and $C_{n,s}$, we see that $B_{n,s} = C_{n,s+1}$ for all n. The intervals $A_{n,s}$ are called the j-conjugates of A_s. Note that if $\Phi_j[S_1] = F_t[S_1]$ then all the conjugates coincide, we have $C_s = B_s$, and so the situation for this interval is similar to the one in the proof of Theorem 2.4. Otherwise, all conjugates are disjoint (without loss of generality, $\Phi_j^{-1}(F_t(x)) \in S_1$ for some $x \in S_2$, in which case x separates $A_{0,s}$ and $A_{1,s}$). See Figure 4.

Once we have these conjugate intervals, we act on them (for descattering, saturating, etc.) at the same time. For descattering, the action is identical on all conjugates. If we wish to destroy adjacencies in some block D_0 of $C_{0,s}$, then at the same time we destroy the corresponding adjacencies of the corresponding blocks $D_n = (F_s \circ \Phi_j^{-1})^n[D_0]$ of $C_{n,s}$. This gives a multiplying effect, ensuring that by the next stage in J_j, not only has each adjacency in each D_n been broken, but in

[1] Defining F_s at other limit stages s is made simpler by various guesses R_i imposing further restraint on weaker guesses. This allows us then to pick a sequence of stages t cofinal in s on which F_t is increasing. See cases (A), (B) and (C) in the formal construction below, page 20.

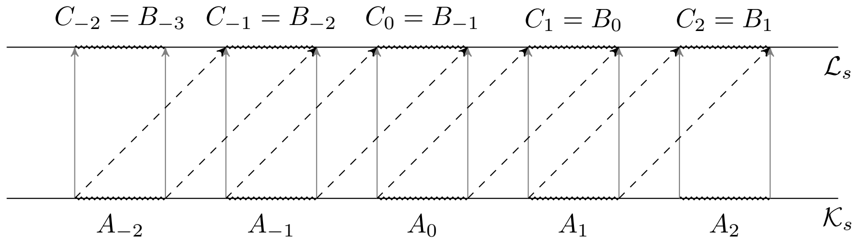

FIGURE 4. The conjugates of A_s.

fact infinitely many points must be inserted in between (see the proof of Claim 3.10 on page 23). In turn, this means that in further instances of descattering we can fix the points which have been affected before, and so not have to change F_s on them again. This gives the desired convergence of F_{s_n} above in the case that the intervals under discussion are scattered and infinite.

A new case not present in the proof of Theorem 2.4 is that of finite intervals. Let A_s be an interval as above. Consider the simple case that $\Phi_1[S_1] = F_t[S_1]$, so A_s coincides with all of its conjugates. At some stage u we may see that $|A_u| = |C_u| = m$ for some $m < \omega$, and then wish to enumerate $(\Phi_j[S_1], \Phi_j[S_2])$ into V_m, the set of cuts of Q which contains all the cuts that define intervals of $\mathcal{L}$ of size m. We only do this if there is evidence that this situation is stable; we want $u \in J_j$. We then need to ensure that if some points are going to be later added to C_{ω_1}, then C_{ω_1} is infinite. The idea is similar to the strategy for descattering and saturating. Restricted to A_u, Φ_1 and F_s agree and are the unique isomorphism between A_u and C_u. If at some stage $v > u$ some point is added to C_{v+1}, then to maintain isomorphism we must add a new point to A_{v+1}, but we can add this point in a place which doesn't match the new point in C_{v+1}. This precludes Φ_j from being an isomorphism between A_{v+1} and C_{v+1} (we need to change F_{v+1} on A_v, however). Since R_j is correct, this means that yet more points must be added later to C_s. Once this symmetry has been broken we can repeat this strategy until C_s is infinite. In fact, once the symmetry is broken we can keep matching the opponent without changing the values of F_s again. This allows the maps $F_{s_n} \upharpoonright A_{s_n}$ above to reach a limit in the case that each A_{s_n} is finite (but A_s is not).

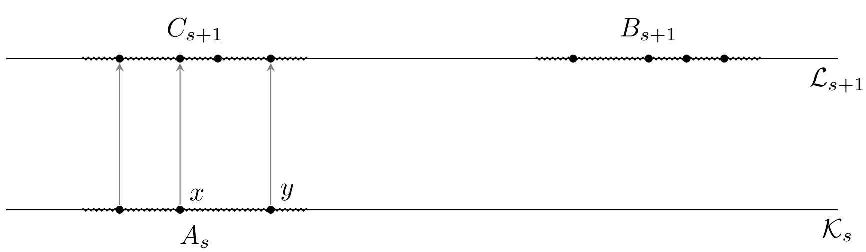

FIGURE 5. To diagonalize, a point should be added to A_{s+1} anywhere except between x and y. $F_{s+1} \upharpoonright A_{s+1}$ will be the unique order-preserving bijection between A_{s+1} and B_{s+1}.

Note, however, that implementing this strategy requires immediate action. When we observe the new point in C_{v+1} we must quickly respond, even if $v \notin J_j$. We cannot allow a weaker guess to act first, since that guess might reply to the new point in C_{v+1} by adding a matching point in A_{v+1}, restoring symmetry and thus allowing C_{ω_1} to be finite, but of size different from the one we guessed at first.

In the more complicated case that the conjugates of A_s are distinct, we again need to use the strategy of working with all conjugates simultaneously. An extra difficulty is, however, that at stages $v \notin J_j$, the various conjugate intervals $B_{n,s}$ need not be isomorphic. We did not have this problem when the $B_{n,s}$ are infinite since once they are infinite, we may restrict all action to stages in J_j. When we act, we need $|B_{v+1}| = |C_{v+1}| > |A_v|$ see Figure 5). We need to balance the need to limit action, so that F does not change on A_s too often (in fact, more than once) while A_s is finite; and the need to act quickly enough so that symmetry can be broken and never repaired. The correct mix is described in Definition 3.7.

These are the ideas behind the construction; we are now ready for the formalities.

Construction: Given an ω_1-computably categorical linear ordering $\mathcal{L}$, we define an increasing, continuous, and ω_1-computable sequence $\langle \mathcal{K}_s \rangle_{s<\omega_1}$ of countable linear orderings. For each $s < \omega_1$, we also define an embedding $F_s \colon \mathcal{K}_s \to \mathcal{L}_s$. If s is a successor ordinal, then F_s will actually be an isomorphism between $\mathcal{K}_s$ and $\mathcal{L}_s$.

Before we describe what we do at each stage, we define some auxiliary notions. At each stage s, we will decide (Definition 3.8) which guess R_j *requires attention* at stage s. The guess R_j will usually require attention at a stage s if $\Phi_{j,s}$ is an isomorphism between $\mathcal{K}_s$ and $\mathcal{L}_s$.

Definition 3.2. We let I_j be the collection of stages at which R_j requires attention.

We will require the sets I_j to be closed. This means that if s is a limit of stages at which R_j requires attention, then R_j requires attention at stage s as well. For simplicity, no guess R_j requires attention at a stage $s \le j$, so $I_j \subseteq (j, \omega_1)$. A guess R_j *receives attention* at stage s if j is least such that R_j requires attention at stage s. If a guess R_j requires attention at stage t, then all guesses R_i for $i > j$ are *initialized* at that stage. Auxiliary notions defined for R_i before that stage are abandoned and may be redefined at a later stage. One of the effects of this initialization is that until R_j itself is initialized, F_s will extend F_{t+1}. Formally, at a stage $s < \omega_1$ we let, for each $j \le s$,

$$r_{j,s} := \sup \left\{ t+1 \,:\, t \in \bigcup_{i<j} (I_i \cap s) \right\};$$

so $r_{j,s} \le s$. For $j > s$ we let $r_{j,s} = s$. The map $F_{r_{j,s}}$ is the restraint imposed on R_j at stage s. We will show (see page 21):

Claim 3.3. For all $s, j < \omega_1$, F_s extends $F_{r_{j,s}}$.

If R_j receives attention at stage s, then it is R_j's task to define F_{s+1}; the guess R_j must let F_{s+1} extend $F_{r_{j,s}}$.

Next we will describe the auxiliary object $S_{j,s}$. This is the stage s approximation to R_j's version of the eventual image of Q in $\mathcal{K}$. Once $S_{j,t}$ is defined, it remains fixed until R_j is initialized; so $S_{j,s} = S_{j,t}$ for all $s > t$ such that $r_{j,s} = r_{j,t}$. When R_j is initialized, $S_{j,s}$ becomes undefined. It will possibly be redefined at a later stage (at which R_j receives attention); at that stage s, $\Phi_{j,s}$ will be an isomorphism from $\mathcal{K}_s$ to $\mathcal{L}_s$. If s is a limit of stages at which R_j is initialized then $S_{j,s}$ is not defined at the beginning of stage s (but may be defined during that stage).

In general, consider a stage s at which $\Phi_{j,s}$ is an isomorphism from $\mathcal{K}_s$ to $\mathcal{L}_s$. Then $\Phi_{j,s}^{-1} \circ F_s$ is a self-embedding of $\mathcal{K}_s$. It may be a proper self-embedding because F_s may fail to be onto $\mathcal{L}_s$. Let

$$N_{j,s} := \bigcap_{n<\omega} \left(\Phi_{j,s}^{-1} \circ F_s\right)^n [\mathcal{K}_s].$$

So $N_{j,s}$ is the largest subset of $\mathcal{K}_s$ restricted to which $\Phi_{j,s}^{-1} \circ F_s$ is an automorphism. For brevity, we let

$$h_{j,s} := \left(\Phi_{j,s}^{-1} \circ F_s\right) \restriction N_{j,s}.$$

Dually, we let

$$M_{j,s} := F_s[N_{j,s}] = \Phi_{j,s}[N_{j,s}]$$

be the largest subset of $\mathcal{L}_s$ restricted to which the self-embedding $F_s \circ \Phi_{j,s}^{-1}$ of $\mathcal{L}_s$ is an automorphism; we let

$$g_{j,s} := \left(F_s \circ \Phi_{j,s}^{-1}\right) \restriction M_{j,s}.$$

The set $S_{j,s}$ has to contain $\mathcal{K}_{r_{j,s}} = \operatorname{dom} F_{r_{j,s}}$, but also be a subset of $N_{j,s}$. Hence we define the following.

Definition 3.4. Let $j < \omega_1$. We let J_j be the set of stages $s > j$ at which:

(1) $\Phi_{j,s}$ is an isomorphism from $\mathcal{K}_s$ to $\mathcal{L}_s$; and
(2) $\mathcal{K}_{r_{j,s}} \subseteq N_{j,s}$.

If $s \in J_j$ and R_j receives attention at stage s then unless already defined, R_j will define $S_{j,s}$ at that stage. Thus, if $s > j$ and R_j is not initialized at stage s, then $S_{j,s}$ is defined if and only if $J_j \cap [r_{j,s}, s]$ is nonempty. We will ensure (see page 21):

Claim 3.5. Let $j < t < s < \omega_1$. Suppose that $S_{j,t}$ is defined, and that R_j is not initialized between stages t and s (so $S_{j,s} = S_{j,t}$). Then F_s and F_t agree on $S_{j,t}$.

Claim 3.6. Let $j < s < \omega_1$. Suppose that $S_{j,s}$ is defined. Then:

(1) $\mathcal{K}_{r_{j,s}} \subseteq S_{j,s}$;
(2) If $\Phi_{j,s}$ is an isomorphism from $\mathcal{K}_s$ to $\mathcal{L}_s$, then $s \in J_j$, $S_{j,s} \subseteq N_{j,s}$ and $h_{j,s}[S_{j,s}] = S_{j,s}$.

Suppose that $S_{j,s}$ is defined for some $j < s < \omega_1$. We give notation to $S_{j,s}$-intervals of $\mathcal{K}_s$ and the corresponding intervals in $\mathcal{L}_s$ as was informally mentioned above. Let (S_1, S_2) be a cut of $S_{j,s}$. We let:

$$\begin{aligned} A_s(j, S_1, S_2) &:= (S_1, S_2)_{\mathcal{K}_s}, \\ B_s(j, S_1, S_2) &:= (F_s[S_1], F_s[S_2])_{\mathcal{L}_s}, \text{ and} \\ C_s(j, S_1, S_2) &:= (\Phi_j[S_1], \Phi_j[S_2])_{\mathcal{L}_s}. \end{aligned}$$

When j is understood from the context, we write $A_s(S_1, S_2)$. If also (S_1, S_2) is fixed then we simply write A_s for $A_s(j, S_1, S_2)$. We similarly write B_s and C_s. If $t < s$, $S_{j,t}$ is defined and R_j is not initialized between stages t and s (i.e., $r_{j,s} = r_{j,t}$), then $A_t = A_s \cap \mathcal{K}_t$ and $C_t = C_s \cap \mathcal{L}_t$; and since F_s and F_t agree on $S_{j,s} = S_{j,t}$, we also have $B_t = B_s \cap \mathcal{L}_t$. If $s \in J_j$ then $C_s = \Phi_j[A_s]$. We always have $B_s \supseteq F_s[A_s]$.

The correspondence between A_s, B_s and C_s can be iterated, again as mentioned above. Let $A_{0,s}(S_1,S_2) = A_s(S_1,S_2)$. There is a unique interval $A_s(S_1',S_2')$ such that $B_s(S_1,S_2) = C_s(S_1',S_2')$; the cut (S_1',S_2') is defined by $S_1' = \Phi_j^{-1}[F_s[S_1]]$, in other words, $S_1' = h_{j,t}[S_1]$ for any $t \in J_j \cap [r_{j,s}, s]$. Here we use Claim 3.6. We let $A_{1,s}(S_1,S_2) = A_s(S_1',S_2')$ and iterate, so $A_{2,s}(S_1,S_2) = A_{1,s}(S_1',S_2')$ and so on; and $A_{-1,s}(S_1',S_2') = A_s(S_1,S_2)$ and so on. Formally, choosing any $t \in J_j \cap [r_{j,s}, s]$, for $n \in \mathbb{Z}$,

$$A_{n,s}(j,S_1,S_2) := A_s\left(j,(h_{j,t})^n[S_1],(h_{j,t})^n[S_2]\right);$$

we again usually omit j and even (S_1,S_2). We call the intervals $A_{n,s}$ the *j-conjugates* of A_s. If $s \in J_j$ and $A_s \subseteq N_{j,s}$, then $A_{n,s} = (h_{j,s})^n[A_s]$, so in this case, the j-conjugates of A_s are precisely the elements of the orbit of A_s under the action of $h_{j,s}$ on the subsets of $N_{j,s}$. We let $B_{n,s}$ and $C_{n,s}$ be the intervals corresponding to $A_{n,s}$; we have $B_{n,s} = C_{n+1,s}$ for all $n \in \mathbb{Z}$.

There are two possibilities: (1) $(S_1',S_2') = (S_1,S_2)$; in this case, for all n, $A_{n,s}(S_1,S_2) = A_s(S_1,S_2)$ (and for all n, $B_{n,s} = C_{n,s} = B_s = C_s$); or (2) either $S_2 \cap S_1'$ or $S_1 \cap S_2'$ is nonempty; in this case the intervals $\{A_{n,s} : n \in \mathbb{Z}\}$ are pairwise disjoint (and the intervals $\{B_{n,s} : n \in \mathbb{Z}\}$ are also pairwise disjoint).

Having defined the conjugates of an interval A_s, we discuss the instances at which a guess R_j would like to diagonalize on a finite $S_{j,s}$-interval A_s.

Definition 3.7. Let $j < s < \omega_1$, and suppose that $S_{j,s}$ is defined. Let A_s be a nonempty $S_{j,s}$-interval of $\mathcal{K}_s$. We say that R_j *diagonalizes on A_s, with m points* (at stage s), if R_j receives attention at stage s, $m = |B_{s+1}| = |C_{s+1}| > |A_s|$, and $F_{s+1} \restriction A_{s+1}$ does not extend $F_s \restriction A_s$. This happens because R_j adds points to A_s so that $\Phi_{j,s}$ cannot be extended to an isomorphism between A_{s+1} and C_{s+1}.

We say that R_j *has an opportunity to diagonalize on A_s (with m points)* if $m = |B_{s+1}| = |C_{s+1}| > |A_s|$, there is a stage $t \in J_j \cap [r_{j,s}, s]$ at which $A_t = A_s \cap \mathcal{K}_t$ is nonempty, and:

- R_j did not diagonalize on $A_r = A_s \cap \mathcal{K}_r$ (with any number of points) at any stage $r \in [r_{j,s}, s)$; and
- for any j-conjugate A_s' of A_s, R_j did not diagonalize on $A_r' = A_s' \cap \mathcal{K}_r$ with m points at any stage $r \in [r_{j,s}, s)$.

Finally, we can now describe when a guess R_j requires attention.

Definition 3.8. Let $j < s < \omega_1$. A guess R_j requires attention at stage s if one of the following holds:

- $\Phi_{j,s}$ is an isomorphism from $\mathcal{K}_s$ to $\mathcal{L}_s$; or
- s is a limit ordinal and $I_j \cap s$ is unbounded in s; or
- $S_{j,s}$ is defined, and R_j has an opportunity to diagonalize on some finite $S_{j,s}$-interval at stage s.

Having described most of the auxiliary notions, we can now describe the construction. We start with $\mathcal{K}_0 := \mathcal{L}_0$ being the empty ordering, and F_0 being the empty function. At stage s of the construction we define $\mathcal{K}_{s+1}$ and F_{s+1}. If s is a successor ordinal then $\mathcal{K}_s$ and F_s will have been defined at the previous stage. If s is a limit ordinal then before defining $\mathcal{K}_{s+1}$ and F_{s+1} we first need to define F_s, letting $\mathcal{K}_s = \bigcup_{t<s} \mathcal{K}_t$.

In the description of what we do at stage s we, of course, use the auxiliary notions described above, which in turn requires the claims we have stated (3.3, 3.5 and 3.6; more will be stated below). This means that both the construction and the claims are defined and verified by simultaneous induction on the stage. At stage s, we assume that the construction has been defined up to that stage, and that the claims hold up to that stage, and then define what we do at that stage; after we specify these instructions, we will verify that the claims continue to hold at the end of the stage.

We first describe how to define $\mathcal{K}_{s+1}$ and F_{s+1} assuming that both $\mathcal{K}_s$ and F_s have already been defined.

If no guess requires attention at stage s, then we let $\mathcal{K}_{s+1}$ be an extension of $\mathcal{K}_s$ such that there is some isomorphism $F_{s+1}\colon \mathcal{K}_{s+1} \to \mathcal{L}_{s+1}$ extending F_s. Otherwise, let R_j be the guess which receives attention at stage s.

Now there are a couple of cases. If $S_{j,s}$ is not yet defined, and $s \notin J_j$, then we act as if R_j did not receive attention: we again let $\mathcal{K}_{s+1}$ be an extension of $\mathcal{K}_s$ such that there is some isomorphism $F_{s+1}\colon \mathcal{K}_{s+1} \to \mathcal{L}_{s+1}$ extending F_s. The reason for R_j officially receiving attention at this stage is merely to impose restraint on weaker guesses.

Next, suppose that $S_{j,s}$ is not yet defined, but that $s \in J_j$. In this case we define $S_{j,s}$ to conform with Claim 3.6:

$$S_{j,s} := \bigcup_{n\in\mathbb{Z}} (h_{j,s})^n[\mathcal{K}_{r_{j,s}}].$$

This is the smallest subset of $N_{j,s}$ containing $\mathcal{K}_{r_{j,s}}$ which is closed under the action of $h_{j,s}$. In this case, too, we end the stage and let F_{s+1} extend F_s.

Now suppose that $S_{j,s}$ is already defined. We will let F_{s+1} agree with F_s on $S_{j,s}$. To define $\mathcal{K}_{s+1}$ and F_{s+1}, we will define, for every nonempty $S_{j,s}$-interval A_s of $\mathcal{K}_s$, an isomorphism $F_{s+1} \restriction A_{s+1}$ between A_{s+1} (which we define) and the corresponding B_{s+1} (which the opponent plays). [Note that by definition of B_s, $F_s \restriction A_s$ is an embedding of A_s into B_s.] Exactly how to define A_{s+1} and $F_{s+1} \restriction A_{s+1}$ depends on the order-type of A_s. We will consider all j-conjugates of an interval simultaneously. That we can do so relies on the following claim.

Claim 3.9. Let $j < s < \omega_1$. Suppose that $S_{j,s}$ is defined (in particular, R_j is not initialized at stage s); let A_s be an $S_{j,s}$-interval. If A_s is scattered (either finite or infinite) then $F_s \restriction A_s$ is onto B_s.

Since $F_s \restriction A_s$ is always an embedding of A_s into B_s, this means that if A_s is scattered then $F_s \restriction A_s$ is an isomorphism of A_s with B_s. Suppose in addition that $s \in J_j$. Consider the j-conjugates of A_s. Suppose that $A_{n,s}$ is scattered for some $n \in \mathbb{Z}$. Then $F_s \restriction A_{n,s}$ is an isomorphism of $A_{n,s}$ to $B_{n,s}$, and so $(\Phi_j^{-1} \circ F_s) \restriction A_{n,s}$ is an isomorphism from $A_{n,s}$ to $A_{n+1,s}$. It follows, of course, that $A_{n+1,s}$ is scattered, too. Similarly, $C_{n,s} = B_{n-1,s}$ is isomorphic to $A_{n,s}$ by Φ_j, and so $B_{n-1,s}$ is scattered. Since $F_s \restriction A_{n-1,s}$ is an embedding of $A_{n-1,s}$ into $B_{n-1,s}$, it follows that $A_{n-1,s}$ is scattered. Thus, for all $n \in \mathbb{Z}$, $A_{n,s}$ is scattered; $A_{n,s} \subseteq N_{j,s}$ for all $n \in \mathbb{Z}$, and similarly, $B_{n,s} \subseteq M_{j,s}$ for all $n \in \mathbb{Z}$; all the intervals $A_{n,s}$ are isomorphic by repeatedly applying $h_{j,s}$, and all the $B_{n,s}$ are isomorphic by repeatedly applying $g_{j,s}$.

Fix an $S_{j,s}$-interval A_s and its j-conjugates $A_{n,s}$.

0. Unless one of the cases below holds, we will extend *simply*, that is, let A_{s+1} be any extension of A_s for which there is an isomorphism from A_{s+1} to B_{s+1} extending F_s, and let $F_{s+1} \upharpoonright A_{s+1}$ be any such isomorphism. In particular, we will extend simply if $s \notin J_j$ and A_s is infinite.

1. Suppose that $s \in J_j$ and that A_s is nonscattered. By Claim 3.9 and the discussion which follows it, each $A_{n,s}$ and each $B_{n,s}$ is nonscattered. For each $n \in \mathbb{Z}$, let $(X_{n,s}, Y_{n,s})$ be the $<_{\omega_1}$-least pair of countable subsets (X, Y) of $C_{n,s}$ such that $X <_{\mathcal{L}_s} Y$ and $(X, Y)_{\mathcal{L}_s}$ is empty. For each n, enumerate a new point into $A_{n,s+1}$ between $\Phi_j^{-1}[X_{n,s}]$ and $\Phi_j^{-1}[Y_{n,s}]$. Since $B_{n,s}$ is nonscattered, we can find an embedding of $A_{n,s}$, with the extra point added, into $B_{n,s}$. We can thus add points to define $A_{n,s+1}$ to be isomorphic to $B_{n,s+1}$ and let $F_{s+1} \upharpoonright A_{n,s+1}$ be any isomorphism between $A_{n,s+1}$ and $B_{n,s+1}$.

2. Suppose that R_j has an opportunity to diagonalize on A_s with m points at stage s (so B_s is finite), and that A_s is the $<_{\omega_1}$-least such interval among its j-conjugates.

As $|A_s| < m = |C_{s+1}|$, the map $\Phi_{j,s} \upharpoonright A_s$ is not onto C_s. We can extend A_s to an ordering A_{s+1} of size m such that $\Phi_{j,s}$ cannot be extended to an isomorphism between A_{s+1} and C_{s+1}. To see this, by the definition of having an opportunity to diagonalize (Definition 3.7), we take a stage $t \in J_j \cap [r_{j,s}, s]$ such that A_t is nonempty. As $t \in J_j$, we know that $\Phi_{j,t}$ is an isomorphism of $\mathcal{K}_t$ and $\mathcal{L}_t$, and so $A_t \subseteq \operatorname{dom} \Phi_{j,t}$; as $\Phi_{j,s}$ extends $\Phi_{j,t}$, we see that $A_s \cap \operatorname{dom} \Phi_{j,s}$ is nonempty.

Since $|C_{s+1}| > |A_s|$, there is some cut (D, E) of A_t such that $(D, E)_{A_s}$ is smaller than $(\Phi_j[D], \Phi_j[E])_{C_{s+1}}$. Since A_t is nonempty, $(D, E)_{A_s}$ is not the only interval of A_s, so we can add points to A_{s+1} elsewhere, so that A_{s+1} contains m points, but $(D, E)_{A_{s+1}} = (D, E)_{A_s}$. Then $\Phi_{j,t} \upharpoonright A_t$ cannot be extended to an isomorphism between A_{s+1} and C_{s+1}. See Figure 5.

This defines A_{s+1}; since $|A_{s+1}| = |B_{s+1}| = m$, we let $F_{s+1} \upharpoonright A_{s+1}$ be the unique isomorphism between A_{s+1} and B_{s+1}.

It is important that at stage s, we do not let R_j diagonalize on any j-conjugate of A_s with m points other than A_s itself; this is why we demanded that A_s be the $<_{\omega_1}$-least such interval among its j-conjugates. So if A'_s is a j-conjugate of A_s, distinct from A_s, and at stage s, R_j has the opportunity to diagonalize on A'_s with m points, then we do not let R_j do so, but rather extend A'_s simply as in (**0**) above. During stage s, we may diagonalize with a different number of points on other j-conjugates of A_s.

3. If $s \in J_j$ and A_s is infinite and scattered, we again treat all of the conjugates $A_{n,s}$ of A_s in one step. Again, these are all isomorphic by powers of $h_{j,s}$.

We let $t^{\mathtt{inf}}_{j,s}(A_s)$ be the least $t \in J_j \cap [r_{j,s}, s]$ such that A_t is infinite. Note that the argument following Claim 3.9 shows that this does not depend on the choice of A_n among its j-conjugates: for all n, $t^{\mathtt{inf}}_{j,s}(A_{n,s}) = t^{\mathtt{inf}}_{j,s}(A_s)$.

Let $D_{0,s}$ be the $<_{\omega_1}$-least *maximal* infinite block of $B_{0,s}$. For $n \in \mathbb{Z}$, let $D_{n,s} = (g_{j,s})^n[D_{0,s}]$; so $D_{n,s}$ is a maximal infinite block of $B_{n,s}$.

Claim 3.10. There are self-embeddings $f_{n,s}$ of $D_{n,s}$ with the following four properties.

- Coherence: The functions $f_{n,s}$ are coherent with respect to $g_{j,s}$: For all n and m, $f_{n+m,s} = (g_{j,s})^m \circ f_{n,s} \circ (g_{j,s})^{-m}$.

- Fixed Points: For all n, the set $E_{n,s} := \{a \in D_{n,s} : f_{n,s}(a) = a\}$ is a finite (possibly empty) convex subset of $D_{n,s}$. Note that the coherence of the functions $f_{n,s}$ shows that for all n and m, $E_{n+m,s} = (g_{j,s})^m[E_{n,s}]$.
- Historical Responsibility: For all n, if $a \in D_{n,s}$ and there is some stage $u \in I_j \cap [t^{\mathtt{inf}}_{j,s}(A_s), s)$ such that $a \in B_{n,u}$ and $F^{-1}_{u+1}(a) \neq F^{-1}_u(a)$, then $a \in E_{n,s}$. Dually, if $F_s(x) \in D_{n,s}$ and there is some stage $u \in I_j \cap [t^{\mathtt{inf}}_{j,s}(A_s), s)$ such that $x \in A_{n,s}$ and $F_{u+1}(x) \neq F_u(x)$, then $F_s(x) \in E_{n,s}$.
- Interpolation: For all n, $a \in D_{n,s} \setminus E_{n,s}$ and $b \in D_{n,s}$ distinct from a, $f_{n,s}(a)$ and $f_{n,s}(b)$ are not adjacent in $D_{n,s}$.

For the third property, note that if $u \in I_j \cap [t^{\mathtt{inf}}_{j,s}(A_s), s)$ and $a \in B_{n,u}$ then by Claim 3.9 (as $A_{n,u}$ is scattered), $a \in \operatorname{range} F_u$.

We fix maps $f_{n,s}$ as given by Claim 3.10, and extend them to all of $B_{n,s}$ by the identity on $B_{n,s} \setminus D_{n,s}$; this is a self-embedding of $B_{n,s}$ since $D_{n,s}$ is a convex subset of $B_{n,s}$. For all $n \in \mathbb{Z}$, we let $A_{n,s+1}$ be an extension of $A_{n,s}$ and $F_{s+1} \upharpoonright A_{n,s+1}$ an extension of $f_{n,s} \circ (F_s \upharpoonright A_{n,s})$ to an isomorphism from $A_{n,s+1}$ to $B_{n,s+1}$.

This completes the instructions for stage s, given $\mathcal{K}_s$ and F_s. At limit stages s, we need to define F_s; we already stipulated that $\mathcal{K}_s = \bigcup_{t<s} \mathcal{K}_t$.

There are four cases.

A. Suppose that $r_{s,s} < s$. So between stages $r_{s,s}$ and s, no guess requires attention. Then our instructions show that for all $t < t'$ in $(r_{s,s}, s)$, $F_{t'}$ extends F_t. In this case we let $F_s = \bigcup_{t \in (r_{s,s}, s)} F_t$.

B. If (A) fails, we let j be the least ordinal $j \leq s$ such that $r_{j,s} = s$. Suppose that j is a limit ordinal. Let $T = \{r_{i,s} : i < j\}$. The set T is unbounded in s. Let $t < t'$ be elements of T, and let i be such that $t = r_{i,s}$. Since $t' \in (r_{i,s}, s)$, $t = r_{i,t'}$, and so Claim 3.3 says that $F_{t'}$ extends F_t. Hence we can let $F_s = \bigcup_{u \in T} F_u$.

C. If both (A) and (B) fail, then there is some (unique) $j < s$ such that $r_{j,s} < s$ but $I_j \cap s$ is unbounded in s. Suppose that $J_j \cap [r_{j,s}, s)$ is empty: $S_{j,t}$ is not defined for any $t \in [r_{j,s}, s)$. Let $t \in I_j \cap [r_{j,s}, s)$. Since R_j receives attention at stage t, the instructions show that F_{t+1} extends F_t. By Claim 3.3, if t' is the next element of I_j beyond t, then $F_{t'}$ extends F_{t+1}, as $r_{j+1,t'} = t+1$. Hence, for all $t < t'$ in $I_j \cap [r_{j,s}, s)$, $F_t \subseteq F_{t+1} \subseteq F_{t'} \subseteq F_{t'+1}$. We thus let $F_s = \bigcup_{t \in I_j \cap [r_{j,s}, s)} F_t = \bigcup_{t \in I_j \cap [r_{j,s}, s)} F_{t+1}$.

D. Otherwise, we again take $j < s$ such that $r_{j,s} < s$ but $I_j \cap s$ is unbounded in s; and now we suppose that $J_j \cap [r_{j,s}, s)$ is nonempty.

Thus, $S_{j,t}$ is defined at stage $w = \min J_j \cap [r_{j,s}, s)$ and we have $S_{j,t} = S_{j,w}$ for all $t \in [w, s)$. Recall that Claim 3.5 says that $F_t \upharpoonright S_{j,w}$ is constant for $t \in [w, s)$. We then let F_s extend this map. To define the rest of F_s, we need to define F_s on any nonempty $S_{j,w}$-interval A_s of $\mathcal{K}_s$.

Let A_s be an $S_{j,w}$-interval of $\mathcal{K}_s$. If B_s is nonscattered then we let $F_s \upharpoonright A_s$ be any embedding of A_s into B_s. Suppose that B_s is scattered. Then for all $u \in [w, s)$, $B_u = B_s \cap \mathcal{L}_u$ is scattered. Since $F_u \upharpoonright A_u$ is an embedding of A_u into B_u, A_u is scattered as well. By Claim 3.9, for $u \in J_j \cap [w, s)$, A_u is isomorphic to B_u by $F_u \upharpoonright A_u$. We require the following fact.

Claim 3.11. Let $j < t < s < \omega_1$, with $t \in I_j$. Suppose that $S_{j,t}$ is defined and that R_j is not initialized between stages t and s. Let A_s be a scattered $S_{j,w}$-interval.

(1) For each $x \in A_t$ there are at most two stages $u \in [t, s)$ at which $F_{u+1}(x) \neq F_u(x)$.
(2) For each $a \in B_t$ there are at most two stages $u \in [t, s)$ at which $F_{u+1}^{-1}(a) \neq F_u^{-1}(a)$.

(Again note that in (2), for all $u \in [t, s)$, $a \in \operatorname{range} F_u$ by Claim 3.9).

Because for all $x \in A_s$ we can find a stage $t \in I_j \cap [w, s)$ such that $x \in A_t$, we see that for all $x \in A_s$ we can let $F_s(x)$ be the limit $\lim_{u \to s} F_u(x)$. It is easy to see that $F_s \restriction A_s$ is order-preserving (and in fact, onto B_s; this will help us prove Claim 3.9).

This completes the construction of $\mathcal{K}$ and of the sequence $\langle F_s \rangle_{s < \omega_1}$.

Promises Were Made: As discussed above, to carry out the construction, we relied on various facts about the construction itself. We now establish these facts, by a global induction on the stages. We begin with an observation.

Lemma 3.12. *Let $j < i < s$ and suppose that both $S_{j,s}$ and $S_{i,s}$ are defined. Then $S_{j,s} \subseteq S_{i,s}$.*

Proof. Let $t = \min J_j \cap [r_{j,s}, s]$ be the stage at which $S_{j,s}$ is defined. Since R_j receives attention at stage t, $r_{i,s} > t$. So $S_{j,s} = S_{j,t} \subseteq \mathcal{K}_t \subseteq \mathcal{K}_{r_{i,s}}$. By Claim 3.6, $\mathcal{K}_{r_{i,s}} \subseteq S_{i,s}$. □

Proof of Claim 3.5. Suppose first that s is a successor stage. By induction, F_{s-1} and F_t agree on $S_{j,t}$, so in this case we just need to show that F_{s-1} and F_s agree on $S_{j,t}$. If F_s extends F_{s-1} we are, of course, done. Suppose, then, that F_s is not an extension of F_{s-1}. This means that at stage $s-1$, some requirement R_i received attention, and $s-1 \in J_i$. Since R_j was not initialized at stage $s-1$, we must have $i \geq j$. By Lemma 3.12, $S_{j,t} = S_{j,s-1} \subseteq S_{i,s-1}$. The instructions for R_i at stage $s-1$ ensure that F_s and F_{s-1} agree on $S_{i,s-1}$, and so agree on $S_{j,t}$.

Suppose that s is a limit stage. In cases (A), (B) and (C) of the definition of F_s, we let F_s be the union of F_u for some u in a set cofinal in s. In these cases, as for all $u \in (t, s)$, F_u and F_t agree on $S_{j,t}$, we have F_s and F_t agree on $S_{j,t}$. In case (D), suppose that R_i defined F_s, that is, $r_{i,s} < s$ but $I_i \cap s$ is unbounded in s. Since $r_{j,s} < s$, we must have $j \leq i$, so $S_{j,t} \subseteq S_{i,s}$, and $F_s \restriction S_{i,s} = F_u \restriction S_{i,s}$ for a set of u cofinal in s. For such u, by induction, F_u agrees with F_t on $S_{j,t}$ and so F_s also agrees with F_t on $S_{j,t}$. □

Proof of Claim 3.3. Of course, if $r_{j,s} = s$ then we are done. Hence, we assume that $r_{j,s} < s$.

First, suppose that s is a successor stage. Then $r_{j,s-1} = r_{j,s}$. By induction, F_{s-1} extends $F_{r_{j,s}}$. If F_s extends F_{s-1}, then the claim holds at s. Suppose that F_s does not extend F_{s-1}. Let R_i be the guess which receives attention at stage $s-1$; then $S_{i,s-1}$ is defined. Since $r_{j,s} < s$, $i \geq j$. Hence $r_{i,s-1} \geq r_{j,s-1} = r_{j,s}$. As argued in the proof of Lemma 3.12, this means that $\mathcal{K}_{r_{j,s}}$ is contained in $S_{i,s-1}$, so $F_{s-1} \restriction S_{i,s-1}$ extends $F_{r_{j,s}}$. At stage $s-1$, R_i is instructed to let F_s agree with F_{s-1} on $S_{i,s-1}$; so F_s extends $F_{r_{j,s}}$.

Next, suppose that s is a limit ordinal. Again we consider the cases defining F_s. If $F_s = \bigcup_{t \in T} F_t$, where T is cofinal in T, then F_s extends $F_{r_{j,s}}$, as by induction, F_t extends $F_{r_{j,s}}$ for $t \in [r_{j,s}, s)$. In case (D), let R_i be the guess which is responsible for defining F_s. Since $r_{j,s} < s$, we have $i \geq j$. Let $t \in J_i \cap [r_{i,s}, s)$. F_s agrees with F_t

on $S_{i,s} = S_{i,t}$. Since $r_{i,s} \geq r_{j,s}$, the set $S_{i,s}$ contains $\mathcal{K}_{r_{j,s}}$. By induction, F_t extends $F_{r_{j,s}}$. So $F_t \upharpoonright S_{i,t}$ extends $F_{r_{j,s}}$, and so F_s extends $F_{r_{j,s}}$. □

Proof of Claim 3.6. Let $t = \min(J_j \cap [r_{j,s}, s))$. This is the stage at which $S_{j,s}$ was defined. We have $r_{j,s} = r_{j,t}$. At stage t, we define $S_{j,t}$ to be a superset of $\mathcal{K}_{r_{j,t}}$. This establishes (1).

By definition, $S_{j,t} \subseteq N_{j,t}$, and $(\Phi_{j,t}^{-1} \circ F_t)[S_{j,t}] = S_{j,t}$. By Claim 3.5, $F_s \upharpoonright S_{j,t} = F_t \upharpoonright S_{j,t}$. Of course, $\Phi_{j,s}$ extends $\Phi_{j,t}$. Thus $(\Phi_{j,s}^{-1} \circ F_s)[S_{j,t}] = S_{j,t}$. This shows that $S_{j,t} \subseteq N_{j,s}$. Since $\mathcal{K}_{r_{j,s}} \subseteq S_{j,s}$, this shows that $s \in J_j$. □

Proof of Claim 3.11. Let $t \in I_j$ with $t < s$. We first note that if $u \in I_j \cap [t, s)$ and $u' := \min(I_j \cap (u, s])$ is the successor of u in I_j, then $r_{j+1,u'} = u + 1$ and so (Claim 3.3) $F_{u'}$ extends F_{u+1}. Also, if $u \in (t, s]$ is a limit ordinal, then the claim holds at u by induction. It suffices, then, to show:

(1) For all $x \in A_t$ there are at most two stages $u \in I_j \cap [t, s)$ such that $F_{u+1}(x) \neq F_u(x)$.
(2) For all $a \in B_t$ there are at most two stages $u \in I_j \cap [t, s)$ such that $F_{u+1}^{-1}(x) \neq F_u^{-1}(x)$.

Both follow directly from our instructions. If $u \in I_j \cap [t, s)$ then (as $w \geq r_{j,s}$) R_j receives attention at stage u. Let $x \in A_t$ and $a \in B_t$.

We first note that there is at most one stage $u \in I_j \cap [t, s)$ at which B_u is finite and at which $F_{u+1} \upharpoonright A_u$ does not extend $F_u \upharpoonright A_u$. This is by the definition of having an opportunity to diagonalize on a finite interval (Definition 3.7).

The "historical responsibility" property of the functions $f_{n,u}$ shows that there is at most one stage $u \in I_j \cap [t, s)$ such that B_u is infinite and such that $F_{u+1}^{-1}(a) \neq F_u^{-1}(a)$. To see this, let u be such a stage. At every stage $v \in I_j \cap (u, s)$, as $u \geq t_{j,v}^{\mathtt{inf}}(A_v)$, this property ensures that $f_{n,v}(a) = a$, where $F_{v+1} \upharpoonright A_{v+1}$ extends $f_{n,v} \circ (F_v \upharpoonright A_v)$, so $F_{v+1}^{-1}(a) = F_v^{-1}(a)$. Similarly, there is at most one stage $u \in I_j \cap [t, s)$ such that B_u is infinite and $F_{u+1}(x) \neq F_u(x)$. This completes the proof of the claim.

We note that it is not necessarily the case that for $x \in A_s$, there are at most two stages $u \in [r_{j,s}, s)$ such that $F_{u+1}(x) \neq F_u(x)$. This is because $F_u(x)$ could change often on the interval $[v, r)$, where v is the least stage such that $x \in A_v$ and $r := \min(I_j \cap (v, s))$ is v's successor in I_j. It is true that $F_u(x)$ changes at most finitely many times on this interval, but we do not need this fact. □

Proof of Claim 3.9. Let A_s be an $S_{j,s}$-interval and let B_s be the corresponding interval in $\mathcal{L}_s$. We need to show that $F_s \upharpoonright A_s$ is onto B_s, equivalently $B_s \subseteq \operatorname{range} F_s$. If s is a successor stage, then F_s is onto $\mathcal{L}_s$ by construction. Since F_s is order-preserving, it follows that $F_s \upharpoonright A_s$ is onto B_s. Suppose then that s is a limit stage.

Consider the cases defining F_s. We claim that in cases (A), (B) and (C), F_s is the union of maps F_u where u ranges over a set T of *successor* stages cofinal in s; this would imply that in these cases, too, F_s is onto $\mathcal{L}_s$. In case (A) we can let T be the collection of all successor ordinals in $(r_{s,s}, s)$. In case (C) we let T be the set of successor ordinals in $I_i \cap [r_{i,s}, s)$. In case (B) let i be the least such that $r_{i,s} = s$. In the construction, we let $T' = \{r_{k,s} : k < i\}$ and we let F_s be the union of F_u for $u \in T'$. While T' may contain limit stages, we show that T' contains a cofinal subset T consisting of successor stages. For $k < i$, let $\alpha_{k,s} = \sup_{v\in I_k\cap s}(v+1) = \max(I_k \cap s)+1$ (recall that I_k is closed). For all $k \le i$, $r_{k,s} = \sup_{k'<k}\alpha_{k',s}$. Let T be the set of stages $\alpha_{k,s}$ such that $\alpha_{k,s} > r_{k,s}$. Certainly T consists of successor ordinals. The fact that $s = \sup_{k<i}\alpha_{k,s}$ shows that T is unbounded in s. And if $\alpha_{k,s} \in T$ then $r_{k+1,s} = \alpha_{k,s}$ and $r_{k+1,s} \in T'$. Thus $T \subseteq T'$.

We discuss case (D). Let $i < s$ such that $r_{i,s} < s$ but $I_i \cap s$ is unbounded in s. Hence $s \in I_i$. By the assumption that R_j is not initialized at stage s, we must have $j \le i$. Let A'_s be a scattered $S_{i,s}$-interval. Let $w := \min(J_i \cap [r_{i,s}, s))$. We define $F_s \restriction A'_s$ to be the limit of $F_u \restriction A'_u$ for $u \in [w, s)$. Let $a \in B'_s$. There is a stage $v \in [w, s)$ such that $a \in B'_v$. Since for all $u \in [w, s)$, A'_u is scattered, Claim 3.11 implies that $F_u^{-1}(a)$ (which exists by induction) is constant on a final segment of s, and so $a \in \operatorname{range} F_s$. Thus $F_s \restriction A'_s$ is onto B'_s. Hence if $i = j$ then we are done.

Suppose that $j < i$. In this case the point is that every scattered $S_{j,s}$-interval is the union of scattered $S_{i,s}$-intervals and some points from $S_{i,s}$. By Lemma 3.12, $S_{j,s} \subseteq S_{i,s}$. Let $a \in B_s$. If $a \in F_s[S_{i,s}]$ then, of course, $a \in \operatorname{range} F_s$. Otherwise, there is an $S_{i,s}$-interval $A'_s \subseteq A_s$ such that $a \in B'_s$. Since $A'_s \subseteq A_s$, A'_s is scattered. In the previous paragraph we observed that $F_s[A'_s] = B'_s$ so in this case, too, $a \in \operatorname{range} F_s$. □

Proof of Claim 3.10. Let U_s be the set of $x \in \bigcup_n A_{n,s}$ such that there is some $t \in I_j \cap [t^{\mathtt{inf}}_{j,s}(A_s), s)$ such that $F_{t+1}(x) \ne F_t(x)$; and dually, let V_s be the set of $a \in \bigcup_n B_{n,s}$ such that there is some $t \in I_j \cap [t^{\mathtt{inf}}_{j,s}(A_s), s)$ such that $F^{-1}_{t+1}(a) \ne F^{-1}_t(a)$; noting again, of course, that for all n and m, $t^{\mathtt{inf}}_{j,s}(A_{n,s}) = t^{\mathtt{inf}}_{j,s}(A_{m,s})$ as that stage is in J_j. We claim that V_s is invariant under $g_{j,s}$, that $V_s = \Phi_{j,s}[U_s] = F_s[U_s]$, and that for all n, $V_s \cap D_{n,s}$ is at most a singleton.

For this, consider a stage $t \in J_j \cap [t^{\mathtt{inf}}_{j,s}(A_s), s)$ (assuming that $t^{\mathtt{inf}}_{j,s}(A_s) < s$) at which $F_{t+1}(x) \ne F_t(x)$ for some $x \in \bigcup_n A_{n,t}$ (equivalently at which $F^{-1}_{t+1}(a) \ne F^{-1}_t(a)$ for some $x \in \bigcup_n B_{n,t}$). Let $u := \min(J_j \cap (t, s])$ be t's successor in J_j. If $v \in I_j \cap (t, u)$ then at stage v, R_j is instructed to let $F_{v+1} \restriction A_{v+1}$ extend $F_v \restriction A_v$. It follows that $F_u \restriction A_u$ extends $F_{t+1} \restriction A_{t+1}$. Since $t \ge t^{\mathtt{inf}}_{j,s}(A_s)$, at stage t we act for A_t and its j-conjugates as in option (3), so we make use of maximal blocks $D_{n,t}$ and self-embeddings $f_{n,t}$ as given by this claim at stage t. As above, for all $n \in \mathbb{Z}$ and all $a \in E_{n,t}$, $F^{-1}_{t+1}(a) = F^{-1}_t(a)$.

We show that for all $n \in \mathbb{Z}$, and all distinct $a < b \in D_{n,t}$ which are not both in $E_{n,t}$, the interval $(a, b)_{B_{n,u}}$ is infinite. We prove, by induction on $m \ge 0$, that each such interval contains at least m points; the base case is vacuous. Assume we showed this for $m \ge 0$. Let $n \in \mathbb{Z}$ and let $a < b \in D_{n,t}$, not both in $E_{n,t}$. Let $x := \Phi^{-1}_{j,t}(a)$ and $y := \Phi^{-1}_{j,t}(b)$; so $x < y$ are elements of $A_{n+1,t}$; and $g_{j,t}(a) = F_t(x)$, and $g_{j,t}(b) = F_t(y)$ are elements of $D_{n+1,t}$. Let $a' := F_{t+1}(x)$ and $b' := F_{t+1}(y)$. Since $\Phi_{j,u}$ extends $\Phi_{j,t}$, and $a' = F_u(x)$, $b' = F_u(y)$, we see that $a' = g_{j,u}(a)$ and $b' = g_{j,u}(b)$.

The coherence property of the functions $f_{n,t}$ shows that not both of $g_{j,t}(a)$ and $g_{j,t}(b)$ are in $E_{n+1,t}$. The definition of F_{t+1} shows that $a' = f_{n+1,t}(g_{j,t}(a))$ and $b' = f_{n+1,t}(g_{j,t}(b))$. The interpolation property of the functions $f_{n,t}$ shows that there is some $c' \in (a', b')_{D_{n+1,t}}$. Now since the function $f_{n+1,t}$ is injective, the definition of the set $E_{n+1,t}$ implies that either a' or b' are not elements of $E_{n+1,t}$. By induction, either the interval $(a', c')_{B_{n+1,u}}$ or the interval $(c', b')_{B_{n+1,u}}$ contains at

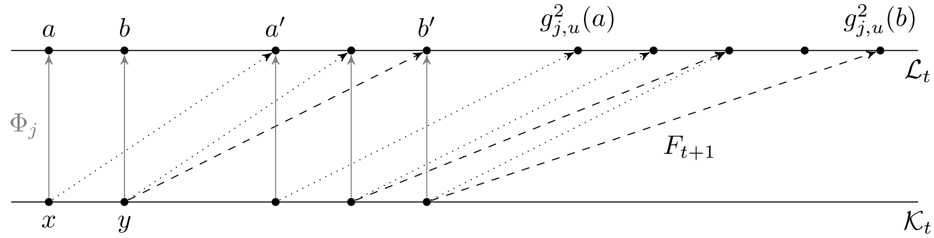

FIGURE 6. The interval $(g^2_{j,u}(a), g^2_{j,u}(b))_{\mathcal{L}_u}$ contains at least three points, and so $(a,b)_{\mathcal{L}_u}$ must contain at least three points. The dotted arrows denote F_t. The dashed arrows denote $F_{t+1} = F_u$. For simplicity $g_{j,t}(a) = g_{j,u}(a)$ and $g^2_{j,t}(a) = g^2_{j,u}(a)$.

least m points; so the interval $(a', b')_{B_{n+1,u}}$ contains at least $m+1$ points. Since $g_{j,u}$ is an isomorphism from $B_{n,u}$ to $B_{n+1,u}$, we see that $(a,b)_{B_{n,u}}$ also contains at least $m+1$ points, as required. See Figure 6.

We note that this proof works if the conjugates $B_{n,s}$ are all identical, and also if they are pairwise disjoint.

We return to the sets U_s and V_s. Let $a \in V_s$; let t witness this fact. So $b = f_{n,t}(a) \neq a$ (where $b \in B_{n,t}$). Let $a' := g_{j,t}(a)$. The coherence of $f_{m,t}$ shows that $b' := f_{n+1,t}(a') = g_{j,t}(b)$, so $b' \neq a'$. Let $x := \Phi^{-1}_{j,t}(a)$; we define $F_{t+1}(x) := b'$, so $b' = g_{j,t+1}(a)$. The fact that $F_t(x) = b'$ and $b' \neq a'$ means that for all $u \in J_j \cap (t,s)$ we have $b' \in V_u$, so inductively $F_u(x) = F_{u+1}(x) = b'$; so $b' = g_{j,s}(a)$. This shows that $g_{j,s}(a) \in V_s$ as well. An identical argument shows that $(g_{j,s})^{-1}(a) \in V_s$; so V_s is invariant under $g_{j,s}$.

This argument also shows that if $a \in V_s$, witnessed by t, then $\Phi^{-1}_{j,t}(a) \in U_s$. If $a \in B_{n,s}$ and $a \notin V_s$, then for all $t \in J_j \cap [t^{\mathtt{inf}}_{j,s}(A_s), s)$ such that $a \in B_{n,t}$, the coherence property shows that $F_{t+1}(x) = F_t(x)$ for $x = \Phi^{-1}_{j,t}(a)$; so $x \notin U_s$. Hence $U_s = \Phi^{-1}_{j,s} V_s$. Since V_s is invariant under $g_{j,s}$, we also have $V_s = F_s[U_s]$.

Suppose, for a contradiction, that $n \in \mathbb{Z}$, and that $a, b \in V_s \cap D_{n,s}$ and $a < b$. Let t_a witness that $a \in V_s$ and t_b witness that $b \in V_s$. Without loss of generality, $t_b \geq t_a$. Then $b \in D_{n,t_b}$. Since $(a,b)_{B_{n,s}}$ is finite, so is $(a,b)_{B_{n,t_b}}$. Since D_{n,t_b} is a *maximal* block, $a \in D_{n,t_b}$ as well. Since $b \notin E_{n,t_b}$, the argument above shows that the interval $(a,b)_{B_{n,u}}$, where $u = \min(J_j \cap (t_b, s))$, is infinite, contradicting that $(a,b)_{B_{n,s}}$ is finite.

This tells us how to define the functions $f_{n,s}$. The order-type of $D_{0,s}$ is either ζ, ω or ω^*. If $\operatorname{otp}(D_{0,s}) = \zeta$, then we let $f_{0,s}$ be a self-embedding of $D_{0,s}$ which fixes the unique element of $V_s \cap D_{0,s}$, if that element exists, moves every other element, and satisfies the interpolation property; so $E_{0,s} = V_s \cap D_{0,s}$. If $\operatorname{otp}(D_{0,s}) = \omega$, then we let $f_{0,s}$ be a self-embedding of $D_{0,s}$ which fixes the initial segment of $D_{0,s}$ determined by the unique element of $V_s \cap D_{0,s}$, and moves every other element; this initial segment is, of course, finite; we can again define $f_{0,s}$ to satisfy interpolation. The case $\operatorname{otp}(D_{0,s}) = \omega^*$ is symmetrical. We then define $f_{n,s}$ for $n \neq 0$ so that coherence holds. The fact that V_s is invariant under $g_{j,s}$, and that $V_s = F_s[U_s]$, shows that this definition of $f_{n,s}$ satisfies the historical responsibility property. $\square$

The Correct Guess: We show that some guess is correct, and is eventually able to act as it wishes. We first note that the arguments in cases (A) or (B) for defining F_s for limit stages s show that if for all $j < \omega_1$, $r_{j,\omega_1} < \omega_1$, that is, if for all $j < \omega_1$, I_j is bounded below ω_1, then we can define an isomorphism F_{w_1} from $\mathcal{K} = \mathcal{K}_{\omega_1}$ to $\mathcal{L} = \mathcal{L}_{\omega_1}$ by taking the union of maps F_t where t ranges over some set cofinal in ω_1. This isomorphism is in fact Δ^0_2. The assumption that $\mathcal{L}$ is ω_1-computably categorical then implies that there is an ω_1-computable isomorphism from $\mathcal{K}$ to $\mathcal{L}$.

On the other hand, let $j < \omega_1$, and suppose that I_j is unbounded in ω_1. Then J_j is also unbounded in ω_1. For suppose otherwise; let $t := \max J_j$. The guess R_j requires attention at stage $s > t$ only if $r_{j,s} \le t$ and R_j has the opportunity, at stage s, to diagonalize on some finite $S_{j,t}$-interval A_s such that A_t is nonempty. Since $\mathcal{K}_t$ is countable, there are only countably many nonempty $S_{j,t}$-intervals of $\mathcal{K}_t$. For each cut (S_1, S_2) of $S_{j,t}$ such that $A_t(j, S_1, S_2)$ is nonempty, there are at most countably many stages $s > t$ at which R_j diagonalizes on $A_s(j, S_1, S_2)$. This is by construction – we never diagonalize at the same cut twice. So R_j *receives* attention at most countably many times after stage t. If $r_{j,\omega_1} > t$, then after stage r_{j,ω_1}, R_j never requires attention, so I_j is bounded below ω_1. Otherwise, R_j receives attention at every stage $s \in I_j \cap [t, \omega_1)$, so again I_j is bounded below ω_1.

Certainly, if J_j is unbounded in ω_1, then Φ_j is an isomorphism from $\mathcal{K}$ to $\mathcal{L}$. We have established, therefore, that in either case, $\mathcal{K}$ and $\mathcal{L}$ are ω_1-computably isomorphic. Let j be the least index such that Φ_j is an ω_1-computable isomorphism from $\mathcal{K}$ to $\mathcal{L}$. The minimality of j shows that for all $i < j$, J_i is bounded below ω_1; we just argued that this implies that for all $i < j$, I_i is bounded below ω_1. Hence $r_{j,\omega_1} < \omega_1$.

We show that J_j is unbounded in ω_1. Let $r = r_{j,\omega_1}$. We know that the set H of stages $s \ge r$ such that $\Phi_{j,s}$ is an isomorphism from $\mathcal{K}_s$ to $\mathcal{L}_s$ is closed and unbounded in ω_1. Claim 3.6 implies that to show that J_j contains a final segment of H, it is sufficient to show that $J_j \cap [r, \omega_1)$ is nonempty. Suppose, for a contradiction, that $J_j \subseteq r$. Let s be the least limit point of H. As $H \subseteq I_j$, case (C) shows that F_s is the union of maps F_t where $t \in H \cap [r, s)$, and that F_s is onto $\mathcal{L}_s$. Hence $N_{j,s} = \mathcal{K}_s$, so $\mathcal{K}_r$ is contained in $N_{j,s}$; so $s \in J_j$ after all, for the desired contradiction.

We have thus established the existence of $j < \omega_1$ such that $r_{j,\omega_1} < \omega_1$ but J_j is unbounded in ω_1. The guess R_j is the “correct guess” with which we work to establish the structure theorem for $\mathcal{L}$.

Enumerating Finite Intervals: From now, we fix j such that $r_{j,\omega_1} < \omega_1$ but J_j is unbounded in ω_1. Let $r := r_{j,\omega_1}$. Let $S := S_{j,\omega_1} = S_{j,s}$ for all $s \in J_j \setminus r$, and let $Q := \Phi_j[S] = F_s[S]$ for such s.

The arguments of the proof of Theorem 2.4 show that every infinite Q-interval of $\mathcal{L}$ is $\aleph_1$-saturated. In slightly more detail, let B_{ω_1} be an infinite Q-interval of $\mathcal{L}$. To show that B_{ω_1} is nonscattered, assume otherwise. For $n \in \mathbb{Z}$, let D_n be the $<_{w_1}$-least maximal infinite block of the j-conjugate B_{n,ω_1} of B_{ω_1}. For sufficiently late $s \in J_j$, for all n, D_n is the $<_{\omega_1}$-least maximal infinite block of $B_{n,s}$. Let $s \in J_j$ be sufficiently late. Then if $A_{n,s}$ is $<_{\omega_1}$-least among its j-conjugates, then $D_{n,s} = D_n$, and at stage s, we add points to $A_{n+1,s}$ to ensure that $D_{n,s}$ is in fact not a convex subset of B_{n,ω_1}; this follows from the fact that $E_{n,s} \neq D_{n,s}$, as $E_{n,s}$ is finite. This is a contradiction, and so B_{ω_1} is nonscattered. Then, an argument identical to the one in Theorem 2.4 shows that B_{ω_1} is $\aleph_1$-saturated.

It remains to deal with finite intervals. For $n > 0$, at stage $s \in J_j \setminus r$ we enumerate a cut (Q_1, Q_2) of Q into V_n if the interval $B_s = (Q_1, Q_2)_{\mathcal{L}_s}$ contains exactly n points. Certainly, if $(Q_1, Q_2)_{\mathcal{L}}$ has size $n > 0$ then $(Q_1, Q_2) \in V_n$. We need to show that the sets V_n are pairwise disjoint. That is, we show that if $u \in J_j$, $u \geq r$, B_u is finite and nonempty, and $s > u$ is also in J_j, then either B_s is infinite, or $B_s = B_u$. Fix such u, s and B_u, and suppose, for contradiction, that B_s is finite but that $B_s \neq B_u$. Let t be the least stage in $J_j \setminus r$ such that B_t is nonempty.

The proof bifurcates into two cases. Either the j-conjugates $B_{n,s}$ of B_s are all identical, or they are pairwise disjoint. First, suppose they are identical. In this case, we first show that if $v \geq t$ and at stage v, R_j diagonalizes on $A_v = \Phi_{j,v}^{-1}(B_v)$, then B_w is infinite, where $w = \min J_j \cap (v, \omega_1)$. For at stage v, we ensure that $\Phi_{j,v} \restriction A_v$ cannot be extended to the unique isomorphism $F_{v+1} \restriction A_{v+1}$ from A_{v+1} to $C_{v+1} = B_{v+1}$; so $\Phi_{j,v}$ and F_{v+1} disagree on some element of A_v. If B_w is finite, then $F_w \restriction A_w = \Phi_{j,w} \restriction A_w$ is an isomorphism between A_w and B_w. However, $\Phi_{j,w}$ extends $\Phi_{j,v}$, and $F_w \restriction A_w$ extends $F_{v+1} \restriction A_{v+1}$, because the instructions don't allow R_j to change F on A between stages v and w; we also use Claim 3.3. This is impossible.

This implies that R_j did not diagonalize on B_v at any stage $v \in [t, s)$. However, let v be the least stage in $[u, s)$ such that $B_{v+1} = B_s$; so $B_{v+1} \neq B_v$. Then R_j has the opportunity to diagonalize on A_v at stage v, because it did not do so at an earlier stage, and $|A_v| = |B_v| < |B_{v+1}| = |C_{v+1}|$ (as $C_{v+1} = B_{v+1}$). This is a contradiction.

Suppose now that the intervals $B_{n,s}$ are pairwise disjoint. Let $m = |B_s|$. Since $s \in J_j$, $m = |B_{n,s}|$ for every j-conjugate $B_{n,s}$ of B_s. We first show that there is some stage $v \in [u, s)$ at which R_j diagonalizes on some conjugate $A_{n,u}$ with m points. Suppose otherwise. As for all $n \in \mathbb{Z}$, $|B_{n,u}| = |B_u| < m$, certainly R_j does not diagonalize on any $A_{n,v}$ with m points at any stage $v \in [t, u)$. For each $m' < m$, there is at most one n such that R_j diagonalizes with m' points on $A_{n,v}$ (at any stage $v \in [t, s)$). Certainly R_j does not diagonalize on any conjugate $A_{n,v}$ with more than m points at any stage before s. Let k be the maximal integer such that R_j diagonalized on $A_{k,v}$ at some $v \in [t, s)$. This is well-defined as the conjugates $A_{n,s}$ are pairwise disjoint. Let v_0 be the least stage $v < s$ at which $|B_{l,v+1}| = m$ for some $l \geq k$. Let l be the least integer $l \geq k$ such that $|B_{l,v_0+1}| = m$. Let v_1 be the least stage $v < s$ at which $|B_{l+1,v+1}| = m$; so $v_1 \geq v_0$. Then $|B_{l,v_1+1}| = |B_{l+1,v_1+1}| = m$ and $|A_{l+1,v_1}| = |B_{l+1,v_1}| < m$. So at stage v_1, R_j has the opportunity to diagonalize on A_{l+1,v_1} with m points, which is impossible.

Let $v \in [u, s)$ be a stage at which R_j diagonalizes on some $A_{n,v}$ with m points. At stage v, we ensure that $\Phi_{j,v} \restriction A_{n,v}$ cannot be extended to an isomorphism of $A_{n,v+1}$ and $C_{n,v+1}$. However, $A_{n,v+1} = A_{n,s}$ and $C_{n,v+1} = C_{n,s}$, and $\Phi_{j,s} \restriction A_{n,s}$ is an extension of $\Phi_{j,v} \restriction A_v$ to precisely such an isomorphism. This yields the desired contradiction, with which we conclude the proof of Theorem 3.1. □

4. Open Questions

If every infinite linear order had proper self-embeddings, it would seem possible to generalize the characterization of computable categoricity to higher cardinals. As this is not the case for linear orders of size $\aleph_1$ (see [2]), we ask for a characterization of computable categoricity at the next cardinal.

Question 4.1. Which ω_2-computable linear orders are ω_2-computably categorical?

The precise analogue of Theorem 3.1 does not hold for ω_2-computably categorical linear orders. One of the obstacles is the existence of linear orders of size $\aleph_1$ which have no proper self-embeddings. A linear ordering obtained by taking a set of parameters of size $\aleph_1$ with $\aleph_2$ many cuts, and inserting a fixed linear order with no self-embedding in each cut, will be ω_2-computably categorical. The anonymous referees gave an even more compelling example. They take the ω_2-sum of linear orders R_α ($\alpha < \omega_2$), no interval of which can be embedded into another (these can be taken to be subsets of $\mathbb{R}$). This counter-example (which is in fact *uniformly* ω_2-computably categorical) has cofinality ω_2 but every proper initial segment has size $\aleph_1$.

We also pose a methodological question:

Question 4.2. What effects do combinatorial principles such as $\diamondsuit$ have on the effectiveness properties of uncountable linear orders (either of size $\aleph_1$ or even $\aleph_2$)? We note that Jensen's original proof of $\diamondsuit$ shows the existence of an ω_1-computable $\diamondsuit$-sequence.

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