INDEX SETS FOR n-DECIDABLE STRUCTURES CATEGORICAL RELATIVE TO m-DECIDABLE PRESENTATIONS

EKATERINA FOKINA, SERGEY GONCHAROV, VALENTINA HARIZANOV, OLEG KUDINOV, AND DANIEL TURETSKY

Abstract. We call a structure *categorical relative to n-decidable presentations* (or *autostable relative to n-constructivizations*) if if any two n-decidable copies of $\mathcal{A}$ are computably isomorphic. For $n = 0$, we get the classical definition of a computably categorical (autostable) structure. Downey, Kach, Lempp, Lewis, Montalbán, and Turetsky proved that there is no simple syntactic characterization of computable categoricity. More formally, they showed that the index set of computable categorical structures is Π^1_1-complete.

In this paper we study the complexity of index sets of structures that are n-decidable and categorical relative to m-decidable presentations, for various $m, n \in \omega$. If $m \geq n \geq 0$, then the index set is again Π^1_1 complete, i.e., there is no nice description of the class of structures that are n-decidable, categorical relative to m-decidable presentations, for $m \geq n$. In the case $m = n - 1 \geq 0$, the index set is Π^0_4 complete, while if $0 \leq m \leq n - 2$, the index set is Σ^0_3 complete.

1. Introduction and preliminaries

We consider only countable structures for computable language. We call such a structure $\mathcal{A}$ is *computable* if its universe can be identified with the set ω of natural numbers in such a way that the relations and operations of $\mathcal{A}$ are uniformly computable. A finite structure is always computable. A structure $\mathcal{A}$ is called *n-decidable*, for $n \geq 0$, if the Σ_n-diagram of $\mathcal{A}$ is decidable. In particular, a structure is 0-decidable iff it is computable.

Throughout the paper we denote by $\mathcal{M}_i$ the (partial) computable structure computed by the ith Turing machine, where $i \in \omega$. Given a class of structures K, the *index set* of K is the set

$$I(K) = \{i \in \omega : \mathcal{M}_i \in K\}$$

of all indices of computable structures from K. If the index set is hyperarithmetical, we say that the class K has a characterization. The idea is that index sets are a way to describe computable members of K, and so classes with hyperarithmetical index sets are nicely describable.

We are interested in complexity of isomorphisms between computable presentations of a countable structure. The main notion here is that of *computable categoricity.* This notion has been part of computable model theory since Frohlich and Shepherdson first produced an example of two computable fields which were

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isomorphic but not computably isomorphic (see [7]). Mal'cev in [23] studied the question of uniqueness of a constructive enumeration for a model and introduced the notion of a recursively stable model. Later in [24] he built isomorphic computable infinite-dimensional vector spaces that were not computably isomorphic. In the same paper he introduced the notion of an *autostable* model, which is equivalent to that of a computably categorical model. Since then, the definition of computable categoricity has been standardized and relativized to arbitrary Turing degrees $\mathbf{d}$, and has been the subject of much study (see, e.g., surveys [11, 5]).

Definition 1. A computable structure $\mathcal{M}$ is $\mathbf{d}$-*computably categorical* (also called $\mathbf{d}$-*autostable*) if, for every computable structure $\mathcal{A}$ isomorphic to $\mathcal{M}$, there exists a $\mathbf{d}$-computable isomorphism from $\mathcal{M}$ onto $\mathcal{A}$. In case $\mathbf{d} = \mathbf{0}$, we simply say that $\mathcal{M}$ is *computably categorical*.

A computable structure $\mathcal{M}$ is *relatively computably categorical* if for every its countable isomorphic copy $\mathcal{A}$ there exists an isomorphism computable in the atomic diagram of $\mathcal{A}$.

Downey, Kach, Lempp, Lewis, Montalbán, and Turetsky [2] proved that there is no simple syntactic characterization of computable categoricity. More formally, they showed that the index set of computable categorical structures is Π^1_1-complete. Combining the methods from [2] and from [10], Bazhenov, Goncharov and Marchuk showed that also the index set of computable structures of algorithmic dimension $n > 1$ is Π^1_1 complete [18]. On the other hand, the index set of relatively computably categorical structures is Σ^0_3-complete (see [2]).

More recently, Goncharov introduced the notion of categoricity restricted to decidable structures [12, 13, 14].

Definition 2. A structure $\mathcal{A}$ is called *decidably categorical* (also called *autostable relative to strong constructivizations*) if any two decidable copies of $\mathcal{A}$ are computably isomorphic.

Goncharov and Marchuk in [20] showed that the index set of computable, decidably categorical structures is $\Sigma^0_{\omega+2}$ complete, while for decidable, decidably categorical structures the index set is a complete Σ^0_3 set. Index sets for decidably categorical structures with particular algebraic, model-theoretic and algorithmic properties were further studied in [15, 17, 19, 18].

In this paper we consider n-decidable structures and their categoricity with respect to m-decidable copies, where $m \leq n \in \omega$.

Definition 3. We call a structure *categorical relative to m-decidable presentations* (or *autostable relative to m-constructivizations*) if if any two m-decidable copies of $\mathcal{A}$ are computably isomorphic.

In particular, being computably categorical is the same as being categorical relative to 0-decidable presenations.

We summarize the results of this paper in the following table.

n-decidable $n \geq 2$	m-decidably categorical $m \leq n-2$	Σ^0_3 complete
n-decidable $n \geq 1$	$(n-1)$-decidably categorical	Π^0_4 complete
n-decidable $n \geq 0$	m-decidably categorical $m \geq n$	Π^1_1 complete

2. Complexity of the Index Sets

We first consider n-decidable structures that are categorical relative to n-decidable presentations.

Theorem 1. *The index set of n-decidable structures that are categorical relative to n-decidable presentations is Π^1_1 complete.*

Proof. Recall that by the result of Downey, Kach, Lempp, Lewis, Montalbán, and Turetsky [2] the index set of computably categorical structures is Π^1_1 complete. This means that for every Π^1_1 set S there is a uniformly computable sequence of structures $\{\mathcal{A}_i\}_{i\in\omega}$ such that $i \in S \iff \mathcal{A}_i$ is computably categorical.

Marker in [25] defined $\forall$- and $\exists$-extensions, $\mathcal{A}_\forall$ and $\mathcal{A}_\exists$, respectively, of an arbitrary structure $\mathcal{A}$. The main property is that the domain and the basic relations of $\mathcal{A}$ are definable in $\mathcal{A}_\forall, \mathcal{A}_\exists$ by universal or existential formulas, respectively. One can iteratively apply the extensions in the obvious way. Define B_i to be be the result of the application of Marker's $(\forall\exists)$-extension n-times. As follows from [1] or [16], if $\mathcal{A}_i$ was computable, then B_i is n-decidable. And from properties of the Marker's extensions proved in [6], A_i is computably categorical iff B_i is categorical relative to n-decidable presentations. The claim follows immediately. $\square$

Corollary 1. *For all $m \geq n \geq 0$, the index set of n-decidable structures that are categorical relative to m-decidable presentations is Π^1_1 complete.*

We now consider 1-decidable, computably categorical structures, i.e. we do not impose additional effectiveness conditions on the copies of the structure except of being computable.

Theorem 2. *The index set of 1-decidable, computably categorical structures is Π^0_4 complete.*

Proof. We first show that the index set is Π^0_4. Recall that $\langle \mathcal{M}_i \rangle_{i\in\omega}$ is a fixed effective listing of all partial computable structures.

The relation "$\mathcal{M}_i$ is n-decidable" is Σ^0_3, as it states that there is a partial computable $\{0,1\}$-valued function f defined on pairs $(\phi(\overline{x}), \overline{a})$ with $\phi(\overline{x})$ a Σ_n formula in the language of $\mathcal{M}_i$ and $\overline{a} \in \mathcal{M}_i^{<\omega}$ such that:

- f is total;
- For $\phi(\overline{x})$ quantifier-free, $f(\phi(\overline{x}), \overline{a}) = 1 \iff \mathcal{M}_i \models \phi(\overline{a})$;
- For $\phi(\overline{x}, \overline{y})$ a Π_{n-1} formula, $f(\exists \overline{y}\, \phi(\overline{x}, \overline{y}), \overline{a}) = 1 \iff \exists \overline{b}\, f(\neg\phi(\overline{x}, \overline{y}), \overline{a}\overline{b}) = 0$.

Consider the following relations on pairs (i, j):

$(i, j) \in E \iff$ $\mathcal{M}_i$ and $\mathcal{M}_j$ are total structures and there is a computable isomorphism between them

$(i, j) \in F \iff$ $\mathcal{M}_i$ and $\mathcal{M}_j$ are total structures and there is a Δ^0_2 isomorphism between them

It is straightforward to show that E is Σ^0_3, while F is Σ^0_4.

Now consider the following property of a computable structure $\mathcal{A}$:

> For every computable structure $\mathcal{B}$, if there is a Δ^0_2 isomorphism from $\mathcal{A}$ to $\mathcal{B}$, then there is a computable isomorphism from $\mathcal{A}$ to $\mathcal{B}$. (†)

As a relation on i, this can be written as

$$\forall j \, F(i, j) \to E(i, j),$$

and so this is Π^0_4.

Note that property (†) is a weakening of computable categoricity. Downey, Kach, Lempp and Turetsky [3] showed that if a structure is computably categorical and 1-decidable, then it is relatively Δ^0_2-categorical. Inspection of their proof reveals that they did not use the full power of computable categoricity; instead, they only used property (†). Thus they showed the following:

Lemma 1. *If a structure is 1-decidable and has property (†), then it is relatively Δ^0_2-categorical.*

Note that a structure which is simultaneously relatively Δ^0_2-categorical and has property (†) is necessarily computably categorical. Thus we have the following: a structure is 1-decidable and computably categorical if and only if it is 1-decidable and has property (†). So the relation "$\mathcal{M}_i$ is 1-decidable and computably categorical" can be written as the conjunction of a Σ^0_3 formula and a Π^0_4 formula, and so is Π^0_4.

To show the completeness at the level Π^0_4, we use a known method to code computable families of functions in 1-decidable unars (for short, S is coded in $\mathcal{M}_S$), as exposed in [4, 21]. The main feature of the construction is the following: S admits exactly one computable numbering up to equivalence iff the unar $\mathcal{M}_S$ is computably categorical. So, the index set of computable families of functions with exactly one computable numbering is m-reducible to required index set. And the first index set was investigated in [22], where its Π^0_4-completeness was proven. The theorem is proven.

□

Using the technique of Marker's extensions, it is not hard to show:

Corollary 2. *For any $n \geq 1$, the index set of n-decidable, categorical relative to $n-1$-decidable presentations structures is Π^0_4 complete.*

Goncharov [9] proved that a 2-decidable computably categorical structure is relatively computably categorical. Downey, Kach, Lempp and Turetsky [3] showed that the index set of relatively computably categorical structures is Σ^0_3 complete. In fact, they show that the index set of 2-decidable computably categorical structures is Σ^0_3 complete. Applying Marker's extensions,

Proposition 1. *For any $n \geq 2$ and $m \leq n-2$, the index set of n-decidable, categorical relative to m-decidable presentations structures is Σ^0_3 complete.*

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