DEGREE SPECTRA OF STRUCTURES RELATIVE TO EQUIVALENCE RELATIONS

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Abstract. A standard way to capture the inherent complexity of the isomorphism type of a countable structure is to consider the collection of all Turing degrees relative to which a given structure has a computable isomorphic copy. This set is called the degree spectrum of structure. Similarly, to characterize the complexity of models of a theory, one may consider the collection of all degrees relative to which the theory has a computable model. In this case we get the spectrum of the theory.

In this paper we generalize these two notions to arbitrary equivalence relations. For a structure $\mathcal{A}$ and an equivalence relation E, we define the degree spectrum $DgSp(\mathcal{A}, E)$ of $\mathcal{A}$ relative to E to be the set of all degrees capable of computing a structure $\mathcal{B}$ that is E-equivalent to $\mathcal{A}$. Then the standard degree spectrum of $\mathcal{A}$ is $DgSp(\mathcal{A}, \cong)$ and the degree spectrum of the theory of $\mathcal{A}$ is $DgSp(\mathcal{A}, \equiv)$. We consider the relations $\equiv_{\Sigma_n}$ ($\mathcal{A} \equiv_{\Sigma_n} \mathcal{B}$ iff the Σ_n theories of $\mathcal{A}$ and $\mathcal{B}$ coincide) and study degree spectra with respect to $\equiv_{\Sigma_n}$.

1. Introduction

For a countable structure $\mathcal{A}$, its degree spectrum $DgSp(\mathcal{A})$ was defined by Richter in [10] and consists of the Turing degrees of all isomorphic copies of $\mathcal{A}$. As shown by Knight in [9], in all nontrivial cases, the degree spectrum of a structure is closed upward. Degree spectra of structures with various model-theoretic and algebraic properties have

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been widely studied; an overview of the current situation can be found, e.g., in [3]. Probably the simplest example of a degree spectrum is a cone above a Turing degree $\mathbf{d}$. On the other hand, no non-degenerate finite or countable union of cones can be a degree spectrum [12]. Slaman and Wehner in [11, 13] gave examples of structures with the degree spectrum consisting of exactly the non-computable degrees. In [8] Kalimullin constructed an example of a structure with its degree spectrum equal to all the non-Δ^0_2 degrees. Greenberg, Montalbán and Slaman showed that non-hyperarithmetical degrees form a spectrum of a structure in [5].

For a theory T, the degree spectrum of T was defined in [1]. It consists of all degrees of countable models of T. Some of the known examples of the spectra of theories include [1]: cones, a non-degenerate union of two cones, exactly the PA degrees, exactly the 1-random degrees. On the other hand, the authors of [1] prove that the collection of non-hyperarithmetical degrees is not the spectrum of a theory. In particular, these examples show that not every spectrum of a structure is a spectrum of a theory and, vice versa, not every spectrum of a theory is a spectrum of a structure.

In this paper we suggest to consider the following generalization of these notions to arbitrary equivalence relations.

Definition 1. The *degree spectrum* of a countable structure $\mathcal{A}$ with universe ω *relative to the equivalence relation* E is

$$DgSp(\mathcal{A}, E) = \{\mathbf{d} \mid \text{there exists a } \mathbf{d}\text{-computable } \mathcal{B}\ E{-}\text{equivalent to } \mathcal{A}\}.$$

A related notion was independently introduced by L. Yu in [14]: for an equivalence relation E, a reduction $\leqslant_r$ over 2^ω and a real $x \in 2^\omega$, the (E, r)-spectrum of x is the set $Spec_{E,r}(x) = \{y \in 2^\omega : \exists z \leqslant_r y(E(z, x))\}$. This definition is related to our Definition 1 as follows:

$$DgSp(\mathcal{A}, E) = \{\deg_T(y) : y \in Spec_{E,T}(D(\mathcal{A}))\},$$

where $D(\mathcal{A})$ is the atomic diagram of $\mathcal{A}$.

The classical degree spectrum of $\mathcal{A}$ is $DgSp(\mathcal{A}, \cong)$, the degree spectrum of $\mathcal{A}$ under isomorphism, while the degree spectra of the theory of $\mathcal{A}$ is $DgSp(\mathcal{A}, \equiv)$, the degree spectrum of $\mathcal{A}$ under elementary equivalence.

In this paper, instead of considering the full theory of a structure, as for theory spectra, we consider Σ_n-fragments of theories and the corresponding equivalence relations $\equiv_{\Sigma_n}$ (two structures are $\equiv_{\Sigma_n}$-equivalent if their Σ_n-theories coincide). We also write $\mathcal{A} \equiv_{\Sigma_n} \mathcal{B}$ when $\mathcal{A}$ and $\mathcal{B}$ are Σ_n-equivalent. We call $DgSp(\mathcal{A}, \equiv_{\Sigma_n})$ the Σ_n*–spectrum* of $\mathcal{A}$. We will study what kinds of spectra are possible with respect to these equivalence relations.

Degree spectra with respect to another natural equivalence relation, that of bi-embeddability, are considered in [4].

2. Two cones

It is well-known that the degree spectrum of a structure cannot be the union of two cones [12]. On the other hand, the authors of [1] built a theory T whose spectrum is equal to a non-degenerate union of two cones. For Σ_n-spectra, the situation depends on n.

We start with a simple observation.

Lemma 2. *Two relational structures $\mathcal{A}$ and $\mathcal{B}$ are Σ_1–equivalent iff they have the same finite substructures (in finite sublanguages).*

Proof. Suppose $\mathcal{A} \equiv_{\Sigma_1} \mathcal{B}$. Choose an arbitrary finite substructure $\mathcal{A}_0$ of $\mathcal{A}$ of a finite sublanguage. As its language is finite, we can write its atomic diagram $D(\mathcal{A}_0)$ as a single first order sentence $\varphi(\overline{a})$ with parameters $\overline{a}$ from A_0. Then $\mathcal{A} \models \exists \overline{x} \varphi(\overline{x})$, where $|\overline{x}| = |\overline{a}|$. By Σ_1–equivalence, $\mathcal{B} \models \exists \overline{x} \varphi(\overline{x})$. Let $\overline{b}$ witness φ in $\mathcal{B}$. Then the finite substructure $\mathcal{B}_0$ of $\mathcal{B}$ with domain $\overline{b}$ and with relation symbols that appear in φ is isomorphic to $\mathcal{A}_0$.

Suppose now that $\mathcal{A}$ and $\mathcal{B}$ have the same finite substructures in finite sublanguages. Assume $\mathcal{A} \models \exists \overline{x} \varphi(\overline{x})$. Let $\overline{a}$ be a witness. Consider the finite substructure $\mathcal{A}_0$ of $\mathcal{A}$ with the universe $\overline{a}$ and the language consisting of the relation symbols used in φ. By assumption, there is a finite substructure $\mathcal{B}_0$ of $\mathcal{B}$ in the same language which is isomorphic to $\mathcal{A}_0$. Then $\mathcal{B}_0 \models \exists \overline{x} \varphi(\overline{x})$, and thus $\mathcal{B} \models \exists \overline{x} \varphi(\overline{x})$. $\square$

Theorem 3. *No Σ_1–spectrum of a structure can be a non-degenerate union of two cones.*

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be Σ_1–equivalent structures that have degrees $\mathbf{a}$ and $\mathbf{b}$, respectively, where $\mathbf{a}$ and $\mathbf{b}$ are incomparable. For simplicity, we use the standard assumption that the language of the structures is relational. We build a Σ_1–equivalent structure $\mathcal{C}$ of degree $\mathbf{c}$, such that $\mathbf{c}$ is neither above $\mathbf{a}$ nor above $\mathbf{b}$.

The universe of $\mathcal{C}$ will be ω. At each stage s we define a finite substructure $\mathcal{C}_s$ with the universe an initial segment of ω. To make sure that $\mathcal{C}$ computes neither $\mathcal{A}$ nor $\mathcal{B}$, we as usually consider the list of requirements of the form $\Phi_e^{\mathcal{C}} \neq \mathcal{A}$ and $\Phi_e^{\mathcal{C}} \neq \mathcal{B}$. Assume that the next requirement is of the form $\Phi_e^{\mathcal{C}} \neq \mathcal{A}$, so we want to diagonalize against $\mathcal{C}$ computing $\mathcal{A}$. Let $\{\mathcal{N}_j\}_{j \in \omega}$ be a list of finite structures, such that each $\mathcal{N}_j$:

- extends $\mathcal{C}_s$,
- has the universe an initial segment of ω,
- is isomorphic to a finite substructure of $\mathcal{B}$ in a finite language,
- every such substructure of $\mathcal{B}$ appears in the list.

Obviously, we can construct such a list computable in $\mathcal{B}$. Now we ask if there are n and $\mathcal{N}_j$ such that $\Phi_e^{\mathcal{N}_j}(n) \downarrow \neq \mathcal{A}(n)$. If the answer is positive, we let $\mathcal{C}_{s+1}$ be equal to such $\mathcal{N}_j$. So the requirement $\Phi_e^{\mathcal{C}} \neq \mathcal{A}$ will be satisfied.

On the other hand, if the answer is negative, then for all n and $\mathcal{N}_j$ either $\Phi_e^{\mathcal{N}_j}(n) \uparrow$ or $\Phi_e^{\mathcal{N}_j}(n) \downarrow = \mathcal{A}(n)$. Suppose that in the end of the construction $\Phi_e^{\mathcal{C}}$ is everywhere defined. Then for every n there exists an $\mathcal{N}_j$ such that $\Phi_e^{\mathcal{N}_j}(n)\downarrow=\mathcal{A}(n)$. So we can compute $\mathcal{A}$ from $\mathcal{B}$, which is a contradiction. Therefore, in this case $\Phi_e^{\mathcal{C}}$ must be partial, and the requirement is again satisfied.

Note that the above construction guarantees that every substructure of $\mathcal{C}$ in a finite sublanguage appears in $\mathcal{A}$ and $\mathcal{B}$. To ensure that $\mathcal{C}\equiv_{\Sigma_1}\mathcal{A},\mathcal{B}$, we also add stages where we extend the previously built $\mathcal{C}_s$ to include the next finite substructure of $\mathcal{A}$ or $\mathcal{B}$. □

Theorem 4. *There is a structure $\mathcal{A}$ with $DgSp(\mathcal{A},\equiv_{\Sigma_2})$ equal to the union of two non-degenerate cones.*

Proof. If we allow infinite languages, the statement follows directly from the result of Andrews and Miller [1], where they build a theory T with the spectrum of T consisting of exactly two cones. Let $\mathcal{A}$ be a model of T and let $\mathcal{B}\equiv_{\Sigma_2}\mathcal{A}$. The theory T is a complete theory that can be axiomatized using Σ_2- and Π_2-sentences. Thus, $\mathcal{B}$ is also a model of T. In other words, $DgSp(\mathcal{A},\equiv_{\Sigma_2})=DgSp(\mathcal{A},\equiv)$, which is the union of two cones.

The result is also true for finite languages, for example, using the transformation from [6] of arbitrary structures into graphs. It is not hard to show that the transformation preserves Σ_2-equivalence. □

3. All but computable

According to [11] and [13], there exist structures with the classical degree spectrum containing exactly all the non–computable degrees. Moreover, as the structure from [11] is not elementary equivalent to a computable structure, the built example actually shows that the degree spectrum of the theory of the constructed structure consists of all the non–computable degrees.

The theory of the structure built in [11] is Σ_3- and Π_3-axiomatizable, however minor modifications can make it axiomatizable using Σ_2- and Π_2-sentences.

Theorem 5. *There exists a countable structure $\mathcal{A}$, such that $DgSp(\mathcal{A}, \equiv_{\Sigma_2})$ consists of exactly all the non–computable Turing degrees. The same is also true for $DgSp(\mathcal{A}, \equiv_{\Sigma_n})$, for all $n \geqslant 2$.*

On the other hand, for Σ_1-spectra this is again not true:

Proposition 6. *No structure $\mathcal{A}$ may have its Σ_1-spectrum consisting of exactly the non-computable degrees.*

Proof. The Σ_1-spectrum of any structure $\mathcal{A}$ has the form $\{\mathbf{d} \mid X \text{ is } \mathbf{d}\text{-c.e.}\}$, where X is the set of Gödel indices of the sentences from the Σ_1-theory of $\mathcal{A}$. As shown in [2], if the collection of oracles that enumerate any set X has positive measure, then X is c.e. So, if $DgSp(\mathcal{A}, \equiv_{\Sigma_1})$ contains all non–computable degrees, then the Σ_1-theory of $\mathcal{A}$ is c.e. It is not hard to show that if a Σ_1-theory is c.e., then it has a computable model (see Theorem 10 below for a more general statement). This completes the proof of the proposition. $\square$

Similar considerations prove the following:

Corollary 7.

(1) *If $DgSp(\mathcal{A}, \equiv_{\Sigma_1})$ contains all non–computable c.e. degrees, it also contains $\mathbf{0}$.*
(2) *If $DgSp(\mathcal{A}, \equiv_{\Sigma_1})$ contains all low degrees, it also contains $\mathbf{0}$.*
(3) *If $DgSp(\mathcal{A}, \equiv_{\Sigma_1})$ contains all high degrees, it also contains $\mathbf{0}$.*
(4) *If $DgSp(\mathcal{A} \equiv_{,\Sigma_1})$ contains all PA degrees, it also contains $\mathbf{0}$.*
(5) *If $DgSp(\mathcal{A}, \equiv_{\Sigma_1})$ contains all degrees above $\mathbf{a}$, it also contains $\mathbf{a}$.*

Proposition 6 and Corollary 7 can also be proved by coding a special kind of a minimal pair of degrees into the above collections of degrees.

Definition 8. The sets X and Y form a Σ_1-*minimal pair* if $\Sigma_1(X) \cap \Sigma_1(Y) = \Sigma^0_1$.

For example, if the set of all non-computable degrees were a Σ_1-spectrum, there would exist structures $\mathcal{A}, \mathcal{B}$ of degrees $\mathbf{a}, \mathbf{b}$, respectively, where $\mathbf{a}$ and $\mathbf{b}$ form a Σ_1-minimal pair. As the Σ_1-theory T_{Σ_1} is c.e. in $\mathcal{A}$ and in $\mathcal{B}$, it must be c.e. In this case it must have a computable model, so the Σ_1-spectrum must contain $\mathbf{0}$. Analogously for results from Corollary 7. A similar idea was used in [1] to prove that certain collections of degrees are not structure spectra.

We use Σ_1-minimal pairs to prove that further collections of degrees cannot be Σ_n-degree spectra, for suitable $n \in \omega$. We need the following two facts.

Observation 9. For any C, if $A \oplus B$ is sufficiently generic, then $A \oplus C$ and $B \oplus C$ form a Σ^0_1-minimal pair over C. That is, $\Sigma^0_1(A \oplus C) \cap \Sigma^0_1(A \oplus C) = \Sigma^0_1(C)$.

Theorem 10. *If T is a complete consistent theory in computable language $\mathcal{L}$, and S is the Σ_n-fragment of T (equivalently, S is the Σ_n-theory of a structure), and S is c.e., then S has a computable model.*

Proof. We perform an effective Henkin construction. Let our universe be $\{c_i\}_{i \in \omega}$, and let $\{\exists \overline{x} \varphi_i(\overline{x})\}_{i \in \omega}$ be an enumeration of all Σ_n-sentences in $\mathcal{L}$, where φ_i is a Π_{n-1}-formula. Let $\{\theta_i\}_{i \in \omega}$ be an enumeration of all Σ_{n-1}-sentences in $\mathcal{L} \cup \{c_i\}_{i \in \omega}$. We will compute the $(n-1)$-diagram of our structure.

During the construction, we will have a set of sentences δ_s, which is the fragment of the diagram we have committed to so far. We begin with $\delta_0 = \emptyset$. We also keep a stage t_s which is the stage we have enumerated S to. We begin with $t_0 = 0$.

At stage $s+1$, let $\hat{\delta}_s$ be made from δ_s by replacing the constant for c_i with the new variable y_i, and similarly $\hat{\theta}_s(\overline{y})$ (where the same substitution $c_i \mapsto y_i$ is made).

Define the following:

$$\begin{aligned}
\psi_t^{s,+} &= \exists\overline{y}\exists\overline{z}\left[\begin{array}{c} \hat{\theta}_s(\overline{y}) \wedge \left(\bigwedge_{\rho\in\hat{\delta}_s} \rho(\overline{y})\right) \wedge \left(\bigwedge_{\exists\overline{x}\tau(\overline{x},\overline{y})\in\hat{\delta}_s} (\exists\overline{w}\in\overline{z})\tau(\overline{w},\overline{y})\right) \\ \wedge\left(\bigwedge_{\substack{i<s \\ \exists\overline{x}\varphi_i(\overline{x})\in S_t}} (\exists\overline{w}\in\overline{yz})\varphi_i(\overline{w})\right) \wedge \left(\bigwedge_{\substack{i<s \\ \exists\overline{x}\varphi_i(\overline{x})\notin S_t}} (\forall\overline{w}\in\overline{yz})\neg\varphi_i(\overline{w})\right)\end{array}\right], \\
\psi_t^{s,-} &= \exists\overline{y}\exists\overline{z}\left[\begin{array}{c} \neg\hat{\theta}_s(\overline{y}) \wedge \left(\bigwedge_{\rho\in\hat{\delta}_s} \rho(\overline{y})\right) \wedge \left(\bigwedge_{\exists\overline{x}\tau(\overline{x},\overline{y})\in\hat{\delta}_s} (\exists\overline{w}\in\overline{z})\tau(\overline{w},\overline{y})\right) \\ \wedge\left(\bigwedge_{\substack{i<s \\ \exists\overline{x}\varphi_i(\overline{x})\in S_t}} (\exists\overline{w}\in\overline{yz})\varphi_i(\overline{w})\right) \wedge \left(\bigwedge_{\substack{i<s \\ \exists\overline{x}\varphi_i(\overline{x})\notin S_t}} (\forall\overline{w}\in\overline{yz})\neg\varphi_i(\overline{w})\right)\end{array}\right].
\end{aligned}$$

where "$\exists\overline{w}\in\overline{yz}$" means there is a tuple of the appropriate length made from the elements of the tuples $\overline{y}$ and $\overline{z}$, and similarly for "$\forall\overline{w}\in\overline{yz}$". Note that both $\psi_t^{s,+}$ and $\psi_t^{s,-}$ are Σ_n-sentences in $\mathcal{L}$. We enumerate S until we see some $\psi_t^{s,+}$ or $\psi_t^{s,-}$ enumerated with $t > t_s$. We will argue in the verification that this must eventually occur.

Suppose we have seen $\psi_t^{s,+}$ be enumerated. Fix some tuple $\overline{c} \in \{c_i\}_{i\in\omega}$ with $|\overline{c}| = |\overline{z}|$ and none of $\overline{c}$ occurring in δ_s or θ_s. Fix a bijection between $\overline{c}$ and $\overline{z}$. Define the map f such that for $z \in \overline{z}$, $f(z)$ follows this bijection, and for y_j, $f(y_j) = c_j$. Note that this is an injection from the variables occurring in $\overline{yz}$ into $\{c_i\}_{i\in\omega}$.

For every sentence $\exists\overline{x}\varphi_i(\overline{x}) \in S_t$, fix a witnessing tuple $\overline{w}_i$. Note that we can identify such $\overline{w}$ effectively: since "$\exists\overline{w}\in\overline{yz}$" is a finite disjunction, we can make more specific versions of $\psi_t^{s,+}$ by retaining only a single disjunct for every φ_i. Eventually, one of these more specific sentences must be enumerated. Similarly, for every sentence $\exists\overline{x}\tau(\overline{x},\overline{y}) \in \delta_s$, fix a witnessing tuple $\overline{w}_\tau$.

Define $t_{s+1} = t$ and

$$\begin{aligned}
\delta_{s+1} &= \delta_s \cup \{\theta_s\} \cup \{\tau(f(\overline{w}_\tau), f(\overline{y})) : \exists\overline{x}\tau(\overline{x},\overline{y}) \in \hat{\delta}_s\} \\
&\quad \cup\{\varphi_i(f(\overline{w_i})) : i < s \ \&\ \exists\overline{x}\varphi_i(\overline{x}) \in S_t\} \\
&\quad \cup\{\neg\varphi_i(f(\overline{w})) : i < s \ \&\ \overline{w}\in\overline{yz} \ \&\ \exists\overline{x}\varphi_i(\overline{x}) \notin S_t\}.
\end{aligned}$$

If instead $\psi_t^{s,-}$ is enumerated, proceed similarly except define δ_{s+1} with $\neg\theta_s$ instead of θ_s. Once t_{s+1} and δ_{s+1} are defined, proceed on to stage $s+2$.

Verification:

Claim 10.1. *For every s, $\exists\overline{y}\bigwedge_{\rho\in\hat{\delta}_s}\rho(\overline{y})\in S$.*

Proof. Induction. □

In particular, the diagram $D=\{\delta_s\}_{s\in\omega}$ we build is consistent.

Claim 10.2. *For every s, we will eventually see some $\psi_t^{s,+}$ or $\psi_t^{s,-}$ enumerated into S.*

Proof. We know that $\exists\overline{y}\hat{\delta}_s(\overline{y})$ is in S and thus in T. Since T is complete, at least one of $\exists\overline{y}(\hat{\delta}_s(\overline{y})\wedge\hat{\theta}_s(\overline{y}))$ or $\exists\overline{y}(\hat{\delta}_s(\overline{y})\wedge\neg\hat{\theta}_s(\overline{y}))$ is in T, and by counting quantifiers, must thus be in S.

Let t be such that $S_t\restriction_s= S\restriction_s$. Then at least one of $\psi_t^{s,+}$ or $\psi_t^{s,-}$ is in T, and thus is in S. □

Claim 10.3. *D is computable.*

Proof. We decide θ_s at stage s. □

Let $\mathcal{M}$ be the structure with universe $\{c_i\}_{i\in\omega}$ determined by the quantifier-free fragment of D.

Claim 10.4. *$\mathcal{M}\models D$.*

Proof. Induction on sentence complexity. For quantifier-free sentences, this is immediate.

Suppose $\exists\overline{x}\tau(\overline{x},\overline{b})\in D$. Then at some sufficiently large stage, we act to put $\tau(\overline{c},\overline{b})\in D$ for some $\overline{b}$. By the inductive hypothesis, $\mathcal{M}\models\tau(\overline{c},\overline{b})$, so $\mathcal{M}\models\exists\overline{x}\tau(\overline{x},\overline{b})$.

Suppose $\forall\overline{x}\tau(\overline{x},\overline{b})\in D$. Then for any $\overline{c}$, it cannot be that $\neg\tau(\overline{c},\overline{b})\in D$, as that would violate the consistency of D. Since we eventually act to decide $\theta = \tau(\bar{c}, \bar{b})$, it must be that $\tau(\bar{c}, \bar{b}) \in D$. By the inductive hypothesis, $\mathcal{M} \models \tau(\bar{c}, \bar{b})$. Since $\bar{c}$ was arbitrary, $\mathcal{M} \models \forall \bar{x} \tau(\bar{x}, \bar{b})$. □

Claim 10.5. $\mathcal{M} \models S$.

Proof. If $\exists \bar{x} \varphi_i(\bar{x}) \in S_t$, then at any stage with $i < s$ and $t < t_s$, we will place the sentence $\varphi_i(\bar{c})$ in D for some $\bar{c}$, and thus $\mathcal{M} \models \exists \bar{x} \varphi_i(\bar{x})$.

If $\exists \bar{x} \varphi_i(\bar{x}) \notin S$, then at every stage with $i < s$, we will place the sentence $\neg \varphi_i(\bar{c})$ in D for every $\bar{c}$ mentioned so far in the construction. Thus $\mathcal{M} \not\models \varphi_i(\bar{c})$ for any $\bar{c}$, and so $\mathcal{M} \not\models \exists \bar{x} \varphi_i(\bar{x})$. □

This completes the proof. □

We now use Observation 9 und Theorem 10 to prove that non-Δ^0_n-degrees cannot be a Σ_n-spectrum.

Theorem 11. *The non-Δ^0_n degrees are not the Σ_n-spectrum of any structure.*

Proof. Suppose there were a structure $\mathcal{M}$ with $\mathrm{Spec}_{\Sigma_n}(\mathcal{M})$ consisting precisely of the non-Δ^0_n degrees. Using Observation 9, fix degrees $\mathbf{a}$ and $\mathbf{b}$ forming a Σ^0_1-minimal pair over $\mathbf{0}^{(n-1)}$, with $\mathbf{a}$ and $\mathbf{b}$ not arithmetical. By jump inversion, there are degrees $\mathbf{c}$ and $\mathbf{d}$ with $\mathbf{c}^{(n-1)} = \mathbf{a}$ and $\mathbf{d}^{(n-1)} = \mathbf{b}$, and neither $\mathbf{c}$ nor $\mathbf{d}$ are arithmetical.

By assumption, $\mathbf{c}, \mathbf{d} \in \mathrm{Spec}_{\Sigma_n}(\mathcal{M})$. Let S be the Σ_n-theory of $\mathcal{M}$. Then $S \in \Sigma^0_n(\mathbf{c}) = \Sigma^0_1(\mathbf{a})$ and also $S \in \Sigma^0_n(\mathbf{d}) = \Sigma^0_1(\mathbf{b})$. Since $\mathbf{a}$ and $\mathbf{b}$ form a Σ^0_1-minimal pair over $\mathbf{0}^{(n-1)}$, $S \in \Sigma^0_1(\mathbf{0}^{(n-1)})$, and thus by, Theorem 10, $\mathbf{0}^{(n-1)}$ can compute a model of S. This model has Δ^0_n-degree, contrary to the assumption. □

4. A non–trivial spectrum for Σ_1–equivalence

In view of the results about Σ_1-spectra from the previous two sections, it is natural to ask whether there exist Σ_1–spectra that are not cones. The next theorem answers this question positively.

Theorem 12. *There exists a structure $\mathcal{A}$ such that its Σ_1–spectrum $DgSp(\mathcal{A}, \equiv_{\Sigma_1})$ cannot be presented as a cone above a degree* $\mathbf{a}$.

Proof. As we already noted above, Σ_1-spectra must have the form $\{\mathbf{d} \mid X \text{ is } \mathbf{d}\text{-c.e.}\}$, where X is the set of Gödel indices of the sentences from the Σ_1-theory. On the other hand, every set of degrees of the form $\{\mathbf{d} \mid X \text{ is } \mathbf{d}\text{-c.e.}\}$, for some X, is a Σ_1-spectrum of a structure $\mathcal{A}_X$: the structure $\mathcal{A}_X$ contains an ω–chain $x_0, x_1, \dots$ using a binary predicate $P(x_n, x_{n+1})$ (and a constant that fixes x_0 as the first element of the chain). Whenever n is enumerated into X, we define $Q(x_n, y_n)$, where y_n is a new element that from now on witnesses $n \in X$. It is clear that $DgSp(\mathcal{A}, \equiv_{\Sigma_1}) = \{\mathbf{d} \mid X \text{ is } \mathbf{d}\text{-c.e.}\}$.

Richter studied sets of this form in [10]. She constructed a non-computably enumerable set X, which is computably enumerable in sets B and C forming a minimal pair. Thus, the degrees that enumerate X do not form a cone. The corresponding structure $\mathcal{A}_X$, built as described above, witnesses the statement of the theorem. □

5. Relations between Σ_n-spectra

In this section we study relations between Σ_n-spectra, for various n.

Proposition 13. *If S is a Σ_n–spectrum then $\{\mathbf{d} \mid \mathbf{d}' \in S\}$ is a Σ_{n+1}–spectrum.*

Proof. The proof is essentially the same as the proof of Lemma 2.8 in [1] which is based on Marker's construction. In that lemma it is proved that if S is a theory spectrum, then so is $\{\mathbf{d} \mid \mathbf{d}' \in S\}$. The idea of the Marker's construction is to build a new theory T' in such a way that every predicate of the original theory T is interpreted by both Σ_2- and Π_2-formula in T'. Using this, one can make sure that for an arbitrary sentence φ from T, the number of quantifier alternations in its interpretation φ' in T' increases only by one. Therefore, if the original theory is axiomatizable by Σ_n- or Π_n-sentences, then the new theory is axiomatizable by Σ_{n+1}- or Π_{n+1}-sentences. $\square$

This result allows us to prove that some collections of degrees are Σ_n-spectra.

Proposition 14. *Non-low$_n$ degrees form a Σ_{n+2}-spectrum.*

Proof. By Theorem 5, the set of degrees $\{\mathbf{d} : \mathbf{d} \nleq_T \mathbf{0}^{(n)}\}$ is a Σ_2-spectrum. Applying Proposition 13 n times we get the desired result. $\square$

Proposition 15. *The hign$_n$ degrees form a Σ_{n+1}-spectrum of a structure.*

Proof. We build a structure $\mathcal{A}$ with its Σ_{n+1}-spectrum consisting of exactly the high$_n$ degrees. Let $\mathcal{B}$ be a structure that has the Σ_1-spectrum of the form $\{\mathbf{d} : \mathbf{d} \geqslant_T \mathbf{0}^{(n+1)}\}$. Applying Proposition 13 n times, we get $\mathcal{A}$ with the desired Σ_{n+1} spectrum. $\square$

Recall that by Corollary 7, high degrees do not form a Σ_1–spectrum. We are going to extend this result by showing that high$_n$ degrees never form a Σ_n-spectrum.

Theorem 16. *The high$_n$ degrees do not form a Σ_n-spectrum of a structure.*

The proof follows from Proposition 17 and Theorem 18, where we compare the descriptive complexity of $\{X \in \omega^\omega : X \text{ is high}_n\}$ and $\{X \in \omega^\omega : X \in S\}$, for a Σ_n-spectrum S.

Proposition 17. *Let T be a Σ_n-fragment of a (complete) theory. Then $\{X : X$ computes (the atomic diagram of) a model of $T\}$ is a $\boldsymbol{\Sigma}^0_{n+2}$-class.*

Proof. X computes a model of T iff

$$\exists\Phi\forall\varphi \in \Sigma_n[\varphi \in T \iff \Phi^X \models \varphi].$$

Here Φ^X is the X-computable structure computed by Φ with oracle X. Then for a Σ_n sentence φ, the complexity of "$\Phi^X \models \varphi$" is $\Sigma_n^{0,X}$. Considering T as a parameter, we get the desired complexity $\mathbf{\Sigma}^0_{n+2}$. $\square$

Theorem 18. *$\{X \in \omega^\omega \ : \ X$ is high$_n\}$ is not a $\mathbf{\Sigma}^0_{n+2}$-class.*

The proof will follow from several claims. The goal is, for every $\mathbf{\Sigma}^0_{n+2}$-class $\mathcal{C}$, to build a function f such that $f \in \mathcal{C} \iff f$ is not high$_n$.

Definition 19. Define a notion of forcing $(\mathbb{P}, \leqslant_{\mathbb{P}})$ where the conditions are $(\sigma_0, \ldots, \sigma_{n-1}) \in (\omega^{<\omega})^n$, and $\overline{\sigma} \geq_{\mathbb{P}} \overline{\tau}$ if and only if the following hold:

(1) $\sigma_m \subseteq \tau_m$ for all $m < n$; and
(2) For every $m < n-1$ and every $x \in \mathrm{dom}(\sigma_{m+1})$, if $\langle x, t\rangle \in (\mathrm{dom}(\tau_m) - \mathrm{dom}(\sigma_m))$, then $\tau_m(\langle x, t\rangle) = \sigma_{m+1}(x)$.

For a function h, define $\mathbb{P}_h = \{\overline{\sigma} \in \mathbb{P} \ : \ \forall x \in \mathrm{dom}(\sigma_{n-1})\, [\sigma_{n-1}(x) \geq h(x)]\}$.

For G a filter, define $f^G_m = \bigcup_{\overline{\sigma}\in G} \sigma_m$.

Note that if G is sufficiently generic, then the f^G_m will be total functions with $f^G_{m+1}(x) = \lim_t f^G_m(\langle x, t\rangle)$ for all x and all $m < n-1$. Intuitively, f^G_{m+1} is the jump of f^G_m. We will not actually verify this, but it guides our intuition.

Claim 19.1. *Fix h.*

For $\mathcal{A}$ a $\mathbf{\Sigma}^0_m$-class with $m < n$, if $\overline{\sigma} \Vdash_{\mathbb{P}_h} [f_0 \in \mathcal{A}]$, then there is $\overline{\tau} \leq_{\mathbb{P}} \overline{\sigma}$ with $\overline{\tau} \in \mathbb{P}_h$ and $(\tau_0, \ldots, \tau_{m-1}, \emptyset, \ldots, \emptyset) \Vdash_{\mathbb{P}} [f_0 \in \mathcal{A}]$.

For $\mathcal{B}$ a $\mathbf{\Pi}^0_m$-class with $m < n$, if $\overline{\sigma} \Vdash_{\mathbb{P}_h} [f_0 \in \mathcal{B}]$, then $(\sigma_0, \ldots, \sigma_m, \emptyset, \ldots, \emptyset) \Vdash_{\mathbb{P}} [f_0 \in \mathcal{B}]$.

Proof. We prove the two parts of the claim simultaneously, by induction.

For $\mathcal{A}$ open, if $\overline{\sigma} \Vdash_{\mathbb{P}_h} [f_0 \in \mathcal{A}]$, then it must be that for every extension $\overline{\rho} \leq_{\mathbb{P}} \overline{\sigma}$ with $\overline{\rho} \in \mathbb{P}_h$, there is an extension $\overline{\tau} \leq_{\mathbb{P}} \overline{\rho}$ with $\overline{\tau} \in \mathbb{P}_h$ and $[\tau_0] \subseteq \mathcal{A}$. Then $(\tau_0, \emptyset, \dots, \emptyset) \Vdash_{\mathbb{P}} [f_0 \in \mathcal{A}]$, as desired.

For $\mathcal{B}$ closed, if $\overline{\sigma} \Vdash_{\mathbb{P}_h} [f_0 \in \mathcal{B}]$, then we claim $(\sigma_0, \sigma_1, \emptyset, \dots, \emptyset) \Vdash_{\mathbb{P}} [f_0 \in \mathcal{B}]$. For suppose not. Then there is an extension $\overline{\rho} \leq_{\mathbb{P}} (\sigma_0, \sigma_1, \emptyset, \dots, \emptyset)$ with $\overline{\rho} \in \mathbb{P}$ and $[\rho_0] \cap \mathcal{B} = \emptyset$. But note that $(\rho_0, \sigma_1, \dots, \sigma_{n-1}) \leq_{\mathbb{P}} \overline{\sigma}$ and $(\rho_0, \sigma_1, \dots, \sigma_{n-1}) \in \mathbb{P}_h$. Since $(\rho_0, \sigma_1, \dots, \sigma_{n-1}) \Vdash_{\mathbb{P}_h} [f_0 \notin \mathcal{B}]$, this contradicts our assumption for $\overline{\sigma}$.

For $\mathcal{A}$ a $\mathbf{\Sigma}^0_{m+1}$-class, write $\mathcal{A} = \bigcup_j \mathcal{B}_j$, where each $\mathcal{B}_j$ is a $\mathbf{\Pi}^0_m$-class. If $\overline{\sigma} \Vdash_{\mathbb{P}_h} [f_0 \in \mathcal{A}]$, then it must be that for every $\overline{\rho} \leq_{\mathbb{P}} \overline{\sigma}$ with $\overline{\rho} \in \mathbb{P}_h$, there is an extension $\overline{\tau} \leq_{\mathbb{P}} \rho$ with $\overline{\tau} \in \mathbb{P}_h$ and a j with $\overline{\tau} \Vdash_{\mathbb{P}_h} [f_0 \in \mathcal{B}_j]$. By induction, $(\tau_0, \dots, \tau_m, \emptyset, \dots, \emptyset) \Vdash_{\mathbb{P}} [f_0 \in \mathcal{B}_j]$. Such a $\overline{\tau}$ suffices for the claim.

For $\mathcal{B}$ a $\mathbf{\Pi}^0_{m+1}$-class, write $\mathcal{B} = \bigcap_j \mathcal{A}_j$, where each $\mathcal{A}_j$ is a $\mathbf{\Sigma}^0_m$-class. If $\overline{\sigma} \Vdash_{\mathbb{P}_h} [f_0 \in \mathcal{B}]$, then we claim $(\sigma_0, \dots, \sigma_{m+1}, \emptyset, \dots, \emptyset) \Vdash_{\mathbb{P}} [f_0 \in \mathcal{B}]$. For suppose not. Then there is an extension $\overline{\rho} \leq_{\mathbb{P}} (\sigma_0, \dots, \sigma_{m+1}, \emptyset, \dots, \emptyset)$ with $\overline{\rho} \in \mathbb{P}$ and some j with $\overline{\rho} \Vdash_{\mathbb{P}} [f_0 \notin \mathcal{A}_j]$.

Consider $(\rho_0, \dots, \rho_m, \sigma_{m+1}, \dots, \sigma_{n-1})$, which is an extension of $\overline{\sigma}$ and an element of $\mathbb{P}_h$. By choice of $\overline{\sigma}$, there must be a $\overline{\nu} \leq_{\mathbb{P}} (\rho_0, \dots, \rho_m, \sigma_{m+1}, \dots, \sigma_{n-1})$ with $\overline{\nu} \in \mathbb{P}_h$ and $\overline{\nu} \Vdash_{\mathbb{P}_h} [f_0 \in \mathcal{A}_j]$. By induction, there is a $\overline{\tau} \leq_{\mathbb{P}} \overline{\nu}$ with $(\tau_0, \dots, \tau_{m-1}, \emptyset, \dots, \emptyset) \Vdash_{\mathbb{P}} [f_0 \in \mathcal{A}_j]$. But then $(\tau_0, \dots, \tau_{m-1}, \rho_m, \dots, \rho_{n-1})$ extends both $(\tau_0, \dots, \tau_{m-1}, \emptyset, \dots, \emptyset)$ and $\overline{\rho}$, and thus $\mathbb{P}$-forces both $[f_0 \in \mathcal{A}_j]$ and $[f_0 \notin \mathcal{A}_j]$, a contradiction. □

Claim 19.2. *Fix h. For $\mathcal{B}$ a $\mathbf{\Pi}^0_m$-class with $m < n$ and $\overline{\sigma} \in \mathbb{P}_h$, if $\overline{\sigma} \Vdash_{\mathbb{P}} [f_0 \in \mathcal{B}]$, then $(\sigma_0, \dots, \sigma_m, \emptyset, \dots, \emptyset) \Vdash_{\mathbb{P}_h} [f_0 \in \mathcal{B}]$.*

Proof. Suppose not. Then there is some $\overline{\rho} \leq_{\mathbb{P}} (\sigma_0, \dots, \sigma_m, \emptyset, \dots, \emptyset)$ with $\overline{\rho} \in \mathbb{P}_h$ and $\overline{\rho} \Vdash_{\mathbb{P}_h} [f_0 \notin \mathcal{B}]$. By Claim 19.1 applied to the complement of $\mathcal{B}$, there is a $\overline{\tau} \leq_{\mathbb{P}} \overline{\rho}$ with $\overline{\tau} \in \mathbb{P}_h$ and $(\tau_0, \dots, \tau_{m-1}, \emptyset, \dots, \emptyset) \Vdash_{\mathbb{P}} [f_0 \notin \mathcal{B}]$. So $(\tau_0, \dots, \tau_{m-1}, \emptyset, \dots, \emptyset)$ and $\overline{\sigma}$ $\mathbb{P}$-force incompatible statements, but $(\tau_0, \dots, \tau_{m-1}, \sigma_m, \dots, \sigma_{n-1})$ is a common extension, which is a contradiction. □

Fix $h \in \Delta^0_n$. Note that if h were computable, $\mathbb{P}_h$ and $\mathbb{P}$ would be computably isomorphic, and so the following claim would be immediate. As it is, $\mathbb{P}_h$ and $\mathbb{P}$ are only Δ^0_n-isomorphic, and the claim does not hold for arbitrary notions of forcing which are Δ^0_n-isomorphic to $\mathbb{P}$—consider $\mathbb{P}$ with the added requirement that $\sigma_0(\langle x, 0\rangle) = \emptyset'(x)$.

Recalling our intuition, the claim holds in this case because the Δ^0_n-information of $\mathbb{P}_h$ only occurs in f^G_{n-1}, which is the $(n-1)$st jump of f^G_0.

Claim 19.3. *If h is Δ^0_n, and G is sufficiently $(\mathbb{P}_h, \leqslant_{\mathbb{P}})$-generic, then f^G_0 is not high$_n$.*

Proof. We begin with the following:

Claim 19.3.1. $\left(f^G_0\right)^{(n)} \leqslant_T \emptyset^{(n)} \oplus \bigoplus_{m<n} f^G_m$.

Proof. It suffices to show that our oracle can uniformly decide $[f^G_0 \in \mathcal{A}]$ for any Σ^0_n-class $\mathcal{A}$. Fix an effective list of Π^0_{n-1}-classes $(\mathcal{B}_j)_{j\in\omega}$ with $\mathcal{A} = \bigcup_j \mathcal{B}_j$.

By Claims 19.1 and 19.2,

$$\begin{aligned} \overline{\sigma} \Vdash_{\mathbb{P}_h} [f_0 \notin \mathcal{A}] &\iff \forall j\, \forall \overline{\tau} \in \mathbb{P}_h \; (\overline{\tau} \leq_{\mathbb{P}} \overline{\sigma} \to \overline{\tau} \nVdash_{\mathbb{P}_h} [f_0 \in \mathcal{B}_j]) \\ &\iff \forall j\, \forall \overline{\tau} \in \mathbb{P}_h \; (\overline{\tau} \leq_{\mathbb{P}} \overline{\sigma} \to \overline{\tau} \nVdash_{\mathbb{P}} [f_0 \in \mathcal{B}_j]) \, . \end{aligned}$$

Since $\mathcal{B}_j$ is Π^0_{n-1}, and $\mathbb{P}$ is a computable notion of forcing, the sentence $\overline{\tau} \Vdash_{\mathbb{P}} [f_0 \in \mathcal{B}_j]$ is uniformly Π^0_{n-1}. Thus $\overline{\sigma} \Vdash_{\mathbb{P}_h} [f_0 \notin \mathcal{A}]$ is uniformly Π^0_n.

On the other hand, if $f_0^G \in \mathcal{A}$, then for some $\overline{\sigma} \in G$, $\exists j\ (\overline{\sigma} \Vdash_{\mathbb{P}_h} [f_0 \in \mathcal{B}_j])$. By Claims 19.1 and 19.2 again,

$$\exists j\ (\overline{\sigma} \Vdash_{\mathbb{P}_h} [f_0 \in \mathcal{B}_j]) \iff \exists j\ (\overline{\sigma} \Vdash_{\mathbb{P}} [f_0 \in \mathcal{B}_j]),$$

which is uniformly Σ_n^0.

Clearly $\bigoplus_{m<n} f_m^G$ computes G, and so $\emptyset^{(n)} \oplus \bigoplus_{m<n} f_m^G$ can decide $[f_0 \in \mathcal{A}]$ by enumerating $\overline{\sigma} \in G$ until it finds $\overline{\sigma}$ with $\overline{\sigma} \Vdash_{\mathbb{P}_h} [f_0 \notin \mathcal{A}]$ or $\exists j\ (\overline{\sigma} \Vdash_{\mathbb{P}_h} [f_0 \in \mathcal{B}_j])$. □

It now suffices to show that $\emptyset^{(n+1)} \not\leqslant_T \emptyset^{(n)} \oplus \bigoplus_{m<n} f_m^G$. Suppose not, and let $\Gamma(\emptyset^{(n)}, f_0^G, \ldots, f_{n-1}^G) = \emptyset^{(n+1)}$. Then consider

$$D = \left\{\overline{\rho} \in \mathbb{P}_h\ :\ \exists x\, \Gamma\left(\emptyset^{(n)}, \overline{\rho}\right)(x){\downarrow} \neq \emptyset^{(n+1)}(x)\right\}.$$

By assumption, G does not meet D, and so G avoids D. So fix $\overline{\sigma} \in G$ such that for all $\overline{\rho} \leqslant_{\mathbb{P}} \overline{\sigma}$, $\overline{\rho} \notin D$. But then $\emptyset^{(n)}$ can compute $\emptyset^{(n+1)}$ via the following algorithm: on input x, enumerate $\overline{\rho} \in \mathbb{P}_h$ extending $\overline{\sigma}$ until finding one with $\Gamma(\emptyset^{(n)}, \overline{\rho})(x){\downarrow}$. Since no such $\overline{\rho}$ is in D, necessarily $\Gamma(\emptyset^{(n)}, \overline{\rho})(x) = \emptyset^{(n+1)}(x)$. Further, there will always be such a $\overline{\rho}$, since there must be one in G.

This is a contradiction, and so it must be that $\emptyset^{(n)} \oplus \bigoplus_{m<n} f_m^G$, and so $\left(f_0^G\right)^{(n)}$, does not compute $\emptyset^{(n+1)}$. □

Fix $\mathcal{C} = \bigcup_i \bigcap_j \bigcup_k \mathcal{C}_{i,j,k}$ a $\mathbf{\Sigma}_{n+2}^0$-class, where each $\mathcal{C}_{i,j,k}$ is $\mathbf{\Pi}_{n-1}^0$. Let $\mathrm{Tot}(\Delta_n^0)$ denote the collection of Δ_n^0 indices that describe total functions. Given $e \in \mathrm{Tot}(\Delta_n^0)$, let φ_e be the corresponding function.

Definition 20. Define a notion of forcing $(\mathbb{Q}, \leq_{\mathbb{Q}})$ where the conditions are pairs $(\overline{\sigma}, g)$ with $\overline{\sigma} \in \mathbb{P}$ and $g : \mathrm{Tot}(\Delta_n^0) \to \omega$ a finite partial function.

Define $(\overline{\sigma}, g) \geq_{\mathbb{Q}} (\overline{\rho}, \hat{g})$ if and only if the following hold:

(1) $\overline{\sigma} \geq_{\mathbb{P}} \overline{\rho}$;
(2) $\mathrm{dom}(g) \subseteq \mathrm{dom}(\hat{g})$;
(3) For all $e \in \mathrm{dom}(g)$, $\hat{g}(e) \geq g(e)$;

(4) For all $e \in \mathrm{dom}(g)$ and all $x \in (\mathrm{dom}(\rho_{n-1}) - \mathrm{dom}(\sigma_{n-1}))$, if $g(e) = \hat{g}(e)$, then $\rho_{n-1}(x) \geq \varphi_e(x)$; and

(5) For all $e \in \mathrm{dom}(g)$, one of the following holds:
  (a) $\hat{g}(e) = g(e)$; or
  (b) There is an $i \leq e$ such that $(\forall j < g(e))\, \exists k\ (\overline{\rho} \Vdash_{\mathbb{P}} [f_0 \in \mathcal{C}_{i,j,k}])$.

For G a filter, define $f_i^G = \bigcup_{(\overline{\sigma},g)\in G} \sigma_i$.

Claim 20.1. *For $\mathcal{A}$ a $\mathbf{\Sigma}^0_m$-class with $m < n$, if $(\overline{\sigma}, g) \Vdash_{\mathbb{Q}} [f_0 \in \mathcal{A}]$, then there is $(\overline{\tau}, g) \leq_{\mathbb{Q}} (\overline{\sigma}, g)$ with $(\tau_0, \ldots, \tau_{m-1}, \emptyset, \ldots, \emptyset) \Vdash_{\mathbb{P}} [f_0 \in \mathcal{A}]$.*

For $\mathcal{B}$ a $\mathbf{\Pi}^0_m$-class with $m < n$, if $(\overline{\sigma}, g) \Vdash_{\mathbb{Q}} [f_0 \in \mathcal{B}]$, then $(\sigma_0, \ldots, \sigma_m, \emptyset, \ldots, \emptyset) \Vdash_{\mathbb{P}} [f_0 \in \mathcal{B}]$.

Proof. As Claim 19.1, mutatis mutandis. □

Claim 20.2. *For $\mathcal{B}$ a $\mathbf{\Pi}^0_m$-class with $m < n$ and $(\overline{\sigma}, g) \in \mathbb{Q}$, if $\overline{\sigma} \Vdash_{\mathbb{P}} [f_0 \in \mathcal{B}]$, then $(\sigma_0, \ldots, \sigma_m, \emptyset, \ldots, \emptyset, g) \Vdash_{\mathbb{Q}} [f_0 \in \mathcal{B}]$.*

Proof. As Claim 19.2, mutatis mutandis. □

Now fix G a sufficiently generic filter for $(\mathbb{Q}, \leq_{\mathbb{Q}})$ ($\Delta^0_\omega(\mathcal{C})$-generic should suffice).

Claim 20.3. *If ℓ is such that for every $i \leq \ell$, $f_0^G \notin \bigcap_j \bigcup_k \mathcal{C}_{i,j,k}$, then there is $(\overline{\sigma}, g) \in G$ such that for all $(\overline{\tau}, \dot{g}) \leq_{\mathbb{Q}} (\overline{\sigma}, g)$ and all $e \leq \ell$ with $e \in Tot(\Delta^0_n)$, $\dot{g}(e) = g(e)$.*

Proof. For every $i \leq \ell$, there some j_i and some $(\overline{\sigma}, g) \in G$ with $(\overline{\sigma}, g) \Vdash_{\mathbb{Q}} [f_0 \notin \bigcup_k \mathcal{C}_{i,j_i,k}]$. By taking a common extension, there is a single $(\overline{\sigma}, g) \in G$ that serves for all $i \leq \ell$. Now suppose there were some $(\overline{\tau}, \dot{g}) \leq_{\mathbb{Q}} (\overline{\sigma}, g)$, $i \leq \ell$ and k such that $\overline{\tau} \Vdash_{\mathbb{P}} [f_0 \in \mathcal{C}_{i,j_i,k}]$. Then by Claim 20.2, we would have $(\overline{\tau}, \dot{g}) \Vdash_{\mathbb{Q}} [f_0 \in \mathcal{C}_{i,j_i,k}]$, a contradiction.

Let $j_0 = \max_{i\leq\ell}\{j_i\}$. Then for each $(\overline{\tau}, \dot{g}) \leq_{\mathbb{Q}} (\overline{\rho}, \hat{g}) \leq_{\mathbb{Q}} (\overline{\sigma}, g)$ and each $e < i_0$ with $e \in \mathrm{Tot}(\Delta^0_n)$, if $e \in \mathrm{dom}(\hat{g})$ and $\hat{g}(e) > j_0$, then $\dot{g}(e) = \hat{g}(e)$. For if this were not the case, by definition we would have $\overline{\tau} \Vdash_{\mathbb{P}} [f_0 \in C_{i,j_i,k}]$ for some $i \leq \ell$ and some k, contrary to the previous paragraph. So for each $e \leq \ell$ with $e \in \mathrm{Tot}(\Delta^0_n)$, the set $\{\hat{g}(e) \ : \ (\overline{\rho}, \hat{g}) \in G\}$ has a maximum. By replacing $(\overline{\sigma}, g)$ with some extension, if necessary, we may assume that $g(e)$ is defined and achieves this maximum. □

Claim 20.4. *If $f_0^G \in \mathcal{C}$, then $G_1 = \{\overline{\sigma} \ : \ \exists g\, (\overline{\sigma}, g) \in G\}$ is $(\mathbb{P}_h, \leq_{\mathbb{P}})$-generic for some Δ^0_n function h.*

Proof. Fix i_0 least with $f_0^G \in \bigcap_j \bigcup_k \mathcal{C}_{i_0,j,k}$. Let $(\overline{\sigma}, g)$ be as in Claim 20.3 with $\ell = i_0 - 1$.

Now, define $h \succ \sigma_{n-1}$ as

$$h(x) = \begin{cases} \min\{\max\{\varphi_e(x) \ : \ e < i_0 \wedge e \in \mathrm{Tot}(\Delta^0_n)\}, \sigma_{n-1}(x), \} & \text{if } x < |\sigma_{n-1}|, \\ \max\{\varphi_e(x) \ : \ e < i_0 \wedge e \in \mathrm{Tot}(\Delta^0_n)\} & \text{otherwise.} \end{cases}$$

Note that $h \in \Delta^0_n$. This is the desired function.

Since for any $(\overline{\tau}, \dot{g}) \leq_{\mathbb{Q}} (\overline{\sigma}, g)$, we know $\dot{g}(e) = g(e)$ for all $e < i_0$ with $e \in \mathrm{Tot}(\Delta^0_n)$, then by definition we have that $\overline{\tau} \in \mathbb{P}_h$. Thus $G_1 \subseteq \mathbb{P}_h$.

Suppose now that $D \subseteq \mathbb{P}_h$ is such that every condition in G_1 can be extended to a condition in D. It suffices to show that for any condition $(\overline{\rho}, \hat{g}) \in G$ extending $(\overline{\sigma}, g)$, there is a condition $(\overline{\tau}, \dot{g}) \in \mathbb{Q}$ with $\overline{\tau} \in D$.

Since $f_0^G \in \bigcup_k \mathcal{C}_{i_0,j,k}$ for all j, choose $(\overline{\nu}, g') \leq_{\mathbb{Q}} (\overline{\rho}, \hat{g})$ in G such that

$$(\forall j < \max\{\hat{g}(e) \ : \ e \in \mathrm{dom}(\hat{g})\})\, \exists k\, ((\overline{\nu}, g') \Vdash_{\mathbb{Q}} [f_0 \in \mathcal{C}_{i_0,j,k}]).$$

Then by Claim 20.1,

$$(\forall j < \max\{\hat{g}(e) \ : \ e \in \mathrm{dom}(\hat{g})\})\, \exists k\, (\overline{\nu} \Vdash_{\mathbb{P}} [f_0 \in \mathcal{C}_{i_0,j,k}]).$$

Choose $\overline{\tau} \in D$ extending $\overline{\nu}$. Define $\dot{g}$ as:

$$\dot{g}(e) = \begin{cases} \hat{g}(e) & e < i_0 \text{ and } e \in \mathrm{dom}(\hat{g}), \\ \hat{g}(e) + 1 & \geq i_0 \text{ and } e \in \mathrm{dom}(\hat{g}). \end{cases}$$

Note that by our choice of $\overline{\nu}$, $(\overline{\tau}, \dot{g}) \leq_{\mathbb{Q}} (\overline{\rho}, \hat{g})$.

This demonstrates that every condition in G can be extended to a condition $(\overline{\tau}, \dot{g}) \in \mathbb{Q}$ with $\overline{\tau} \in D$. So if G is sufficiently generic relative to D, then G_1 must meet D. $\square$

It follows that if $f_0^G \in \mathcal{C}$, then f_0^G is not high$_n$.

Claim 20.5. *If $f_0^G \notin \mathcal{C}$, then f_{n-1}^G dominates all total Δ_n^0 functions.*

Proof. Fix $e \in \mathrm{Tot}(\Delta_n^0)$. Let $(\overline{\sigma}, g)$ be as in Claim 20.3 with $\ell = e$.

Then by definition, for all $(\overline{\rho}, \hat{g}) \leq_{\mathbb{Q}} (\overline{\sigma}, g)$ and all $x \in (\mathrm{dom}(\rho_{n-1}) - \mathrm{dom}(\sigma_{n-1}))$, we have $\rho_{n-1}(x) \geq \varphi_e(x)$. So $f_{n-1}^G(x) \geq \varphi_e(x)$ for all $x \geq |\sigma_{n-1}|$. $\square$

By the limit lemma, $f_{n-1}^G \leqslant_T \left(f_0^G\right)^{(n-1)}$. It follows that if $f_0^G \notin \mathcal{C}$, then f_0^G is high$_n$.

Proof of Theorem 18. For any $\boldsymbol{\Sigma}_{n+2}^0$-class $\mathcal{C}$, the above forcing produces a function f_0^G such that $f_0^G \in \mathcal{C} \iff f_0^G$ is not high$_n$. $\square$

Theorem 21. *There is a Σ_{n+1}-spectrum that is not a Σ_n-spectrum of any structure.*

Proof. Follows directly from Proposition 15 and Theorem 18. $\square$

6. Σ_n-SPECTRA VS THEORY SPECTRA

We now prove that there is a theory spectrum that is not a Σ_n-spectrum, for any $n \geqslant 1$.

Definition 22. Let $\mathcal{F} = \{X \in 2^\omega : (\exists\Phi)(\forall n)[\Phi(X^{(n)} \oplus \{n\}) = \emptyset^{(2n)}]\}$.

Theorem 23. *$\mathcal{F}$ is not the Σ_k-spectrum of any structure $\mathcal{M}$ for any $k \in \omega$.*

Proof. Suppose not, and fix witnessing M and k. By a standard Friedberg jump inversion construction, fix $\mathbf{a}$ and $\mathbf{b}$ forming a minimal pair over $\mathbf{0}^{(3k)}$ with $\mathbf{a}' = \mathbf{b}' = \mathbf{0}^{(\omega)}$. By jump inversion again, there are $\mathbf{c}$ and $\mathbf{d}$ both above $\mathbf{0}^{(2k)}$ with $\mathbf{c}^{(k)} = \mathbf{a}$ and $\mathbf{d}^{(k)} = \mathbf{b}$.

Note that $\mathbf{c} \in \mathcal{F}$: for $C \in \mathbf{c}$, if $n \leq k$, $C^{(n)} \geqslant_T C \geqslant_T \emptyset^{(2k)} \geqslant_T \emptyset^{(2n)}$; if $n > k$, $C^{(n)} \geqslant_T C^{(k+1)} = \emptyset^{(\omega)} \geqslant_T \emptyset^{(2n)}$. Further, all of these reductions are uniform. Similarly, $\mathbf{d} \in \mathcal{F}$. Thus there is an $M_{\mathbf{c}} \in \mathbf{c}$ and an $M_{\mathbf{d}} \in \mathbf{d}$ with

$$\mathrm{Th}_{\Sigma_k}(\mathcal{M}_{\mathbf{c}}) = \mathrm{Th}_{\Sigma_k}(\mathcal{M}_{\mathbf{d}}) = \mathrm{Th}_{\Sigma_k}(\mathcal{M}).$$

Then $\mathrm{Th}_{\Sigma_k}(\mathcal{M}) \in \Sigma^0_k(\mathbf{c}) \subset \Delta^0_1(\mathbf{a})$, and $\mathrm{Th}_{\Sigma_k}(\mathcal{M}) \in \Sigma^0_k(\mathbf{d}) \subset \Delta^0_1(\mathbf{b})$. By our choice of $\mathbf{a}$ and $\mathbf{b}$, $\mathrm{Th}_{\Sigma_k}(\mathcal{M}) \in \Delta^0_1(\mathbf{0}^{(3k)})$, and so there is a $\mathbf{0}^{(3k)}$-computable model of $\mathrm{Th}_{\Sigma_k}(\mathcal{M})$. But clearly no arithmetical degree can be in $\mathcal{F}$, which is a contradiction. $\square$

Theorem 24. *There is a structure $\mathcal{M}$ with $DgSp(\mathcal{M}, \cong) = DgSp(\mathcal{M}, \equiv) = \mathcal{F}$.*

Proof. Our structure will be an effective disjoint union $\mathcal{M} = \bigsqcup_{n\in\omega} \mathcal{M}_n$. In $\mathcal{M}_n$, we will code $\emptyset^{(2n)}$ in a manner than can be decoded by the nth jump. Our language for $\mathcal{M}_n$ will be $\{P_i, N_i\}_{i\in\omega} \cup \{\to\}$, where the P_i and N_i are unary relations, and $\to$ is a binary relation.

We recall the following trees (in the language of directed graphs), originally due to Hirschfeldt and White [7]:

- A_1 is the tree consisting of only the root;
- E_1 is the tree where the root has infinitely many children, and all of these children are leaves;

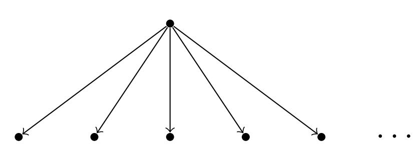

FIGURE 1. The tree E_1.

- A_{k+1} is the tree where the root has infinitely many children all of whose subtrees are a copy of E_k;

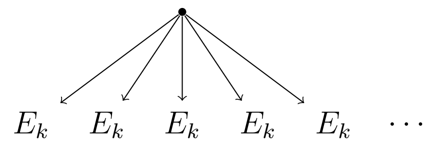

FIGURE 2. The tree A_{k+1}.

- E_{k+1} is the tree where the root has infinitely many children whose subtrees are a copy of E_k, and also has infinitely many children whose subtrees are a copy A_k.

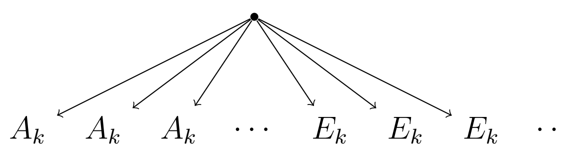

FIGURE 3. The tree E_{k+1}.

Hirschfeldt and White showed that given a Σ^0_k predicate, one can computably construct a tree T which is isomorphic to E_k if the predicate holds, and is isomorphic to A_k if it fails, and further this construction is uniform in an index for the predicate.

Also, there is a first-order Σ_k formula that holds of the root of the E_k tree, but does not hold of the root of the A_k tree. We define these recursively: define $\varphi_1(x) : \exists z[x \to z]$; define $\varphi_{k+1}(x) : \exists z[x \to z \wedge \neg\varphi_k(z)]$.

We now construct $\mathcal{M}_n$ as follows: for each i, there is a unique element x with $\mathcal{M} \models P_i(x)$, and x is the root of a tree of type E_{n+1} if $i \in \emptyset^{(2n)}$ and of type A_{n+1} if $i \notin \emptyset^{(2n)}$; conversely there is a unique element y with $\mathcal{M} \models N_i(y)$, and y is the root of a tree of type A_{n+1} if $i \in \emptyset^{(2n)}$ and of type E_{n+1} if $i \notin \emptyset^{(2n)}$.

We claim that if $X \in \mathcal{F}$, then X uniformly computes a copy of $\mathcal{M}_n$. For $\emptyset^{(2n)} \in \Delta^0_{n+1}(X)$, and thus for the x and y with $P_i(x)$ and $N_i(y)$, we can construct the trees rooted at x and y computably relative to X as described above.

Conversely, we claim that if X uniformly computes structures $(L_n)_{n\in\omega}$ with L_n elementarily equivalent to $\mathcal{M}_n$, then $X \in \mathcal{F}$. For

$i \in \emptyset^{(2n)} \iff (\exists x \in L_n)[P_i(x) \wedge \varphi_{n+1}(x)] \iff (\forall y \in L_n)[N_i(y) \Rightarrow \neg\varphi_{n+1}(y)]$.

Thus $\emptyset^{(2n)} \in \Delta^0_{n+1}(X)$, and further the code is obtained uniformly. $\square$

References

[1] Uri Andrews and Joseph S. Miller. Spectra of theories and structures. *Proc. Amer. Math. Soc.*, 143(3):1283–1298, 2015.

[2] K. de Leeuw, E. F. Moore, C. E. Shannon, and N. Shapiro. Computability by probabilistic machines. In *Automata studies*, Annals of mathematics studies, no. 34, pages 183–212. Princeton University Press, Princeton, N. J., 1956.

[3] Ekaterina B. Fokina, Valentina Harizanov, and Alexander Melnikov. Computable model theory. In Rod Downey, editor, *Turing's Legacy*, pages 124–194. Cambridge University Press, 2014. Cambridge Books Online.

[4] Ekaterina B. Fokina, Dino Rossegger, and Luca San Mauro. Bi-embeddability spectra of structures. preprint.

[5] Noam Greenberg, Antonio Montalbán, and Theodore A. Slaman. Relative to any non-hyperarithmetic set. *J. Math. Log.*, 13(1):1250007, 26, 2013.

[6] D. Hirschfeldt, B. Khoussainov, R. Shore, and A. Slinko. Degree spectra and computable dimensions in algebraic structures. *Ann. Pure Appl. Logic*, 115(1-3):71–113, 2002.

[7] Denis R. Hirschfeldt and Walker M. White. Realizing levels of the hyperarithmetic hierarchy as degree spectra of relations on computable structures. *Notre Dame J. Formal Logic*, 43(1):51–64 (2003), 2002.

[8] I. Sh. Kalimullin. Almost computably enumerable families of sets. *Mat. Sb.*, 199(10):33–40, 2008.

[9] Julia F. Knight. Degrees coded in jumps of orderings. *J. Symbolic Logic*, 51(4):1034–1042, 1986.

[10] Linda Jean Richter. Degrees of structures. *J. Symbolic Logic*, 46(4):723–731, 1981.

[11] T. Slaman. Relative to any nonrecursive set. *Proc. Amer. Math. Soc.*, 126(7):2117–2122, 1998.

[12] Ivan N. Soskov. Degree spectra and co-spectra of structures. *Annuaire Univ. Sofia Fac. Math. Inform.*, 96:45–68, 2004.

[13] S. Wehner. Enumerations, countable structures and Turing degrees. *Proc. Amer. Math. Soc.*, 126(7):2131–2139, 1998.

[14] Liang Yu. Degree spectra of equivalence relations. In *Proceedings of the 13th Asian Logic Conference*, pages 237–242. World Sci. Publ., Hackensack, NJ, 2015.

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