LINEAR ORDERS REALIZED BY C.E. EQUIVALENCE RELATIONS

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Abstract. Let E be a computably enumerable (c.e.) equivalence relation on the set ω of natural numbers. We say that the quotient set ω/E (or equivalently, the relation E) *realizes* a linearly ordered set $\mathcal{L}$ if there exists a c.e. relation $\trianglelefteq$ respecting E such that the induced structure $(\omega/E; \trianglelefteq)$ is isomorphic to $\mathcal{L}$. Thus, one can consider the class of all linearly ordered sets that are realized by ω/E; formally, $\mathcal{K}(E) = \{\mathcal{L} \mid \text{the order-type } \mathcal{L} \text{ is realized by } E\}$. In this paper we study the relationship between computability-theoretic properties of E and algebraic properties of linearly ordered sets realized by E. One can also define the following pre-order $\leqslant_{lo}$ on the class of all c.e. equivalence relations: $E_1 \leqslant_{lo} E_2$ if every linear order realized by E_1 is also realized by E_2. Following the tradition of computability theory, the lo-degrees are the classes of equivalence relations induced by the pre-order $\leqslant_{lo}$. We study the partially ordered set of lo-degrees. For instance, we construct various chains and antichains and show the existence of a maximal element among the lo-degrees.

1. Introduction

In this paper we are interested in countable linearly ordered sets, their computably enumerable (c.e.) representations, and dependency of these representations on the underlying domains. By linear orders we always mean reflexive, transitive, and anti-symmetric binary relations such that any two elements in their domain are comparable (note our use of reflexive rather than irreflexive relations). To explain our set-up, we start with the following known folklore result about countable linearly ordered sets. For every countable linearly ordered set $\mathcal{L} = (L; \leqslant_L)$ there exists a mapping h from the linearly ordered set $(\mathbb{Q}; \leqslant)$ of rational numbers onto $\mathcal{L}$, $h : \mathbb{Q} \to L$, such that $h(x) \leqslant_L h(y)$ whenever $x \leqslant y$. Consider the kernel E of this homomorphism:

$$E = \{(x, y) \mid h(x) = h(y)\}.$$

The natural order relation $\leqslant$ *respects* E in the following sense: for all $x, y, x', y' \in \mathbb{Q}$, if xEx', yEy' and $\neg(xEy)$, then $x \leqslant y$ if and only if $x' \leqslant y'$; in addition, the induced quotient linearly ordered set $(Q/E; \leqslant)$ is isomorphic to $\mathcal{L}$. Furthermore, $\leqslant$ is a computable relation. Note that if a relation $\trianglelefteq$ respects E in the above sense, then $\trianglelefteq$ naturally induces a relation on the quotient set ω/E; we use the same notation $\trianglelefteq$ for the induced relation on ω/E. Further, if $(\omega; \trianglelefteq)$ is a linear ordering, then so is $(\omega/E; \trianglelefteq)$.

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In the above treatment, we are not concerned with whether or not $x \trianglelefteq y$ for xEy. If $(\omega; \trianglelefteq)$ is a linear ordering and E contains a non-trivial equivalence class, then there will always be xEy with $x \trianglelefteq y$ and $y \not\trianglelefteq x$. If we worried about such x and y, we would be forced to conclude that no linear ordering respects E. Instead, we have chosen to simply ignore those pairs xEy and to assume that the induced relation is always symmetric. Alternatively, we could replace all instances of $\trianglelefteq$ with $\trianglelefteq \cup E$.

From the above we conclude that for every countable linearly ordered set $\mathcal{L}$ there exists an equivalence relation E on ω and a computable relation $\trianglelefteq$ such that $\trianglelefteq$ respects E and the induced structure $(\omega/E; \trianglelefteq)$ is isomorphic to $\mathcal{L}$. Thus, in this sense, every linear order can be viewed as a linear order on the domain of the type ω/E for an appropriate E. This observation suggests that one might study the isomorphism types of linear orders over the domains of the form ω/E. We give the following definition central to this paper, and refer to the isomorphism types of linearly ordered sets as *order-types*.

Definition 1. Let E be an equivalence relation on ω and let $\mathcal{L}$ be an order-type.

(1) An *E-linear order* is a structure of the type $(\omega/E; \trianglelefteq)$ where $\trianglelefteq$ is a c.e. relation respecting E such that the induced structure $(\omega/E; \trianglelefteq)$ is linearly ordered. We sometimes call $(\omega/E, \trianglelefteq)$ a *linear ordering on E*.

(2) We say that E *realizes* $\mathcal{L}$ if there exists an E-linear order isomorphic to $\mathcal{L}$. Otherwise, we say that E *omits* the order-type $\mathcal{L}$.

In order to consider effective linearly ordered sets we consider domains of the type (ω/E) where E is a c.e. equivalence relation. We formalize this as follows:

Definition 2. A linear ordered set $\mathcal{L}$ is *computably enumerable* (*c.e.*) if $\mathcal{L}$ is an E-linear order for some computably enumerable equivalence relation E. Often we abuse this definition, and refer to an order-type as c.e. if it is isomorphic to a c.e. linearly ordered set.

From now on throughout the paper all our equivalence relations E are computably enumerable; furthermore, the domains ω/E are infinite. In particular, any E-linear order is a c.e. linear order with infinite domain ω/E.

Given a c.e. equivalence relation E, the natural class of linear orders associated with E is the following:

$$\mathcal{K}(E) = \{\mathcal{L} \mid \text{the order-type } \mathcal{L} \text{ is realized by } E\}.$$

Informally, the class $\mathcal{K}(E)$ represents the algebraic content of the domain ω/E in terms of the linearly ordered sets realized by E. A typical question one might now ask is to describe the isomorphism types of order types realized by E.

We now provide several simple notations and results that explain the definitions given above. Some of the examples are taken from [5]. In [5] it is proved that for any E-linear order $(\omega/E; \trianglelefteq)$ there exists a computable linear order $(\omega; \trianglelefteq')$ such that $\trianglelefteq'$ respects E and $(\omega/E; \trianglelefteq)$ is the ordering induced by $\trianglelefteq'$. It is not too hard to see that if each equivalence class of E is an infinite set then the order $(\omega; \trianglelefteq')$ can be made isomorphic to the order of the rational numbers.

Let $X_1, \ldots, X_n$ be pairwise disjoint c.e. sets such that $\omega \setminus (X_1 \cup \ldots \cup X_n)$ is infinite. Define the following c.e. equivalence relation:

$$(i, j) \in E(X_1, \ldots, X_n) \iff (i = j) \vee (i, j \in X_1) \vee \ldots \vee (i, j \in X_n).$$

Thus, the equivalence classes of $E(X_1,\ldots,X_n)$ are either sets $X_1,\ldots,X_n$ or singletons $\{k\}$, where $k \notin X_1 \cup \ldots \cup X_n$. Equivalence relations of the form $E(X_1,\ldots,X_n)$ have been widely studied, though in a different context, e.g., in [1, 3, 4].

Note that in every $E(X_1,\ldots,X_n)$-linear order the sets X_1, …, X_n represent pairwise distinct points of the order. In [5] it is proved that every linear order realized by $E(X_1,\ldots,X_n)$ is also realized by $E(X_1,\ldots,X_{n-1})$. Furthermore, it is shown that the converse of this implication does not hold. As a consequence, one obtains that every linear order realized by $E(X_1,\ldots,X_n)$ is (isomorphic to) a computable linear order.

Consider the equivalence relation $E(X)$. In [5], it is shown that $E(X)$ realizes a linear order if and only if X is one-reducible to a join of two semi-recursive sets. Semi-recursive sets are introduced by Jockusch [7], and a set X is called semi-recursive if there exists a computable total function f of two variables such that for all $x, y \in \omega$ we have the following: $f(x,y) \in \{x,y\}$ and if $\{x,y\} \cap X \neq \emptyset$ then $f(x,y) \in X$. The mentioned result, for instance, implies that the equivalence relation $E(X)$ omits linear orders in each of the following cases: X is maximal, r-maximal, simple and not hypersimple, and creative.

Based on the characterization result mentioned above, it turns out that one can give a full description of linearly ordered sets realized over $E(X)$ in case X is a simple set [5]. Namely, in case X is a simple set any of the following three cases occurs: (1) $E(X)$ realizes no linear order; (2) The only linear order realized over $E(X)$ is of the type $\omega+1+\omega^\star$; (3) The linear orders realized over $E(X)$ are precisely those of the form $\omega+1+\omega^\star$, $\omega+n$ or $n+\omega^\star$, where $n \in \omega$. There are simple sets that exhibit each of these three cases. Case (2) implies that for every $n > 1$ there exists an equivalence relation E_n such that the only linear order realized over E is $n \cdot (\omega+1+\omega^\star)$.

Let $\mathcal{L} = (L; \leqslant_L)$ be a linear order. An element $a \in L$ is called *discrete* if either a is the rightmost element and a has an immediate predecessor, or a is the leftmost element and a has an immediate successor, or a has both immediate successor and predecessor. Otherwise, we say that a is a *limit point* of $\mathcal{L}$. The following is easy to note [5]. If a c.e. equivalence relation E has a c.e. and non-computable equivalence class, say A, then any linear order realized over E must have at least one limit point; in fact the equivalence class A represents a limit point of the order.

We now can compare equivalence relations in terms of order-types that they realize. The following definition first appeared in [5]:

Definition 3. Let E_1 and E_2 be c.e. equivalence relations. We say that E_1 is *lo-reducible* to E_2, written $E_1 \leqslant_{lo} E_2$, if $\mathcal{K}(E_1) \subseteq \mathcal{K}(E_2)$. In other words, E_1 is lo-reducible to E_2 if every order-type realized by E_1 is also realized by E_2.

Intuitively, $E_1 \leqslant_{lo} E_2$ tells us that, in terms of realising the linear order types, the relation E_2 possesses at least as much as algebraic content as E_1. The notation $E_1 \leqslant_{lo} E_2$ is also consistent with the set-theoretic inclusion $\subseteq$ as, by the definition, $E_1 \leqslant_{lo} E_2$ if and only if $\mathcal{K}(E_1) \subseteq \mathcal{K}(E_2)$.

It is clear that $\leqslant_{lo}$ is transitive and reflexive. Two equivalence relations E_1 and E_2 are lo-equivalent, written $E_1 \equiv_{lo} E_2$, if $E_1 \leqslant_{lo} E_2$ and $E_2 \leqslant_{lo} E_1$. Following the terminology from computability theory, we refer to the equivalence classes of this relation as *lo-degrees*. Thus, $\leqslant_{lo}$ induces a partial order on lo-degrees, and thus determines a degree structure on c.e. equivalence relations. This partial order has a least element, consisting of c.e. equivalence relations realizing no order-types. Such equivalence relations exist as mentioned above, e.g. $E(X)$ where X is a maximal set. This partial order is infinite as it has an infinite anti-chain. Indeed, above we mentioned that for each n there exists an equivalence relation E_n such that the only linear order realized over E is $n \cdot (\omega + 1 + \omega^\star)$. This sequence E_n of equivalence relations is clearly an anti-chain. Moreover, these are atoms in the partial order of *lo*-degrees. Furthermore, the partial order $\leqslant_{lo}$ has an infinite chain. This is witnessed by the result that we mentioned above: there exists a sequence of c.e. sets X_1, X_2, ... such that $E(X_1, \dots, X_n) <_{lo} E(X_1, \dots, X_{n-1})$ [5]. In this paper we continue our study of the partial order of *lo*-degrees and its sub-orders. We note that paper [6] investigates a related topic for various classes of graphs.

2. An $\leqslant_{LO}$-anti-chain on equivalence relations $E(X)$

In this section we study the partial order $\leqslant_{lo}$ restricted to the *lo*-degrees of equivalence relations of the type $E(X)$, where X is an infinite and co-infinite c.e. set. Namely, we consider the partial order $\leqslant_{lo}$ restricted to the following set

$$\mathcal{P} = \{E(X)/\equiv_{lo} \mid X \text{ is an infinite and co-infinite c.e. set}\}.$$

Our goal is to study some of the properties of this partial order. For instance, we will prove that this set has infinite chains and anti-chains.

We start with the following two simple observations. Consider the following subset $\mathcal{S}$ of *lo*-degrees from $\mathcal{P}$:

$$\mathcal{S} = \{E(X)/\equiv_{lo} \mid X \text{ is simple}\}.$$

The results mentioned in the previous section give us the following corollary:

Corollary 4. The partial order $\leqslant_{lo}$ on $\mathcal{S}$ is a 3-element linear order.

Proof. Let X_1, X_2, and X_3 be simple sets such that (1) $E(X_1)$ realizes no linear order (for example, a maximal set); (2) The only linear order realized over $E(X_2)$ is of the type $\omega+1+\omega^\star$; (3) The only linear orders realized over $E(X_3)$ are of the form, $\omega+1+\omega^*$, $\omega+n$ or $n+\omega^\star$, where $n \in \omega$. Clearly, $E(X_1) <_{lo} E(X_2) <_{lo} E(X_3)$. As mentioned above, these are the only possibilities that occur for simple sets. □

Another corollary that follows from the introduction is this:

Corollary 5. The partial order $\leqslant_{lo}$ on $\mathcal{P}$ has a least and greatest element.

Proof. The least element is witnessed by $E(X)$, where X is a maximal set. The largest element $E(Y)$ is witnessed by a computable set Y, as every linear order realized over any $E(Z)$ has a computable copy, that is, realized over $E(Y)$. □

Corollary 6. $\mathcal{S}$ is an initial segment of $\mathcal{P}$.

Proof. Suppose X is a non-simple set such that $(\omega/E(X), \trianglelefteq)$ is a linear order. Fix infinite computable $C \subseteq \omega \setminus X$. We can construct a new linear order by restricting to $\trianglelefteq$ on $\omega \setminus C$ and placing C to the right of $\omega \setminus C$. That is, we define $a \hat{\trianglelefteq} b \iff a \trianglelefteq b$ for $a, b \in \omega \setminus C$, and $a \hat{\trianglelefteq} b$ for $a \in \omega \setminus C$ and $b \in C$. Then we can define $\hat{\trianglelefteq}$ on C to be any computable linear order we like. Thus $E(X)$ realizes infinitely many linear orders, and so is not $\leqslant_{lo}$-below any *lo*-degree in $\mathcal{S}$. □

We would like to say a few words by comparing the standard degree structures (e.g. m-degrees and T-degrees) with the lo-degrees. First, the definition of lo-reducibility is a Π^1_1-definition while the definition of m-reducibility is an arithmetic definition on c.e. sets. Second, all the c.e. m-degrees (and T-degrees) are m-reducible to the m-degree of every creative set. In contrast, if X is creative then $E(X)$ forms the least lo-degree. From this view point, lo-reducibility behaves somewhat orthogonally to m-reducibility. Third, every non-computable T-degree is witnessed by a simple set. In contrast, the simple sets X, via the mapping $X \to E(X)$, exhibit only three elements in the partial order $\mathcal{P}$ by our first corollary.

Now our goal is to show that the set $\mathcal{P}$ is infinite. We prove this by exhibiting that the partial order $\mathcal{P}$ contains an infinite anti-chain with respect to lo-reducibility. We start with the following definition about linear orders.

Definition 7. A *block* in a linear order L is a subset $B \subseteq L$ which is maximal with the property that for any two points $x, y \in B$, the interval $[x, y]_L$ is finite.

Note that singletons have the described property, and thus every point from L is contained in a block. Note also that blocks are necessarily convex, and they partition L such that the ordering of L induces an ordering on the blocks. Finally, observe that every block is either finite or has order-type ω, ω^* or ζ (here ζ is the order-type of the integers).

Definition 8. For $A \subseteq \{\omega, \omega^*, \zeta\} \cup \omega/\{0\}$, the *shuffle-sum of* A, denoted $\sigma(A)$, is the countable order-type in which:

- Every block has an order-type appearing in A;
- There is no greatest or least block; and
- For every $\nu \in A$, and every pair of distinct blocks $B_0 < B_1$, there is a block B_2 with order-type ν such that $B_0 < B_2 < B_1$.

Theorem 9. *The partial order $\mathcal{P}$ contains an infinite anti-chain.*

Proof. Let $A \subseteq \omega/\{0\}$ be non-empty and computable. Let K be any nice presentation of $1 + \sigma(A) + 1 + \sigma(A) + 1$. By nice, we mean that K is computable, and given a point we can effectively determine the order-type of its block, and given two points we can effectively determine whether they are in the same block. Such a presentation exists since A is computable.

We will construct a set X as a c.e. interval of K. Our goal is to ensure that in any linear ordering $(\omega/E(X), \trianglelefteq)$ on $E(X)$, there is an interval to the immediate left of X and an interval to the immediate right of X both of order-type $\sigma(A)$. We construct X using a finite-injury priority construction, which we now describe.

Let 0 denote the central 1-block of K. At stage 0, let $X_0 = \{0\}$. We will grow X by expanding to the left and right of 0, always maintaining a 0-1-law on blocks. That is, for any block of K, either that block is a subset of X or it is disjoint from X. So when we enumerate a point into X, we will be sure to also enumerate any other points in the same block. Indeed, at every stage s, X_s will consist of two blocks from K and all the points between them.

We have strategies for dealing with each c.e. relation $\trianglelefteq_e$. Each strategy will receive a restraint (B_L, B_R) consisting of two blocks, one to the left of X and one to the right, indicating blocks that that strategy is forbidden from enumerating into X.

Given a restraint (B_L, B_R), our strategy for $\trianglelefteq_e$ begins at stage s_0 by choosing arbitrary blocks $C_L, C_R \not\subseteq X_{s_0}$ with $B_L <_K C_L <_K 0 <_K C_R <_K B_R$ and setting (C_L, C_R) as the restraint for lower priority strategies. Let H be the eth block of K, in some effective numbering. If $H \not\subseteq X_{s_0}$, then we choose C_L and C_R such that either $H <_K C_L$ or $C_R <_K H$. Note that such blocks always exist, since X_{s_0} has a leftmost and rightmost block.

At stage s, we search for a pair of points (a, b) such that:

- $a, b \in \omega \backslash X_s$;
- a and b are not in the same block;
- $0 \trianglelefteq_e b \trianglelefteq_e a$ or $a \trianglelefteq_e b \trianglelefteq_e 0$; and
- We can enumerate a into X without enumerating b. That is, we must have $B_L <_K a <_K B_R$ and one of:
  - $0 <_K a <_K b$;
  - $b <_K a <_K 0$;
  - $b <_K 0 <_K a$; or
  - $a <_K 0 <_K b$.

Having found such points, we enumerate a into X_{s+1}, along with all other points in the same block as a, and all points between X_s and a. We let B be the block of b. We choose new blocks $C'_L, C'_R \not\subseteq X_{s+1}$ such that $B_L <_K C'_L <_K 0 <_K C'_R <_K B_R$ and either $B <_K C'_L$ or $C'_R <_K B$. If $H \not\subseteq X_{s+1}$, then we again require that either $H <_K C'_L$ or $C'_R <_K H$. We injure all lower priority strategies and set (C'_L, C'_R) as their new restraint. We then end the action for this strategy.

We arrange these strategies into a standard priority construction. Let $-\infty$ and ∞ be the leftmost and rightmost points of K. We begin by giving the highest priority strategy restraint $(\{-\infty\}, \{\infty\})$. Note that between injuries, each strategy acts and injures lower priority strategies at most once, and so this is finite injury.

Claim 9.1. *If our strategy for $\trianglelefteq_e$ acts and is then never injured, $(\omega/E(X), \trianglelefteq_e)$ is not a linear order on $E(X)$.*

Proof. Let (a, b) be the pair the strategy found which caused it to act. Then $a \in X$, and so $a \sim_{E(X)} 0$. Meanwhile, since the strategy is never again injured, no higher priority strategy will enumerate b into X. By the choice of restraint, no lower priority strategy will enumerate b into X. Thus $b \notin X$, but since either $0 \trianglelefteq_e b \trianglelefteq_e a$ or $a \trianglelefteq_e b \trianglelefteq_e 0$, $\trianglelefteq_e$ cannot be a linear order on $E(X)$. □

Claim 9.2. *If $(\omega/E(X), \trianglelefteq_e)$ is a linear order on $E(X)$, then there is an interval of $(\omega/E(X), \trianglelefteq_e)$ of order-type $\sigma(A) + 1 + \sigma(A)$ with X as the central 1-block.*

Proof. Let (B_L, B_R) be the final restraint imposed on the strategy for $\trianglelefteq_e$. By the previous claim, the strategy for $\trianglelefteq_e$ never acts after this restraint is imposed.

Let $I_L = \{a \in K \ : \ B_L <_K a <_K X\}$ and $I_R = \{a \in K \ : \ X <_K a <_K B_R\}$. Note that $C_L \subset I_L$ and $C_R \subset I_R$, so in particular these are nonempty. Moreover, if (B^f_L, B^f_R) is the final restraint imposed *by* the strategy for $\trianglelefteq_f$, and $H \not\subseteq X$ is some block of K, then for some f either $H <_K B^f_L$ or $B^f_R <_K H$. Thus I_L has no rightmost element, and I_R no leftmost, and so $(I_L, <_K)$ and $(I_R, <_K)$ both have order-type $\sigma(A)$.

Suppose there were $a \in I_L$ and $b \in I_R$ with $a \trianglelefteq_e X$ and $b \trianglelefteq_e X$. Without loss of generality, $a \trianglelefteq_e b \trianglelefteq_e X$. But then (a, b) would be of one of the forms we are searching for, and the strategy would eventually see them and act, contrary to hypothesis.

Symmetric reasoning holds for $X \trianglelefteq_e a$ and $X \trianglelefteq_e b$. So it must be that the $\trianglelefteq_e$ ordering places all elements of I_L on one side of X, and all elements of I_R on the other. Without loss of generality, $I_L \trianglelefteq_e X \trianglelefteq_e I_R$.

Now, suppose there were some $b \in \omega/(I_L \cup X \cup I_R)$ and $a \in I_R$ with $X \trianglelefteq_e b \trianglelefteq_e a$. Then either $I_R <_K b$ and so $a <_K b$, or $b <_K I_L$, and so $b <_K 0$. Again, (a, b) is of one of the forms we are searching for, contrary to hypothesis. Symmetric reasoning holds for $a \in I_L$. So $I_L \cup \{X\} \cup I_R$ is convex in $(\omega/E(X), \trianglelefteq_e)$.

Now, fix $a \in I_R$, and let D be the block of K containing a, and let G be the block of $(\omega/E(X); \trianglelefteq_e)$ containing a.

Claim 9.2.1. $X \notin G$.

Proof. Suppose not. Since I_R has no leftmost element, there are infinitely many points $c \in I_R$ with $c <_K a$. Since the interval $[X, a]_{\leqslant_e}$ is finite, choose such a c which is not in this interval, and is not in D. Since $c \in I_R$, as earlier argued it must be that $X \leqslant_e c$. Thus it must be that $a \leqslant_e c$. Then (a, c) is of one of the forms we are searching for, contrary to hypothesis. □

Claim 9.2.2. $D = G$.

Proof. Suppose there is $b \in D \backslash G$. Since b is in the same K-block as a, b is also in I_R. Suppose $a \trianglelefteq_e b$. Since $b \notin G$, there must be infinitely many points c with $a \trianglelefteq_e c \trianglelefteq_e b$. Choose such a $c \notin D$. As we have argued that I_r is convex in $(\omega/E(X); \trianglelefteq_e)$, it must be that $c \in I_r$, and so $0 <_K c <_K B_r$. If $c <_K D$, then the pair (c, a) is of one of the forms we are searching for, contrary to hypothesis. If $D <_K c$, then (b, c) is of one of the forms we are searching for, contrary to hypothesis. Symmetric reasoning holds when $b \trianglelefteq_e a$.

Suppose instead there is $b \in G \backslash D$ with $a <_K b$. Since $b \notin D$, and $a \in I_R$, there must be infinitely many points c with $a <_K c <_K b$ and $a <_K c <_K B_R$. Since the interval $[a, b]_{\trianglelefteq_e}$ is finite, choose such a c which is not in this interval, and is in neither D nor b's block in K. If $c \trianglelefteq_e a$, then (a, c) is of one of the forms we are searching for, contrary to hypothesis. If $b \trianglelefteq_e c$, then (c, b) is of one of the forms we are searching for, contrary to hypothesis. Since $c \notin [a, b]_{\leqslant_e}$, these are the only two possibilities.

Now suppose instead that there is $b \in G \backslash D$ with $b <_K a$. Since $b \in G$, $b \notin X$. Since $b \notin D$, and $a \in I_R$, there must be infinitely many points c with $b <_K c <_K a$ and $X <_K c <_K a$. Since the interval $[a, b]_{\trianglelefteq_e}$ is finite, choose such a c which is not in this interval, and is in neither D nor b's block in K. If $c \trianglelefteq_e b$, then (b, c) is of one of the forms we are searching for, contrary to hypothesis. If $a \trianglelefteq_e c$, then (c, a) is of one of the forms we are searching for, contrary to hypothesis. Since $c \notin [a, b]_{\leqslant_e}$, these are the only two possibilities. □

The symmetric claim holds for $a \in I_L$.

Thus the set of blocks in $(I_R, <_K)$ equals the set of blocks in $(I_R, \trianglelefteq_e)$. Consider now the ordering induced on these blocks by $<_K$ and $\trianglelefteq_e$. If there were blocks H, J with $H <_K J$ and $J \trianglelefteq_e H$, then for any $a \in H$ and $b \in J$, (a, b) would be of one of the forms we are searching for, contrary to hypothesis. So the two orders induce the same orderings on the blocks. Thus $(I_R, \trianglelefteq_e)$ has order-type $\sigma(A)$.

By symmetric reasoning, $(I_L, \trianglelefteq_e)$ has order-type $\sigma(A)$. □

Now, take any infinite collection of distinct computable sets $(A_i : i \in \omega)$, and let X_i be the c.e. set obtained by running the above construction for A_i. Then for each i, $\sigma(A_i)+1+\sigma(A_i)$ is realized on $E(X_i)$ by $(K_i/E(X_i), \leqslant_{K_i})$, where K_i is the K of the above construction.

Further, any linear ordering on $E(X_j)$ contains an interval around X_j of order-type $\sigma(A_j)+1+\sigma(A_j)$, where X_j is the central 1-block. So for $i \neq j$, $\sigma(A_i)+1+\sigma(A_i)$ is not realized on $E(X_j)$. So for $i \neq j$, $E(X_i)$ and $E(X_j)$ are incomparable in $\mathcal{P}$. □

Remark 10. If X is non-computable, then every linear order realized by $E(X)$ has a limit point represented by X. Thus, each such linear order must have at least one limit point. The linear orders constructed in this section give examples of equivalence relations of the form $E(X)$ that only realize linear orders with infinitely many limit points.

In fact, one can show that if a relation $E(X)$ does not realize a linear order with exactly one limit point, it only realizes linear orders with infinitely many limit points—if $(\omega/E(X); \trianglelefteq)$ has only finitely many limit points, then there are $a, b \notin X$ such that $[a, b]_{\trianglelefteq}$ contains X and contains no limit points other than X. Since $E(X)$ is trivial outside of $[a, b]_{\trianglelefteq}$, we can modify $\trianglelefteq$ to remove all limit points outside of $[a, b]_{\trianglelefteq}$. Thus $E(X)$ realizes a linear order with exactly one limit point.

Note that if $E(X_i)$ is from the constructed anti-chain, and Y is simple with $E(Y)$ realizing some linear order, then $E(X_i)$ and $E(Y)$ form a minimal pair. The authors had originally believed that if $i \neq j$, then $E(X_i)$ and $E(X_j)$ form a minimal pair. In fact, this is false, as the following lemma shows:

Lemma 11. *If $E(Z_0)$ and $E(Z_1)$ form a minimal pair in $\mathcal{P}$, then exactly one of Z_0 or Z_1 is simple.*

Proof. As previously discussed, if Z_0 and Z_1 are both simple with $E(Z_0)$ and $E(Z_1)$ realizing linear orders, then $E(Z_0)$ and $E(Z_1)$ both realize $\omega+1+\omega^*$, and thus are not a minimal pair.

If instead neither of Z_0 or Z_1 are simple, we can fix computable sets C_0 and C_1 such that $Z_i \subseteq C_i$, and such that $C_i \backslash Z_i$ and $\omega \backslash C_i$ are infinite, for $i \in \{0, 1\}$. Fix $\trianglelefteq_i$ such that $(\omega/E(Z_i), \trianglelefteq_i)$ realizes a linear order on $E(Z_i)$, for $i \in \{0, 1\}$. Let α_i be the order-type of $(C_i/E(Z_i), \trianglelefteq_i)$. As previously mentioned, α_i must be computable [5]. Then $E(Z_0)$ realizes $\alpha_0 + \alpha_1$ (by using the elements of $\omega \backslash C_0$ to realize α_1), and $E(Z_1)$ does as well (by using the elements of $\omega \backslash C_1$ to realize α_0). Thus $E(Z_0)$ and $E(Z_1)$ do not form a minimal pair. □

3. An $\leqslant_{LO}$-chain on equivalence relations $E(X)$

In the first paragraph of the proof of Claim 9.2.2, we made implicit use of the fact that all order-types in A were finite, which allowed us to find $c \notin D$. Our construction of an infinite chain will be similar, but we will use sets A containing ζ.

Theorem 12. *The partial order $\mathcal{P}$ contains an infinite chain of lo-degrees.*

Proof. Fix $(A_i : i \in \mathbb{Z})$ a uniformly computable sequence of sets with $A_j \supset A_i$ for $j < i$. For definiteness, let $A_i = \{2n : n \geqslant i\}$ and $A_{-i} = \{2n : n \in \omega\} \cup \{2n+1 : n < i\}$, for $i \in \omega$.

Let K_i be a nice presentation of $1 + \sigma(A_i \cup \{\zeta\}) + 1 + \sigma(A_i \cup \{\zeta\}) + 1$ uniformly in i. By nice, we again mean that given a point we can effectively determine the order-type of its block, and given two points we can effectively determine whether they are in the same block. We let 0_i denote the central 1-block of K_i.

We will uniformly construct X_i as a c.e. interval of K_i containing 0_i, similar to the construction for the previous theorem. Simultaneously, we will construct functions f_i mapping blocks of K_i to intervals of K_{i-1} satisfying the following properties:

(1) f_i is total on the blocks of $K_i - X_i$;
(2) f_i maps blocks of $K_i - X_i$ to intervals of $K_{i-1} - X_{i-1}$;
(3) If B_0 and B_1 are distinct blocks of $K_i - X_i$ with $B_0 <_{K_i} B_1$, then $f(B_0)$ and $f(B_1)$ are disjoint intervals of $K_{i-1} - X_{i-1}$ with $f(B_0) <_{K_{i-1}} f(B_1)$;
(4) If B is a block of $K_i - X_i$ with $0_i <_{K_i} B$, then $0_{i-1} <_{K_{i-1}} f_i(B)$, where 0_i and 0_{i-1} are the central points of K_i and K_{i-1}, respectively;
(5) If B is a block of $K_i - X_i$ with $B <_{K_i} 0_i$, then $f_i(B) <_{K_{i-1}} 0_{i-1}$;
(6) Every block of $K_{i-1} - X_{i-1}$ is contained in some $f_i(B)$ for B a block of $K_i - X_i$;
(7) If B is a block of $K_i - X_i$ with $|B| < \infty$, then $f_i(B)$ is a block of $K_{i-1} - X_{i-1}$ of the same size; and
(8) If B is a block of $K_i - X_i$ of order-type ζ, then $f_i(B)$ has a leftmost block and a rightmost block and infinitely many elements (we allow the possibility that $f(B)$ is a single block of order-type ζ).

The function f_i will be used to demonstrate that every order-type realized by $E(X_{i-1})$ is also realized by $E(X_i)$. The idea is that we will pull-back linear orderings on $E(X_{i-1})$ using f_i.

Note that properties 7 and 8 together say that for every block B, $|B| = |f_i(B)|$; since we intend to pull-back linear orderings along f_i, this will be an essential requirement. Recall that $A_i \subsetneq A_{i-1}$, so fix $n \in A_{i-1} - A_i$, and suppose that C is a block of $K_{i-1} - X_{i-1}$ of size n. By property 6, there will be some block B of $K_i - X_i$ with $C \subseteq f_i(B)$. By property 7, if B is finite, then $|B| = |C| = n$. But since $n \notin A_i$, this is impossible. So it must be that B is infinite. This is why we require the ζ-blocks; they allow f_i to capture the blocks which are of a size missing from A_i.

Most of the desired properties of f_i will be satisfied simply by how we choose to define $f_i(B)$. The only two properties that cause concern are properties 2 and 6. We will use strategies to satisfy property 6. Every block of K_{i-1} will have some strategy which waits until that block is contained in $f_i(B)$ for some B, then restrains B from entering X_i. Of course, this restraint may be injured by higher priority strategies, but this will occur only finitely many times.

To satisfy property 2, when we define $f_i(B)$, we will make the promise that if X_{i-1} contains any block from $f_i(B)$, then $B \subset X_i$. So any strategy that enumerates part of $f_i(B)$ into X_{i-1} will simultaneously enumerate B into X_i. This means that when strategies place restraint on K_i, they are also placing restraint on K_{i-1}: if some strategy restrains B from entering X_i, it also restrains all $C \subseteq f_i(B)$ from entering X_{i-1}. And in turn, this restrains all $D \subseteq f_{i-1}(C)$ from entering X_{i-2}, etc.

To properly track this restraint, we will introduce additional functions $g_{i,j}$ for $i \geqslant j$:

- We define $g_{i,i}(B) = B$ for all blocks B of K_i;
- If $0_i <_{K_i} B$, we define $g_{i,j-1}(B)$ to be the leftmost block of $f_j(g_{i,j}(B))$;
- If $B <_{K_i} 0_i$, we define $g_{i,j-1}(B)$ to be the rightmost block of $f_j(g_{i,j}(B))$.

As we are defining the various f_i during the construction, this will simultaneously define the various $g_{i,j}$. By induction, $g_{i,j}$ will be defined on all blocks of $K_i - X_i$, and if $0_i <_{K_i} B$ then $0_j <_{K_j} g_{i,j}(B)$. If a strategy restrains B from entering X_i, then it will simultaneously restrain $g_{i,j}(B)$ from entering X_j.

Note that restraint only propagates down the K_i: a restraint on K_i is simultaneously a restraint on K_j for all $j < i$, but never on any K_j with $j > i$. Contrapositively, enumerating elements into X_i can force the enumeration of elements into X_j for $j > i$, but never forces enumeration into any X_j with $j < i$.

Strategy for defining all the f_i:

We have a single global strategy responsible for defining all the f_i. We begin by defining $f_i(\{\infty_i\}) = \{\infty_{i-1}\}$ for all i, where ∞_i is the rightmost point of K_i. Similarly, we define $f_i(\{-\infty_i\}) = \{-\infty_{i-1}\}$ for the leftmost points.

At stage s, we let B be the least block (by Gödel number) in any $K_i - X_{i,s}$ with $f_i(B)$ not yet defined. We choose a block C of $K_{i-1} - X_{i-1,s}$ of the same order-type as B and positioned so as to maintain properties 3, 4 and 5, and define $f_i(B) = C$. Such a C can always be found because f_i has only been defined on finitely many of the blocks by this stage, and because $A_i \subset A_{i-1}$, and because of the nature of shuffle-sums.

We then let A be the least block (again by Gödel number) in any $K_i - X_{i,s}$ with A not yet contained in $f_{i+1}(C)$ for some C a block of $K_{i+1} - X_{i+1,s}$. Note that A may be contained in $f_{i+1}(C)$ for some $C \subseteq X_{i+1,s}$. We choose a block B with $A <_{K_i} B$ and such that no point x with $A <_{K_i} x <_{K_i} B$ is contained in $f_{i+1}(C)$ for any C a block of $K_{i+1} - X_{i+1,s}$. We choose a block C of $K_{i+1} - X_{i+1,s}$ of order-type ζ and positioned so as to maintain properties 3, 4 and 5, and define $f_{i+1}(C) = A \cup \{x \ : A <_{K_i} x <_{K_i} B\} \cup B$. Such a C can always be found because f_{i+1} has only been defined on finitely many blocks by this stage, and because of the nature of shuffle-sums.

Strategy for ensuring property 6 for block B:

For B a block of K_i, we wait until a stage s when either we see $B \subseteq f_{i+1}(C)$ for some $C \nsubseteq X_{i+1,s}$, or we see $B \subset X_{i,s}$. In the former case, we restrain C from entering X_{i+1}. In the latter case, we do nothing. While we are waiting, we do not permit any lower priority strategy to act.

Strategy for $\trianglelefteq_e$ and X_j:

This is much the same as in the anti-chain construction. We treat the restraints slightly differently, however. If there is some $i \geqslant j$ and some B a block of $K_i - X_{i,s}$ which some higher priority strategy has restrained from entering X_i, and $g_{i,j}(B)$ is not yet defined, we do nothing and do not permit any lower priority strategy to act. Otherwise, consider the (finitely many) blocks $g_{i,j}(B)$ for such blocks B with $0_i <_{K_i} B$. We let B_R be the leftmost such block, or $B_R = \{\infty_j\}$ if the collection of blocks is empty. Similarly, we consider $g_{i,j}(B)$ for blocks B restrained by higher priority strategies with $B <_{K_i} 0_i$ and let B_L be the rightmost such block or $\{-\infty_j\}$. This gives us our restraint (B_L, B_R).

The strategy then proceeds as in the anti-chain construction, so we merely summarize. We choose arbitrary blocks C_L and C_R of $K_j - X_{j,s}$ with $B_L <_{K_j} C_L <_{K_j} 0_i <_{K_j} C_R <_{K_j} B_R$ and restrain them from entering X_j. We then search for a pair of points (a, b) from $K_j - X_{j,s}$ satisfying the same properties as before. Having found such points, we enumerate the block containing a and all points between a and 0_j into $X_{j,s+1}$.

Now we must take additional action to satisfy our promises and maintain property 2. For every block B of K_{j+1} such $g_{j+1,j}$ is defined and we have just enumerated $g_{j+1,j}(B)$ into X_j, we must enumerate B into X_{j+1}. Simultaneously, we must enumerate all points between B and 0_{j+1} into X_{j+1}. Then, for every block B' of K_{j+2} such that $g_{j+2,j+1}$ is defined and we have just enumerated $g_{j+2,j+1}(B')$ into X_{j+1}, we must enumerate B' into X_{j+2}. Simultaneously, we must enumerate all points between B' and 0_{j+2}, etc.

Note that this is a finite process: our restraints ensure that we will never need to enumerate ∞_i or $-\infty_i$, and since we only ever extend the definition of a single f_i at any given stage, there are only finitely many i such that f_i has been defined on any point other than these. Thus this process will terminate. Once it does, we once more proceed as in the anti-chain construction: we choose new blocks C'_L and C'_R to restrain, being careful that this restraint prevents b from being enumerated into X_j, and then injure all lower priority strategies and end the action.

Full Construction:

We arrange the various strategies into a standard priority construction, placing the global strategy responsible for defining the f_i as the highest priority strategy. Note that this highest strategy never injures lower priority strategies. All other strategies act and injure lower priority strategies at most once, and so this is finite injury.

Verification:

Claim 12.1. *The functions f_i satisfy the stated properties.*

Proof. That f_i is defined on all the blocks of $K_i - X_i$ is immediate by the fact that the global strategy acts at every stage, and always chooses the Gödel least block not yet handled. Similarly, properties 3, 4 and 5 and properties 7 and 8 follow by the action of said strategy.

For property 2, note that we only enumerate into an X_i because of the action of the strategy for some $\trianglelefteq_e$ and some X_j, and when we act for such a strategy, we explicitly ensure that we maintain this property.

Suppose that property 6 were not satisfied. Then some strategy for ensuring property 6 fails to ensure its requirement. Consider the highest priority strategy which fails, and let B a block of $K_i - X_i$ be the block it was targeted for. Let s_0 be a stage after which this strategy is never again injured. Then at every stage $s > s_0$, it must be that B is never contained in any $f_{i+1}(C)$ for some $C \not\subseteq X_{i+1,s}$. For if there were such a C, the strategy would restrain it from entering X_{i+1}, and since the strategy is never injured, this C would witness property 6 for B, contrary to hypothesis.

So then after stage s_0, no lower priority strategy is ever permitted to act. Since the strategy is never again injured, no higher priority strategy will enumerate elements into any X_j at any stage after s_0. So for every block D, when the global strategy considers D after stage s_0 and chooses a C and defines $f_j(C) \supset D$, that C will never be enumerated into X_j, and so the global strategy will never again consider this D. So eventually the global strategy will consider B and define such a C, contrary to hypothesis. $\square$

Claim 12.2. *If the strategy for $\trianglelefteq_e$ and X_j acts and is then never injured, then $(\omega/E(X_j), \trianglelefteq_e)$ is not a linear order on $E(X_j)$.*

Proof. As in the anti-chain construction. □

Now, fix a c.e. relation $\trianglelefteq$ and assume $(\omega/E(X_j), \trianglelefteq)$ is a linear order on $E(X_j)$. Let (B_L, B_R) be the restraint on the strategy for $\trianglelefteq$ and X_j. Define $I_R = \{a \in K_j \ : \ X_j <_{K_j} a <_{K_j} B_R\}$ and $I_L = \{a \in K_j \ : \ X_j <_{K_j} a <_{K_j} B_R\}$.

Claim 12.3. *I_L and I_R both have order-type $\sigma(A \cup \{\zeta\})$ in K_j.*

Proof. As in the anti-chain construction. □

Claim 12.4. *$\trianglelefteq$ places all elements of I_L on one side of X_j and all elements of I_R on the other.*

Proof. As in the anti-chain construction. □

By replacing $\trianglelefteq$ with $\trianglelefteq^*$ if necessary, assume $I_L \trianglelefteq X_j \trianglelefteq I_R$.

Claim 12.5. *$I_L \cup X_j \cup I_R$ is an interval of $(\omega/E(X_j), \trianglelefteq)$.*

Proof. As in the anti-chain construction. □

Now, fix $a \in I_R$, and let D be the block of K_j containing a, and let G be the block of $(\omega/E(X_j), \trianglelefteq)$ containing a.

Claim 12.6. *$X \notin G$*

Proof. As in the Claim 9.2.1. □

Claim 12.7. *If D is finite, $D = G$. If D is infinite, then $D \supseteq G$.*

Proof. As in Claim 9.2.2, ignoring the first half of the proof for infinite D. □

Claim 12.8. *$(\omega/E(X_j), \trianglelefteq) \ncong (\omega/E(X_{j+1}), <_{K_{j+1}})$.*

Proof. Fix $n \in A_j - A_{j+1}$. In $(\omega/E(X_j), \trianglelefteq)$, X is an accumulation point for blocks of size n. In $(\omega/E(X_{j+1}), <_{K_{j+1}})$, there are no blocks of size n. □

Claim 12.9. *$\trianglelefteq$ induces a well-defined linear ordering on the blocks of $(I_R, <_{K_j})$ and $(I_L, <_{K_j})$, and it is the same as the ordering induced by $<_{K_j}$.*

Proof. As in the anti-chain construction. □

Claim 12.10. *There is a c.e. relation $\hat{\trianglelefteq}$ such that $(\omega/E(X_{j+1}), \hat{\trianglelefteq}) \cong (\omega/E(X_j), \trianglelefteq)$.*

Proof. Let C_L and C_R be blocks of $K_{j+1} - X_{j+1}$ such that $B_L \subseteq f_{j+1}(C_L)$ and $B_R \subseteq f_{j+1}(C_R)$. Define

$$O_{j+1} = \{a \ : \ a <_{K_{j+1}} C_L\} \cup C_L \cup C_R \cup \{a \ : \ C_R <_{K_{j+1}} a\},$$

and

$$O_j = \{a \ : \ a <_{K_j} g_{j+1,j}(C_L)\} \cup g_{j+1,j}(C_L) \cup g_{j+1,j}(C_R) \cup \{a \ : \ g_{j+1,j}(C_R) <_{K_j} a\}.$$

Note that these are computable sets, and either these are both infinite, or $(B_L, B_R) = (\{-\infty_j\}, \{\infty_j\})$, in which case they both have size 2. In either case, fix $h : O_{j+1} \to O_j$ a computable bijection.

Our construction of $\hat{\trianglelefteq}$ will be based on the action of the strategy for $\trianglelefteq$ and X_j. Specifically, we will rely on Claim 12.9, which says that $\trianglelefteq$ induces the same order on the blocks of $K_j - X_j$ between B_L and B_R as $<_{K_j}$. Of course, we cannot rely on this behavior for the blocks to the left of B_L or to the right of B_R, and so we will need a different approach for those. Our partition of K_j into O_j and $K_j - O_j$ (and the corresponding partition of K_{j+1}) allows us to effectively determine which of these cases we are in; note that $K_j - O_j$ contains X and is entirely contained between B_L and B_R.

For every block B of $K_{j+1} - O_{j+1}$ with $f_{j+1}(B)$ defined, fix h_B a computable bijection between B and $f_{j+1}(B)$ (uniformly in B). Now, we define $\hat{\trianglelefteq}$ by pulling back along these bijections. Specifically:

(1) If $a, b \in O_{j+1}$, when we see $h(a) \trianglelefteq h(b)$ we enumerate $a \hat{\trianglelefteq} b$;
(2) If $a, b \in K_{j+1} - O_{j+1}$, a and b are in different blocks and $a <_{K_{j+1}} b$, we enumerate $a \hat{\trianglelefteq} b$;
(3) If $a, b \in B$ a block of $K_{j+1} - O_{j+1}$, when we see $h_B(a) \trianglelefteq h_B(b)$ we enumerate $a \hat{\trianglelefteq} b$;
(4) If $a \in K_{j+1} - O_{j+1}$ and $b \in O_{j+1}$, when we see $0_j \trianglelefteq h(b)$ we enumerate $a \hat{\trianglelefteq} b$, and when we see $h(b) \trianglelefteq 0_j$ we enumerate $b \hat{\trianglelefteq} a$;
(5) If $a, b \in X_{j+1}$, we enumerate $a \hat{\trianglelefteq} b$ and $b \hat{\trianglelefteq} a$.

Claim 12.10.1. *$\hat{\trianglelefteq}$ is well-defined on $\omega/E(X_{j+1})$.*

Proof. Suppose $a_0, a_1 \in X_{j+1}$ and $b \in K_{j+1}$. We must show that $a_0 \hat{\trianglelefteq} b \iff a_1 \hat{\trianglelefteq} b$, and $b \hat{\trianglelefteq} a_0 \iff b \hat{\trianglelefteq} a_1$.

If $b \in X_{j+1}$, this is by case 5 of the definition.

If $b \in O_{j+1}$, this is by case 4.

If $b \in K_{j+1} - X_{j+1} - O_{j+1}$, then b cannot be in the same block as either a_0 or a_1. So case 2 applies, and $a_0 \hat{\trianglelefteq} b \iff a_0 <_{K_{j+1}} b$ and $a_1 \hat{\trianglelefteq} b \iff a_1 <_{K_{j+1}} b$. Since X_{j+1} is an interval of K_{j+1}, this follows. □

Claim 12.10.2. *$(\omega/E(X_{j+1}), \hat{\trianglelefteq})$ is antisymmetric.*

Proof. Given $a, b \in K_{j+1}$ distinct and not both in X_{j+1}, we must show that precisely one of $a \hat{\trianglelefteq} b$ or $b \hat{\trianglelefteq} a$ holds.

If $a, b \in O_{j+1}$, then this is by case 1 and the fact that $\trianglelefteq$ is a linear order on $\omega/E(X_j)$ (and O_j is disjoint from X_j).

If $a, b \in K_{j+1} - O_{j+1}$ are in different blocks, then case 2 applies, and this is by the fact that $<_{K_{j+1}}$ is a linear order.

If $a, b \in K_{j+1} - O_{j+1}$ are in the same block B, then B is disjoint from X_{j+1}, and $f_{j+1}(B)$ is disjoint from X_j. Then this is by case 3 and the fact that $\trianglelefteq$ is a linear order on $\omega/E(X_j)$.

If $a \in K_{j+1}$ and $b \in O_{j+1}$, then case 4 applies, and this is by the fact that $\trianglelefteq$ is a linear order on $\omega/E(X_j)$. □

Claim 12.10.3. *$(\omega/E(X_{j+1}), \hat{\trianglelefteq})$ is a linear order.*

Proof. It remains to show that $\hat{\trianglelefteq}$ is transitive. Suppose $a \hat{\trianglelefteq} b$ and $b \hat{\trianglelefteq} c$. We must show that $a \hat{\trianglelefteq} c$. What follows is an exhaustive case analysis.

If $b \in O_{j+1}$, then by case 4, it cannot be that both $a, c \in K_{j+1} - O_{j+1}$. If both $a, c \in O_{j+1}$, then this is by case 1 and the fact that $\trianglelefteq$ is transitive.

If $a \in K_{j+1} - O_{j+1}$ and $b, c \in O_{j+1}$, then by case 1 we see that $h(b) \trianglelefteq h(c)$, while by case 4 we see that $0_j \trianglelefteq h(b)$. Since $\trianglelefteq$ is transitive, we have that $0_j \trianglelefteq h(c)$, and so by case 4 again we have $a \hat{\trianglelefteq} c$.

If $a, b \in O_{j+1}$ and $c \in K_{j+1} - O_{j+1}$, the reasoning is symmetric to the above.

If $b \in K_{j+1} - O_{j+1}$ and $a, c \in O_{j+1}$, then by case 4 we have that $h(a) \trianglelefteq 0_j$ and $0_j \trianglelefteq h(c)$, so by transitivity of $\trianglelefteq$ we have $h(a) \trianglelefteq h(c)$, and so by case 1 we have $a \hat{\trianglelefteq} c$.

If $a, b \in K_{j+1} - O_{j+1}$ and $c \in O_{j+1}$, then by two applications of case 4 we see that $a \hat{\trianglelefteq} c$.

If $a \in O_{j+1}$ and $b, c \in K_{j+1} - O_{j+1}$, the reasoning is symmetric to the above.

If $a, b, c \in K_{j+1} - O_{j+1}$, and a and c are in the same block, then by case 2 and antisymmetry, either b is in the same block or $a, b, c \in X_{j+1}$. The former case follows by case 3 and the transitivity of $\trianglelefteq$, while the latter case follows by case 5.

If $a, b, c \in K_{j+1} - O_{j+1}$, $a, b \in X_{j+1}$ and $c \notin X_{j+1}$, then by case 2 the block of c is to the right of the block of b, so c is to the right of X_{j+1}. By case 2, $a \hat{\trianglelefteq} c$.

If $a, b, c \in K_{j+1} - O_{j+1}$, $b, c \in X_{j+1}$ and $a \notin X_{j+1}$, the reasoning is symmetric to the above.

If $a, b, c \in K_{j+1} - O_{j+1}$, a and c are in different blocks, and no more than one of a, b or c is in X_{j+1}, then by case 2 the block of a is to the left of the block of b or is the same block, and the block of b is to the left of the block of c or is the same block, so the block of a is to the left of the block of c, and so $a \hat{\trianglelefteq} c$ by case 2.

If $a, c \in X_{j+1}$, then $a \hat{\trianglelefteq} c$ by case 5. $\square$

Now it remains only to show that $(\omega/E(X_{j+1}), \hat{\trianglelefteq}) \cong (\omega/E(X_j), \trianglelefteq)$. We define the isomorphism g as follows:

- $g(X_{j+1}) = X_j$;
- For $x \in O_{j+1}$, $g(x) = h(x)$;
- For $x \in B$ a block of $K_{j+1} - O_{j+1} - X_{j+1}$, $g(x) = h_B(x)$.

By property 6 of f_{j+1} and our definitions of h and the various h_B, this map is a bijection. It remains to show that for $a, b \in \omega/E(X_{j+1})$, we have $a \hat{\trianglelefteq} b \iff g(a) \trianglelefteq g(b)$.

If $a, b \in B$ a block of $K_{j+1} - O_{j+1} - X_{j+1}$, this is by definition.

If $a, b \in O_{j+1}$, this is again by definition.

If $a = X_{j+1}$ and $b \in O_{j+1}$, this is again by definition.

If $a = X_{j+1}$ and $b \in B$ for B a block of $K_{j+1} - O_{j+1} - X_{j+1}$, then without loss of generality we may assume $X_{j+1} <_{K_{j+1}} B$, and thus $a \hat{\trianglelefteq} b$ by case 2. By property 7, $f_{j+1}(B) \subseteq I_R$. By assumption, $X_j \trianglelefteq I_R$, so since $g(b) \in f_{j+1}(B)$ and $g(a) = X_j$, $g(a) \trianglelefteq g(b)$. By antisymmetry, this suffices.

If $a \in K_{j+1} - O_{j+1} - X_{j+1}$ and $b \in O_{j+1}$, this is by the fact that $K_j - O_j$ is an interval in $(\omega/E(X_j), \trianglelefteq)$ and contains 0_j.

If $a \in B_0$ and $b \in B_1$ are distinct blocks of $K_{j+1} - O_{j+1} - X_{j+1}$, then using the fact that $\trianglelefteq$ induces the same ordering on the blocks of $I_L \cup I_R \supseteq K_j - O_j - X_j$ as $<_{K_j}$ does:

$$
\begin{aligned}
a \hat{\trianglelefteq} b &\Rightarrow a <_{K_{j+1}} b \\
&\Rightarrow B_0 <_{K_{j+1}} B_1 \\
&\Rightarrow f_{j+1}(B_0) <_{K_j} f_{j+1}(B_1) \\
&\Rightarrow f_{j+1}(B_0) \trianglelefteq f_{j+1}(B_1) \\
&\Rightarrow h_{B_0}(a) \trianglelefteq h_{B_1}(b) \\
&\Rightarrow g(a) \trianglelefteq g(b).
\end{aligned}
$$

By antisymmetry, this suffices. ☐

It follows that $E(X_j) \leqslant_{lo} E(X_{j+1})$, while $(\omega/E(X_{j+1}), <_{K_{j+1}})$ witnesses that $E(X_j) \not\equiv_{lo} E(X_{j+1})$. This completes the proof. ☐

4. The partial order $\leqslant_{lo}$ on equivalence relations $E(X_1, \dots, X_n)$

In this section we study basic properties of the partial order $\leqslant_{lo}$ restricted to the following set:

$$\mathcal{P}_n = \{E(X_1, \dots, X_n) \mid X_1, \dots, X_n \text{ are disjoint c.e. sets with co-infinite union}\}.$$

We first observe a necessary condition for $E(X_1, \dots, X_n)$ realizing a linear order.

Proposition 13. If $E(X_1, \dots, X_n)$ realizes a linear order, then the sets $X_1, \dots, X_n$ are pairwise computably separable—that is, for each $i \neq j$ there exists a computable set $R_{i,j}$ such that $X_i \subseteq R_{i,j}$ and $R_{i,j} \cap X_j = \emptyset$.

Proof. Suppose $(\omega/E(X_1, \dots, X_n), \trianglelefteq)$ realizes a linear order. By renumbering if necessary, we may assume $X_i \trianglelefteq X_{i+1}$ for all i. It suffices to show that for each i, there is a computable set R_i with $X_j \subseteq R_i$ for $j \leqslant i$ and $R_i \cap X_j = \emptyset$ for $j > i$.

There are two cases. First, suppose there is an $a \in \omega - X_i - X_{i+1}$ with $X_i \trianglelefteq a \trianglelefteq X_{i+1}$. Then let $R_i = \{b \in \omega \ : \ b \trianglelefteq a\}$. Clearly R_i is as desired, and R_i is c.e.. On the other hand, the complement of R_i is $\{b \in \omega \ : \ a \trianglelefteq b \wedge a \neq b\}$, which is also c.e., and so R_i is computable.

The second case is when there is no such a. Then fix $c \in X_i$ and let $R_i = \{b \in \omega \ : \ b \trianglelefteq c\} \cup X_i$. Clearly R_i is c.e.. Fix $d \in X_{i+1}$. Then the complement of R_i is $\{b \in \omega \ : \ d \trianglelefteq b\} \cup X_j$, which is also c.e., and so R_i is computable. ☐

Now we characterize when $E(X_1, \dots, X_n)$ realizes a linear order.

Proposition 14. Let $E(X_1, \dots, X_n) \in \mathcal{P}_n$ such that the sets $X_1, \dots, X_n$ are pairwise computably separable. The following are equivalent:

(1) $E(X_1, \dots, X_n)$ realizes a linear order.
(2) Each $E(X_1)$, …, $E(X_n)$ realizes a linear order.
(3) The disjoint sum $E(X_1) \oplus \dots \oplus E(X_n)$ realizes a linear order (note that this sum is an equivalence relation on $\omega \times \omega$).

Proof. (1) $\to$ (2). Let $\mathcal{L}$ be an $E(X_1, \dots, X_n)$-linear order $(\omega/E(X_1, \dots, X_n); \trianglelefteq)$. By finite intersection, there is a computable set $R_i \supseteq X_i$ such that R_i is disjoint from every X_j for $j \neq i$. We define a new c.e. relation by restricting to $\trianglelefteq$ on R_i and placing all the points of $\omega \setminus R_i$ to the right of R_i in their natural ordering. That is, we define $\hat{\trianglelefteq}$ as follows: for $a, b \in R_i$, $a \hat{\trianglelefteq} b \iff a \trianglelefteq b$; for $a, b \notin R_i$, $a \trianglelefteq b \iff a \leqslant b$; for $a \in R_i$ and $b \notin R_i$, $a \hat{\trianglelefteq} b$ and $\neg(b \hat{\trianglelefteq} a)$. Clearly this is a linear order on $\omega/E(X_i)$.
(2) $\to$ (3). Let $\trianglelefteq_i$ be a linear order realized by $E(X_i)$, $i = 1, \dots, n$. Then if we define

$$(i, a) \trianglelefteq (j, b) \iff [(i < j) \vee (i = j \wedge a \trianglelefteq_i b)],$$

this realizes a linear order on $E(X_i) \oplus \dots \oplus E(X_n)$.
(3) $\to$ (1). Consider $R_i = \bigcap_{j \neq i} R_{i,j}$ and assume that these partition ω. Then the map $(i, a) \mapsto a$ is a bijection between $R_1 \oplus \dots \oplus R_n$ and ω which induces a bijection between the quotient sets $R_1 \oplus \dots \oplus R_n/E(X_1) \oplus \dots \oplus E(X_n)$ and $\omega/E(X_1, \dots, X_n)$. By taking the linear order on $E(X_1) \oplus \dots \oplus E(X_n)$, restricting and pushing forward, we obtain a linear order on $E(X_1, \dots, X_n)$. ☐

As mentioned previously, an equivalence relation $E(X_1, \dots, X_n)$ realizes only linear orders with computable copies. We now show that this can fail for an equivalence relation of the form $E(X_1, X_2, \dots)$.

Definition 15. An infinite linear order is called *η-like* if every block it contains is finite.

For an η-like linear order $\mathcal{L}$, define

$$\text{blocks}(\mathcal{L}) = \{n \ : \ n \text{ is the size of a block of } \mathcal{L}.\}$$

Note that blocks($\mathcal{L}$) necessarily omits 0.

The following result is implicit in work of Lerman. There it is stated for a special class of linear orders (the η-representations), but the only hypothesis used in the proof is that all blocks are finite, i.e. the linear order is η-like.

Theorem 16 (Lerman [8])**.** *There is a non-empty Δ^0_3-set $Z \subseteq \omega \setminus \{0,1\}$ with $Z \neq$ blocks($\mathcal{L}$) for every computable η-like linear order $\mathcal{L}$.*

Theorem 17. *For every non-empty Σ^0_3-set $Z \subseteq \omega \setminus \{0\}$, there is a uniformly c.e. sequence $(X_i)_{i\in\omega}$ such that $E(X_1, X_2, \dots)$ realizes an η-like linear order $\mathcal{L}$ with blocks($\mathcal{L}$) $= Z$.*

Proof. We simultaneously construct $(X_i)_{i\in\omega}$ and $\mathcal{L}$. Nonuniformly fix some $k \in Z$. Fix a quantifier-free formula $\varphi(n, x, s)$ such that $n \in Z \iff \exists x \exists^\infty s\, \varphi(n, x, s)$. We will describe a strategy which, given $n, x \in \omega$, constructs $k \cdot \mathbb{Q} + n$ when $\exists^\infty s\, \varphi(n, x, s)$, and constructs $k \cdot \mathbb{Q} + k$ when $\neg\exists^\infty s\, \varphi(n, x, s)$ (we use multiplication to denote the reverse-lexicographic product, so $k \cdot \mathbb{Q}$ has blocks of size k arranged densely). $\mathcal{L}$ will be formed by simply concatenating the results of all of these strategies. Note that as described, we will necessarily have $Z = \text{blocks}(\mathcal{L})$.

Our strategy begins by choosing points $y_0, \dots, y_{n-1}$ and indices $i_0, \dots, i_{n-1}$, which it lays claim to. It also lays claim to an infinite subset of the domain ω to work with; all future points placed by this strategy will be drawn from this subset, and we will be certain to place every point in the subset. We order the points $y_0 \lhd y_1 \lhd \cdots \lhd y_{n-1}$ and immediately enumerate y_j into X_{i_j} for $j < n$. We then place $k-1$ points to the immediate right of each y_j. We begin building a copy of $k \cdot \mathbb{Q}$ to the immediate left of each y_j. We then proceed in stages:

At stage s, if $\varphi(n, x, s)$ holds, for every $j < n-1$ we enumerate into $X_{i_j,s+1}$ every point z between y_j and y_{j+1}. For y_{n-1}, we enumerate into $X_{i_{n-1},s+1}$ the $k-1$ points to the immediate right of $X_{i_{n-1},s}$. Thus the n points $[y_0] \lhd [y_1] \lhd \cdots \lhd [y_{n-1}]$ are currently adjacent. We then add a new set of $k-1$ points to the immediate right $X_{i_j,s+1}$ for each j. Also, for each $j > 0$, we start building a new copy of $k \cdot \mathbb{Q}$ to the immediate left of y_j; the copy of $k \cdot \mathbb{Q}$ to the immediate left of y_0 was unaffected by this process, and we continue building it at this stage.

At stage s, if $\varphi(n, x, s)$ does not hold, we simply continue constructing the copies of $k \cdot \mathbb{Q}$.

Note that if there are infinitely many s with $\varphi(n, x, s)$ holding, then every point placed between y_j and y_{j+1} will be enumerated into X_{i_j}, and all the points placed to the right of y_{n-1} will be enumerated into $X_{i_{n-1}}$. So after identifying equivalent points, we have constructed a copy of $k \cdot \mathbb{Q} + n$, as desired.

If instead there are only finitely many s with $\varphi(n, x, s)$ holding, then eventually $k-1$ points are placed to the immediate right of each y_j and those points are never enumerated into X_{i_j} (or any other X_i). Simultaneously, a copy of $k \cdot \mathbb{Q}$ is begun to the immediate left of each y_j and the points in that copy are never enumerated into any X_i. So after identifying equivalent points, we have constructed a copy of $(k \cdot \mathbb{Q} + k) \cdot n \cong k \cdot \mathbb{Q} + k$, as desired. $\square$

Corollary 18. There is a uniformly c.e. sequence $(X_i)_{i \in \omega}$ such that $E(X_1, X_2, \dots)$ realizes a linear order with no computable copy.

Now we give a full description of linearly ordered sets realized by certain equivalence relations from the class $\mathcal{P}_n$.

Definition 19. An equivalence relation $E(X_1, \dots, X_n)$ is *n-simple* if there is a computable partition $\omega = R_1 \sqcup \dots \sqcup R_n$ such that for each i, $X_i \subseteq R_i$ and X_i is simple inside R_i—that is, X_i intersects every infinite c.e. subset of R_i.

Lemma 20. *If $E(X_1, \dots, X_n)$ is n-simple, then the choice of partition is unique mod finite: for every computable partition $\omega = S_1 \sqcup \dots \sqcup S_n$ with $X_i \subseteq S_i$, S_i is determined up to a finite difference.*

Proof. Let $\omega = R_1 \sqcup \dots \sqcup R_n$ be a partition witnessing n-simplicity, and $\omega = S_1 \sqcup \dots \sqcup S_n$ be any other partition with $X_i \subseteq S_i$. If there is some i with $R_i \Delta S_i$ infinite, then there is some j with $R_j \setminus S_j$ infinite. But then this is an infinite computable subset of R_j disjoint from X_j, contrary to hypothesis. $\square$

Using results from [5] mentioned in the introduction we can fully characterize the order-types of linear orders realized over n-simple equivalence relations. For this we give the following definition:

Definition 21. Call an order-type *basic* if it is isomorphic to either $\omega + i$ or $i + \omega^\star$ or $\omega + 1 + \omega^\star$. Call an order type (n, s)*-basic* if it is isomorphic to the sum $\mathcal{L}_1 + \dots + \mathcal{L}_n$ of n basic linear orders of which at least s have the order type $\omega + 1 + \omega^\star$.

Theorem 22. *Let $E(X_1, \dots, X_n)$ be an n-simple equivalence relation that realizes a linear order. There exists an $s \leqslant n$ such that a linear order $\mathcal{L}$ is realized over $E(X_1, \dots, X_n)$ if and only if $\mathcal{L}$ is (n, s)-basic.*

Proof. Given a linear order $\mathcal{L}$ realized over $E(X_1, \dots, X_n)$, partition ω into $R_1 \sqcup \dots \sqcup R_n$ with $X_i \subseteq R_i$ and the R_i convex under $\mathcal{L}$'s ordering (we can obtain such R_i as the difference of the sets from the proof of Proposition 13). Then $\mathcal{L} = \mathcal{L}_{\sigma(1)} + \dots + \mathcal{L}_{\sigma(n)}$, where each $\mathcal{L}_i$ is realized over $R_i / E(X_i)$, for some permutation σ of $\{1, \dots, n\}$. Since X_i is simple inside R_i, from the results in [5] as mentioned in the introduction, $\mathcal{L}_i$ is basic. The number s is the number of i such that the only linear order realized over $R_i / E(X_i)$ has the order type $\omega + 1 + \omega^\star$ (since R_i is unique up to finite difference, this does not depend on the choice of partition). $\square$

5. The partial order $\leqslant_{lo}$ has a maximal element

In this section we show that $\leqslant_{lo}$ has a maximal element.

Lemma 23. *If E realizes both $(\omega, <)$ and $(\mathbb{Q}, <)$, then E is computable.*

Proof. Fix $\trianglelefteq_0$ such that $(\omega/E, \trianglelefteq_0)$ realizes $(\omega, <)$. Since ω is discrete, every equivalence class in E is computable. For example, if $[e]$ is the equivalence class of the $\trianglelefteq_0$-leftmost point, and $[h]$ is the equivalence class of the next $\trianglelefteq_0$-leftmost point, then the complement of $[e]$ is $\{a \ : \ h \trianglelefteq_0 a\} \cup [h]$, which is c.e.. Of course, we must show that these classes are uniformly computable.

Nonuniformly fix a computable index for $[e]$, the equivalence class of the $\trianglelefteq_0$-leftmost point. Fix $\trianglelefteq_1$ such that $(\omega/E, \trianglelefteq_1)$ realizes $(\mathbb{Q}, <)$. We must show that we can co-enumerate E.

Given $a, b \in \omega$, without loss of generality we may assume that $a \trianglelefteq_1 b$. Since we have a computable index for $[e]$, we may also assume that $a, b \notin [e]$. If $a \trianglelefteq_1 e \trianglelefteq_1 b$, then necessarily a and b are in different equivalence classes.

Suppose instead that $a \trianglelefteq_1 b \trianglelefteq_1 e$ (the reverse case is symmetric). If $(a, b) \notin E$, then the $\trianglelefteq_1$-interval $([a], [b])$ contains infinitely many equivalence classes. Of course, the $\trianglelefteq_1$-interval $([e], \infty)$ does as well. Thus we will eventually locate points $c, d, f \in \omega$ with $a \trianglelefteq_1 c \trianglelefteq_1 d \trianglelefteq_1 b \trianglelefteq_1 e \trianglelefteq_1 f$, $c \trianglelefteq_0 f \trianglelefteq_0 d$ and $f \notin [e]$.

Note that such a triple (c, d, f) proves that $(a, b) \notin E$, for since equivalence classes must be convex in both $\trianglelefteq_0$ and $\trianglelefteq_1$, if aEb, then $aEcEdEb$, and thus fEd, and thus fEe, which would be a contradiction. So upon finding such a triple, we may enumerate $(a, b) \notin E$. $\square$

As a corollary of this lemma one gets the following result:

Theorem 24. *There is a $\leqslant_{lo}$-degree containing precisely the computable equivalence relations, and this degree is maximal.*

Proof. Necessarily a computable equivalence relation realizes every computable linear order and only the computable linear orders. Thus all computable equivalence relations share the same $\leqslant_{lo}$-degree. Denote this degree by $\mathbf{c}$. Then for any $E \geqslant_{lo} \mathbf{c}$, E must realize all computable linear orders, and in particular must realize both $(\omega, <)$ and $(\mathbb{Q}, <)$. It follows that E is computable, and so $E \in \mathbf{c}$. $\square$

Corollary 25. The set of all lo-degrees is not an upper semi-lattice.

Proof. Let E be a computable equivalence relation. Let E' be a c.e. equivalence relation that realizes a linear order with no computable copy. The existence of such equivalence relations was originally shown by Feiner [2], but an alternate proof is given in Section 4. From the above, no equivalence relation F exists such that $E \leqslant_{lo} F$ and $E' \leqslant_{lo} F$. $\square$

Note that Lemma 23 could be extended by replacing $(\omega, <)$ by any linear order of Hausdorff rank 1 (a finite sum of ω, ω^* and $n \in \omega$), and by replacing $(\mathbb{Q}, <)$ by any linear order with only finitely many adjacencies. However, we will now show that the lemma fails if we try to extend to Hausdorff rank 2.

A *transversal* of an equivalence relation is a c.e. set $X \subseteq \omega$ such that X contains precisely one point from every equivalence class. It is not hard to see that a c.e. equivalence relation has a transversal if and only if it is computable.

We now prove the following result:

Theorem 26. *There is a noncomputable c.e. equivalence relation E that realizes both $(\omega^2, <)$ and $(\mathbb{Q}, <)$.*

Proof. Let $\omega^{[i]} = \{\langle i, n\rangle : n \in \omega\}$. On each $\omega^{[i]}$ we define an equivalence E_i and relations $\trianglelefteq_1^i$, $\trianglelefteq_2^i$ that respect E_i such that $(\omega^{[i]}/E_i, \trianglelefteq_1^i) \cong (\omega, \leqslant)$ and $(\omega^{[i]}/E_i, \trianglelefteq_2^i) \cong (\mathbb{Q}, \leqslant)$. Then we let E be the disjoint union of E_i's and define $\trianglelefteq_k$, for $k = 1, 2$, as follows: if $a \in \omega^{[i]}$ and $b \in \omega^{[j]}$ then

$$a \trianglelefteq_k b \iff i < j \text{ or } i = j \text{ and } a \trianglelefteq_k^i b.$$

In this case we will have $(\omega/E, \trianglelefteq_1) \cong (\omega^2, \leqslant)$ and $(\omega/E, \trianglelefteq_2) \cong (\mathbb{Q}, \leqslant)$.

Consider the requirements

$$R_i : \text{the range of } \varphi_i \text{ is not a transversal of } E.$$

Each R_i will work inside $\omega^{[i]}$ and will not interfere with the other requirements. Each requirement will act at most once and is never injured. If R_i never acts then E_i will be the identity relation on $\omega^{[i]}$. Otherwise, E_i will have the form $E(X)$ for some finite $X \subseteq \omega^{[i]}$.

Let $(\omega, \leqslant_\omega)$ and $(\omega, \leqslant_\mathbb{Q})$ be some fixed computable presentations of $(\omega, \leqslant)$ and $(\mathbb{Q}, \leqslant)$, respectively. At stage s we define E_i^s and $\trianglelefteq_k^{i,s}$, for $k = 1, 2$, on some finite subset $D_i^s \subseteq \omega^{[i]}$ such that $(D_i^s/E_i^s, \trianglelefteq_1^{i,s}) \cong (\{0, \dots, s\}, \leqslant_\omega)$ and $(D_i^s/E_i^s, \trianglelefteq_2^{i,s}) \cong (\{0, \dots, s\}, \leqslant_\mathbb{Q})$. In the end of the construction we will have $E_i = \bigcup_s E_i^s$ and $\trianglelefteq_k^i = \bigcup_s \trianglelefteq_k^{i,s}$.

At stage 0 we let all D_i^0, E_i^0 and $\trianglelefteq_k^{i,0}$ be empty sets. At stage $s+1$ we consider all $i \leqslant s+1$ and for each R_i check if there are $x, y \leqslant s+1$ such that $\varphi_i^{s+1}(x)\downarrow = \langle i, 0\rangle$ and $\varphi_i^{s+1}(y)\downarrow = \langle i, 1\rangle$. If no such x, y exist or if R_i has already acted, then we extend D_i^s, E_i^s, $\trianglelefteq_k^{i,s}$ to D_i^{s+1}, E_i^{s+1}, $\trianglelefteq_k^{i,s+1}$ in such a way that $(D_i^{s+1}/E_i^{s+1}, \trianglelefteq_1^{i,s+1}) \cong (\{0, \dots, s+1\}, \leqslant_\omega)$ and $(D_i^{s+1}/E_i^{s+1}, \trianglelefteq_2^{i,s+1}) \cong (\{0, \dots, s+1\}, \leqslant_\mathbb{Q})$. While doing so we also make sure that if E_i^s was the identity relation on D_i^s, then E_i^{s+1} is the identity on D_i^{s+1}. Similarly, if E_i^s had the form of $E(X)$ for some finite X on domain D_i^s, then E_i^{s+1} also has the same form $E(X)$ but on a larger domain D_i^{s+1}.

If we find such x, y and R_i has not acted yet, then we collapse all elements of $D_i^s = \{0, \dots, s\}$ into one equivalence class so that E_i^t at all later stages $t \geqslant s$ will have the form $E(D_i^s)$. After that again extend D_i^s, E_i^s, $\trianglelefteq_k^{i,s}$ to D_i^{s+1}, E_i^{s+1}, $\trianglelefteq_k^{i,s+1}$ in such a way that $(D_i^{s+1}/E_i^{s+1}, \trianglelefteq_1^{i,s+1}) \cong (\{0, \dots, s+1\}, \leqslant_\omega)$ and $(D_i^{s+1}/E_i^{s+1}, \trianglelefteq_2^{i,s+1}) \cong (\{0, \dots, s+1\}, \leqslant_\mathbb{Q})$.

This completes the construction. It is not hard to see that E and $\trianglelefteq_k$, for $k = 1, 2$, are c.e. relations. Moreover, each requirement R_i is satisfied, which implies that E is not computable. □

Much more needs to be done in the study of properties of *lo*-degrees. For instance, we do not know if there exist infinitely many maximal elements among the *lo*-degrees. We do not know if the set of *lo*-degrees is dense upward, that is, if it is true that for every non-maximal *lo*-degree x there exists a non-maximal *lo*-degree y such that $x <_{lo} y$.

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