COMPUTABLE BI-EMBEDDABLE CATEGORICITY

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We study the algorithmic complexity of isomorphic embeddings between computable structures. Suppose that L is a language. We say that L-structures $\mathcal{A}$ and $\mathcal{B}$ are *bi-embeddable* (denoted by $\mathcal{A} \approx \mathcal{B}$) if there are isomorphic embeddings $f \colon \mathcal{A} \hookrightarrow \mathcal{B}$ and $g \colon \mathcal{B} \hookrightarrow \mathcal{A}$. The systematic investigation of the bi-embeddability relation in computable structure theory was initiated by Montalbán [1, 2]: he proved that any hyperarithmetical linear order is bi-embeddable with a computable one. In [3], similar results were obtained for abelian p-groups, Boolean algebras, and compact metric spaces. The paper [4] studies degree spectra with respect to bi-embeddability.

Definition 1. Let $\mathbf{d}$ be a Turing degree. We say that a computable structure $\mathcal{S}$ is $\mathbf{d}$*-computably bi-embeddably categorical* if for any computable structure $\mathcal{A} \approx \mathcal{S}$, there are $\mathbf{d}$-computable isomorphic embeddings $f \colon \mathcal{A} \hookrightarrow \mathcal{S}$ and $g \colon \mathcal{S} \hookrightarrow \mathcal{A}$. The *bi-embeddable categoricity spectrum* of $\mathcal{S}$ is the set

$$CatSpec_{\approx}(\mathcal{S}) = \{\mathbf{d} : \mathcal{S} \text{ is } \mathbf{d}\text{-computably bi-embeddably categorical}\}.$$

A degree $\mathbf{c}$ is the *degree of bi-embeddable categoricity* of $\mathcal{S}$ if $\mathbf{c}$ is the least degree in the spectrum $CatSpec_{\approx}(\mathcal{S})$.

Definition 1 is similar to the notions of categoricity spectrum and degree of categoricity which were introduced in [5]. The *categoricity spectrum* of a computable structure $\mathcal{S}$ is the set of all Turing degrees which are capable of computing isomorphisms among arbitrary computable isomorphic copies of $\mathcal{S}$. The *degree of categoricity* of $\mathcal{S}$ is the least degree from the categoricity spectrum of $\mathcal{S}$.

Our first result gives examples of degrees of bi-embeddable categoricity. It shows that every degree of categoricity known in the literature [5, 8] can be realized as a degree of bi-embeddable categoricity. We make use of the following notion. A structure $\mathcal{A}$ is called *bi-embeddably trivial* (or *b.e. trivial* for short) if for any $\mathcal{B}$ bi-embeddable with $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{A}$ are isomorphic.

Theorem 1. *Let α be a computable non-limit ordinal. Suppose that $\mathbf{d}$ is a Turing degree such that $\mathbf{d}$ is d.c.e. in $\mathbf{0}^{(\alpha)}$ and $\mathbf{d} \geq \mathbf{0}^{(\alpha)}$. There is a computable, bi-embeddably trivial structure $\mathcal{S}$ with degree of bi-embeddable categoricity $\mathbf{d}$.*

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Proof Sketch. We build two b.e. trivial computable structures $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A} \cong \mathcal{B}$, $\mathcal{A}$ is $\mathbf{d}$-computably categorical, and any embedding from $\mathcal{A}$ into $\mathcal{B}$ must compute $\mathbf{d}$. Here we give a construction for the case when $\mathbf{d}$ is d.c.e. over $\mathbf{0}^{(2\beta+1)}$, where β is an infinite ordinal.

Ash's characterization of the back-and-forth relations for linear orders and his pairs of structures theorem, see Chapters 11 and 16 in [6], tells us that for any $\Sigma^0_{2\beta+1}$ set S, there is a computable sequence $(C_e)_{e\in\omega}$ of linear orders such that

$$C_e \cong \begin{cases} \omega^\beta \cdot 2, & \text{if } e \in S, \\ \omega^\beta, & \text{if } e \notin S. \end{cases} \tag{1}$$

A relativized version of the argument from [5, Theorem 3.1] shows that one can choose a set $D \in \mathbf{d}$ such that D is d.c.e. in $\mathbf{0}^{(2\beta+1)}$ and for any oracle X, we have:

$$(\overline{D} \text{ is c.e. in } X) \ \Rightarrow\ D \leq_T X \oplus \mathbf{0}^{(2\beta+1)}.$$

The language of our structures contains an equivalence relation $\sim$, a partial order $\leq$, a unary predicate T, and a unary predicate P_e, for each $e \in \omega$. We have that $D = U \setminus V$ for U and V c.e. in $\mathbf{0}^{(2\beta+1)}$, where $V \subset U$. We first describe the construction of $\mathcal{A}$. For every e, we choose elements a_e and b_e in $\mathcal{A}$, and for every P_e, $P_e(A)$ is infinite and includes a_e, b_e.

For a fixed e, we give a construction for the substructure on $P_e(A)$. We let $P_e(A)$ consist of two infinite equivalence classes (with respect to $\sim$) such that $a_e \not\sim b_e$. The two classes $[a_e]$ and $[b_e]$ will both contain pairs of linear orders, i.e., structures of the form (L_1, L_2) where L_1 and L_2 are linear orders (with respect to $\leq$), any $x \in L_1$ and $y \in L_2$ are incomparable, and $T([a_e]) = L_1$.

If $e = 2m$, then we encode the information whether or not m is an element of D in $P_e(A)$. There are three cases:

1. $m \notin U$: we build $T([a_e]), \neg T([a_e]), T([b_e]) \cong \omega^\beta$, and $\neg T([b_e]) \cong \omega^\beta \cdot 2$;
2. $m \in U \setminus V$: we build $T([b_e]) \cong \omega^\beta$ and $T([a_e]), \neg T([a_e]), \neg T([b_e]) \cong \omega^\beta \cdot 2$;
3. $m \in V$: we build $T([a_e]), T([b_e]), \neg T([a_e]), \neg T([b_e]) \cong \omega^\beta \cdot 2$.

Analyzing this construction, we see that

$$[a_e] \cong \begin{cases} (\omega^\beta \cdot 2, \omega^\beta \cdot 2), & \text{if } m \in U, \\ (\omega^\beta, \omega^\beta), & \text{if } m \notin U; \end{cases} \quad \text{and} \quad [b_e] \cong \begin{cases} (\omega^\beta \cdot 2, \omega^\beta \cdot 2), & \text{if } m \in V, \\ (\omega^\beta, \omega^\beta \cdot 2) & \text{if } m \notin V. \end{cases}$$

If $e = 2m+1$, then we let $[b_e] \cong (\omega^\beta, \omega^\beta \cdot 2)$, and for $[a_e]$ we let

$$[a_e] \cong \begin{cases} (\omega^\beta \cdot 2, \omega^\beta \cdot 2), & \text{if } m \in \emptyset^{(2\beta+1)}, \\ (\omega^\beta, \omega^\beta), & \text{if } m \notin \emptyset^{(2\beta+1)}. \end{cases}$$

The existence of the uniformly computable sequence of structures $(C_e)_{e\in\omega}$ from (1) implies that we can do the construction computably.

For $\mathcal{B}$, we again choose elements $\hat{a}_e$, $\hat{b}_e$ for every e, and we build $\mathcal{B}$ like $\mathcal{A}$ with the difference that the roles of $\hat{a}_e$ and $\hat{b}_e$ are switched. Clearly, $\mathcal{B}$ and $\mathcal{A}$ are isomorphic and computable. It is not hard to show that they are b.e. trivial: Indeed, every embedding of $\mathcal{A}$ into a bi-embeddable copy $\hat{\mathcal{A}}$ must map elements in $P_e(A)$ to elements in $P_e(\hat{A})$, for every $e \in \omega$. Every $P_e(\hat{A})$ must have exactly 2 equivalence classes as otherwise $P_e(\hat{A}) \not\approx P_e(A)$. Moreover, the pairs of structures that we use are pairs of well-orders, and thus b.e. trivial.

Following the lines of the proof of [7, Theorem 4], it is not hard to obtain that $\mathcal{A}$ is $\mathbf{d}$-computably categorical. It remains to show that for every $f\colon \mathcal{A} \hookrightarrow \mathcal{B}$, $f \geq_T D$. We have that $f \geq_T \mathbf{0}^{(2\beta+1)}$ because

$$m \in \emptyset^{(2\beta+1)} \Leftrightarrow f(a_{2m+1}) \sim \hat{b}_{2m+1} \quad \text{and} \quad m \notin \emptyset^{(2\beta+1)} \Leftrightarrow f(a_{2m+1}) \sim \hat{a}_{2m+1}.$$

Similarly, we have that

$$m \notin U \setminus V \Leftrightarrow (f(a_{2m}) \sim \hat{a}_{2m}) \text{ or } (m \in V).$$

Thus, $\overline{D}$ is c.e. in $f \oplus \mathbf{0}^{(2\beta+1)}$. Hence, $D \leq_T (f \oplus \mathbf{0}^{(2\beta+1)}) \equiv_T f$.

The construction for the case $\alpha = 2\beta + 2$ is nearly the same. The only difference is that in place of (1), we use the following fact: For any $\Sigma^0_{2\beta+2}$ set S, there is a computable sequence $(C_e)_{e\in\omega}$ of linear orders such that

$$C_e \cong \begin{cases} \omega^{\beta+1} + \omega^{\beta}, & \text{if } e \in S, \\ \omega^{\beta+1}, & \text{if } e \notin S. \end{cases}$$

The proof for finite α can be obtained by minor modifications. □

The rest of the paper is devoted to bi-embeddable categoricity for structures from familiar algebraic classes. Recall that $\mathcal{A} = (A, E^2)$ is an *equivalence structure* if E is an equivalence relation on the domain of $\mathcal{A}$.

Theorem 2 ([9]). *Any computable equivalence structure has degree of bi-embeddable categoricity* $\mathbf{d} \in \{\mathbf{0}, \mathbf{0}', \mathbf{0}''\}$.

Note that a similar result for degrees of categoricity was proved by Csima and Ng (unpublished).

Theorem 3. (a) *A computable Boolean algebra is computably bi-embeddably categorical if and only if it is finite.*
(b) *A computable linear order is computably bi-embeddably categorical if and only if it is finite.*

Note that Theorem 3 contrasts with the characterizations of computably categorical Boolean algebras [10, 11] and computably categorical linear orders [10, 12]: In particular, a computable Boolean algebra is computably categorical iff its set of atoms is finite.

An undirected graph is *strongly locally finite* if each of its components is finite. It is easy to show that every computable, strongly locally finite graph is $\mathbf{0}'$-computably categorical.

Theorem 4. (a) *There exists a computable, strongly locally finite graph which is not hyperarithmetically bi-embeddably categorical.*
(b) *The index set of* $\mathbf{0}'$*-computably bi-embeddably categorical, strongly locally finite graphs is* Π^1_1*-complete.*

Proof. Ad (a). Let $H \subseteq \omega^{<\omega}$ be a computable tree without hyperarithmetic paths. We build a strongly locally finite graph G_H such that the partial ordering under embeddability of its components is computably isomorphic to H.

For any $\sigma \in H$, G_H contains the component C_σ: A ray of length $|\sigma| + 1$ where the first vertex has a loop connected to it and the $(i + 2)^{th}$ vertex for $i < |\sigma|$ has a cycle of length $\sigma(i) + 2$ attached. Clearly the partial ordering of the components is computably isomorphic to H by $C_\sigma \mapsto \sigma$. Now G_H has a bi-embeddable copy $\tilde{G}$

that skips a fixed C_σ such that σ lies on a path in H. Now consider embeddings $\mu : G_H \to G$ and $\nu : G \to G_H$, then $C_\sigma \subset \mu(C_\sigma) \subset \nu(\mu(C_\sigma)) \subset \dots$ and thus there is $f \in [H]$ hyperarithmetic in $\mu \oplus \nu$. Hence, $\mu \oplus \nu$ itself can not be hyperarithmetic.

Ad (b). Let $(T_i)_{i\in\omega}$ be a uniformly computable sequence of trees such that T_i is well-founded iff $i \in \mathcal{O}$. For two strings σ, τ of the same length let $\sigma \star \tau = \sigma_0\tau_0\sigma_1\tau_1 \dots \sigma_{|\sigma|-1}\tau_{|\tau|-1}$, and consider the sequence of trees $(S_i)_{i\in\omega}$

$$S_i = \{\xi : \xi \subseteq \sigma \star \tau,\ |\sigma| = |\tau|,\ \sigma \in T_i,\ \tau \in H\}.$$

Clearly, it is uniformly computable, and S_i is well-founded iff $i \in \mathcal{O}$. Furthermore, no path in $[S_i]$ is hyperarithmetical. Using the same coding as above we get that if $i \in \mathcal{O}$, then G_{S_i} is b.e. trivial and thus $\mathbf{0}'$-computably bi-embeddably categorical. If $i \notin \mathcal{O}$, then G_{S_i} is not $\mathbf{0}^{(\alpha)}$-computably bi-embeddably categorical for $\alpha < \omega_1^{\mathrm{CK}}$. □

Note that in [13], it was shown that the index set of computably categorical structures is Π^1_1-complete. We leave open whether a similar result can be obtained for computably bi-embeddably categorical structures.

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