AAA 108

FRÄISSÉ'S CONJECTURE AND BIG RAMSEY DEGREES OF STRUCTURES ADMITTING FINITE MONOMORPHIC DECOMPOSITIONS

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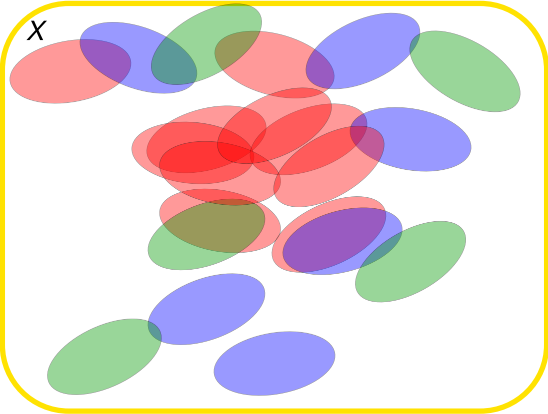
COLORING THE INFINITE

THEOREM 1 (RAMSEY 1930)

Let X be a countably infinite set. For any finite coloring of $[X]^n$, there exists an infinite subset $M \subseteq X$ such that $[M]^n$ is monochromatic.



Frank P. Ramsey (1903–1930)

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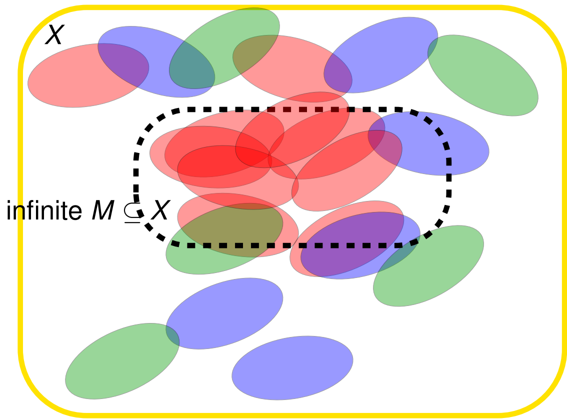

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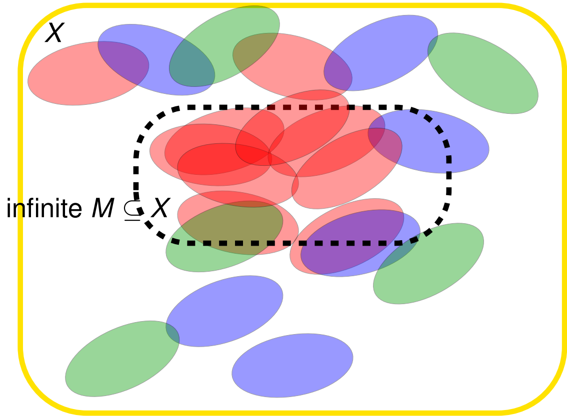

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COLORING THE INFINITE

THEOREM 2 (RAMSEY 1930)

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Theorem 2 (Ramsey 1930)

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If we impose an order type of ω on X, this theorem becomes a **structural Ramsey theorem** about infinite linear orders.

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COLORING THE INFINITE

THEOREM 3 (RAMSEY 1930)

~~Let ω be a countably infinite set.~~ For any finite coloring of $[\omega]^n$, there exists an ~~infinite subset~~ $L \subset \omega$, $L \cong \omega$, such that $[L]^n$ is monochromatic.

If we impose an order type of ω on X, this theorem becomes a **structural Ramsey theorem** about infinite linear orders.

We could, of course, say we're coloring all n-element chains instead of $[L]^n$.

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Let A be a finite, and $\mathfrak{B}$ an infinite $\mathcal{L}$-structrue over some language $\mathcal{L}$. Denote by $\binom{X}{Y}$ the set of substructures of X isomorphic to Y and by $\mathrm{Emb}(Y, X)$ the set of embeddings from Y to X.

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For every function(k-coloring) $\chi : \binom{\mathfrak{B}}{A} \longrightarrow [k]$, we seek to find $B \subset \mathfrak{B}$, $B \cong \mathfrak{B}$ such that $|\chi[\binom{B}{A}]| = 1$

$$\Longleftrightarrow \text{(more or less)}$$

For every $\chi : \mathrm{Emb}(A, \mathfrak{B}) \longrightarrow [k]$, we seek to find $f \in \mathrm{Emb}(\mathfrak{B}, \mathfrak{B})$ such that $|\chi[f \circ \mathrm{Emb}(A, \mathfrak{B})]| = 1$.

Note that the latter version lets us swhitch completely to category theory. ([Mašulović ∼2015],[Solecki 2022] and many more)

STRUCTURES- RARELY MONOCHROMATIC

THEOREM 4 (GALVIN)

For every coloring of 2-element chains in $(\mathbb{Q}, <)$ into finitely many colors, there is a $S \subset \mathbb{Q}, S \cong \mathbb{Q}$ such that $\chi[\binom{S}{[2]}] \leq 2$. This bound is tight!

- Expectation: monochromatic copy
- Reality: **oligochromatic** copy
- ologochromatic- not depending on the initial number of colors.

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DEFINITION 5 (KECHRIS–PESTOV–TODORCEVIC)

Let S be a structure and let A be a finite substructure of S.

- The big Ramsey degree of A in S is the least $t \in \mathbb{N}$ such that for every finite coloring
$$\chi : \mathrm{Emb}(A, S) \to [k],$$
there exists an isomorphic copy $C \leq S$ satisfying
$$|\chi(\mathrm{Emb}(A, C))| \leq t.$$
- We write $T(A, S) = t$, or $T(A, S) = \infty$ if no such t exists.

KNOWN RESULTS ON ORDERS

- **Ordinals**
  - $T(1, \omega^\alpha) = 1$ for every ordinal α [Fraïssé]
  - $T(1, \alpha) < \infty$ for every infinite ordinal α [Fraïssé]
- **Scattered linear orders**
  - $T(1, A) = 1$ for every additively indecomposable A [Laver]
  - $T(1, S) < \infty$ for every scattered S [Laver]
- **Non-scattered linear orders**
  - $\mathbb{Q} \nrightarrow (\mathbb{Q})^2_2$ [Galvin]
  - $T(n, \mathbb{Q}) < \infty$+ formula for every $n \in \mathbb{N}$ [Galvin, Laver, Devlin]

KNOWN RESULTS ON ORDERS

- **Countable linear orders**
  - α . . . a countable ordinal
  - S . . . a countable linear order
- **Big Ramsey spectrum**

$$\mathrm{Spec}(S) = \big(T(1,S),\ T(2,S),\ T(3,S),\ \dots\big)$$

- **Classification for all countable linear orders**
  - **Theorem** [Mašulović, Šobot] $\mathrm{Spec}(\alpha)$ is finite if and only if $\alpha < \omega^\omega$.
  - **Theorem** [Galvin, Laver, Devlin] For every non-scattered S, $\mathrm{Spec}(S)$ is finite.
  - **Theorem** [Da Silva Barbosa, Mašulović, Nenadov] For scattered S, $\mathrm{Spec}(S)$ is finite if and only if $\mathrm{rk}_{\mathrm{Hausd}}(S) < \infty$.
- **We can even calculate spectra**
  - **Theorem**[Boyland, Gasarch, Hurtig,Rust] Formula for Big Ramsey degrees of all countable ordinals.
- **Partial orders**
  - **Theorem**[Balko, Chodounský, Dobrinen, Hubička, Konečný, Vena, Zucker] Found spectrum for the Random poset
  - **Theorem**[Mašulović,T] Generic 2-dimensional partial order $\mathbb{P}_2$ has finite spectrum.

Monomorphic Structures

Definition 6 (Fraïssé)

A structure S is monomorphic if all finite substructures of S of the same size are isomorphic.

Examples

- Linear orders
- Hausdorff topological spaces [Raghavan, Todorčević]

Characterization

Theorem 7 (Fraïssé; Pouzet)

A countable relational structure $\mathcal{M} = (M, \ldots)$ is monomorphic if and only if it is quantifier-free definable in some linear order $(M, <)$.

In this case, we say that the linear order $(M, <)$ chains $\mathcal{M}$.

Monomorphic Structures and Big Ramsey Spectra

- **Setup**
  - $\mathcal{M}$ ... a countable monomorphic structure
  - $T(n, \mathcal{M})$ ... the big Ramsey degree of the unique n-element substructure of $\mathcal{M}$
- **Big Ramsey spectrum**

$$\mathrm{Spec}(\mathcal{M}) = \big(T(1, \mathcal{M}),\ T(2, \mathcal{M}),\ T(3, \mathcal{M}),\ \ldots\big)$$

Theorem 8 (Mašulović, T)

$\mathrm{Spec}(\mathcal{M})$ is finite if and only if $\mathrm{Spec}(\mathcal{M}, \prec)$ is finite for some (and hence for every) **minimal** linear order $\prec$ that chains $\mathcal{M}$.

$$\Uparrow$$

Theorem 9 (Fraïssé's Conjecture; Laver)

The class of all countable linear orders is a well-quasi-order under embeddability.

PROFILE= T.P.T.O.E.N.I.S.O.S.*n*

- Profile of a monomorphic structure: $(1, 1, 1, 1, ...)$.
- Can we exitend our results for slowly-growing structures?
- Not so painful for ordered structures!
- In this case, we can classify spectra for all structures of polynomial growth(with finite signature).

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THEOREM 10 (OUDRAR,POUZET)

Let $\mathcal{C}$ be a hereditary class of finite ordered relational structures with a finite **restricted** signature μ.
Then exactly one of the following holds:

- There exists an integer k such that every member of $\mathcal{C}$ admits an **interval decomposition** into at most $k + 1$ blocks. In this case, $\mathcal{C}$ is a finite union of ages of ordered relational structures, each having an interval decomposition into at most $k + 1$ blocks, and the profile of $\mathcal{C}$ is polynomial.
- Otherwise, the profile of $\mathcal{C}$ grows exponentially.

FMD- KEEPING A LOW PROFILE

- $\mathcal{S} = (S, \ldots)$ ... a relational structure
- $\{E_\alpha : \alpha < \kappa\}$ ... a partition of S

DEFINITION 11 (POUZET, THIÉRY)

The partition $\{E_\alpha : \alpha < \kappa\}$ is a **monomorphic decomposition** of $\mathcal{S}$ if for all finite substructures $A, B \leq \mathcal{S}$ of the same size,

$$A \cong B \iff |A \cap E_\alpha| = |B \cap E_\alpha| \text{ for all } \alpha < \kappa.$$

THEOREM 12 (POUZET, THIÉRY)

Every relational structure admits a coarsest monomorphic decomposition, called the **minimal monomorphic decomposition**.

FMD- KEEPING A LOW (RAMSEY)

- **Setup**
  - $\mathcal{S} = (S, \dots)$ . . . a countable relational structure
  - $\{E_1, \dots, E_m\}$ . . . a finite monomorphic decomposition of $\mathcal{S}$(polynomial growth)
  - $\mathcal{S}[E]$ . . . the substructure of $\mathcal{S}$ induced by $E \subseteq S$(obviously monomprphic)
- **Main result(though not a surprising one)**

> **THEOREM 13 (MAŠULOVIĆ, T)**
>
> $\mathcal{S}$ has finite big Ramsey degrees if and only if each $\mathcal{S}[E_i]$ does, for $1 \leq i \leq m$.

$\Uparrow$

- **Underlying principle(a surprising result and technique)**
  A product Ramsey statement for linear orders

A PRODUCT RAMSEY THEOREM FOR LINEAR ORDERS

- $L_1, \dots, L_m$ ... countable linear orders with finite big Ramsey spectra

THEOREM 14 (MAŠULOVIĆ, T)

For every choice of $n_1, \dots, n_m \in \mathbb{N}$ there exists $t \in \mathbb{N}$ such that for every finite coloring

$$\chi : \mathrm{Emb}(n_1, L_1) \times \cdots \times \mathrm{Emb}(n_m, L_m) \to \{1, \dots, k\},$$

there exist suborders $C_i \leq L_i$ with $C_i \cong L_i$ for $1 \leq i \leq m$ such that

$$\left|\chi\big(\mathrm{Emb}(n_1, C_1) \times \cdots \times \mathrm{Emb}(n_m, C_m)\big)\right| \leq t.$$

$$T\big((n_1, \dots, n_m), (L_1, \dots, L_m)\big) < \infty.$$

Afetrmath

As a nice consequence of the fact that $T((n, m), (\mathbb{Q}, \mathbb{Q})) < \infty$, we prove that Cameron's generic permutation has finite spectrum, and from there:

Theorem 15 (Mašulović,T)

Generic permutation $(\mathbb{Q}, <, \sqsubset)$ has finite big Ramsey degrees.

Theorem 16 (Mašulović,T)

The **generic 2-dimensional poset** is quantifier-free definable in the generic permutation:

$$x \preceq y \quad \text{iff} \quad x = y \ \text{ or } \ (x < y \ \text{ and } \ x \sqsubset y).$$

$\mathbb{P}_2$ is a weak Fräissé limit of all posets embeddable into a product of two chains.

Weak Fräissé limits are precisely ...

How it's done

- Many proofs in Ramsey theory use various color transfer principles, to "steal" Ramsey properties from other structures.
- It's difficult(or in some cases impossible) to prove finite big Ramsey degrees on product categories just by color transfer from the structures themselves
- It turns out to be a problem of book-keeping!
- **Strategy**
  - Find a strong enough catgorical notion of color transfer(a lot of reading involved)
  - Prove it respects products
  - Find a Big Ramsey structure/category whose products reduce to itself.
  - Hope it will be strictly stronger than all nice chains.
- The category in question is $\mathbb{Q}$ with partial set-functions and $(\mathbb{Q}, <)$ self-embeddings.
- particuarly tough are scattered chains, where we mimic the proof of da Silva Barbosa,Mašulović, Nenadov

COLOR-STEALING MAP

DEFINITION 17

Let $\mathcal{A}$ and $\mathcal{B}$ be locally small categories. For $A, X \in \mathrm{Ob}(\mathcal{A})$ and $B, Y \in \mathrm{Ob}(\mathcal{B})$, we write

$$(A, X)_{\mathcal{A}} \prec (B, Y)_{\mathcal{B}}$$

to denote that there exist:

- a subset $M \subseteq \hom(B, Y)$, and
- a set-function $\phi : M \to \hom(A, X)$

such that for every $h \in \hom(Y, Y)$, there exists $g \in \hom(X, X)$ satisfying

$$g \circ \hom(A, X) \subseteq \phi\big(M \cap (h \circ \hom(B, Y))\big).$$