Graph Representation Between Some Polynomial Clones

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108^{th} Workshop on General Algebra
Vienna, Austria

6 – 8 February 2026

Funded by the European Union
NextGenerationEU

Funded by the EU NextGenerationEU through the Recovery and Resilience Plan for Slovakia under the project No. 09I03-03-V05-00008.

Introduction

The lattice of clones on a set A:

If $|A| = 1$, then the clone lattice consists of one clone.

If $|A| = 2$, then the clone lattice consists of countably many clones (Post's Lattice).

If $|A| > 2$, then the clone lattice consists of uncountable many clones.

Investigated interval

$J_n = \langle \mathrm{P}(\mathbb{Z}_n, +), \mathrm{P}(\mathbb{Z}_n, +, \cdot) \rangle$

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The elements of $\mathrm{P}(\mathbb{Z}_n, +)$ *are all linear functions in the following form*

$$p(x_1, \ldots, x_n) = a_0 + a_1 x_1 + a_2 x_2 + \cdots + a_n x_n,$$

where $a_0 \in \mathbb{Z}_n$*,* $a_1, \ldots, a_n \in \mathbb{Z}$*.*

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The elements of $\mathrm{P}(\mathbb{Z}_n, +, \cdot)$ *are all polynomial functions in the following form*

$$q(\mathbf{x}) = \sum_{\alpha} a_\alpha \mathbf{x}^\alpha,$$

where $\mathbf{x} = (x_1, \ldots, x_n)$*, the sum consists of finitely many tuples* $\alpha = (\alpha_1, \ldots, \alpha_n)$ *of natural numbers, coefficients* a_α *belong to the set* $\mathbb{Z}_n$ *and* $\mathbf{x}^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$*.*

What is already known about the interval $J_n = \langle \mathrm{P}(\mathbb{Z}_n, +), \mathrm{P}(\mathbb{Z}_n, +, \cdot) \rangle$?

Solved cases:

- $n =$ prime number p (Rosenberg, 1970)

$\mathfrak{J}_p$

Lemma

Let $m, n \in \mathbb{N}$. If gcd$(m, n) = 1$, then

$$J_{mn} \cong J_m \times J_n.$$

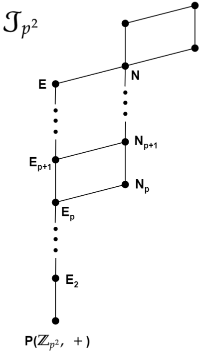
Corollary

It suffices to investigate the case $n = p^k$, $k \in \mathbb{N}$, where p is a prime number.

What is already known about the interval $J_n = \langle \mathsf{P}(\mathbb{Z}_{p^k}, +), \mathsf{P}(\mathbb{Z}_{p^k}, +, \cdot) \rangle$?

Solved cases:

- $n = p^2$, where p is prime number (Krokhin et al. 1997; Idziak and Bulatov, 2003)



Investigated interval $\langle \mathsf{P}(\mathbb{Z}_{p^k}, +), \mathsf{P}(\mathbb{Z}_{p^k}, +, \cdot) \rangle$

Open Problem

What is the structure of the clone lattice of the interval $\langle \mathrm{P}(\mathbb{Z}_{p^3}, +), \mathrm{P}(\mathbb{Z}_{p^3}, +, \cdot) \rangle$*?*

Investigated interval $\langle \mathrm{P}(\mathbb{Z}_{p^k}, +), \mathrm{P}(\mathbb{Z}_{p^k}, +, \cdot) \rangle$

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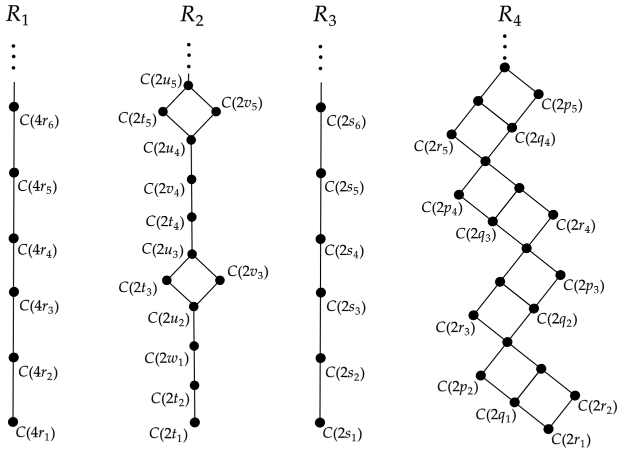
What is the structure of the clone lattice of the interval $\langle \mathrm{P}(\mathbb{Z}_{p^3}, +), \mathrm{P}(\mathbb{Z}_{p^3}, +, \cdot) \rangle$*?*

$\downarrow$

Open Problem

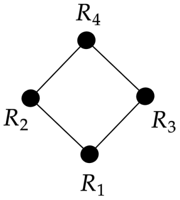
What is the structure of the clone lattice of the interval $\langle \mathrm{P}(\mathbb{Z}_8, +), \mathrm{P}(\mathbb{Z}_8, +, \cdot) \rangle$*?*

Lattice of clones generated by operations with even coefficients:



Ivestigated interval $\langle \mathsf{P}(\mathbb{Z}_8, +), \mathsf{P}(\mathbb{Z}_8, +, \cdot) \rangle$

Lattice of clones generated by operations with even coefficients:



Clones generated by some operations with odd coefficients

Let $\mathrm{C}(f_E) \subseteq \mathrm{P}(\mathbb{Z}_8, +, \cdot)$ *be a clone generated by an operation*

$$f_E(x_1, \dots, x_n) = x_1 \dots x_n \sum_{ij \in E} (x_i - 1)(x_j - 1),$$

where $E \subseteq [n]^2$ *(the set of all pairs from* $\{1, 2, \dots, n\}$*).*

Problem

The description of clones $\mathrm{C}(f_E) \subseteq P(\mathbb{Z}_8, +, \cdot)$ *generated by the operation*

$$f_E(x_1, \ldots, x_n) = x_1 \ldots x_n \sum_{ij \in E} (x_i - 1)(x_j - 1),$$

where $E \subseteq [n]^2$ *(the set of all pairs from* $\{1, 2, \ldots, n\}$*).*

Graph(n) - the set of all graphs on the vertex set $\{1, 2, \ldots, n\}$

$E \in$ Graph(n) - a graph on n vertices with an edge set $E \subseteq [n]^2$

Lemma

For every $E, D \in$ *Graph*(n), $f_E + f_D = f_{E+D}$.

$[E]$ - a class of graphs generated by E closed under isomorphism and symmetric difference
Equivalently,

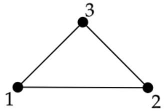
$$[E] = \{E_1 + \cdots + E_m | \; E_i \cong E \text{ for every } i\}, \text{ where } + \text{ is the symmetric difference}$$

$[E_1, \ldots, E_m]$ - a class of graphs generated by $E_1, \ldots, E_m$ closed under isomorphism and symmetric difference

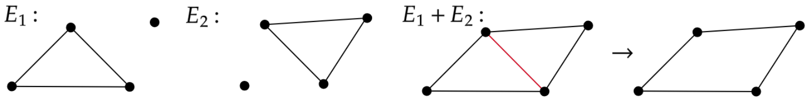
Examples

$$f_E = x_1x_2x_3((x_1-1)(x_2-1)+(x_1-1)(x_3-1)+(x_2-1)(x_3-1))$$

$E:$



The symmetric difference of graphs:



Problem

The description of clones $\mathrm{C}(f_E) \subseteq \mathrm{P}(\mathbb{Z}_8, +, \cdot)$ *generated by the operation*

$$f_E(x_1, \ldots, x_n) = x_1 \ldots x_n \sum_{ij \in E} (x_i - 1)(x_j - 1),$$

where $E \subseteq [n]^2$ *(the set of all pairs from* $\{1, 2, \ldots, n\}$*).*

Theorem

$f_D \in C(f_E)$ *if and only if* $D \in [E]$*.*

Theorem

$f_D \in C(f_{E_1}) \vee C(f_{E_2}) \ldots C(f_{E_m})$ *if and only if* $D \in [E_1, E_2, \ldots, E_m]$*.*

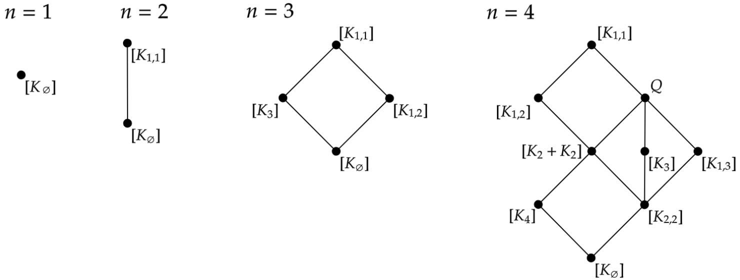
Problem

The description of all subsets of Graph(n) *that are closed under isomorphism and symmetric difference.*

The lattice of the graph classes according to the number of vertices n

Denotations

K_m - *a complete graph on m vertices and $n-m$ isolated vertices*
$K_m + K_{n-m}$ - *a disjoint union of K_m and K_{n-m}*
$K_{m,n-m}$ - *a complete bipartite graph (with $n-m-k$ isolated vertices)*



Quasi-complete graphs

Definition

*A graph $E = (V, E)$ is **quasi-complete** if any of the following conditions is satisfied for any pair of vertices $u, v \in V(E)$:*

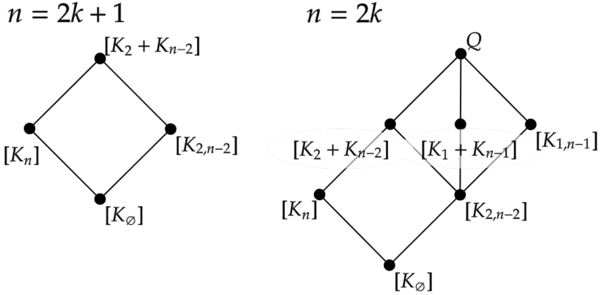
(a) $N(u) \cup \{u, v\} = N(v) \cup \{u, v\}$,

(b) $N(u) \cap N(v) = \emptyset$ *and* $N(u) \cup N(v) \cup \{u, v\} = V(E)$.

Lemma

A graph E of the order n is quasi-complete, iff it is isomorphic to the graph $K_{m,n-m}$ or $K_m + K_{n-m}$ for some $m \in \{0, 1, \ldots, \lfloor \frac{n}{2} \rfloor\}$.

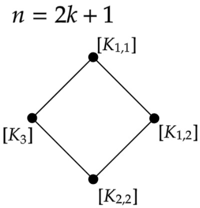
Quasi-complete graphs $n > 4$ $(k > 1)$

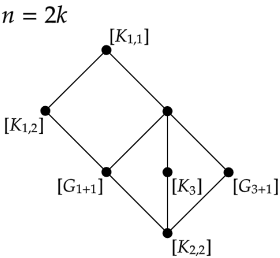


Non-quasi-complete graphs $n > 4$ $(k > 1)$

Lemma

If E is not quasi-complete, then $[E]$ contains $K_{2,2}$.





Duality

$E \perp D \Leftrightarrow (E' \cap D' \mid$ is even whenever $E' \cong E,\; D' \cong D)$

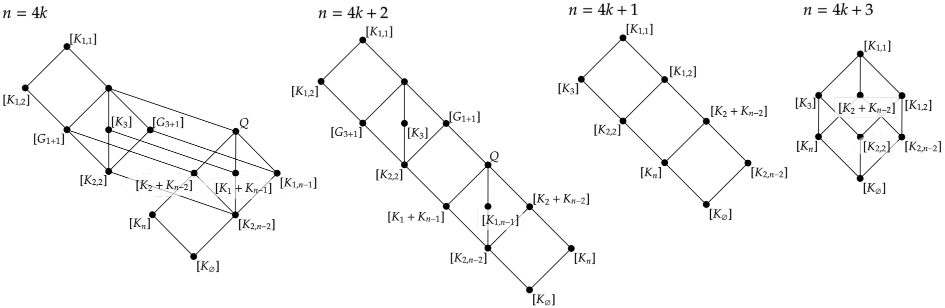
For $M \subseteq \mathsf{Graph}(n)$,

$$M^{\perp} = \{E \in \mathsf{Graph}(n) \mid E \perp D \text{ for every } D \in M\}$$

Lemma

$D \in [E]$ *if and only if* $E^{\perp} \subseteq D^{\perp}$.

If $n > 4$ and $k \in \mathbb{N}$, then



Next aims

1. Compare $C(f_E)$ for different n
2. Investigate operations:

$$x_1 x_2 \dots x_n \sum (x_i^2 - 1)(x_j - 1)$$

$$x_1 x_2 \dots x_n \sum (x_i^2 - 1)(x_j^2 - 1)$$

3. Compare $C(f_E)$ with previously investigated clones

Thank you for your attention :)