On the isomorphism problem for procountable groups and oligomorphic groups

Gianluca Paolini
University of Torino

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Outline

1. Introduction
2. Procountable groups
3. Oligomorphic groups
4. The epimorphism relation on countable groups

The structure of the talk

In this talk we present results from the following cluster of papers.

- S. Gao, A. Nies, G. Paolini. Procountable groups are not classifiable by countable structures. arXiv: 2512.12256.
- G. Paolini, S. Shelah. $\aleph_1$-free abelian non-Archimedean Polish groups. arXiv:2410.02485.
- A. Nies, G. Paolini. Oligomorphic groups, their automorphism groups, and the complexity of their isomorphism. arXiv:2410.02248.
- G. Paolini. The isomorphism problem for oligomorphic groups with weak elimination of imaginaries. Bull. Lond. Math. Soc. **56** (2024).
- G. Paolini, S. Shelah. Torsion-free abelian groups are faithfully Borel complete and pure embeddability is a complete analytic quasi-order, with Saharon Shelah. Sci. China Math. **68** (2025).
- S. Gao, F. Li, A. Nies, G. Paolini. The epimorphism relation among countable groups is a complete analytic quasi-order. arXiv:2511.06517.

Borel reducibility

Definition

Let X_1 and X_2 be two standard Borel spaces, and let also E_1 be an equivalence relation defined on X_1 and E_2 be an equivalence relation defined on X_2. E_1 is Borel reducible to E_2, denoted as $E_1 \leqslant_B E_2$, when there is a Borel map $F : X_1 \to X_2$ such that for every $x, y \in X_1$ we have:

$$xE_1y \Leftrightarrow F(x)E_2F(y).$$

F is Borel if for every Borel $Y_2 \subseteq X_2$ we have that $f^{-1}(Y_2) \subseteq X_1$ is Borel.

Complexity of subsets

Definition

Let X_1 and X_2 be two standard Borel spaces, and let also $Y_1 \subseteq X_1$ and $Y_2 \subseteq X_2$. We say that Y_1 is reducible to Y_2, denoted as $Y_1 \leqslant_R Y_2$, when there is a Borel map $\mathbf{B} : X_1 \to X_2$ such that for every $x \in X_1$ we have:

$$x \in Y_1 \Leftrightarrow \mathbf{B}(x) \in Y_2.$$

Definition

Let X_1 be a Borel space and $Y_1 \subseteq X_1$. We say that Y_1 is complete analytic (resp. complete co-analytic) if for every Borel space X_2 and analytic subset (resp. co-analytic subset) Y_2 of X_2 we have that $Y_2 \leqslant_R Y_1$.

Borel reducibility for countable structures

Fact

The set $\mathcal{K}^L_\omega$ of structures with domain ω in a given countable language L is endowed with a standard Borel space structure $(\mathcal{K}^L_\omega, \mathcal{B})$. Every Borel subset of this space $(\mathcal{K}^L_\omega, \mathcal{B})$ is naturally endowed with the Borel structure induced by $(\mathcal{K}^L_\omega, \mathcal{B})$.

Examples, $L = \{e, \cdot, ()^{-1}\}$, and we let $\mathcal{K}'$ to be one of the following:

(a) the set of elements of $\mathcal{K}^L_\omega$ which are groups;

(b) the set of elements of $\mathcal{K}^L_\omega$ which are abelian groups;

(c) the set of elements of $\mathcal{K}^L_\omega$ which are torsion-free abelian groups;

(d) the set of elements of $\mathcal{K}^L_\omega$ which are *n*-nilpotent groups ($n < \omega$).

Complexity of isomorphism on countable structures

The two previous slides allow us to ask questions like the following.

Question

Is $\cong$ on countable groups as complicated as $\cong$ on countable graphs?

Question

Is $\cong$ on countable **abelian** groups as complicated as $\cong$ on countable graphs?

The second question had been open since 1989, it was settled in:

- G. Paolini, S. Shelah. Torsion-free abelian groups are Borel complete. Ann. of Math. (2) **199** (2024), no. 3, 1177-1224.

Closed subgroups of $\mathrm{Sym}(\mathbb{N})$

A topological group is called **non-archimedean** if it has a neighbourhood basis of the identity consisting of open subgroups. Such groups are totally disconnected. A topological group is said to be **Polish** if it is separable (there is a countable dense subset) and completely metrizable.

Non-archimedean Polish groups are precisely the **closed subgroups of** $\mathrm{Sym}(\mathbb{N})$, where (as usual) the basic open subgroups of $\mathrm{Sym}(\mathbb{N})$ are the pointwise stabilizers of finite subsets of $\mathbb{N}$.

We denote by **CSg**$(\mathrm{Sym}(\mathbb{N}))$ the space of closed subgroups of $\mathrm{Sym}(\mathbb{N})$.

Fact

CSg$(\mathrm{Sym}(\mathbb{N}))$ admits a Borel space structure.

The program from KNT

- A. S. Kechris, A. Nies and K. Tent. The complexity of topological group isomorphism. J. Symb. Log. **83** (2018), no. 3, 1190–1203.

In the paper above the following programme was initiated:

- For natural classes of closed subgroups of $\mathrm{Sym}(\mathbb{N})$, determine whether they are Borel.
- If a class $\mathcal{C}$ is Borel, study the Borel complexity of topological isomorphism on $\mathcal{C}$.

Main Question

What is the complexity of topological isomorphism on $\mathbf{CSg}(\mathrm{Sym}(\mathbb{N}))$?

State of the art # 1

- The compact non-archimedean Polish groups, or equivalently, the **profinite groups**, have isomorphism which is Borel bireducible with **graph isomorphism** on countable graphs (abbreviated as GI).

- One can also consider the subclass of **abelian profinite groups**. This is strictly above the smooth equivalence relation (i.e., $=_{\mathbb{R}}$) and strictly below GI. In particular, it is not above the equivalence relation E_0.

State of the art # 2

- Several results on **oligomorphic groups**, see later.
- The isomorphism relation of **extremely amenable** non-archimedean Polish groups was studied by Etedadialiabadi, S. Gao and F. Li, who also showed that this class is Borel. More recent work of R. Li (unpublished) shows that this relation is Borel bi-reducible with GI.
- Paolini and Shelah proved that the following properties are complete co-analytic subsets of the space of closed **abelian subgroups** of $\mathrm{Sym}(\mathbb{N})$: separability, torsionlessness, $\aleph_1$-freeness and $\mathbb{Z}$-homogeneity.

An abelian group is $\aleph_1$-freeness is every countable subgroup is free abelian.

Outline

Procountable groups

Definition

A topological group G is called **procountable** if G is topologically isomorphic to the inverse limit of an inverse system $(G_n, p_n)_{n\in\mathbb{N}}$ of countable groups and onto homomorphisms $p_n\colon G_{n+1} \to G_n$, where the topology of G is the subspace topology given by the product $\prod_n G_n$, where each G_n is equipped with the discrete topology.

Proposition

The procountable closed subgroups of $\mathrm{Sym}(\mathbb{N})$ form a Borel class $\mathcal{C}$.

Discrete groups

Since each countable discrete group is procountable, we know that GI is Borel reducible to the isomorphism of procountable groups through any of:

- The isomorphism of countable locally finite groups that are nilpotent of class 2 and exponent 3 is Borel bireducible with GI (Mekler).
- The isomorphism of countable torsion-free abelian groups is Borel bireducible with GI (Paolini & Shelah).

In particular, we get that the isomorphism relation of procountable groups is above GI and so a complete analytic set and hence non-Borel.

Isomorphism on structures as an orbit equivalence relation

Let $\mathcal{K}_\omega$ be a class of structures in the same language with domain ω.

The Polish group $\mathrm{Sym}(\mathbb{N})$ acts on $\mathcal{K}_\omega$ in the obvious way: given $g \in \mathrm{Sym}(\mathbb{N})$ we let $g.M = N$ if and only if $g : N \cong M$.

Thus, isomorphism on $\mathcal{K}_\omega$ can be seen as an orbit equivalence relation induced by a Borel action of a Polish group on $\mathcal{K}_\omega$.

The relation ℓ_∞

Recall that E_1 is the Borel equivalence relation of eventual equality between sequences of real numbers. By a celebrated theorem of Kechris and Louveau, E_1 is **not Borel reducible to any orbit equivalence relation induced by a Borel action of a Polish group**. It follows that any equivalence relation which Borel reduces E_1 has the same property.

Recall that ℓ_∞ is the Borel equivalence relation on $\mathbb{R}^\mathbb{N}$ given by

$$(x_n)\ell_\infty(y_n) \iff \exists M \in \mathbb{R}\ \forall n \in \mathbb{N}\ |x_n - y_n| < M.$$

By a theorem of Rosendal, E_1 is Borel reducible to ℓ_∞.

Conclusion

The equivalence relation ℓ_∞ is not Borel reducible to any orbit equivalence relation induced by a Borel action of a Polish group.

The main theorem

Theorem (Gao, Nies, Paolini)

The equivalence relation ℓ_∞ is Borel reducible to the topological isomorphism relation on procountable groups. Consequently, the topological isomorphism relation on procountable groups is not Borel reducible to any orbit equivalence relation induced by a Borel action of a Polish group. In particular, it is not Borel reducible to GI.

It is known that being Borel reducible to GI is equivalent to being below the orbit equivalence relation of a Borel action of $\mathrm{Sym}(\mathbb{N})$, which is also equivalent to being **classifiable by countable structures (CCS)**.

Corollary (Gao, Nies, Paolini)

Topological isomorphism on procountable groups is not CCS.

The two steps of the proof

Definition

Let X and Y be Polish metric spaces. A bijection Φ between X and Y is called a **uniform homeomorphism** if both Φ and its inverse are uniformly continuous. X and Y are said to be **uniformly homeomorphic** (denoted $U \cong_u T$) if there is a uniform homeomorphism between X and Y.

- We first show that ℓ_∞ is Borel reducible to the relation of uniform homeomorphism between path spaces $[T]$, where T is a pruned subtree of $\mathbb{N}^{<\mathbb{N}}$ and $[T]$ carries the standard ultrametric.
- We reduce this uniform homeomorphism relation to the topological isomorphism relation on the space of procountable groups.

A few details on the second step

For a countably infinite set V, we define the free product of cyclic groups of size 2 with generators in V by (known as free Coxeter group of rank $\aleph_0$):

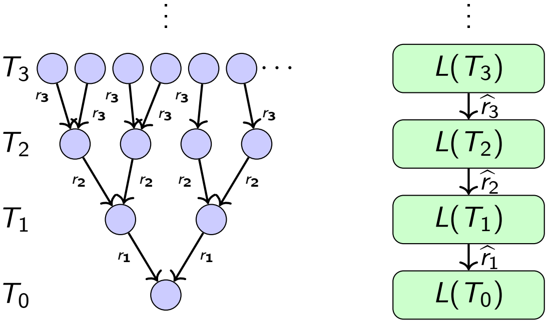
$$L(V) = \langle s \mid s^2 = e \ \ (s \in V) \rangle.$$

If $r\colon V \to W$, then we have a map $\widehat{r}\colon L(V) \to L(W)$ induced by $v \mapsto r(v)$.

Given a pruned tree T, let $T_n = \{s \in T : |s| = n\}$ be the n-th level of the tree T, and let $r_n\colon T_{n+1} \to T_n$ be the predecessor map. Let

$$G_T = \varprojlim_{n \in \mathbb{N}} (L(T_{n+1}), \widehat{r}_{n+1}).$$

The construction



$$G_T = \varprojlim_{n \in \mathbb{N}} (L(T_{n+1}), \widehat{r}_{n+1})$$

A few details on the second step (cont.)

Theorem (Gao, Nies, Paolini)

Let T and U be pruned trees on $\mathbb{N}$. Then $[T] \cong_u [U]$ (uniformly homeomorphic) if and only if G_T is topologically isomorphic to G_U.

In order to do this we rely on an analysis of isomorphisms of **inverse systems** of countable groups, in the sense of **"Shape Theory"**, cf.:

- S. Mardešić and J. Segal. Shape Theory: The Inverse System Approach. 1982.

An improved version of the main theorem

Theorem

Fix an odd prime p. The equivalence relation ℓ_∞ is Borel reducible to the topological isomorphism relation on 2-nilpotent procountable groups of exponent p. Consequently, the topological isomorphism relation on such procountable groups is not Borel reducible to any orbit equivalence relation induced by a Borel action of a Polish group. In particular, it is not Borel reducible to GI.

Open questions

Question

Is the relation of topological isomorphism on procountable groups a complete analytic equivalence relation (i.e., any analytic equivalence relation is Borel reducible to it)?

Question

Are the abelian procountable groups classifiable by countable structures?

By the Borel completeness of countable torsion-free abelian groups, GI is a lower bound for topological isomorphism on procountable abelian groups.

Outline

Oligomorphic groups

A **permutation group** on a countably infinite set X is a subgroup of the group $\mathrm{Sym}(X)$ of permutations of X. Such a group is called **oligomorphic** if $\forall n \in \omega$, the canonical action of G on X^n has only finitely many orbits.

Oligomorphic groups are precisely the automorphisms groups of ω-**categorical** countably infinite structures, i.e., those countable models M such that $\mathrm{Th}(M)$ has only one model in $\aleph_0$ (up to isomorphism).

The set of oligomorphic subgroups of $\mathrm{Sym}(\mathbb{N})$ (denoted as **Oligo**$(\mathbb{N})$) form a Borel subset of **CSg**$(\mathrm{Sym}(\mathbb{N}))$. The main question here is:

Main Question

What is the complexity of topological isomorphism on **Oligo**$(\mathbb{N})$?

We sometime say "isomorphism" instead of "topological isomorphism".

- A. Nies, P. Schlicht, and K. Tent. Coarse groups, and the isomorphism problem for oligomorphic groups. J. Math. Log. **22** (2022).

In the paper above it was shown that topological isomorphism on **Oligo**($\mathbb{N}$) is Borel reducible to a Borel equivalence relation with all classes countable.

Main Question

Is topological isomorphism on **Oligo**($\mathbb{N}$) smooth (reducible to $=_{\mathbb{R}}$)?

N.B. it has long been known that $=_{\mathbb{R}}$ is a lower bound.

A few definitions from model theory

Definition

We say that a structure M has no algebraicity if for every $A \subseteq M$ we have that $\mathrm{acl}_M(A) = A$ (where $\mathrm{acl}_M(A)$ is the model theoretic algebraic closure).

Definition

Let M be ω-categorical, then we say that M has **weak elimination of imaginaries (WEI)** if for every finite algebraically closed sets $A, B \subseteq M$,

$$G_{(A \cap B)} = \langle G_{(A)} \cup G_{(B)} \rangle_G$$

where $G = \mathrm{Aut}(M)$ and $G_{(C)}$ denotes the pointwise stabilizer of $C \subseteq M$.

With some abuse of notation we say that an oligomorphic group G has weak elimination of imaginaries if $G \cong \mathrm{Aut}(M)$, with M with WEI.

Main results on isomorphism on oligomorphic groups

Theorem (Paolini)

Isomorphism on oligomorphic groups with **WEI** is smooth.

Theorem (Nies, Paolini)

Isomorphism on oligomorphic groups with **no algebraicity** is smooth.

For other partial results in this vein see the paper:

- A. Nies and G. Paolini. Oligomorphic groups, their automorphism groups, and the complexity of their isomorphism. arXiv:2410.02248.

Let G be a Polish group. We denote by $\mathrm{Aut}(G)$ the group of **topological automorphisms** of G (group automorphisms which are homeomorphisms).

Proposition (Nies, Paolini)

Let $G \leqslant \mathrm{Sym}(\mathbb{N})$ be oligomorphic. Then $\mathrm{Aut}(G)$ has a unique Polish group topology that makes the action $\mathrm{Aut}(G) \times G \to G$ continuous.

Nies and Schlicht strenghtened the above to **Roelcke precompact groups**.

The first result on outer automorphisms

For G a Polish group, we define $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$ (recall that $\mathrm{Aut}(G)$ denotes the group of **topological** automorphisms of G).

A topological group is said to be **t.d.l.c.** if it is totally disconnected and locally compact (the identity element has a compact neighborhood).

Theorem (Nies, Paolini)

Let $G \leqslant \mathrm{Sym}(\mathbb{N})$ be oligomorphic. Let N_G be its normaliser in $\mathrm{Sym}(\mathbb{N})$.

(a) $\mathrm{Inn}(G)$ is closed in $\mathrm{Aut}(G)$.

(b) The Polish group N_G/G is profinite.

(c) The Polish group $\mathrm{Out}(G)$ is t.d.l.c..

The second result on outer automorphisms

$\mathrm{Aut}(G)$ is quasi-oligomorphic if it is isomorphic to an oligomorphic group.

Theorem (Nies, Paolini)

Let $\mathcal{C}$ be either

(1) the class of oligomorphic groups with no algebraicity, or

(2) the class of oligomorphic groups with WEI.

Then

(a) The topological isomorphism relation on $\mathcal{C}$ is smooth.

(b) If $G \in \mathcal{C}$ then $\mathrm{Out}(G)$ is profinite, and $\mathrm{Aut}(G)$ is quasi-oligomorphic.

The main open question on outer automorphisms

Main Question

Is there an oligomorphic group $G \leqslant \mathrm{Sym}(\mathbb{N})$ such that the Polish group $\mathrm{Out}(G)$ is not profinite (recall that this is always t.d.l.c.)?

In my view, our inability to answer this question is the primary obstacle to determining whether topological isomorphism on $\mathbf{Oligo}(\mathbb{N})$ is smooth.

Outline

Complete analytic quasi-orders on countable groups

Definition

A quasi-order is said to be complete analytic if any analytic quasi-order is Borel reducible to it.

Fact (Louveau, Rosendal)

The quasi-order on countable graphs "there exists an homomorphism" is a complete analytic quasi-order.

Fact (Calderoni, Thomas)

The embeddability relation between countable torsion-free abelian groups is a complete analytic quasi-order.

The case of torsion-free abelian groups

For $A \leqslant B$ abelian, A is pure in B if for every $0 < n < \omega$ and $a \in A$

$$B \models n|a \;\Rightarrow\; A \models n|a.$$

Theorem (Paolini, Shelah)

The pure embeddability relation on countable torsion-free abelian groups is a complete analytic quasi-order.

Theorem (Paolini, Shelah)

Elementary embeddability (equiv., pure embeddability) between countable models of $\mathrm{Th}(\mathbb{Z}^{(\omega)})$ is a complete analytic quasi-order.

- G. Paolini and S. Shelah. Torsion-free abelian groups are faithfully Borel complete and pure embeddability is a complete analytic quasi-order, with Saharon Shelah. Sci. China Math. **68** (2025).

What about epimorphisms?

Fact (Camerlo)

The quasi-order on countable graphs "there exists an epimorphism (i.e., a surjective homomorphism)" is a complete analytic quasi-order.

Question

What about the relation of epimorphism between countable **groups**?

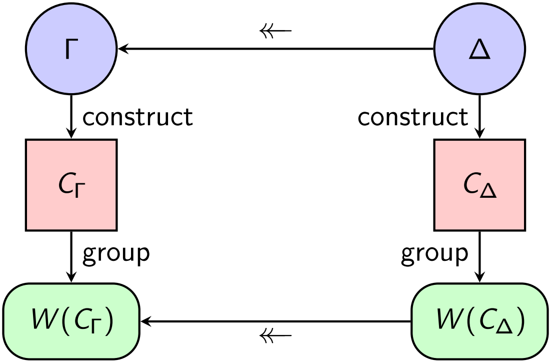
The main theorem on the complexity of epimorphism

For countable groups A, B, we write $A \twoheadleftarrow B$ if there is a surjective homomorphism from B onto A; equivalently, A is $\cong$ to a quotient of B.

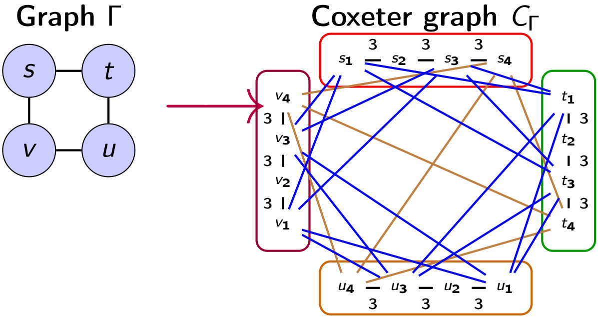
Theorem (Gao, Li, Nies, Paolini)

The relation $\{\langle A, B\rangle \colon A \twoheadleftarrow B\}$ on the Borel space of countable groups is a complete analytic quasi-order. Thus, the relation of bi-epimorphism on the same Borel space is a complete analytic equivalence relation.

The group theoretic construction



$\Gamma \twoheadleftarrow \Delta$ if and only if $W(C_\Gamma) \twoheadleftarrow W(C_\Delta)$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Thank you!

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- S. Gao, A. Nies, G. Paolini. Procountable groups are not classifiable by countable structures. arXiv: 2512.12256.
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