Morphisms of coherent configurations

AAA 108

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(joint work with A. Jenčová and G. Jenča)

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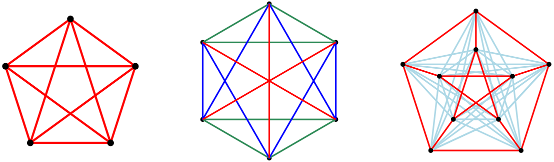
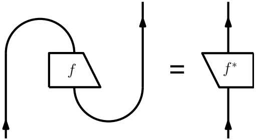
Coherent configurations are combinatorial structures encoding highly symmetric edge-colourings of complete digraphs.



We (me and my coauthors) are interested in dagger-compact categories used to describe quantum processes $(\mathbf{Rel}, \mathbf{Hilb})$.



Coherent configuration $\rightsquigarrow$ †-Frobenius monoid internal to $\mathbf{Rel}$, $\mathbf{Hilb}$.

Jenča, Jenčová, and Lachman 2025

Warning: some notation introduced on the next slide is not standard.

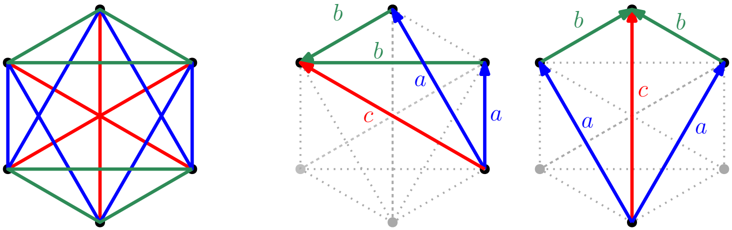
Definition of coherent configurations (association schemes)

A *coherent configuration* is an ordered pair (X, S), where X is a finite set of vertices and S (a set of colors) is a partition of $X \times X$, such that:

(A1) id_X is a union of colors in S (if $\mathrm{id}_X \in S$, it is an *association scheme*),

(A0) for each $a \in S$, its *inverse* $a^{-1} = \{(x', x) \mid (x, x') \in a\}$ is also in S,

(A8) *coherence condition* holds.



Structure constants ∇: e.g., $\nabla^{c}_{a,b} = 2$

Valency: $\|a\|$ = the number of arrows of color a with the same source. E.g., $\|a\| = 2$, $\|b\| = 2$, $\|c\| = 1$, $\|\bullet\| = 1$ (always).

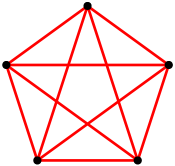
Examples

Let G be a group having an action on a finite set X. Set

$$\begin{aligned} \text{vertices} &= X, \\ \text{colors} &= \text{orbits of the action of } G \rightharpoonup X \times X \to X \times X. \end{aligned}$$

Trivial scheme: action of S_X on X

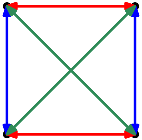
(S_X, X)



$\|a\| \in \{1, n-1\}$

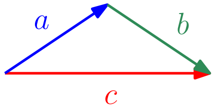
Thin scheme: Cayley action of G on itself

(G, G)

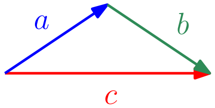


$\|a\| = 1$

(co-)Algebraic (co-)feeling

"$\nabla\colon (a, b) \mapsto c$" iff 

(co-)Algebraic (co-)feeling

"$\nabla\colon ({\color{blue}a}, {\color{green}b}) \mapsto {\color{red}c}$" iff 

We have a monoid $(\mathbb{R}^S; \nabla, e)$ (internal to $\mathbf{Hilb}$):

$$\nabla\colon \mathbb{R}^S \otimes \mathbb{R}^S \to \mathbb{R}^S \qquad |a\rangle \otimes |b\rangle \mapsto \sum_c \nabla^c_{a,b} |c\rangle$$

$$e\colon \mathbb{R}^1 \to \mathbb{R}^S \qquad 1 \mapsto \sum_{a \subseteq \mathrm{id}_X} |a\rangle$$

Associativity: $\nabla \circ (\nabla \otimes \mathrm{id}) = \nabla \circ (\mathrm{id} \otimes \nabla)$

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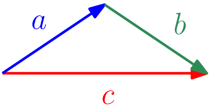
"$\nabla\colon (a, b) \mapsto c$" iff 

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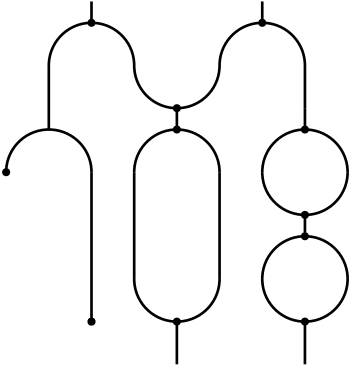
We also have a comonoid $(\mathbb{R}^S; \nabla^\dagger, e^\dagger)$:

$$\nabla^\dagger\colon \mathbb{R}^S \to \mathbb{R}^S \otimes \mathbb{R}^S$$
$$e^\dagger\colon \mathbb{R}^S \to \mathbb{R}^1.$$

Let us switch to a more decorative version

$$\nabla^c_{a,b} \rightsquigarrow \boldsymbol{\nabla}^c_{a,b} = \sqrt{\frac{\|c\|}{\|a\|\|b\|}}\nabla^c_{a,b}$$

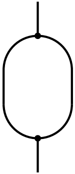
Theorem 1 (Jenča, Jenčová, and Lachman 2025).
$(\mathbb{R}^S; \boldsymbol{\nabla}, e, \boldsymbol{\nabla}^\dagger, e^\dagger)$ *is a dagger-Frobenius monoid in* $\mathbf{Hilb}$.



Any "connected" term composed from $\boldsymbol{\nabla}$, e, $\boldsymbol{\nabla}^\dagger$, and $e^\dagger$ depends only on the number of inputs, outputs, and "holes" (see Heunen and Vicary 2020).

Spectrum

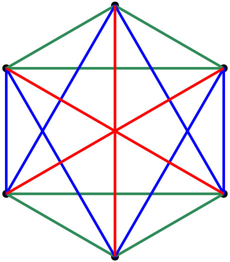
For $(\mathbb{R}^S; \nabla, e, \nabla^\dagger, e^\dagger)$ the following operator



$$= \quad \nabla \circ \nabla^\dagger \colon \mathbb{R}^S \to \mathbb{R}^S$$

is symmetric (self-adjoint), hence has a real, well-behaved spectrum.

Example:



$$\nabla \circ \nabla^\dagger = \begin{bmatrix} 4 & \sqrt{2} & 0 & 0 \\ \sqrt{2} & 5 & 0 & 0 \\ 0 & 0 & 4 & \sqrt{2} \\ 0 & 0 & \sqrt{2} & 5 \end{bmatrix}, \quad \sigma = \{3, 6\}$$

Right notion of morphisms?

Definition 2 (French 2013).
A pair of maps $\phi = (\phi_0, \phi_1)\colon (X,S) \to (Y,T)$ is called *admissible* morphism if

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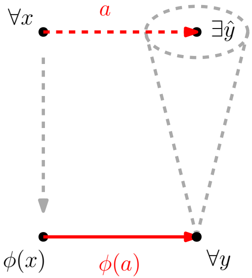
(i) $(x, x') \in a \implies (\phi_0(x), \phi_0(x')) \in \phi_1(a)$ (colour-preserving property)

(ii) For each $(\phi_0(x), y) \in \phi_1(a)$, there exists $\hat{y} \in X$, s.t. $(x, \hat{y}) \in a$. (colour-lifting property)



There is a fully faithful functor $\mathbf{FinGrp} \to \mathbf{AS}$.

Theorem 3 (Jenča, Jenčová, and Lachman 2025).
Let $\phi\colon (X,S) \to (Y,T)$ be a surjective admissible morphism of schemes such that

$$\begin{array}{ll} \textit{Spectrum of } (X,S) & \lambda = (\lambda_1, \dots, \lambda_n) \\ \textit{Spectrum of } (Y,T) & \rho = (\rho_1, \dots, \rho_m) \end{array}$$

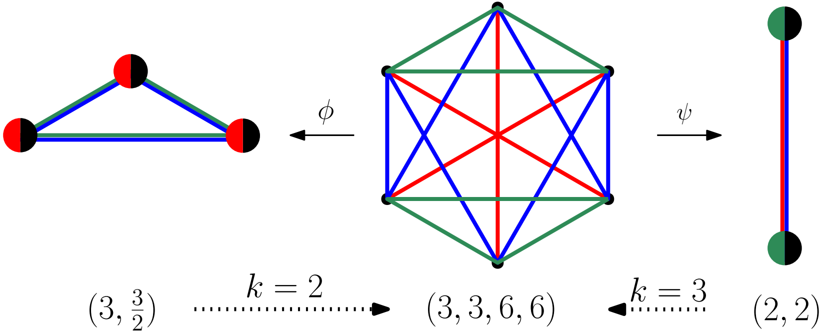
Then there is an integer k, such that

$$k \cdot \rho = (k \cdot \rho_1, \dots, k \cdot \rho_m)$$

is a subvector of λ (i.e., each $k \cdot \rho_i$ is also an eigenvalue of (X,S)).

Example: For a regular scheme corresponding to a group G we have

$$\nabla \circ \nabla^\dagger = \begin{bmatrix} |G| & 0 & \cdots & 0 \\ 0 & |G| & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & |G| \end{bmatrix}, \qquad \lambda = (|G|, \dots, |G|). \tag{1}$$


ϕ
ψ
$(3, \frac{3}{2})$
$k = 2$
$(3, 3, 6, 6)$
$k = 3$
$(2, 2)$

Coherent lifting property

Definition 4.

A *rainbow* is an ordered pair (X, S) satisfying (A1),(A0),~~(A8)~~.

A *morphism of rainbows* is a color-preserving maps between the vertex sets.

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Let $\phi\colon (X, S) \to (Y, T)$, $\pi\colon (Z, R) \to (W, U)$ be two rainbow morphisms. We say ϕ has the left coherent lifting property with respect to π

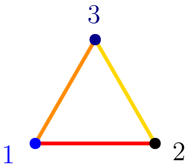
$$\phi \sharp \pi$$

if: Given any α, β, and $\lambda_1\colon T \to R$ fitting into

$$\begin{array}{ccc} (X, S) & \xrightarrow{\alpha} & (Z, R) \\ {\scriptstyle\phi}\downarrow & \overset{(\lambda_0,\lambda_1)}{\nearrow} & \downarrow{\scriptstyle\pi} \\ (Y, T) & \xrightarrow[\beta]{} & (W, U) \end{array} \tag{2}$$

the number $L(\phi, \pi, \lambda_1)$ of λ_0 making the diagram (2) commuting depends only on λ_1 (not on α and β).

Denote $\mathcal{D}_n$ a rainbow, with vertices $\{1, \ldots, n\}$ and each edge having a unique color.



$$(\mathcal{D}_1 \hookrightarrow \mathcal{D}_2) \sharp \phi \quad \text{iff} \quad \phi \text{ is an admissible morphism}$$

$$(\mathcal{D}_2 \hookrightarrow \mathcal{D}_3) \sharp (! \colon (X, S) \to \bullet) \quad \text{iff} \quad (X, S) \text{ is a coherent configuration}$$

Lemma 5.
Let ϕ be a morphism of rainbows. Then

$$\{\pi \mid \phi \sharp \pi\}$$

is closed under compositions, pullbacks, and "certain" retracts.

References I

French, C. (2013). "Functors from association schemes". In: *Journal of Combinatorial Theory, Series A* 120.6, pp. 1141–1165. DOI: https://doi.org/10.1016/j.jcta.2013.03.001.

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THANK YOU FOR YOUR ATTENTION