Multi-sorted partial algebras of tree languages of a fixed variable

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Outline of presentation

1. Introduction and preliminaries
   - Terms
   - Tree languages
   - Question
2. Results
3. Open problems

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Terms

- Let I be a nonempty indexed set and $(f_i)_{i\in I}$ be a sequence of operation symbols.
- The sequence $\tau := (n_i)_{i\in I}$ is said to be a *type* where $n_i \in \mathbb{N}$ is a natural number of the arity of f_i .
- We denote by $X_n := \{x_1, \ldots, x_n\}$ a finite set of *alphabet* and its elements are called *variables* and $X := \{x_1, x_2, \ldots\}$.
- The set $W_\tau(X_n)$ of all *n-ary terms of type τ* is the smallest set which contains X_n inductively defined by
  - $X_n \subset W_\tau(X_n)$ and
  - If $t_1, \ldots, t_{n_i} \in W_\tau(X_n)$ and f_i is an operation symbol of the arity n_i implies $f_i(t_1, \ldots, t_{n_i}) \in W_\tau(X_n)$.

Terms of a fixed variable

- An idempotent algebra is an algebra of type (2) satisfying the identity $f(x,x) \approx x$.
- A subclass of terms called terms of a fixed variable of type τ was introduced in 2020.
- Let us consider the following example. In the set $W^{fv}_{(2)}(X_3)$ with respect to a binary operation symbol f, the elements $x_1,\ x_2,\ x_3,\ f(x_1,x_1),\ f(x_2,x_2),\ f(f(x_3,x_3),x_3)$ are examples of members of this set. However, the terms $f(x_2,x_1)$ and $f(x_1,f(x_2,x_2))$ are not terms of a fixed variable, i.e., they belong to $W_{(2)}(X_3) \setminus W^{fv}_{(2)}(X_3)$.

Outline of presentation

Tree languages

- On the set $W_\tau(X_n)$, one consider its the power set, i.e., $\mathcal{P}(W_\tau(X_n))$. In this matter, each element in $\mathcal{P}(W_\tau(X_n))$ is called a tree language. Actually, it generalizes formal languages, that is, sets of words over a given alphabet.
- The operation

$$\widehat{S}^n_m : \mathcal{P}(W_\tau(X_n)) \times \mathcal{P}(W_\tau(X_m))^n \to \mathcal{P}(W_\tau(X_m))$$

for $m, n \in \mathbb{N}$ is defined as follows:

(1) If $B = \{x_j\}$ for $1 \leq j \leq n$, then $\widehat{S}^n_m\,(B, B_1, \ldots, B_n) := B_j$.

(2) If $B = \{f_i(s_1, \ldots, s_{n_i})\}$ and assume that $\widehat{S}^n_m\,(\{s_j\}, B_1, \ldots, B_n)$ for $1 \leq j \leq n$ are already defined, then $\widehat{S}^n_m\,(B, B_1, \ldots, B_n) :=$ $\{f_i(u_1, \ldots, u_{n_i}) \mid u_j \in \hat{S}^n_m\,(\{s_j\}, B_1, \ldots, B_n)\}$.

(3) If $|B| > 1$, then $\widehat{S}^n_m\,(B, B_1, \ldots, B_n) := \bigcup_{b \in B} \widehat{S}^n_m\,(\{b\}, B_1, \ldots, B_n)$.

(4) If one of the sets $B, B_1, \ldots, B_n$ is empty, then $\widehat{S}^n_m\,(B, B_1, \ldots, B_n) := \emptyset$.

Clones

- The *variety of all abstract clones*, denoted by K_0, is a family of $\mathbb{N}$-sorted algebras satisfying the following three identities:
  - (C1) $\tilde{S}^n_m(\tilde{S}^p_n(\tilde{Z},\tilde{Y}_1,\ldots,\tilde{Y}_p),\tilde{X}_1,\ldots,\tilde{X}_n) \approx \tilde{S}^p_m(\tilde{Z},\tilde{S}^n_m(\tilde{Y}_1,\tilde{X}_1,\ldots,\tilde{X}_n),\ldots,\tilde{S}^n_m(\tilde{Y}_p,\tilde{X}_1,\ldots,\tilde{X}_n)), m,n,p \in \mathbb{N}$;
  - (C2) $\tilde{S}^n_m(\lambda_j,\tilde{X}_1,\ldots,\tilde{X}_n) \approx \tilde{X}_j, n,m \in \mathbb{N}, 1 \leq j \leq n$;
  - (C3) $\tilde{S}^n_n(\tilde{Y},\lambda_1,\ldots,\lambda_n) \approx \tilde{Y}, n \in \mathbb{N}$;

    where $\tilde{S}^n_m$, $\tilde{S}^p_n$, $\tilde{S}^p_m$, $\tilde{S}^n_n$ are operation symbols, $\tilde{Z},\tilde{Y}_1,\ldots,\tilde{Y}_p,\tilde{X}_1,\ldots,\tilde{X}_n,\tilde{Y}$ are variables for terms, and λ_j are symbols for variables.
- In general, (C1) is said to be the *superassociative law* since it generalizes the associative law.
- Each member of the variety K_0 is called an abstract clone.
- Every concrete clone can be considered as an abstract clone.
- For the converse, every abstract clone is isomorphic to some concrete clone.

Outline of presentation

Question

- $(\mathcal{P}(W_\tau^{fv}(X_n))$
- Each element of $\mathcal{P}(W_\tau^{fv}(X_n))$ is called a tree language of a fixed variable.
- Let $\tau = (2)$ be a type with a single binary operation symbol f. We now list several examples of tree languages of a fixed variable of type (2), that is, elements of $\mathcal{P}(W_{(2)}^{fv}(X_3))$:

$$\{x_1, x_2, f(x_1, x_1)\}, \quad \{x_1, x_2, x_3, f(f(x_1, x_1), f(x_1, x_1))\}.$$

- For a tree language A, the symbol var(A) denotes the set of all variables that appear in A. Consider the following examples: var$(\{x_1, x_2, f(x_1, x_1)\}) = \{x_1, x_2\}$ and var$(\{x_3, f(x_2, x_2), f(x_4, x_4)\}) = \{x_2, x_3, x_4\}$.
- $|$var$(A)|$ we mean the number of elements in the set var(A).

Question

- Unfortunately, in general, each set in the family $(\mathcal{P}(W_{\tau}^{fv}(X_n)))_{n\in\mathbb{N}}$ is not closed under a non-deterministic superposition.
- Let $\tau = (2)$ be a type with a binary operation symbol f, and let $X_2 = \{x_1, x_2\}$ and $X_3 = \{x_1, x_2, x_3\}$ be sets of variables. On the sets $\mathcal{P}(W_{(2)}^{fv}(X_2))$ and $\mathcal{P}(W_{(2)}^{fv}(X_3))$, consider the following subsets:

$$A = \{f(x_1, x_1)\}, \quad B_1 = \{x_3, f(x_2, x_2)\}, \quad B_2 = \{x_1, f(x_3, x_3)\}.$$

Then

$$\begin{gathered}
\widehat{S}_3^2(A, B_1, B_2,) = S_3^2(\{f(x_1, x_1)\}, \{x_3, f(x_2, x_2)\}, \{x_1, f(x_3, x_3)\}) \\
= \{f(u_1, u_2) \mid u_1 \in \widehat{S}_3^2(\{x_1\}, \{x_3, f(x_2, x_2)\}, \{x_1, f(x_3, x_3)\}, \\
u_2 \in \widehat{S}_3^2(\{x_1\}, \{x_3, f(x_2, x_2)\}, \{x_1, f(x_3, x_3)\})\} \\
= \{f(u_1, u_2) \mid u_1 \in \{x_3, f(x_2, x_2)\}, u_2 \in \{x_3, f(x_2, x_2)\})\} \\
=\{f(x_3, x_3), f(x_3, f(x_2, x_2)), f(f(x_2, x_2), x_3), f(f(x_2, x_2), f(x_2, x_2))\} \\
\nsubseteq W_{(2)}^{fv}(X_3).
\end{gathered}$$

Results

First, it is easy to see that $\widehat{S}^n_m(A, B_1, \ldots, B_n) \in \mathcal{P}(W^{fv}_\tau(X_m))$ if $A = \emptyset$ or $B_j = \emptyset$ for some $j \in \{1, \ldots, n\}$.

Lemma

Let $u, v \in W^{fv}_\tau(X_n)$ with $|\text{var}(u)| = 1 = |\text{var}(v)|$. Let $B_1, \ldots, B_n$ be subsets of $W^{fv}_\tau(X_m)$ with $|\text{var}(B_1)| = \cdots = |\text{var}(B_n)| = 1$. Then

$$|\text{var}(\widehat{S}^n_m(\{u\}, B_1, \ldots, B_n))| = 1 = |\text{var}(\widehat{S}^n_m(\{v\}, B_1, \ldots, B_n))|.$$

Lemma

Let $B_1, \ldots, B_n \subseteq W^{fv}_\tau(X_m)$ with $|\text{var}(B_1)| = \cdots = |\text{var}(B_n)| = 1$. If $\emptyset \neq A \in \mathcal{P}(W^{fv}_\tau(X_n))$ and $|\text{var}(A)| = 1$, then

$$|\text{var}(\widehat{S}^n_m(A, B_1, \ldots, B_n))| = 1.$$

Results

Lemma

If $A \subseteq W_\tau^{fv}(X_n), B_1, \ldots, B_n \subseteq W_\tau^{fv}(X_m)$ *and if* $|\text{var}(B_1)| = \cdots = |\text{var}(B_n)| = 1$ *or if* $A = \emptyset$*, then*

$$\widehat{S}_m^n(A, B_1, \ldots, B_n) \in \mathcal{P}(W_\tau^{fv}(X_m)).$$

$$\overline{S}_m^n(A, B_1, \ldots, B_n) := \begin{cases} \widehat{S}_m^n(A, B_1, \ldots, B_n), & \text{if } B_j = \emptyset \text{ for some } j \in \{1, \ldots, n\} \\ & \text{or } A = \emptyset \text{ or } |\text{var}(B_j)| = 1 \\ & \text{for every } j = 1, \ldots, n, \\ \text{not defined}, & \text{otherwise.} \end{cases}$$

Results

Theorem

The partial power clone of a fixed variable

$$PC^{fv}(\tau) := ((\mathcal{P}(W_\tau^{fv}(X_n)))_{n\in\mathbb{N}}, (\overline{S}_m^n)_{n,m\in\mathbb{N}}, (\{x_i\})_{i\leq n, n\in\mathbb{N}})$$

belongs to the variety of abstract partial clones.

Results

Theorem

Let $A, A' \in \mathcal{P}(W_\tau^{fv}(X_n)), B_j, B'_j \in \mathcal{P}(W_\tau^{fv}(X_m))$ *and* $|\text{var}(B_j)| = 1$ *for* $j = 1, \ldots, n$. *Then the following statements hold:*

(1) *If* $A' \subseteq A$, *then* $\overline{S}^n_m(A', B_1, \ldots, B_n) \subseteq \overline{S}^n_m(A, B_1, \ldots, B_n)$,

(2) *If* $B'_j \subseteq B_j$ *for all* $j = 1, \ldots, n$, *then* $\overline{S}^n_m(A, B'_1, \ldots, B'_n) \subseteq \overline{S}^n_m(A, B_1, \ldots, B_n)$.

(3) *If* $a \in A, b_j \in B_j$ *for all* $j = 1, \ldots, n$, *then* $\{S^n_m(a, b_1, \ldots, b_n)\} \subseteq \overline{S}^n_m(A, B_1, \ldots, B_n)$.

Results

For any $B \in \mathcal{P}(W_\tau(X_n))$ the set $B^{\mathcal{A}}$ of term operations induced on the algebra $\mathcal{A} := (A, (f_i^A)_{i \in I})$ is defined as follows:

(1) If $B = \{x_j\}$ for $1 \leq j \leq n$, then $B^{\mathcal{A}} := \{e_j^{n,\mathcal{A}}\}$.

(2) If $B = \{f_i(t_1, \ldots, t_{n_i})\}$, then $B^{\mathcal{A}} := \{f_i^{\mathcal{A}}(t_1^{\mathcal{A}}, \ldots, t_{n_i}^{\mathcal{A}})\}$ where $f_i^{\mathcal{A}}$ is the fundamental operation of $\mathcal{A}$ corresponding to the operation symbol f_i and each $t_j^{\mathcal{A}}$ is a term opeation on $\mathcal{A}$ induced by a term t_j.

(3) If $|B| > 1$ then $B^{\mathcal{A}} := \bigcup_{b \in B} \{b\}^{\mathcal{A}}$.

(4) If $B = \emptyset$, then $B^{\mathcal{A}} := \emptyset$.

Let $\mathcal{P}(W_\tau(X_n))^{\mathcal{A}}$ be the collection of all sets of n-ary term operations induced by sets of n-ary terms of type τ on the algebra $\mathcal{A}$.

Results

For any $m, n \in \mathbb{N}$ and for any algebta $\mathcal{A}$ of type τ, multi-sorted partial operations defined on sets of term operations of a fixed variable can be defined in the following way:

$$\bar{\bullet}_m^{n,A} : \mathcal{P}(W_\tau^{fv}(X_n))^{\mathcal{A}} \times (\mathcal{P}(W_\tau^{fv}(X_m))^{\mathcal{A}}))^n \multimap\!\!\rightarrow \mathcal{P}(W_\tau^{fv}(X_m))^{\mathcal{A}}$$

with

$$\bar{\bullet}_m^{n,A}(B^{\mathcal{A}}, B_1^{\mathcal{A}}, \ldots, B_n^{\mathcal{A}}) := \begin{cases} \bullet_m^{n,A}(B^{\mathcal{A}}, B_1^{\mathcal{A}}, \ldots, B_n^{\mathcal{A}}), & \text{if } B_j = \emptyset \text{ for some } j \in \{1, \ldots, n\} \\ & \text{or } A = \emptyset \text{ or } |\text{var}(B_j)| = 1 \\ & \text{for every } j = 1, \ldots, n, \\ \text{not defined,} & \text{otherwise.} \end{cases}$$

Results

Lemma

For any $m, n \in \mathbb{N}$ *and for any algebta* $\mathcal{A}$ *of type* τ, $\overline{\bullet}_m^{n,A}$ *are partial operations.*

Lemma

Let $A \in \mathcal{P}(W_\tau^{fv}(X_n))$ *and* $B_1, \ldots, B_n \in \mathcal{P}(W_\tau^{fv}(X_m))$. *Then the equation*

$$[\overline{S}_m^n(B, B_1, \ldots, B_n)]^{\mathcal{A}} = \overline{\bullet}_m^{n,A}(B^{\mathcal{A}}, B_1^{\mathcal{A}}, \ldots, B_n^{\mathcal{A}})$$

is weak.

Theorem

$PC^{fv}(\tau)$ *is weakly homomorphic to* $PC^{fv}(\mathcal{A})$.

Results

For $1 \leq i \leq j \leq n$, the binary partial operation

$$\cdot_{ij \leq n} : (\mathcal{P}(W_\tau^{fv}(X_n)))^2 \multimap\!\!\rightarrow \mathcal{P}(W_\tau^{fv}(X_n))$$

is defined by
$A \cdot_{ij \leq n} B = \overline{S}_n^n(A, \{x_1\}, \ldots, \{x_{i-1}\}, B, \{x_{i+1}\}, \ldots, \{x_{j-1}\}, B, \{x_{j+1}\}, \ldots \{x_n\})$.

Theorem

$((\mathcal{P}(W_\tau^{fv}(X_n)))_{n\in\mathbb{N}}, (\cdot_{ij\leq n})_{n\in\mathbb{N}})$ *is a partial semigroup.*

Theorem

Let $i, j, k, l \in \{1, \ldots, n\}$. *Then*

$$(\mathcal{P}(W_\tau(X_n)), \cdot_{ij\leq n}) \cong (\mathcal{P}(W_\tau(X_n)), \cdot_{kl\leq n}).$$

Open problems

Questions

- Give characterizations of idempotent and regular elements in $((\mathcal{P}(W_\tau^{fv}(X_n)))_{n\in\mathbb{N}}, (\cdot_{ij\leq n})_{n\in\mathbb{N}})$.
- Explain the sets of tree languages of a fixed variable when substituting finite alphabets X_n with infinite ones, .i.e., $X = \bigcup_{n\in\mathbb{N}} X_n$.

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Thank you